Title	Anisotropic curvature flow in a very thin domain
Author(s)	Arisawa, M.; Giga, Y.
Citation	Hokkaido University Preprint Series in Mathematics, 495, 1-21
Issue Date	2000-10-1
DOI	10.14943/83641
Doc URL	http://hdl.handle.net/2115/69245
Type	bulletin (article)
File Information	pre495.pdf

Anisotropic curvature flow in a very thin domain

Mariko Arisawa and Yoshikazu Giga

Series #495. October 2000

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#471 K. Ito and Y. Kohsaka, Stability of a stationary solution for evolving boundaries of symmetric three-phases driven by surface diffusion, 36 pages. 1999.

#472 Y. Giga and M.-H. Sato, A level set approach to semicontinuous viscosity solutions for Cauchy problems, 32 pages. 1999.

#473 R. Yoneda, Characterization of Bloch space and Besov spaces by oscillations, 31 pages. 1999.

#474 T. Nakazi, Norm inequalities for some singular integral operators, 13 pages. 1999.

#475 A. Inoue and Y. Kasahara, Asymptotics for prediction errors of stationary processes with reflection positivity, 15 pages. 1999.

#476 T. Sano, On affine parallels of generic plane curves, 8 pages. 1999.

#477 T. Nakazi, On an invariant subspace whose common zero set is the zeros of some function, 11 pages. 1999.

#478 M.-H. Giga and Y. Giga, Generalized motion by nonlocal curvature in the plane, 68 pages. 2000.

#479 M.-H. Giga and Y. Giga, Crystalline and level set flow – Convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane, 16 pages. 2000.

#480 A. Arai and M. Hirokawa, Stability of ground states in sectors and its application to the Wigner-Weisskopf model, 16 pages. 2000.

#481 T. Nakazi, Two dimensional Q-algebras, 11 pages. 2000.

#482 N. H. Bingham and A. Inoue, Tauberian and Mercerian theorems for systems of kernels, 16 pages. 2000.

#483 N. H. Bingham and A. Inoue, Abelian, Tauberian and Mercerian theorems for arithmetic sums, 29 pages. 2000.

#484 I. A. Bogaevski and G. Ishikawa, Lagrange mappings of the first open Whitney umbrella, 22 pages. 2000.

#485 A. Arai and H. Kawano, A class of deformations of the Schrödinger representation of the Heisenberg commutation relation and exact solution to a Heisenberg equation and a Schrödinger equation, 22 pages. 2000.

#486 T. Nakazi, Functions in N_+ with the positive real parts on the boundary, 21 pages. 2000.

#487 Y. Shibukawa, Classification of the R-operator, 36 pages. 2000.

#488 A. Inoue, Asymptotic behaviour for partial autocorrelation functions of fractional ARIMA processes, 20 pages. 2000.

#489 S. Ohtani, Construction of unramified Galois extensions over maximal abelian extensions of algebraic number fields, 14 pages. 2000.

#490 T. Nakazi and T. Yamamoto, The real part of an outer function and a Helson-Szegö weight, 13 pages. 2000.

#491 A. Yamagami, On Gouvêás conjecture on controlling the conductor, 11 pages. 2000.

#492 I. Tsuda and M. Hatakeyama, Making sense of internal logic: Theory and a case study, 10 pages. 2000.

#493 I. Tsuda, Towards an interpretation of dynamic neural activity in terms of chaotic dynamical systems, 73 pages. 2000.

#494 T. Mikami, Optimal control for absolutely continuous stochastic processes and the mass transportation problem, 17 pages. 2000.

Anisotropic curvature flow in a very thin domain

Mariko Arisawa and Yoshikazu Giga

Department of Mathematical Sciences, GSIS, Tohoku University Sendai 980-77, Japan.

Department of Mathematics, Hokkaido University, Sapporo 060, Japan.

