Functions In N_+ With The Positive Real Parts On The Boundary, And Extremal Problems In H^1

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Functions In N_+ With The Positive Real Parts On The Boundary,
And
Extremal Problems In H^1

by

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Abstract. An essentially bounded function ϕ on the unit circle gives a continuous linear functional T_ϕ on the Hardy space H^1. $\rho(\phi)$ denotes a set of all complex numbers s such that there exists at least one function which attains the norm of $T_{\phi-s}$. In a previous paper, we showed that $\mathbf{C}\backslash\overline{\rho(\phi)}$ is empty or an open disc. Unfortunately we did not know when $\rho(\phi)$ is open or closed. In this paper, we study when $\rho(\phi)$ is open or closed. Moreover the functions in the Smirnov class N_+ whose real parts are nonnegative on the unit circle are described and studied. Then we give new characterizations of exposed points in the unit ball of H^1 and we determine when the sum of two inner functions is outer. As an result, we can describe all functions which have their Denjoy-Wolff points on the unit circle.

§1. Introduction

Let D be the open unit disc in the complex plane and let ∂D be the boundary of D. An analytic function in D is said to be of class N if the integrals $\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta$ are bounded for $r < 1$. If f is in N, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂D. If

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})| d\theta$$

then f is said to be of class N_+. The set of all boundary functions in N_+ is denoted by N_+. For $0 < p \leq \infty$, the Hardy space H^p, is defined by $N_+ \cap L^p$.

For ϕ in L^∞, we denote by T_ϕ the corresponding functional defined on H^1, that is,

$$T_\phi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta}) d\theta / 2\pi.$$

Let S_ϕ be the set of functions in H^1 which satisfy $T_\phi(f) = \|T_\phi\|$ and $\|f\|_1 \leq 1$. For each positive integer n, $\rho_n(\phi)$ denotes the set of all complex numbers s for which the dimension of $\langle S_{\phi-s}\rangle$, the linear span of $S_{\phi-s}$, is n. Put

$$\rho(\phi) = \bigcup_{n=1}^{\infty} \rho_n(\phi) \ ;$$

then $\rho(\phi)$ is the set of all complex numbers s for which $S_{\phi-s}$ is nonempty. In the previous paper [7], the author used the set $E(\phi)$ in order to describe $\rho(\phi)$, where

$$E(\phi) = \{f(0) : \|\phi - f\|_\infty = \|\phi + H^\infty\| \ \text{ and } \ f \in H^\infty\}.$$

In fact he showed that $\mathbf{C}\backslash E(\phi) \subseteq \rho(\phi)$. Hence if $E(\phi) = \{s\}$ for some complex number s then $\rho(\phi) = \mathbf{C}$ or $\rho(\phi) = \mathbf{C}\backslash\{s\}$. In this paper we show the converse. In the previous paper [7], the author did not consider $\rho_n(\phi)$. In this paper we show that $\rho(\phi) = \rho_1(\phi) \cup \rho_n(\phi)$ for some n with $2 \leq n \leq \infty$ when $\bigcup_{n=2}^{\infty} \rho_n(\phi) \neq \emptyset$. In the previous paper [7], the author showed that

$$\mathbf{C}\backslash E(\phi) \subseteq \rho(\phi) \subseteq \mathbf{C}\backslash E(\phi)^\circ$$

when $\rho(\phi) \neq \mathbf{C}$; here $E(\phi)^\circ$ denotes the interior of $E(\phi)$. In this paper we show that $\rho(\phi) = \mathbf{C}\backslash E(\phi)^\circ$ under a certain condition on ϕ. This depends on a well known theorem of Adamyan, Arov and Krein (cf. Theorem 5.3 in [3, Chapter IV]). Our argument has some connection with a theorem of Sarason in [10]. We give a proof of a weakened version of a Corollary in [10].

There exist a lot of functions in N_+ which are real valued on ∂D. Moreover $\bigcap_{p<1} H^p$ has nonconstant functions that are real valued on ∂D. On the other hand, the Hardy space H^1 does not contain such a function. Put $\mathrm{Arg}N_+ = \{f/|f| : f$ is a nonzero function in $N_+\}$; it is not difficult to see that $\mathrm{Arg}N_+$ is just the set of all unimodular functions in L^∞. In this paper, we are interested in

$$\mathrm{Arg}H^1 = \{f/|f| : f \text{ is a nonzero function in } H^1\}.$$

If k is a nonegative nonzero function in N_+, f is a nonzero function in H^1 and $g = kf$, then $g/|g|$ belongs to $\mathrm{Arg}H^1$.

We call q in N_+ an inner function if $|q| = 1$ *a.e.* on ∂D. A function g in N_+ is called outer if it is not divisible in N_+ by a nonconstant inner function. When g is in H^1, g is outer if and only if f/g belongs to H^∞ whenever $|f| = |g|$ *a.e.* on ∂D and f is a nonzero function in H^1. Then $f = qg$ for some inner function q. A function g in H^1 is called strongly outer if and only if f/g belongs to H^∞ whenever $\dfrac{f}{|f|} = \dfrac{g}{|g|}$ *a.e.* on ∂D and f is a nonzero function in H^1. Hence then $f = ag$ for some positive constant a. It is easy to see that if g is strongly outer then it is outer. If g has norm 1, g is outer if and only if g is an extreme point in the unit ball in H^1, and g is strongly outer if and only if g is an exposed point in the unit ball.

For a function F in H^1,

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}|F(e^{it})|dt \quad (z \in D)$$

is called the Herglotz integral of $|F|$. Then f is an outer function in N_+ and $\mathrm{Re}f = |F|$ *a.e.* on ∂D. Put

$$L^1_+ = \{\mathrm{Re}f : f \text{ is the Helglotz integral of } |F| \text{ and } F \in H^1\}$$

and

$$L_+ = \{\mathrm{Re}f : f \text{ is outer in } N_+ \text{ and } \mathrm{Re}f \geq 0 \ a.e. \text{ on } \partial D\}.$$

Then $L^1_+ \subsetneq L_+$ and $L^1_+ = \{u \in L^1 : u \geq 0 \ a.e. \text{ on } \partial D \text{ and } \log u \in L^1\}$.

In this paper, we show the following : For an outer function G in N_+, $G/|G| \in \mathrm{Arg}H^1$ if and only if $|G| \in L_+$. For a nonzero function F in H^1, F is strongly outer if and only if $|F/(1+q)^2| \notin L_+$ for any nonconstant inner function q. Sarason [10] conjectured that F is strongly outer if and only if $|F/(1+q)^2| \notin L^1_+$ for any nonconstant inner function q. However Inoue [5] gave a counterexample for this conjecture. Thus our result seems to be interesting. A function whose real part is positive on D is in $\cap_{p<1}H^p$ and is an outer function. Such functions are very important and well understood. For example, they are Herglotz integrals of positive measures. On the other hand, a function in N_+ whose real part is nonnegative on ∂D is not necessarily outer. Such functions also seems to be important. In this paper, we study them. We show that if f is a nonzero function in N_+ with $\mathrm{Re}f \geq 0$ on ∂D then $f = (q+k)/(q-k)$ where q is inner, k is a contractive function in H^∞ and $q-k$ is outer. Helson [4] proved this when $\mathrm{Re}f = 0$ on ∂D. Moreover we determine when $q-k$ is outer, using the parametrization theorem of Adamyan, Arov and Krein (cf. Theorem 5.3 [3, Chapter IV]). Sarason [11] considevel the cace where $q = z$

and k is inner using different methods. As a by-product, we describe all functions whose Denjoy-Wolff points are on ∂D.

