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ON GOUVÊA'S CONJECTURE ON CONTROLLING THE CONDUCTOR

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ON GOUVÊA'S CONJECTURE ON CONTROLLING THE CONDUCTOR

ATSUSHI YAMAGAMI

ABSTRACT. In this article, Gouvêa's conjecture on controlling the conductor is proven in a special case.

0. Introduction

Let $p \geq 7$ be a prime number, and $G_{\mathbb{Q}}$ the absolute Galois group of $\mathbb{Q}$. For an absolutely irreducible residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p),$$

we denote by $N = N(\bar{\rho})$ the conductor (for the definition of the conductor, see Section 2). We assume that $\bar{\rho}$ is associated to a classical eigenform f of level N defined over $\mathbb{Z}_p$. Let $S = \{$ the prime divisors l of $Np\} \cup \{\infty\}$. Then we know that $\bar{\rho}$ factors through the Galois group G_S of the maximal Galois extension of $\mathbb{Q}$ unramified outside S, and we consider the deformation problem of the residual representation

$$\bar{\rho} : G_S \to GL_2(\mathbb{F}_p).$$

In [10], Mazur showed that there exist a complete Noetherian local ring $\boldsymbol{R}(\bar{\rho}, S)$ with residue field $\boldsymbol{k}$ and a deformation of $\bar{\rho}$

$$\boldsymbol{\rho}^{\mathrm{univ}} : G_S \to GL_2(\boldsymbol{R}(\bar{\rho}, S))$$

such that any deformation of $\bar{\rho}$

$$\rho : G_S \to GL_2(A)$$

is obtained from $\boldsymbol{\rho}^{\mathrm{univ}}$ via a unique homomorphism $\boldsymbol{R}(\bar{\rho}, S) \to A$.

On the other hand, we regard f as being in $\boldsymbol{V}_{\mathrm{par}}(\mathbb{Z}_p, N)$ which is the ring of parabolic p-adic modular functions of tame level N defined over $\mathbb{Z}_p$. Let $\boldsymbol{T}_0^{\star}(\mathbb{Z}_p, N)$ be the restricted Hecke algebra on $\boldsymbol{V}_{\mathrm{par}}(\mathbb{Z}_p, N)$ (cf. [5]). We denote by $\boldsymbol{T}(\bar{\rho}, N)$ the completion of $\boldsymbol{T}_0^{\star}(\mathbb{Z}_p, N)$ at the maximal ideal $\mathbf{m}_f$ which is the kernel of the map $\boldsymbol{T}_0^{\star}(\mathbb{Z}_p, N) \to \mathbb{F}_p$ determined by f. Then it was shown in [5] that there exists a deformation of $\bar{\rho}$ to $\boldsymbol{T}(\bar{\rho}, N)$

$$\boldsymbol{\rho}^{\mathrm{mod}} : G_S \to GL_2(\boldsymbol{T}(\bar{\rho}, N))$$

which is universal among deformations of $\bar{\rho}$ associated to p-adic modular functions. Thus we have a natural surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N).$$

For a residual modular representation $\bar{\rho}$ of level $N = N(\bar{\rho})$, Gouvêa's conjecture on controlling the conductor is stated as follows (for the details, see [6]). Let S be as above and S^0 the subset of S consisting of primes $l \mid N$ at which $\bar{\rho}$ is "l-ordinary." In [6], Gouvêa showed that the conductor of any "S^0-ordinary" deformation of $\bar{\rho}$ is N. Then he defined the level N universal deformation ring $\boldsymbol{R}(\bar{\rho}, N)$ to be the universal S^0-ordinary deformation ring (for which, see Section 1). He also showed that the surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N)$$

factors through the universal level N deformation ring:

$$\begin{tikzcd} \boldsymbol{R}(\bar{\rho}, S) \arrow[rr] \arrow[dr] & & \boldsymbol{T}(\bar{\rho}, N) \\ & \boldsymbol{R}(\bar{\rho}, N), \arrow[ur] & \end{tikzcd}$$

and he gave the following

Conjecture ([6], Section 4). The surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, N) \to \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism.