§1. Introduction

We study two types of evolutions of surfaces in a very thin region in $\mathbf{R}^n$.

$$\Omega_\varepsilon = \mathbf{R}^{n-1} \times (0, \varepsilon) = \{ \ x = (x', x_n) \ ; \ x' \in \mathbf{R}^{n-1}, \ 0 < x_n < \varepsilon \ \}.$$

The first one is the anisotropic mean curvature flow, and the second one is the constant velocity (in normal direction) motion. The thickness of the region Ω_ε is $\varepsilon > 0$, a very small positive number. The first anisotropic curvature flow equation is of form

$$V = M(\mathbf{n})(-\mathrm{div}_{\Gamma_t}(D\gamma)(\mathbf{n})) + C(x, t, \mathbf{n}) \text{ on } \Gamma_t^\varepsilon.$$

Here V denotes the normal velocity in the direction of the unit normal vector field $\mathbf{n}$ of the hypersurface Γ_t^ε in Ω_ε at time t and div denotes the surface divergence. The functions M, γ, C are given functions ; M is a positive function called a *mobility*. The function γ is originally defined on the unit sphere but it is extended for all $p \in \mathbf{R}^n$ so that $\gamma(p) = \gamma(p/|p|)|p|$; γ is called an *interfacial energy density*. The quantity $D\gamma$ denotes the gradient of γ in $\mathbf{R}^n$. The term C is a driving force term. If $\gamma(p) = |p|, C = 0$ and $M = 1$, the equation becomes the famous mean curvature flow equation. The anisotropic curvature flow equation naturally arises in material sciences and crystal growth problems.

The reader is referred to a nice book of M. E. Gurtin [18] and a review article by J. Tayler, J. Cahn and A. Handwerker [27] for its physical background. In these problems, C is often of form $C(x,t,\mathbf{n}) = M(\mathbf{n})c(x,t)$. We consider the above anisotropic curvature flow in Ω_ε with the right angle boundary condition

$$\Gamma_t^\varepsilon \perp \Omega_\varepsilon.$$

The initial data we consider here is of form $\Gamma_0^\varepsilon = \bar{\Gamma}_0^\varepsilon \times (0,\varepsilon)$ with $\bar{\Gamma}_0^\varepsilon \subset \mathbf{R}^{n-1}$. We are concerned with the problem to find an equation which governs the evolution of the limit $\bar{\Gamma}_t$ of Γ_t^ε as $\varepsilon \to 0$. Roughly speaking, our conclusion says that $\bar{\Gamma}_t \subset \mathbf{R}^{n-1}$ solves the equation

$$V = M(\mathbf{m},0)\ (-\mathrm{div}_{\bar{\Gamma}_t}\ D\bar{\gamma}(\mathbf{m}))\ +\ C(x',\ 0,\ t,\ \mathbf{m},\ 0) \text{ on } \bar{\Gamma}_t$$

with $\bar{\gamma}(\bar{p}) = \gamma(\bar{p},0), \bar{p} \in \mathbf{R}^{n-1}$, where $x' \in \mathbf{R}^{n-1}$ and $\mathbf{m}$ denotes the unit normal vector field of $\bar{\Gamma}_t$ in $\mathbf{R}^{n-1}$. The initial data of $\bar{\Gamma}_t$ is $\bar{\Gamma}_0^\varepsilon$. We shall state our problem and results more rigorously later in this introduction.

The second problem we consider is

$$V = M(\mathbf{n}) \quad \text{on } \Gamma_t^\varepsilon \quad \text{with} \quad \Gamma_t^\varepsilon \perp \partial\Omega_\varepsilon$$

with the initial data which is not necessarily parallel in the directon of x_n. Roughly speaking, we conclude that the limit evolution $\bar{\Gamma}_t$ solves

$$V = M(\mathbf{m},0) \text{ on } \bar{\Gamma}_t$$

with some initial data $\bar{\Gamma}_0$ which allows the existence of an initial layer.

We formulate these problems by the level set approach whose analytic foundation has been established by Y. -G. Chen, Y. Giga and S. Goto [13] and L. C. Evans and J. Spruck [12]. See also Y. Giga [16].