Throughout this paper, for a function F in H^1, we write the Herglotz integral f of $|F|$ in the form :

$$f(z) = \frac{1 + Q_F(z)}{1 - Q_F(z)} \quad (z \in D)$$

where Q_F is a function in H^∞ with $\|Q_F\|_\infty \leq 1$.

§2. $\rho(\phi) = \mathbf{C}$

In the previous paper [7], the following were proved :

(1) If $\|\phi + H^\infty\| \underset{\neq}{>} \|\phi + H^\infty + \mathbf{C}\|$ then $\rho(\phi) = \mathbf{C}$ and hence if ϕ is a function in $H^\infty + C$ then $\rho(\phi) = \mathbf{C}$. Here C is the set of all continuous functions on ∂D.

(2) If S_ϕ contains at least two functions then $\rho(\phi) = \mathbf{C}$.

In this section, we study the set $\rho(\phi) = \bigcup_{n=1}^{\infty} \rho_n(\phi)$ in detail.

Theorem 1. *Let ϕ be a function in L^∞. Then, $\rho(\phi) = \mathbf{C}$ or $\mathbf{C}\backslash\{s\}$ if and only if $E(\phi) = \{s\}$, where s is in $\mathbf{C}$.*

Proof. The 'if' part is known in the previous paper [7, (3) of Theorem 3]. We will prove the 'only if' part. Suppose $\rho(\phi) = \mathbf{C}$. If $\phi \in H^\infty$ then $E(\phi) = \{\phi(0)\}$ clearly and so we may assume that $\|\phi + H^\infty\| = 1$. If $\|\phi - F\|_\infty = 1$ and $F \in H^\infty$ then $F = F(0) + zk$ and $k \in H^\infty$. Put $s = F(0)$, then $\|\phi - s + zH^\infty\| = 1$ and $S_{\phi-s}$ is not empty because $\rho(\phi) = \mathbf{C}$. Hence $\phi - F$ is an extremal kernel of $T_{\phi-s}$ [2, p132] and so

$$\phi - F = \phi - s - zk = \frac{|f|}{f}$$

for some nonzero function f in H^1 [2, p133]. Then

$$E(\phi) - s = E(\phi - s) = E\left(\frac{|f|}{f}\right).$$

Suppose $E(\phi)^\circ \neq \emptyset$ and so $E\left(\frac{|f|}{f}\right)^\circ \neq \emptyset$. Then we may assume that s belongs to $E(\phi)^\circ$ and so $0 \in E\left(\frac{|f|}{f}\right)^\circ$. Since $E\left(\frac{|f|}{f}\right)^\circ \neq \emptyset$, there exists a nonzero complex number $t \in E\left(\frac{|f|}{f}\right)^\circ$ such that $\left\|\frac{|f|}{f} - t - zh\right\|_\infty = 1$ for some $h \in H^\infty$. Since $t + zh \neq 0$, f^{-1}

belongs to H^1 by Lemma 5.4 in [3, Chapter IV]. We may assume that $\|f^{-1}\|_1 = 1$. Since $\left\|\dfrac{|f|}{f} + H^\infty\right\| = \|\phi + H^\infty\| = 1$, $|f|/f = |f^{-1}|/f^{-1}$ and f^{-1} is an exposed point of the unit ball in H^1, by Lemma 5 in [7]

$$E\left(\frac{|f|}{f}\right) = \{z \in \mathbf{C} : |z - f^{-1}(0)| \leq |f^{-1}(0)|\}.$$

This implies that 0 does not belong to $E\left(\dfrac{|f|}{f}\right)^\circ$. This contradiction shows that $E(\phi)^\circ = \emptyset$. By Proposition 6 in [7], if $E(\phi)$ is not a single point then $E(\phi)$ is a closed disc and so $E^\circ(\phi)$ is not empty. This implies that $E(\phi) = \{s\}$ for some $s \in \mathbf{C}$.

Suppose $\rho(\phi) = \mathbf{C}/\{s\}$. If $s \neq F(0) = s'$ in the proof above, then we can show $E(\phi) = \{s'\}$ as when $\rho(\phi) = \mathbf{C}$. By the 'if' part, $\rho(\phi) = \mathbf{C}\backslash\{s'\}$ because $\rho(\phi) \neq \mathbf{C}$. This contradicts that $\rho(\phi) = \mathbf{C}\backslash\{s\}$ because $s \neq s'$. This contradiction implies that $E(\phi) = \{s\}$.

Theorem 2. *Let ϕ be a function in L^∞. If $\bigcup_{n=2}^{\infty} \rho_n(\phi) = \emptyset$ then $\rho(\phi) = \rho_1(\phi)$. If $\bigcup_{n=2}^{\infty} \rho_n(\phi) \neq \emptyset$ then the following (1) $\sim$ (3) are valid.*

(1) $\rho(\phi) = \mathbf{C}$.

(2) $\bigcup_{n=2}^{\infty} \rho_n(\phi) = E(\phi) = \{s\}$ for some s in $\mathbf{C}$ and $\rho_1(\phi) = \mathbf{C}\backslash\{s\}$.

(3) $\rho(\phi) = \rho_1(\phi) \cup \rho_n(\phi)$ for some n with $2 \leq n \leq \infty$.

Proof. By definition $\rho(\phi) = \rho_1(\phi)$ if $\bigcup_{n=2}^{\infty} \rho_n(\phi) = \emptyset$. Suppose $\bigcup_{n=2}^{\infty} \rho_n(\phi) \neq \emptyset$. (1) was proved in the previous paper [7, Theorem 4]. (2) (1) and Theorem 1 imply that $E(\phi) = \{s\}$. If $t \neq s$ then $t \in \mathbf{C}\backslash E(\phi)$ and so by definition of $E(\phi)$ there exists $k \in H^\infty$ such that $\|\phi - t + zH^\infty\| = \|\phi - t + zk\|_\infty$ and $\|\phi - t + zk\|_\infty \gneq \|\phi + H^\infty\|$. If $t \in \bigcup_{n=2}^{\infty} \rho_n(\phi)$ then $S_{\phi-t}$ is not a single point and so by Theorem 9 in [1], $S_{\phi-t} \ni zh$ for some $h \in H^1$. Hence $\|T_{\phi-t}\| = \|T_{z(\phi-t)}\|$ and so $\|\phi - t + zH^\infty\| = \|\phi - t + H^\infty\|$. This contradicts that $\|\phi - t + zk\|_\infty \gneq \|\phi + H^\infty\|$. Thus $t \notin \bigcup_{n=2}^{\infty} \rho_n(\phi)$ and $t \in \rho_1(\phi)$. This implies that $\rho_1(\phi) \supseteq \mathbf{C}\backslash E(\phi) = \mathbf{C}\backslash\{s\}$ and $\bigcup_{n=2}^{\infty} \rho_n(\phi) = \{s\}$. Now (2) follows. (3) is clear by (2).