In this article, we prove this conjecture in a special case. We obtain the following

Main Theorem. *Let $p \geq 7$ be a prime number, $k \geq 2$ an integer. For an absolutely irreducible residual representation*

$$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p),$$

we denote by $N = N(\bar{\rho})$ the conductor. We assume that $\bar{\rho}$ is associated to a classical eigenform f of type $(N, k, 1)_{\mathbb{Z}_p}$. Let $S = \{$ the prime divisors l of $Np\} \cup \{\infty\}$, and we consider the deformation problem of the residual representation

$$\bar{\rho} : G_S \to GL_2(\mathbb{F}_p)$$

where G_S is the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S. We denote by A_p the T_p-eigenvalue of f and suppose that A_p is not a unit of $\mathbb{Z}_p$. Suppose further that the polynomial

$$X^2 - A_p X + p^{k-1}$$

has simple roots in $\mathbb{Z}_p$. If the deformation problem for $\bar{\rho}$ is unobstructed, then the conjecture on controlling the conductor is true for $\bar{\rho}$, i.e., the surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, N) \rightarrow \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism.

Here we mean by "the deformation problem for $\bar{\rho}$ is unobstructed" that

$$\dim_{\mathbb{F}_p} H^2(G_S, \mathrm{Ad}(\bar{\rho})) = 0,$$

where $\mathrm{Ad}(\bar{\rho})$ is the $\mathbb{F}_p$-vector space of 2×2 matrices with entries in $\mathbb{F}_p$, on which G_S-action is given by

$$\sigma \cdot M = \bar{\rho}(\sigma) M \bar{\rho}(\sigma)^{-1} \quad (\sigma \in G_S,\ M \in \mathrm{Ad}(\bar{\rho})).$$

Remark 0.1. Let S' be a finite set of rational places containing S. Using "Inflation-Restriction sequence" of Galois cohomology, we can show that if the deformation problem of $\bar{\rho}$ is unobstructed for S', then the deformation problem of $\bar{\rho}$ is unobstructed for S.

Remark 0.2. (1) We can easily check that if $0 < \mathrm{ord}_p(A_p) < (k-1)/2$ then the polynomial $X^2 - A_p X + p^{k-1}$ has simple roots in $\mathbb{Z}_p$ (cf. [13], Theorem II. 2.1).

(2) In the case that k is odd and $\mathrm{ord}_p(A_p) > (k-1)/2$, we see that the polynomial has simple roots in $\mathbb{Z}_p$ if and only if $p \equiv 1 \pmod 4$ (cf. [13], Theorem II. 3.3).

(3) Coleman and Edixhoven showed that if the crystalline Frobenius of the Grothendieck motive $M(f)$ over $\mathbb{Q}_p$ is semi-simple, then the characteristic polynomial $X^2 - A_p X + p^{k-1}$ has simple roots (which are not necessarily in $\mathbb{Z}_p$, cf. [4]).

The proof of the main theorem is divided into two steps. The first step is to show that the natural homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \rightarrow \boldsymbol{R}(\bar{\rho}, N)$$

is surjective. In Section 1, we show in a general situation that the universal S^0-ordinary deformation ring is a quotient of the universal full-deformation ring using the Lenstra-de Smit construction of universal deformation rings.

The second step is to show that the natural surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \rightarrow \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism. This is a generalization of the main theorem of [8] to the case of an arbitrary tame level. In Section 2, we regard our $\bar{\rho}$ as being associated to the "twins" of level Np coming from f. We then reduce the proof of the isomorphism to the arguments of Gouvêa-Mazur in [8] which use the theory of "infinite ferns."

1. The universal S^0-ordinary deformation ring

In this section, we consider "S^0-ordinary" deformations in a general situation. Let $p \geq 7$ be a prime, $\boldsymbol{k}$ a finite field of characteristic p, S a finite set of rational places containing p and ∞, and G_S the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S. Let

$$\bar{\rho} : G_S \to GL_2(\boldsymbol{k})$$

be an absolutely irreducible residual representation.