(Anisotropic mean curvature flow equation.)

$$(1)\qquad \frac{\partial u_\varepsilon}{\partial t} = |\nabla u_\varepsilon| M\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right) \mathrm{div}(D\gamma(\nabla u_\varepsilon)) - C\left(x, t, \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right)|\nabla u_\varepsilon|,$$
$$t > 0, \qquad x = (x', x_n) \in \mathbf{R}^{n-1} \times (0,\varepsilon)$$

$$(2)\qquad u_\varepsilon(0,x) = u_0(x'), \qquad x = (x', x_n) \in \mathbf{R}^{n-1} \times (0,\varepsilon)$$

$$(3)\qquad \frac{\partial u_\varepsilon}{\partial x_n}(t,x) = 0, \qquad t > 0,\ x = (x',0) \text{ and } (x',\varepsilon)$$

(Constant normal velocity equation)

$$(4)\qquad \frac{\partial u_\varepsilon}{\partial t} = M\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right)|\nabla u_\varepsilon|, \qquad t > 0,\ x = (x', x_n) \in \mathbf{R}^{n-1} \times (0,\varepsilon)$$

$$(5)\qquad u_\varepsilon(0,x) = u_{0\varepsilon}(x), \qquad x = (x', x_n) \in \mathbf{R}^{n-1} \times (0,\varepsilon)$$

$$(6)\qquad \frac{\partial u_\varepsilon}{\partial x_n}(t,x) = 0, \qquad t > 0,\ x = (x',0) \text{ and } (x',\varepsilon)$$

In this formulation our interested surface at time t (denoted by Γ_t^ε) ($t \geq 0$) is given by

$$\Gamma_t^\varepsilon = \{x \ \in \ \mathbf{R}^n \mid u_\varepsilon(t,x) = \text{const} \ \}$$

and its outward unit normal $\mathbf{n}$ is given by $\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$. The function $\gamma(p)$ ($p \in \mathbf{R}^n$) is now assumed to be

(7) $\gamma \geq 0$ is convex, positively homogeneous of degree one and C^2 outside the origin.

The functions $M(p)$ ($p \in \mathbf{R}^n$) in (1) and (4), which we use the same notation without any confusion, is an extension of M(the mobility) to $\mathbf{R}^n$. We assume that

(8) there are constants M_1 and M_2 ($0 < M_1 < M_2$) such that

$$M_1 \leq M\left(\frac{p}{|p|}\right) \leq M_2 \qquad \text{for all } p \in \mathbf{R}^n.$$

In (1), $C(x,t,\frac{p}{|p|})$ is a given bounded function in $\mathbf{R}^n \times \mathbf{R}^+ \times S^{n-1}$ and homogeneous in degree 0 in the last variables $\left(\frac{p}{|p|} \in S^{n-1}\right)$. As for the initial condition, we remark that in (2) u_0 depends only on the first $n-1$ variables x', the same given function for all $\varepsilon > 0$, while in (5) $u_{0\varepsilon}$ depends on $x \in \Omega_\varepsilon$ and may oscillate as $\varepsilon > 0$ goes to 0. The precise condition on $u_{0\varepsilon}(x)$ will be given below in this paper.

These problems are examples of the multi-scale model in the engineering. The interfacial surface Γ_t moves in the region whose thickness in the n-th direction is uncomparably smaller than those of other directions. Therefore, as is stated before in this introduction, we are interested in the limit behaviors of u_ε of (1)-(3) and (4)-(6) as $\varepsilon > 0$ goes to 0, rather than each solutions u_ε themselves. Our questions are the following.

(A) Do the limits $\lim_{\varepsilon \downarrow 0} \ u_\varepsilon(t,x',x_n)$ exist?, and if they exist, are they independent of the last variable x_n ?

(9) $$\lim_{\varepsilon \downarrow 0} u_\varepsilon(t,x',x_n) = \ \bar{u}(t,x')$$

(B) If (9) holds, can we characterize the limits $\bar{u}$ by some effective P.D.E.s?, or in other words, can we find the effective limit laws for the motions of the limit surfaces $\lim_{\varepsilon \downarrow o} \{\Gamma_t^\varepsilon\}_{t \geq 0}$ in $\mathbf{R}^{n-1}$?