§3. $\rho(\phi) \neq \mathbf{C}$

In the previous paper [7], the following were proved :

(1) If $\rho(\phi) = \mathbf{C}$ then $\|\phi + H^\infty\| = \|\phi + H^\infty + C\|$.

(2) If $\rho(\phi) \neq \mathbf{C}$ then $\mathbf{C}\backslash E(\phi) \subseteq \rho(\phi) \subseteq \mathbf{C}\backslash E(\phi)^\circ$.

In this section, we study $E(\phi)$ in detail and using it we show that $\rho(\phi) = \mathbf{C}\backslash E(\phi)$ for some special ϕ with $\rho(\phi) \neq \mathbf{C}$. If $E(\phi)$ is a single point we need to study nothing. Suppose $E(\phi)$ is not a single point. Then there exists an exposed point f in the unit ball of H^1 such that

$$\frac{f}{|f|} \in \frac{\phi}{a} + H^\infty$$

where $a = \|\phi + H^\infty\|$ (see Theorem 5.3 in [3, Chapter IV]). Then $\left\| \frac{f}{|f|} + H^\infty \right\| = 1$, $E(\phi) = aE\left(\frac{f}{|f|}\right) + b$ and $\rho(\phi) = a\rho\left(\frac{f}{|f|}\right) + b$ for some complex number b. The following Proposition 3 is an easy result of known deep results.

Proposition 3. *Let ϕ be a unimodular function in L^∞. Then the following (1) $\sim$ (3) are equivalent.*

(1) $\phi = \dfrac{f}{|f|}$ for some nonzero function f in H^1.

(2) There exists a nonzero function g in H^∞ such that $\|\phi + g\|_\infty \leq 1$.

(3) $E(\phi) \neq \{0\}$.

Proof. (1) $\Rightarrow$ (2) is a result of Lemma 5.5 in [3, Chapter IV]. (2) $\Rightarrow$ (1) is a result of Lemma 5.4 in [3, Chapter IV]. (2) $\Rightarrow$ (3) is a result of Lemma 5.5 in [3, Chapter IV]. (3) $\Rightarrow$ (2) is clear.

Proposition 4. *Suppose $\phi = f/|f|$ for some exposed point f in the unit ball of H^1 and $\|\phi + H^\infty\| = 1$. Let $(1 + Q_f)/(1 - Q_f)$ be the Herglotz integral of $|f|$. Then the following (1) $\sim$ (3) are true.*

(1) $E(\phi) = \{z : |z - f(0)| \leq |f(0)|\}$.

(2) If $s \in E(\phi)$ then there exist an inner function q and an outer function g in H^∞ such that

$$\phi - g = q\,\frac{f}{|f|}\,\frac{1 - Q}{1 - \bar{Q}}\,\frac{1 - \bar{Q}\bar{q}}{1 - Qq}$$

and $g(0) = f(0)(1 - q(0)) = s$ where $Q = Q_f$. Moreover there exists an outer function F_s in H^1 with $\phi - g = F_s/|F_s|$.

(3) In (2), if $s \in \partial E(\phi)$ then q is just constant. The converse is also valid.

Proof. (1) This is known from [7]. We give a proof for completeness. Let $K_\phi = \{g \in H^\infty : \|\phi - g\|_\infty \leq 1\}$ and $K_\phi(0) = \{g(0) : g \in K_\phi\}$. Since $\|\phi + H^\infty\| = 1$, $K_\phi(0) = E(\phi)$. Since $\|f\|_1 = 1$, $Q_f(0) = 0$ and so by Lemma 4 in [7] $K_\phi(0) =$

$\{f(0)(1-w(0)) : w \in H^\infty \text{ and } \|w\|_\infty \leq 1\}$. This implies (1) (see Lemma 5 in [7]). (2) If $s \in E(\phi)$ then by definition there exist a nonzero function g in H^∞ such that $\|\phi - g\|_\infty \leq 1$ and $g(0) = s$. Then by Theorem 5.3 in [3, Chapter IV]

$$g = \frac{f(1-Q)(1-w)}{1-Qw}$$

where $w \in H^\infty$ with $\|w\|_\infty \leq 1$ and $Q = Q_f$. Then g is an outer function in H^∞. Since $Q_f(0) = 0$, $g(0) = f(0)(1-w(0))$. Since there exists an inner function q such that $q(0) = w(0)$, we may assume $w = q$. Hence

$$\begin{aligned}
\phi - g &= \frac{f}{|f|} - f\frac{(1-Q)(1-q)}{1-Qq} \\
&= \frac{f}{|f|}\left(1 - \frac{1-|Q|^2}{|1-Q|^2}\,\frac{(1-Q)(1-q)}{1-Qq}\right) \\
&= \frac{f}{|f|}\,\frac{1-Q}{1-\bar{Q}}\,\frac{q-\bar{Q}}{1-Qq}
\end{aligned}$$

because $|f| = (1-|Q|^2)/|1-Q|^2$. By Lemma 5.4 in [7, Chapter IV], there exists an outer function F_s in H^1 such that $\phi - g = F_s/|F_s|$. (3) If $s \in \partial E(\phi)$ then $|s - f(0)| = |f(0)|$ by (1) and hence $|f(0)q(0)| = |f(0)|$. Therefore q is constant.

Theorem 5. *Suppose $\phi = f/|f|$ for some exposed point f in the unit ball of H^1, $\|\phi + H^\infty\| = 1$ and $\rho(\phi) \neq \mathbf{C}$. Let $(1+Q_f)/(1-Q_f)$ be the Herglotz integral of $|f|$. If $(1-|Q_f|^2)^{-1}$ is in L^1, then $\rho(\phi) = \mathbf{C}\backslash E(\phi)^\circ$ and f^{-1} belongs to H^1.*