First we briefly recall the deformation theory of Galois representations. Let A be a complete Noetherian local ring with residue field $\boldsymbol{k}$. We shall consider homomorphisms $\rho : G_S \to GL_2(A)$ which make the following diagram commutative:

$$\begin{array}{ccc} G_S & \xrightarrow{\rho} & GL_2(A) \\ & \bar{\rho} \searrow & \downarrow \\ & & GL_2(\boldsymbol{k}), \end{array}$$

where the homomorphism $GL_2(A) \to GL_2(\boldsymbol{k})$ is induced from the natural reduction homomorphism $A \to \boldsymbol{k}$. We say two such lifts are *strictly equivalent* if they are conjugate by an element of $\ker(GL_2(A) \to GL_2(\boldsymbol{k}))$, and we refer to a strict equivalence class of lifts of $\bar{\rho}$ as *deformation* of $\bar{\rho}$ to A.

Mazur showed in [10] the following

Theorem 1.1 ([10], Proposition 1). *Let a residual representation*

$$\bar{\rho} : G_S \to GL_2(\boldsymbol{k})$$

be absolutely irreducible. Then there exist a complete Noetherian local ring $\boldsymbol{R}(\bar{\rho}, S)$ with residue field $\boldsymbol{k}$ and a deformation of $\bar{\rho}$

$$\boldsymbol{\rho}^{\mathrm{univ}} : G_S \to GL_2(\boldsymbol{R}(\bar{\rho}, S))$$

such that any deformation ρ of $\bar{\rho}$ to a complete Noetherian local ring A with residue field $\boldsymbol{k}$ is obtained from $\boldsymbol{\rho}^{\mathrm{univ}}$ via a unique homomorphism $\boldsymbol{R}(\bar{\rho}, S) \to A$.

The ring $\boldsymbol{R}(\bar{\rho}, S)$ depends only on $\bar{\rho}$ up to "twist-equivalence," i.e., if $\bar{\rho}'$ is equivalent to a twist of $\bar{\rho}$ by a one-dimensional character, then there exists a canonical isomorphism

$$r(\bar{\rho}, \bar{\rho}') : \boldsymbol{R}(\bar{\rho}, S) \xrightarrow{\sim} \boldsymbol{R}(\bar{\rho}', S).$$

Now we consider the "ordinary" deformations. Let $\rho : G_S \to GL_2(A)$ be a Galois representation and $M = A \times A$ with G_S-action by ρ. Let l be a prime number, and I an inertia group at l. We say ρ is *l-ordinary* if the submodule $M^I \subset M$ of invariants under I is a free A-module of rank one and a direct summand of M.

Let $S^0 \subset S$, and assume that $\bar{\rho} : G_S \to GL_2(\boldsymbol{k})$ is S^0-ordinary, i.e., $\bar{\rho}$ is l-ordinary for all $l \in S^0$. Then we can consider S^0-ordinary deformations of $\bar{\rho}$. The following theorem asserts that the universal S^0-ordinary deformation ring exists and it is a quotient of $\boldsymbol{R}(\bar{\rho}, S)$.

Theorem 1.2 (cf. [6], Theorem 2). *Let $S^0 \subset S$, and assume that $\bar{\rho} : G_S \to GL_2(\boldsymbol{k})$ is absolutely irreducible and S^0-ordinary. Then there exist a complete Noetherian local ring $\boldsymbol{R}(\bar{\rho}, S, S^0)$ with residue field $\boldsymbol{k}$ and an S^0-ordinary deformation of $\bar{\rho}$*

$$\boldsymbol{\rho}(S, S^0) : G_S \to GL_2(\boldsymbol{R}(\bar{\rho}, S, S^0))$$

such that any S^0-ordinary deformation ρ of $\bar{\rho}$ to a complete Noetherian local ring A with residue field $\boldsymbol{k}$ is obtained from $\boldsymbol{\rho}(S, S^0)$ via a unique homomorphism $\boldsymbol{R}(\bar{\rho}, S, S^0) \to A$.

Moreover, the natural homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{R}(\bar{\rho}, S, S^0)$$

is surjective.

Proof. We use the Lenstra-de Smit construction of universal deformation rings to show the existence of the universal S^0-ordinary deformation ring (cf. [9]).

Note that Lenstra and de Smit considered the deformation problem of residual representations in the category of complete local rings (not necessarily Noetherian). However, by Theorem (2.3)(3), (4) in [9], the universal deformation ring that they constructed in [9] is isomorphic to the one that is constructed in the category of complete *Noetherian* local rings, since $\dim_{\mathbb{F}_p} H^1(G_S, \mathrm{Ad}(\bar{\rho})) < \infty$.