The answers to these questions are positive, and will be given below in Theorems 1.3 and 1.4 in this introduction.

As the pure P.D.E.'s problems, the above (A) and (B) may be classified into the framework of the homogenizations and the penalizations. For the homogenizations and the penalizations, we refer the readers to A. Bensoussan, J. L. Lions and G. Papanicolaou

[8], P.-L. Lions, G. Papanicolaou and S. R. S. Varadhan [25], L. C. Evans [10], [11], M. Arisawa [4], [5], K. Horie and H. Ishii [19], R. Jensen and P.-L. Lions [24], F. Bagagiolo and M. Bardi [7], O. Alvarez and M. Bardi [1], and the references therein. Our problems (A) and (B) have two distinct points from the above cited works and the others. The first point is that Ω_ε in (1)-(3) and (4)-(6) are changing as $\varepsilon \downarrow 0$, while most of the existing homogenizations and penalizations results treat the fixed domain. (There are some exception like K. Horie and H. Ishii [19] which treats the shrinking domain as ours, but with a much simpler first-order homogenization than ours.)

The second point is that the operators $|\nabla u|\ M\ (\nabla u\ /\ |\nabla u|)\ \mathrm{div}\ (D\gamma(\nabla u))$ and $M\ (\nabla u\ /\ |\nabla u|)\ |\nabla u|$ are not treated in so far as we know, in particular even the former operator with $M\ (p\ /\ |p|) \equiv\ 1$ and $\gamma = |p|$ (the mean curvature's case) is not studied. In [7] F. Bagagiolo and M. Bardi treated the penalization with the first-order operator written in the form of $\sup_{\alpha\in\mathcal{A}}\{\langle -b(x,\alpha), \nabla u\rangle - f(x,\alpha)\}$ arising in the optimal control problems (singular perturbation of the controlled dynamical system). The first-order operator $M(\nabla u/|\nabla u|)\ |\nabla u|$ are not in their framework, and the motivations to study the penalization problem are different. The existing results for the second-order homogenizations and penalizations concern with the uniformly elliptic operators, and the results for the first-order homogenizations and penalizations concern with the Hamilton-Jacobi operators in the optimal control problem.

Our strategy to answer the questions (A) and (B) for (1)-(3) and (4)-(6) is the following. Put

$$\eta = \frac{x_n}{\varepsilon} \qquad \eta \in (0,1), \quad x_n \in (0,\varepsilon), \tag{10}$$

and define

$$v_\varepsilon(t,x',\eta) = u_\varepsilon(t,x',\varepsilon\eta) \qquad t \geq 0, x' \in \mathbf{R}^{n-1}, \eta \in (0,1) \tag{11}$$

From this re-scaling,

$$\frac{\partial u_\varepsilon}{\partial x_i} = \frac{\partial v_\varepsilon}{\partial x_i}\ (1 \leq i \leq n-1), \quad \frac{\partial u_\varepsilon}{\partial x_n} = \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}, \quad \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} = \frac{\partial^2 v_\varepsilon}{\partial x_i \partial x_j}\ (1 \leq i,j \leq n-1),$$

$$\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_n} = \varepsilon^{-1}\frac{\partial^2 v_\varepsilon}{\partial x_i \partial \eta}\ (1 \leq i \leq n-1), \quad \frac{\partial^2 u_\varepsilon}{\partial x_n^2} = \varepsilon^{-2}\frac{\partial^2 v_\varepsilon}{\partial \eta^2},$$

and (1)-(3), (4)-(6) are rewritten as follows.