Proof. Since $\mathbf{C}\backslash E(\phi) \subseteq \rho(\phi) \subseteq \mathbf{C}\backslash E(\phi)^\circ$ by [7], it is sufficient to show that if $s \in \partial E(\phi)$ then s belongs to $\rho(\phi)$. If $s \in \partial E(\phi)$ then by Proposition 4 there exist an outer function g, a constant e^{it} and an outer function F_s in H^1 such that $\dfrac{f}{|f|} - g = \dfrac{F_s}{|F_s|}$ and $g(0) = f(0)(1-e^{it}) = s$. Then $\rho\left(\dfrac{f}{|f|}\right) - s = \rho\left(\dfrac{F_s}{|F_s|}\right)$. In order to show that $s \in \rho(\phi)$, it is sufficient to show that $0 \in \rho\left(\dfrac{F_s}{|F_s|}\right)$. We will write $Q = Q_f$. If $(1-|Q|^2)^{-1} \in L^1$ then there exists an outer function P such that $|P|^2 + |Q|^2 = 1$. Put

$$h = \frac{P}{1-Q} \quad \text{and} \quad h_t = \frac{P}{1-e^{it}Q}\,;$$

then $|h|^2 = (1-|Q|^2)/|1-Q|^2 = |f|$ and $|h_t|^2 = (1-|Q|^2)/|1-e^{it}Q|^2$. Then h_t is a function in H^2 [10]. Hence we may assume that $f = h^2$ because h and f are outer (Change P to $e^{it}P$ if necessary). Therefore by Proposition 4

$$
\begin{aligned}
\frac{h_t^2}{|h_t|^2} &= \frac{P}{\bar{P}}\frac{1-e^{-it}\bar{Q}}{1-e^{it}Q} \\
&= \frac{f}{|f|}\frac{1-Q}{1-\bar{Q}}\frac{1-e^{-it}\bar{Q}}{1-e^{it}Q} = e^{-it}\frac{F_s}{|F_s|}
\end{aligned}
$$

because

$$
\frac{f}{|f|} = \frac{h^2}{|h|^2} = \frac{P}{\bar{P}}\frac{1-\bar{Q}}{1-Q}.
$$

If $(1-|Q|^2)^{-1} \in L^1$ then $P^{-1} \in H^2$ and so Q/P belongs to H^2 because Q is in H^∞. Hence h_t^{-1} belongs to H^2 because $h_t^{-1} = P^{-1} - e^{it}Q/P$. Now this implies that F_s^{-1} belongs to H^1 because $h_t^2/|h_t|^2 = e^{-it}F_s/|F_s|$. This is equivalent to that $0 \in \rho\left(\frac{F_s}{|F_s|}\right)$.

§4. Argument

If f is a nonzero function in H^1 and $f = qh$ where q is inner and h is outer, then $g = (1+q)^2 h$ is outer and $f/|f| = g/|g|$ *a.e.* on ∂D. Hence $\mathrm{Arg}H^1 = \{f/|f|\ ;\ f \text{ is outer in } H^1\}$. It is interesting to note that $\mathrm{Arg}N_+$ is a very big set. In fact, $\mathrm{Arg}N_+$ is the set of all unimodular functions in L^∞. For, by a theorem of Douglas and Rudin (cf. Theorem 2.1 in [3, Chapter V]), if ϕ is a unimodular function in L^∞ there exist inner functions q_1 and q_2 such that $\|\phi - \overline{q_1}q_2\|_\infty < 1$. and so $\|q_1\phi - q_2\|_\infty < 1$. Hence by Lemma 5.4 in [3, Chapter IV] there exists an outer function f such that $q_1\phi = f/|f|$ on ∂D. Therefore

$$
\phi = \overline{q_1}\frac{f}{|f|} = \frac{F}{|F|}
$$

where $F = f/(1+q_1)^2$. This implies that ϕ belongs to $\mathrm{Arg}N_+$.

Lemma 1. *Let G and F be outer functions in N_+ and $G/|G| = F/|F|$ on ∂D. If there exists an outer function f in N_+ with $|F| = \mathrm{Re}f$ on ∂D, then there exists an outer function g in N_+ with $|G| = \mathrm{Re}g$ on ∂D. Moreover $G/g = F/f$ and $g/f \geq 0$ on ∂D.*

Proof. If $|F| = \mathrm{Re}f$ on ∂D and f is outer, then put $k = 2F/f$ on ∂D. Then k belongs to H^∞ and

$$
\begin{aligned}
&\left|\frac{F}{|F|} - k\right|^2 = \left|\frac{F}{|F|} - \frac{2F}{f}\right|^2 \\
&= 1 - 4\mathrm{Re}\frac{\bar{F}}{|F|}\frac{F}{f} + 4\frac{|F|^2}{|f|^2} = 1 - 4\left(\mathrm{Re}\frac{|F|}{f} - \frac{|F|^2}{|f|^2}\right) \\
&= 1 - 4\frac{|F|}{|f|^2}(\mathrm{Re}\bar{f} - |F|) = 1.
\end{aligned}
$$

If $G/|G| = F/|F|$ on ∂D, then $\left|\frac{G}{|G|} - k\right| = 1$ where $k = 2F/f$. Hence $1 - 2\mathrm{Re}\frac{\bar{G}}{|G|}k + |k|^2 = 1$ and so $|k|^2 = 2\mathrm{Re}\frac{\bar{G}}{|G|}k$. Therefore

$$|G| = 2\mathrm{Re}\frac{\bar{G}k}{|k|^2} = 2\mathrm{Re}\frac{\bar{G}}{\bar{k}} = \mathrm{Re}\frac{2G}{k}.$$

Put $g = 2G/k$ then g is outer because $k = 2F/f$ is outer. Moreover $G/g = F/f$ and $g/f \geq 0$ on ∂D.

Lemma 2. *If G is an outer function in N_+ and there exists an outer function g in N_+ with $|G| = \mathrm{Re}g$ on ∂D, then there exists an outer function F in H^1 such that $G/|G| = F/|F|$ on ∂D.*

Proof. By the proof of Lemma 1, $\left|\frac{G}{|G|} - \frac{2G}{g}\right| = 1$ on ∂D and $2G/g$ is a nonzero function in H^∞. Hence by the proof of Lemma 5.4 in [3, Chapter IV] there exists an outer function F in H^1 such that $G/|G| = F/|F|$ on ∂D.

Theorem 6. *Let G be an outer function in N_+. There exists an outer function F in H^1 such that*

$$\frac{G}{|G|} = \frac{F}{|F|} \quad \textit{a.e. on } \partial D$$

if and only if there exists an outer function g in N_+ with $|G| = \mathrm{Re}g$ a.e. on ∂D. Moreover then $G/g = F/f$ a.e. on ∂D where f is an outer function with $|F| = Ref$ a.e. on ∂D.

Proof. The 'if' part follows from Lemma 2. For the 'only if' part, let f be the Herglotz integral of $|F|$; then $|F| = \mathrm{Re}f$ on ∂D and f is outer in N_+. By Lemma 1, there exists an outer function g in N_+ with $|G| = \mathrm{Re}g$ on ∂D.

§5. Strongly outer functions

Several characterizations of strongly outer functions are known. When g is an outer function in H^1, the author [8] showed the following are equivalent :

(1) g is a strongly outer function.

(2) If f is an outer function in N_+ and $|g| \leq \mathrm{Re}f$ *a.e.* on ∂D, then $|g| \leq \mathrm{Re}f$ on D.

(3) If f is an outer function in N_+ and $|g| \leq \mathrm{Re}f$ *a.e.* on ∂D, then $|g(0)| \leq (\mathrm{Re}f)(0)$.

In this section, we give some new characterizations of strongly outer functions which are related to the ones above.