Let $V = \boldsymbol{k} \times \boldsymbol{k}$ with G_S-action by $\bar{\rho}$. We denote by $\mathrm{Def}(V, A)$ the set of deformations of $\bar{\rho}$ to A. Further we denote by $S^0(A)$ the set of S^0-ordinary deformations and $S_l(A)$ the set of l-ordinary deformations for each $l \in S^0$. Then $S^0(A)$ and $S_l(A)$ are subsets of $\mathrm{Def}(V, A)$, and we have

$$S^0(A) = \bigcap_{l \in S^0} S_l(A).$$

By Proposition (6.1) of [9], we know that the functor $S_l(\cdot)$ satisfies the conditions $(1) - (3)$ in Section 6 of [9]. Therefore the functor $S^0(\cdot)$ also satisfies these conditions, and we know that this is represented by a quotient ring $\boldsymbol{R}(\bar{\rho}, S, S^0)$ of the universal deformation ring $\boldsymbol{R}(\bar{\rho}, S)$. Namely, the natural homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{R}(\bar{\rho}, S, S^0)$$

is surjective. □

We return to Gouvêa's conjecture on controlling the conductor. In [6], Gouvêa defined the level N universal deformation ring $\boldsymbol{R}(\bar{\rho}, N)$ to be the universal S^0-ordinary deformation ring. So the natural homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{R}(\bar{\rho}, N)$$

is surjective by the theorem above.

2. The proof of the main theorem

In Gouvêa's commutative diagram

$$\begin{tikzcd} \boldsymbol{R}(\bar{\rho}, S) \arrow[rr] \arrow[dr] & & \boldsymbol{T}(\bar{\rho}, N) \\ & \boldsymbol{R}(\bar{\rho}, N), \arrow[ur] & \end{tikzcd}$$

we know that the homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{R}(\bar{\rho}, N)$$

is surjective by Section 1. To obtain the main theorem, in this section, we show that the natural surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism for $\bar{\rho}$ of the main theorem.

2.1. The conductor of a residual representation

First we recall the definition of the conductor of a residual representation, and state an essential lemma on modular representations.

Let $p \geq 7$ be a prime number, $\boldsymbol{k}$ a finite field of characteristic p. For a residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\boldsymbol{k}),$$

the *conductor* $N(\bar{\rho})$ of $\bar{\rho}$ is defined to be

$$N(\bar{\rho}) = \prod_{l \neq p} l^{n(l, \bar{\rho})},$$

where the numbers $n(l, \bar{\rho})$ are defined as follows: choose a place of $\bar{\mathbb{Q}}$ over l, and let I be the corresponding inertia group. Let $\bar{V} = \boldsymbol{k} \times \boldsymbol{k}$

with $G_{\mathbb{Q}}$-action given by $\bar{\rho}$, and $\bar{V}_0$ the subspace of $\bar{V}$ fixed by $\bar{\rho}(I)$. Then

$$n(l, \bar{\rho}) = 2 - \dim \bar{V}_0 + \mathrm{sw}(\bar{\rho})$$

where $\mathrm{sw}(\bar{\rho})$ is the Swan conductor of the restriction of $\bar{\rho}$ to I.

Note that $n(l, \bar{\rho}) = 0$ if and only if $\bar{\rho}$ is unramified at l. So the product in the definition is meaningful. Note also that $N(\bar{\rho})$ is prime to p. (For the definition of the conductor of a residual representation, see also [6], [14].)

Now we assume that $\bar{\rho}$ is absolutely irreducible and associated to a classical eigenform f of level $N(\bar{\rho})$ defined over a $W(\boldsymbol{k})$-algebra B with residue field $\boldsymbol{k}$. Then Gouvêa showed the following lemma in [6]. It is very essential to proving $\boldsymbol{R}(\bar{\rho}, S) \cong \boldsymbol{T}(\bar{\rho}, N)$ in the next section.