$$\left\{ \begin{array}{l} \dfrac{\partial v_\varepsilon}{\partial t} = \sqrt{|\nabla_{x'} v_\varepsilon|^2 + \left(\varepsilon^{-1}\dfrac{\partial v_\varepsilon}{\partial \eta}\right)^2}\ M\left(\dfrac{\left(\nabla_{x'} v_\varepsilon, \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right)}{\sqrt{|\nabla_{x'} v_\varepsilon|^2 + \left(\varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right)^2}}\right) \mathrm{Tr}\ G_\varepsilon\ Y \\ -C\left(x', \varepsilon\eta, t, \frac{\left(\nabla_{x'} v_\varepsilon, \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right)}{\sqrt{|\nabla_{x'} v_\varepsilon|^2 + \left(\varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right)^2}}\right) \quad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta \in (0,1), \end{array} \right. \tag{12}$$

$$(13) \qquad v_\varepsilon(0,x',\eta) = u_0(x') \qquad x' \in \mathbf{R}^{n-1}, \eta \in (0,1),$$

$$(14) \qquad \frac{\partial v_\varepsilon}{\partial \eta} = 0 \qquad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta = 0 \text{ and } 1,$$

where Tr denotes the trace of an $n \times n$ matrix, $G_\varepsilon = (g_{ij})_{1 \leq i,j \leq n}$ is the $n \times n$ matrix such that

$$(15) \qquad \begin{cases} g_{ij} = \frac{\partial^2 \gamma}{\partial p_i \partial p_j} \left(\frac{\partial v_\varepsilon}{\partial x'},\ \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right) & 1 \leq i,j \leq n-1, \\ g_{in} = g_{ni} = \varepsilon^{-1}\frac{\partial^2 \gamma}{\partial p_i \partial p_n} \left(\frac{\partial v_\varepsilon}{\partial x'}, \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right) & 1 \leq i \leq n-1, \\ g_{nn} = \varepsilon^{-2}\frac{\partial^2 \gamma}{\partial p_n{}^2} \left(\frac{\partial v_\varepsilon}{\partial x'}, \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right), \end{cases}$$

and Y is the Hessian of $v(x',\eta)$, *i.e.* $Y = (y_{ij})_{1 \leq i,j \leq n}$,
where $y_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j}$ $(1 \leq i,\ j \leq n-1)$, $y_{in} = y_{ni} = \frac{\partial^2 v}{\partial x_i \partial \eta}$ $(1 \leq i \leq n-1)$,
and $y_{nn} = \frac{\partial^2 v}{\partial \eta^2}$.

$$(16) \qquad \frac{\partial v_\varepsilon}{\partial t} = M\left(\frac{(\nabla_{x'}v_\varepsilon, \varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta})}{\sqrt{|\nabla_{x'}v_\varepsilon|^2 + \left(\varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right)^2}}\right)\sqrt{|\nabla_{x'}v_\varepsilon|^2 + \left(\varepsilon^{-1}\frac{\partial v_\varepsilon}{\partial \eta}\right)^2}$$

$$t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta \in (0,1),$$

$$(17) \qquad v_\varepsilon(0,x',\eta) = v_0(x',\eta) \qquad x' \in \mathbf{R}^{n-1},\ \eta \in (0,1),$$

$$(18) \qquad \frac{\partial v_\varepsilon}{\partial \eta} = 0 \qquad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta = 0 \text{ and } 1,$$

where we assume that

$$(19) \qquad \begin{cases} & v_0(x',\eta) \text{ is bounded, Lipschitz in } \mathbf{R}^{n-1} \times (0,1), \\ & v_0(x',\eta) = u_{0\varepsilon}(x',\varepsilon\eta) \\ & \text{for all } x' \in \mathbf{R}^{n-1},\ \text{for all } \eta \in (0,1),\ \text{for all } \varepsilon > 0, \\ \text{and} & \\ & \min_{\eta \in [0,1]} v_0(x',\eta) \text{ is Lipschitz.} \end{cases}$$

Remark that (12)-(14) and (16)-(18) are now the homogenization (or penalization) in the fixed domain $\mathbf{R}^{n-1} \times (0,1)$.

We further assume that

$$(20) \qquad \sup_{x' \in \mathbf{R}^{n-1}} |\nabla_{x'}u_0(x')| + \sup_{x' \in \mathbf{R}^{n-1}} |\nabla_{x'}^2 u_0(x')| < {}^\exists C$$

$$(u_0 \text{ is } C^2 \text{ in } \mathbf{R}^{n-1}),$$

and that

$$(21) \qquad \sum_{i=1}^{n} p_i \frac{\partial C}{\partial x_i}(x,t,\frac{p}{|p|}) \geq 0.$$

The homogenization results for (12)-(14) and (16)-(18) are respectively as follows.