Lemma 3. *Suppose f and g are functions in N_+ whose real parts are positive on D. If f/g is nonnegative a.e. on ∂D, then g is a scalar multiple of f.*

Proof. Since $\mathrm{Re}f$ and $\mathrm{Re}g$ are positive on D, $\mathrm{Re}(f+g)$ is also positive on D. Hence $f+g$ is outer and $f/(f+g)$ is in $N_+ \cap L^\infty$. Then $f/(f+g)$ is a positive constant because $N_+ \cap L^\infty = H^\infty$ and $f/(f+g)$ is nonnegative on ∂D.

Theorem 7. *Let F be an outer function in H^1. Then the following (1) $\sim$ (3) are equivalent.*

(1) F is a strongly outer function.

(2) If $|F| = \mathrm{Re}f$ a.e. on ∂D and f is an outer function in N_+, then $\mathrm{Re}f$ is positive on D.

(3) For each nonconstant inner function q, there does not exist an outer function g in N_+ with

$$\left|\frac{F}{(1+q)^2}\right| = \mathrm{Re}g \quad \textit{a.e.} \text{ on } \partial D.$$

Proof. (1) $\Rightarrow$ (2). Let f_0 be the Helglotz integral of $|F|$. If $|F| = \mathrm{Re}f$ on ∂D, then by the remark preceding this theorem (see Theorem 6 in [8]), $|F| \le \mathrm{Re}f$ on D because F is strongly outer. Hence $\mathrm{Re}f$ is positive on D. (2) $\Rightarrow$ (1). If F is not strongly outer then there exists an outer function G in H^1 such that $F/|F| = G/|G|$ and G is not a positive scalar multiple of F. Let g_0 be the Helglotz integral of $|G|$. By Lemma 1 there exists an outer function f such that $\mathrm{Re}f = |F|$ and $F/f = G/g_0$. By the hypothesis on F, $\mathrm{Re}f$ is positive on D. Now Lemma 3 implies that f is a positive scalar multiple of g_0 because f/g_0 is nonnegative *a.e.* on ∂D. Hence G is a positive scalar multiple of F because $|F| = a|G|$ for some positive constant a, and both F and G are outer. (1) $\Rightarrow$ (3). If $\left|\frac{F}{(1+q)^2}\right| = \mathrm{Re}g$ *a.e.* on ∂D for some outer function g, then by Theorem 6 there exists an outer function h in H^1 such that

$$\bar{q}\frac{F}{|F|} = \frac{F}{(1+q)^2}\frac{|1+q|^2}{|F|} = \frac{h}{|h|} \quad \text{on } \partial D$$

and so $F/|F| = qh/|qh|$ on ∂D. This means that F is not strongly outer. (3) $\Rightarrow$ (1). If F is not strongly outer then there exist a nonconstant inner function q and an outer function G in H^1 such that $\bar{q}\frac{F}{|F|} = \frac{G}{|G|}$ on ∂D. Put $B = F/(1+q)^2$; then B is outer and $B/|B| = G/|G|$. By Lemma 1 there exists an outer function g with $|B| = \mathrm{Re}g$ on ∂D. This contradicts (3).

Proposition 8. *Let F be an outer function in H^1. Then the following (1) $\sim$ (3) are equivalent.*

(1) F is a strongly outer function.

(2) For each nonconstant inner function q, there does not exist an outer function g in N_+ with $\left|\dfrac{F}{(1+q)^2}\right| \leq \mathrm{Re} g$ a.e. on ∂D.

(3) If g is a function in H^∞ and $\left|\dfrac{F}{|F|} - g\right| \leq 1$ a.e. on ∂D, then g is an outer function.

Proof. (1) $\Rightarrow$ (3). Suppose F is strongly outer. If there exists a nonzero function g in H^∞ such that $\left|\dfrac{F}{|F|} - g\right| \leq 1$ *a.e.* on ∂D, then there exists a measurable function α on ∂D such that

$$|\alpha| \leq \pi/2 \;\; a.e. \text{ and } \alpha = \arg \frac{|F|}{F} g \quad (mod\ 2\pi).$$

Let $\phi = e^{\tilde{\alpha} - i\alpha}$; then ϕg belongs to H^1 and $\phi g / F \geq 0$ *a.e.* on ∂D (see the proof of Lemma 5.4.in [3, Chapter IV]). Hence $F = a\phi g$ for some positive constant a because F is strongly outer. This implies that g is outer. (3) $\Rightarrow$ (1). If F is not strongly outer, then there exists a nonconstant inner function q and an outer function F_0 in H^1 such that $\dfrac{F}{|F|} = q\dfrac{F_0}{|F_0|}$ *a.e.* on ∂D. Let f_0 be a Helglotz integral of $|F_0|$; then $|F_0| = \mathrm{Re} f_0$ and f_0 is outer. Let $g_0 = 2F_0/f_0$; then by the proof of Lemma 1 $\left|\dfrac{F_0}{|F_0|} - g_0\right| = 1$ and hence $\left|\dfrac{F}{|F|} - qg_0\right| = 1$. This contradicts (3). (1) $\Rightarrow$ (2). Let F be strongly outer. Suppose there exists an outer function f such that $|F/(1+q)^2| \leq \mathrm{Re} f$ *a.e.* on ∂D for some nonconstant inner function q. Let $G = F/(1+q)^2$; then by the proof of Lemma 1

$$\left|\frac{G}{|G|} - \frac{2G}{f}\right|^2 = 1 - 4|G||f|^2\left(\mathrm{Re} f - |G|\right) \leq 1.$$

Hence if $g = 2G/f$ then g is outer and

$$\left|\frac{F}{|F|} - qg\right| = \left|\frac{G}{|G|} - g\right| \leq 1 \;\; a.e. \text{ on } \partial D.$$

This contradicts that F is strongly outer because (1) $\Leftrightarrow$ (3). (2) $\Rightarrow$ (1) is a result of Theorem 7.

§6. Connection with a theorem of Sarason

Let f be an exposed point of the unit ball of H^1. Put $Q = Q_f$ and let P be an outer function such that $|Q|^2 + |P|^2 = 1$. For a unimodular constant λ, put

$$h_\lambda = \frac{P}{1-\lambda Q}\ ;$$

then $h_1^2 = f$. Sarason [10] showed that h_λ^2 is an exposed point of the unit ball of H^1, using operator theory. In this section, we give a proof of a part of the resut, using results in the previous sections.

For a function f in H^1 with $\|f\|_1 = 1$, put

$$\phi_\lambda = \frac{f}{|f|} - \frac{f(1-Q)(1-\lambda)}{1-\lambda Q}$$

for each unimodular constant λ where $Q = Q_f$. Then

$$\phi_\lambda = \frac{f}{|f|}\cdot\frac{1-Q}{1-\bar{Q}}\cdot\frac{1-\bar{\lambda}\bar{Q}}{1-\lambda Q}\cdot\lambda$$

and so $|\phi_\lambda| = 1$. By Lemma 5.4 in [3, Chapter IV], $\phi_\lambda = f_\lambda/|f_\lambda|$ for some function f_λ in H^1 with $\|f_\lambda\|_1 = 1$.