Lemma 2.1 (cf. [6], Lemma 7). *Let $\bar{\rho}$ and f be as above. Then the eigenform f is a newform of level $N(\bar{\rho})$.*

2.2. Twins, Coleman's families and infinite ferns

Now we prove that the isomorphism $\boldsymbol{R}(\bar{\rho}, S) \cong \boldsymbol{T}(\bar{\rho}, N)$ for $\bar{\rho}$ of the main theorem and finish the proof of the main theorem. Let $\bar{\rho} : G_S \to GL_2(\mathbb{F}_p)$ and f be as in the assumption of the main theorem. Note that $N = N(\bar{\rho})$ is the conductor of $\bar{\rho}$. Then the classical eigenform f associated to $\bar{\rho}$ is of type $(N, k, 1)_{\mathbb{Z}_p}$, i.e., a cusp form of level N, weight k with trivial character whose Fourier expansion is lying in $\mathbb{Z}_p[\![q]\!]$ and whose first Fourier coefficient is equal to 1, and which is an eigenform for the Hecke operators T_l for prime numbers l not dividing Np and for the Atkin operators U_q for prime numbers q dividing Np.

Remark 2.1. We denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. We fix once and for all an isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$. Given a classical modular form f, we transport the Fourier coefficients of f in $\mathbb{C}$ to $\mathbb{C}_p$ via our fixed isomorphism. In this way we regard f as being defined over $\mathbb{C}_p$.

Since the level N of f is the conductor of $\bar{\rho}$, f is a *newform* of level N by Lemma 2.1. From the relation

$$U_p = T_p - p^{k-1} B_p,$$

we know that the characteristic polynomial of the Hecke operator U_p acting on the two-dimensional space of oldforms of level Np spanned by f and $f|B_p$ over $\mathbb{C}_p$ is

$$X^2 - A_p X + p^{k-1}.$$

Here A_p is the T_p-eigenvalue of f, and B_p is the standard "degeneracy operator" which acts on the Fourier expansion of f as $(f|B_p)(q) = f(q^p)$.

By the assumption, this polynomial has simple roots in $\mathbb{Z}_p$. We denote these roots by λ_1, λ_2. Then the following two eigenforms of type $(Np, k, 1)_{\mathbb{Z}_p}$

$$f_1 = f - \lambda_2 \cdot f|B_p,$$
$$f_2 = f - \lambda_1 \cdot f|B_p$$

form a basis of that space. Their T_l-eigenvalues for l not dividing Np and U_q-eigenvalues for q dividing N are equal to those of f, and U_p-eigenvalues are λ_1 and λ_2, respectively. In particular, from the equality of the T_l-eigenvalues for l not dividing Np, we may assume that $\bar{\rho}$ is associated to both of f_1 and f_2. We call f_1 and f_2 the *twins* coming from f. Note that the twins are "new away from p." (We say that a classical eigenform g of level Np is *new away from* p when g is a newform of level Np or g belongs to a space of oldforms spanned by φ and $\varphi|B_p$ for a newform φ of level N.)

Putting $g = f_1$, we may assume that $\bar{\rho}$ is associated to g which is a classical eigenform of type $(Np, k, 1)_{\mathbb{Z}_p}$ and new away from p. Since the U_p-eigenvalue λ_1 of g is a simple root of the polynomial $X^2 - A_pX + p^{k-1}$, we have $\lambda_1^2 \neq p^{k-1}$. We put $\alpha = \mathrm{ord}_p(\lambda_1)$ and call it the *slope* of g. Since A_p is not a unit, we have $0 < \alpha < k-1$, i.e., g is of *non-critical slope*.

Remark 2.2. Note that we do not need the level of f to be the conductor $N(\bar{\rho})$ of $\bar{\rho}$ except when we show that f is a newform in the above arguments concerning eigenforms.

Note also that the assumption that the polynomial $X^2 - A_pX + p^{k-1}$ has simple roots in $\mathbb{Z}_p$ implies that the eigenform g is defined over $\mathbb{Z}_p$. This fact enables us to use the same arguments as in [8] dealing with eigenforms defined over $\mathbb{Z}_p$.

Summing up the arguments above, we know that the main theorem follows from the following theorem which is a generalization of the main theorem of [8] to the case of an *arbitrary* tame level. (From now on, the symbol N denotes any positive integer prime to p.)