Proposition 9. *Let λ be a unimodular constant.*

(1) If f is an exposed point of the unit ball of H^1 then f_λ is also an exposed one.

(2) If f is a strongly exposed point of the unit ball of H^1 then f_λ is also a strongly exposed one.

(3) If $(1-|Q|^2)^{-1}$ is in L^1 then f_λ^{-1} belongs to H^1.

Proof. (1) By Lemma 5 in [7], $K_\phi(0) = \{z : |z - f(0)| \le |f(0)|\}$ where $\phi = f/|f|$. Since

$$K_{\phi_\lambda}(0) = K_\phi(0) - f(0)(1-\lambda)$$

$f(0)(1-\lambda) \in \partial K_\phi(0)$ because $|f(0)(1-\lambda) - f(0)| = |f(0)|$. Thus the interior of $K_{\phi_\lambda}(0)$ does not contain 0 because 0 is not in the interior of $K_\phi(0)$. By Proposition 14 in [7], f_λ is an exposed point of the unit ball of H^1. (2) By [12], f_λ is a strongly exposed point of the unit ball of H^1 if and only if $\|\phi_\lambda + H^\infty + C\| < 1$. Since $\|\phi + H^\infty\| = \|\phi_\lambda + H^\infty\|$, $\|\phi + H^\infty + C\| = \|\phi_\lambda + H^\infty + C\|$. This implies (2). (3) Since $f_\lambda = h_\lambda^1/\|h_\lambda^2\|_1$ and $h_\lambda^{-1} = P^{-1} - \lambda\frac{Q}{P}$, it is clear that $f_\lambda^{-1} \in H^1$.

Now Theorem 7 and Proposition 9 give a weakened version of a Corollary in [10]. That is, $h_\lambda^2/\|h_\lambda\|_1^2$ is an exposed point of the unit ball of H^1 if h_1^2 is an exposed point. For

$$\frac{h_\lambda^2}{|h_\lambda|^2} = \frac{h^2}{|h|^2}\,\frac{1-Q}{1-\bar{Q}}\,\frac{1-\bar{\lambda}\bar{Q}}{1-\lambda Q}$$

and so

$$\phi_\lambda = \frac{f_\lambda}{|f_\lambda|} = \lambda \cdot \frac{h_\lambda^2}{|h_\lambda|^2}.$$

By [6], $\|h_\lambda^2\|_1 \le 1$. By Proposition 9, $h_\lambda^2/\|h_\lambda^2\|_1$ is an exposed point of the unit ball of H^1.

Here we raise a natural question related to (3) of Proposition 9 : Does f_λ^{-1} belong to H^1 for arbitrary λ in ∂D if f^{-1} is in H^1 ? If the answer is yes, we can prove an interesting theorem (related to Theorem 5 in this paper). That is, $0 \in \rho(\phi)$ if and only if $\rho(\phi) = \mathbf{C}\backslash E(\phi)^\circ$ when $\phi = f/|f|$ for some exposed point f in the unit ball of H^1, $\|\phi + H^\infty\| = 1$ and $\rho(\phi) \neq \mathbf{C}$. For the proof, note that $0 \in \rho(\phi)$ if and only if f^{-1} belongs to H^1 when $\|\phi + H^\infty\| = 1$.

§7. Function whose real part is nonnegative on ∂D

In Sections 4 and 5, functions in L_+ were important. It seems that L_+ is a very big set compared to L_+^1. We understand well functions f with Ref in L_+^1. In this section, we study functions f with Ref in L_+. Three sets, $\tilde{H}^1$, $H_+(D)$ and $H_+(\partial D)$ are defined in the following :

$$\tilde{H}^1 = \{F \in N_+ \ ; \ \frac{F}{|F|} \in ArgH^1\},$$

$$H_+(D) = \{f \in N_+ \ ; \ \mathrm{Re} f > 0 \ \text{ on } \ D\},$$

and

$$H_+(\partial D) = \{f \in N_+ \ ; \ \mathrm{Re} f \ge 0 \ \text{ on } \ \partial D\}.$$

Then Re$H_+(\partial D) = L_+$, Re$H_+(D) \supset L_+^1$ and $\tilde{H}^1 \supset H^1$.

If F is a nonzero function in H^1, then $|F| = \mathrm{Re} f$ on ∂D for some function f in $L_+(D)$. If G is a nonzero function in $\tilde{H}^1$, then $|G| = \mathrm{Re} g$ for some function g in $H_+(\partial D)$. For, by definition on G there exists an outer function F in H^1 such that $G/F \ge 0$ on ∂D. Put $s = G/F$ then s is in N_+ and so

$$|G| = s|F| = s\mathrm{Re} f = \mathrm{Re}(sf)$$

where $g = sf$ is in $H_+(\partial D)$. If G is outer then g is also outer because s is outer. If F is a nonzero function in N_+ and $|F| = \mathrm{Re} f$ for some f in $H_+(D)$, then $F \in \bigcap_{p<1} H^p$. If G is a nonzero function in N_+ and $|G| = \mathrm{Re} g$ for some outer g in $L_+(\partial D)$ then G belongs to $\tilde{H}^1$ by Theorem 6.

A function f is in $H_+(D)$ if and only if $f = (1+k)/(1-k)$ for some contractive function k in H^∞ with $k \not\equiv 1$. Then f is outer and $\mathrm{Re} f = (1-|k|^2)/|1-k|^2$ on ∂D. The following Proposition 10 is a similar result for functions in $H_+(\partial D)$ instead of $H_+(D)$.

Proposition 10. *f is a function in N_+ whose real part is nonnegative a.e. on ∂D if and only if there exist an inner function q and a contractive function k in H^∞ such that*

$$f = \frac{q+k}{q-k} \quad \textit{and} \quad q-k \ \textit{ is outer.}$$

Then $\mathrm{Re} f = (1-|k|^2)/|q-k|^2$ *a.e. on* ∂D.

Proof. If $f \in N_+$, $\mathrm{Re} f \geq 0$ *a.e.* on ∂D and $\phi = (f-1)/(f+1)$, then $|\phi| \leq 1$ *a.e.* on ∂D. Write $f+1 = q_1 h_1$ and $f-1 = q_2 h_2$ where q_i is inner and h_i is outer $(i = 1,2)$. Then

$$f = \frac{1+\phi}{1-\phi} = \frac{q+k}{q-k}$$

where $q = q_1$ and $k = q_1 \phi = q_2 h_2 / h_1$. Since $2 = q_1 h_1 - q_2 h_2 = (q-k)h_1$, $q-k$ is outer. It is easy to see that the converse is true because $(q+k)/(q-k) = (1+\bar{q}k)/(1-\bar{q}k)$ and $q-k$ is outer.

If a function f has the form : $f = sg$ for some g in $H_+(D)$ and some nonnegative function s in N_+, then f belongs to $H_+(\partial D)$. The following Theorem 11 gives a partial converse.

Theorem 11. *Suppose q is an inner function and β is a contractive function in H^∞ such that* $\log(1-|\beta|^2)$ *is in L^1. If $f = (q+\beta)/(q-\beta)$ and $q-\beta$ is an outer function, then there exist a nonnegative function s in N_+ and a contractive function α in H^∞ with* $\log(1-|\alpha|^2) \in L^1$ *such that*

$$\frac{q+\beta}{q-\beta} = \frac{1+\alpha}{1-\alpha} \cdot s + t$$

where t is a function in N_+ with $\mathrm{Re} t = 0$ *a.e. on ∂D.*

Proof. Since $\log(1-|\beta|^2) \in L^1$, there exists an outer function a in H^∞ such that $|\beta|^2 + |a|^2 = 1$. Put $F = a^2/(q-\beta)^2$; then

$$|F| = \frac{1-|\beta|^2}{|q-\beta|^2} = \mathrm{Re} f \quad \text{and} \quad f \in N_+.$$

By Lemma 2, there exists an outer function G in H^1 such that $G/|G| = F/|F|$ on ∂D. Let g be the Helglotz integral of $|G|$; then $\mathrm{Re} g = |G|$. By Lemma 1, there exists an outer function f_0 in N_+ such that $\mathrm{Re} f_0 = \mathrm{Re} f$ on ∂D and $f_0/g \geq 0$ on ∂D. Put $s = f_0/g$; then

$$f = sg + (f - f_0).$$

Put $t = f - f_0$; then Proposition 10 implies this theorem.

In Theorem 11, if $\dfrac{q+\beta}{q-\beta} = \dfrac{1+\alpha}{1-\alpha}s$ then

$$\frac{\beta}{q} = \frac{(s+1)\alpha + (s-1)}{(s-1)\alpha + (s+1)}$$

where q is the inner part of the function $(s-1)\alpha + (s+1)$. Proposition 10 describes the functions s in N_+ where $s \geq 0$ *a.e.* on ∂D or Re$s = 0$ *a.e.* on ∂D. In fact, if Re$s = 0$ *a.e.* on ∂D, then by Proposition 10 $s = \dfrac{q+k}{q-k}$ and k is inner. This is a result of Helson [4]. In fact, he described the real valued functions in N_+. If $s \geq 0$ *a.e.* on ∂D, then Re$(is) = 0$ *a.e.* on ∂D. Hence $is = \dfrac{q+k}{q-k}$ and k is inner. Therefore $s = \dfrac{q+k}{i(q-k)}$ and Im$\bar{q}k \leq 0$ (see [8]).

It is known [10] that F is an outer function in H^1 with norm one if and only if $F = \alpha^2/(1-\beta)^2$ where α and β are in H^∞ with $|\alpha|^2 + |\beta|^2 = 1$ on ∂D, α is outer and $\beta(0) = 0$. The following Proposition 12 is a generalization to $\tilde{H}^1$.

Proposition 12. *F is an outer function in $\tilde{H}^1$ if and only if*

$$F = \frac{\alpha^2}{(q-\beta)^2} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1 \quad a.e. \text{ on } \partial D$$

wheere α and $q \pm \beta$ are outer functions and q is an inner function.

Proof. If $F \in \tilde{H}^1$, then by the remark above Proposition 10 there exists an outer function f in $H_+(\partial D)$ with $|F| = $ Ref. By Proposition 10, $f = (q+\beta)/(q-\beta)$ and $q \pm \beta$ are outer. Since $|F| = $ Re$f = (1-|\beta|^2)/|q-\beta|^2$, there exists an outer function α such that $|\alpha|^2 + |\beta|^2 = 1$ on ∂D. Hence $F = \alpha^2/(q-\beta)^2$. Conversely if $F = \alpha^2/(q-\beta)^2$ then $|F| = $ Re$f = (1-|\beta|^2)/|q-\beta|^2$ where $f = (q+\beta)/(q-\beta)$. Theorem 6 and Proposition 10 imply that F belongs to $\tilde{H}^1$.

§8. When $q - k$ is outer ?

In the description of $H_+(D)$, the function $1-k$ is important, where k is a contractive function in H^∞, and it is always outer. In the description of $H_+(\partial D)$ (see Proposition 10 and Theorem 11), it is important to know whether $q-k$ is an outer function. Such functions have been studied by Sarason [11] (see [9]) when k is an inner function. In this section, given an inner function q we determine k such that $q-k$ is an outer function. When $q=z$ and k is an inner function, using the Denjoy-Wolff point of k, Sarason [11] determined k such that $z-k$ is an outer function.

For an inner function q, put

$$\mathcal{E}(q)=\{k\ ;\ q-k \text{ is outer and } \|k\|_\infty \leq 1\}.$$

If $q\equiv 1$, then $\mathcal{E}(q)$ is just the unit ball of H^∞. In general, $\mathcal{E}(q)$ is not empty. It is easy to see that the following functions :

$$\frac{1-q}{2},\ -\frac{1-q}{2},\ \left(\frac{1+q}{2}\right)^2$$

belong to $\mathcal{E}(q)$. By Lemma 1 in [11], if $\|k\|_\infty<1$ then k does not belong to $\mathcal{E}(q)$ for any nonconstant q. In this section, given an inner function q we determine k such that $q-k$ is an outer function.

Theorem 13. *Let q be an inner function. The function k belongs to $\mathcal{E}(q)$ if and only if*

$$k=q-\frac{F(1-Q_F)(1-w)}{1-Q_Fw}=q\frac{1-Q_F}{1-\bar{Q}_F}\frac{w-\bar{Q}_F}{1-Q_Fw}$$

where F is an outer function in H^1 with $q\bar{F}\geq 0$ a.e. on ∂D and norm one, w is a contractive function in H^∞ and $\dfrac{1+Q_F}{1-Q_F}$ is the Herglotz integral of $|F|$. k is an inner function if and only if w is inner. If γ is a positive constant and w is a contractive function in H^∞, then there exists a contractive function w_γ in H^∞ such that

$$\frac{F(1-Q_F)(1-w_\gamma)}{1-Q_Fw_\gamma}=\frac{\gamma F(1-Q_{\gamma F})(1-w)}{1-Q_{\gamma F}w}.$$

Proof. This is essentially a theorem due to Adamyan, Arov and Krein (cf. Theorem 5.3 in [3, Chapter IV]). If $k\in\mathcal{E}(q)$ then there exists an outer function h such that $k=q-h$ and so $\|q-h\|_\infty\leq 1$. By Lemma 6 in [7], there exists an outer function F in H^1 with norm one such that $q\bar{F}\geq 0$ *a.e.* on ∂D and $h=F(1-Q_F)(1-w)/(1-Q_Fw)$ with Q_F as above. This implies the 'only if' part. For the converse, by simple caluculations (see the proof of Proposition 4),

$$k=q-\frac{F(1-Q_F)(1-w)}{1-Q_Fw}=q\frac{1-Q_F}{1-\bar{Q}_F}\ \frac{w-\bar{Q}_F}{1-Q_Fw}$$

because $\bar{q}F=|F|=(1-|Q_F|^2)/|1-Q_F|^2$. This implies that k belongs to $\mathcal{E}(q)$. The second statement in the theorem is clear by the proof of Lemma 6 in [7].