Theorem 2.2. *Let $p \geq 7$ be a prime number, $k \geq 2$ an integer, and N a positive integer prime to p. Let $S = \{$ the prime divisors l of $Np\} \cup \{\infty\}$ and G_S the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S. We assume that an absolutely irreducible residual representation*

$$\bar{\rho} : G_S \to GL_2(\mathbb{F}_p)$$

is associated to a classical eigenform g of type $(Np, k, 1)_{\mathbb{Z}_p}$, non-critical slope α and new away from p. Further, assume that $\lambda_p^2 \neq p^{k-1}$ where λ_p is the U_p-eigenvalue of g. If the deformation problem for $\bar{\rho}$ is unobstructed, then the natural surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism.

To prove this theorem, we need the following two propositions on "twins" and "Coleman's families" respectively.

Proposition 2.3. *Let f be a classical eigenform of type $(Np, k, 1)_{\mathbb{C}_p}$. Suppose that the slope of f is equal to neither $(k-1)/2$ nor $(k-2)/2$ and that f is new away from p. Then f has a twin.*

Proof. If f is a newform, then its Hecke U_p-eigenvalue is $\pm p^{(k-2)/2}$ (cf. [1], Theorem 3). As we assume that the slope of f is not equal to $(k-2)/2$, f is an oldform of level Np. Since f is new away from p, f belongs to a space of oldforms spanned by φ and $\varphi | B_p$ for a newform φ of level N.

Let a_p be the U_p-eigenvalue of f. Then we can check easily that f and $f' = \varphi - a_p \cdot \varphi | B_p$ are the twins coming from φ, since the slope of f is not equal to $(k-1)/2$. □

Remark 2.3. From the proof above, we know that if f is defined over $\mathbb{Z}_p$, then the twin f' of f is also defined over $\mathbb{Z}_p$. This fact is essential to constructing an "infinite fern" in the universal deformation space $X = \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\boldsymbol{R}(\bar{\rho}, S), \mathbb{Z}_p)$ (for which, see [12]).

Proposition 2.4 (cf. [3], Corollary B5. 7. 1). *Let g be a classical eigenform of type $(Np, k_0, 1)_{\mathbb{Z}_p}$ which is new away from p and slope $\alpha = \mathrm{ord}_p(\lambda_p)$, where λ_p the U_p-eigenvalue of g. Suppose that $k_0 > \alpha + 1$ and that $\lambda_p^2 \neq p^{k_0-1}$.*

Then there exist a disk $D \subset \mathbb{Z}_p$ containing k_0 and p-adic analytic functions $a_n(s) : D \to \mathbb{Z}_p$ for $n \geq 1$ such that if $k \in D$ is an integer strictly greater than $\alpha + 1$, then

$$f_k(q) = \sum_{n \geq 1} a_n(k) q^n$$

is a classical eigenform of type $(Np, k, 1)_{\mathbb{Z}_p}$ and slope α which is new away from p, and $f_{k_0} = g$.

Proof. The proof of this proposition can be found in [3], Section B5. Note that Coleman constructed his families of eigenforms using the duality between the space of "p'-new forms" and the Hecke algebra acting on it. This implies the eigenforms f_k are new away from p. □

These propositions enable us to construct an "infinite fern" in the universal deformation space $X = \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\boldsymbol{R}(\bar{\rho}, S), \mathbb{Z}_p)$ by the same way as in [12]. The unobstructedness of the deformation problem for $\bar{\rho}$ implies that X has a structure of $\mathbb{Q}_p$-analytic manifold of dimension 3. Gouvêa and Mazur showed in [8] the isomorphism of tame level 1,

$$\boldsymbol{R}(\bar{\rho}, \{p, \infty\}) \cong \boldsymbol{T}(\bar{\rho}, 1),$$

by showing the density of modular points in the universal deformation space. Recall that, in their arguments, the existence of an infinite fern of level p caused this density on the assumption that the universal deformation space is a 3-dimensional $\mathbb{Q}_p$-analytic manifold. Now we have an infinite fern of level Np in our space X. Therefore Theorem 2.2 can be proven by the same arguments as in [8], and we omit the details of the proof here.

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