Corollary 1. *The function k belongs to $\mathcal{E}(z)$ if and only if*

$$k=\frac{(3z+2a)w-(z+2a)}{(3+2\bar{a}z)-(1+2\bar{a}z)w}$$

where a is constant with $|a| = 1$ and w is a contractive function in H^∞. k is inner if and only if w is inner.

Proof. In Theorem 13, suppose $q = z$. If $\bar{z}F \geq 0$ or ∂D and $F \in H^1$ is outer then $F = \gamma(z + a)(1 + \bar{a}z)$ where a is constant with $|a| = 1$. By the second statement of Theorem 13, we can assume $\gamma = 1$. The Herglotz integral of $|z + a|^2$ is $2(1 + \bar{a}z)$ and so $Q_F = (1 + 2\bar{a}z)/(3 + 2\bar{a}z)$. By Theorem 13, $k \in \mathcal{E}(z)$ if and only if

$$k = z\frac{1 - Q_F}{1 - \bar{Q}_F}\frac{w - \bar{Q}_F}{1 - Q_F w}$$

where $Q_F = (1 + 2\bar{a}z)/(3 + 2\bar{a}z)$ for some a with $|a| = 1$ and w is a contractive function in H^∞. Then

$$\begin{aligned} & z\frac{1 - Q_F}{1 - \bar{Q}_F}\frac{w - \bar{Q}_F}{1 - Q_F w} \\ = \; & z\frac{1 - \frac{1+2\bar{a}z}{3+2\bar{a}z}}{1 - \frac{z+2a}{3z+2a}}\frac{w - \frac{z+2a}{3z+2a}}{1 - \frac{1+2\bar{a}z}{3+2\bar{a}z}w} \\ = \; & \frac{(3z + 2a)w - (z + 2a)}{(3 + 2\bar{a}z) - (1 + 2\bar{a}z)w}. \end{aligned}$$

This implies the corollary.

Corollary 2. *Suppose q is an inner function. Put*

$$k = \frac{(3q + 2a)w - (q + 2a)}{(3 + 2\bar{a}q) - (1 + 2\bar{a}q)w}$$

where a is a unimodular constant and w is a contractive function ; then k is a contractive function in H^∞ and $q - k$ is an outer function. If w is an inner function then so is k.

Proof. Since $3 + 2\bar{a}q$ is invertible in H^∞, and $(1 + 2\bar{a}q)/(3 + 2\bar{a}q)$ is contractive, $(3 + 2\bar{a}q) - (1 + 2\bar{a}q)w$ is outer and so k belongs to H^∞. Put $Q = (1 + 2\bar{a}q)/(3 + 2\bar{a}q)$; then

$$k = q\frac{1 - Q}{1 - \bar{Q}}\frac{w - \bar{Q}}{1 - Qw}$$

and so k is contractive. If w is inner, then k is also inner. By a simple caluculation,

$$q - k = \frac{2\bar{a}(q + a)^2(1 - w)}{(3 + 2\bar{a}q) - (1 + 2\bar{a}q)w},$$

$q - k$ is outer.

When k is inner and λ is its Denjoy-Wolff point, in Proposition 2 in [11], Sarason showed that $k \in \mathcal{E}(z)$ if and only if $\lambda \in \partial D$. However the proof shows that his result is true for arbitrary contractive k in H^∞. Hence Corollary 1 describes completely those contractive functions in H^∞ whose Denjoy-Wolff points are in ∂D.

§9. Connection with a theorem of Helson

In this section, we are interested in the set $H_+(\partial D) = \{f \in N_+ \; ; \; \mathrm{Re}f \geq 0 \; a.e. \text{ on } \partial D\}$. When f is a function in $H_+(\partial D)$ and $\mathrm{Re}f = 0$ *a.e.* on ∂D, using operator theory, Helson [4] showed the following : If the inner part of $\lambda + f$ is a finite Blaschke product of degree n for some real constant λ, then the inner part of $t + f$ is a finite Blaschke product of degree n for any real constant t. We prove a theorem of this type for arbitrary functions in $H_+(\partial D)$, using a theorem of Adamyan, Arov and Krein (cf. Theorem 5.3 in [3, Chapter IV]).

Theorem 14. *Suppose f is a nonzero function in N_+ and* $\mathrm{Re}f$ *is nonnegative a.e. on ∂D. Let n be a nonnegative integer. If the inner part of $\lambda + f$ is a Blaschke product of degree n for some positive constant λ, then the inner part of $t+f$ is a Blaschke product of degree n for any positive constant t.*

Proof. When $n = 0$, by hypothesis $\lambda + f$ is outer. Hence $k = (\lambda - f)/(\lambda + f)$ is a contractive function in H^∞ because $\mathrm{Re}f \geq 0$ *a.e.* on ∂D. Since $f = \lambda(1-k)/(1+k)$, $\mathrm{Re}f > 0$ on D. Thus $t + f$ is outer for any nonnegative constant t. Suppose $n \geq 1$ and b_t is the inner part of $t+f$ where t is a positive constant. By hypothesis b_λ is a Blaschke product of degree n for some $\lambda > 0$. For any $t > 0$, $k_t = b_t(f-t)/(f+t)$ is a contractive function in H^∞ and $f = t(b_t + k_t)/(b_t - k_t)$ where $b_t - k_t$ is outer, by Proposition 10. Hence for any $t > 0$,

$$f = \lambda\frac{b_\lambda + k_\lambda}{b_\lambda - k_\lambda} = t\frac{b_t + k_t}{b_t - k_t}.$$

Therefore

$$\begin{aligned} \frac{k_t}{b_t} &= \frac{f-t}{f+t} = \frac{\lambda(b_\lambda + k_\lambda) - t(b_\lambda - k_\lambda)}{\lambda(b_\lambda + k_\lambda) + t(b_\lambda - k_\lambda)} \\ &= \frac{(\lambda - t)b_\lambda + (\lambda + t)k_\lambda}{(\lambda + t)b_\lambda + (\lambda - t)k_\lambda} = \frac{\frac{\lambda-t}{\lambda+t}b_\lambda + k_\lambda}{b_\lambda + \frac{\lambda-t}{\lambda+t}k_\lambda} \end{aligned}$$

By Lemma 6 in [7],

$$b_\lambda + \frac{\lambda - t}{\lambda + t}k_\lambda = \frac{F(1-Q)(1-w)}{1-Qw}$$

where F is a function in H^1, $\bar{b}_\lambda F \geq 0$ *a.e.* on ∂D, $Q = Q_F$ and w is a contractive function in H^∞. Since b_λ is a Blaschke product of degree n and $\bar{b}_\lambda F \geq 0$ *a.e.* on ∂D, the inner part of F is a Blaschke product of degree $\leq n$. Since $b_t - k_t$ is outer, b_t and the inner part of k_t are relatively prime. Hence b_t is a Blaschke product of degree $\leq n$. If the degree of b_t is less than n, then by the proof above the degree of $b_\lambda < n$. This contradiction implies that the degree of b_t is exactly n.

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