Theorem 1.1. *Let v_ε $(\varepsilon > 0)$ be the solutions of (12)-(14). Assume that (7),(8),(20) and (21) hold. Then, we have the following.*
(i) If

$$\frac{\partial C}{\partial t}(x,t,\frac{p}{|p|}) = 0 \qquad x \in \mathbf{R}^n, t \geq 0,\ p \in \mathbf{R}^n \tag{22}$$

holds, there exists an unique function $\bar{v}(t,x')$ such that

$$\lim_{\varepsilon \downarrow 0} v_\varepsilon(t,x',\eta) = \bar{v}(t,x') \quad \textit{locally uniformly in} \quad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta \in (0,1) \tag{23}$$

which is the solution of

$$\begin{cases} \frac{\partial \bar{v}}{\partial t} = |\nabla_{x'}\bar{v}|\ M\left(\frac{(\nabla_{x'}\bar{v},0)}{|\nabla_{x'}\bar{v}|}\right) \mathrm{div}_{x'}\left(D_{p'}\bar{\gamma}(\nabla_{x'}\bar{v})\right) - C\left(x',0,t\ \frac{(\nabla_{x'}\bar{v},0)}{|\nabla_{x'}\bar{v}|}\right) \\ t \geq 0,\ x' \in \mathbf{R}^{n-1}, \end{cases}$$
$$\bar{v}(0,x') = u_0(x'), \tag{24}$$

where

$$\bar{\gamma}(p') = \gamma(p',0), \qquad p' \in \mathbf{R}^{n-1}. \tag{25}$$

(ii) If there exists $T > 0$ such that

$$\begin{cases} \frac{\partial C}{\partial t}\left(x,t,\frac{p}{|p|}\right) \geq 0, \qquad 0 \leq t \leq T,\ x \in \mathbf{R}^n,\ p \in \mathbf{R}^n \\ \text{and} \\ u_\varepsilon(t,x',x_n) \textit{ are convex in } (x',x_n) \in \mathbf{R}^{n-1} \times (0,\varepsilon) \\ \textit{for all } t \in (0,T) \textit{ and all } \varepsilon \in (0,1), \end{cases} \tag{26}$$

there exists a unique function $\bar{v}(t,x')$ for which (23) and (24) hold with (25), by replacing $t \geq 0$ in (23) and (24) to $0 \leq t \leq T$.
(iii) If there exists $T > 0$ such that

$$\begin{cases} \frac{\partial C}{\partial t}\left(x,t,\frac{p}{|p|}\right) \leq 0,\ 0 \leq t \leq T,\ x \in \mathbf{R}^n,\ p \in \mathbf{R}^n \\ \textit{and} \\ u_\varepsilon(t,x',x_n) \text{ are concave in } (x',x_n) \in \mathbf{R}^{n-1} \times (0,\varepsilon) \\ \textit{for all } t \in (0,T) \textit{ and all } \varepsilon \in (0,1), \end{cases} \tag{27}$$

there exists a unique function $\bar{v}(t,x')$ for which (23) and (24) hold with (25), by replacing $t \geq 0$ in (23) and (24) to $0 \leq t \leq T$.

Theorem 1.2. *Let $v_\varepsilon(\varepsilon > 0)$ be the solutions of (16)-(18). Assume that (8),(19) hold. Then there exists an unique function $\bar{v}(t,x')$ such that*

$$\lim_{\varepsilon \downarrow 0} v_\varepsilon(t,x',\eta) = \bar{v}(t,x') \quad \textit{locally uniformly in} \quad t > 0,\ x' \in \mathbf{R}^{n-1} \tag{28}$$

and $\bar{v}$ is the solution of

$$\frac{\partial \bar{v}}{\partial t} = M\left(\frac{(\nabla_{x'}\bar{v},0)}{|\nabla_{x'}\bar{v}|}\right)\ |\nabla_{x'}\bar{v}| \qquad t \geq 0, \quad x' \in \mathbf{R}^{n-1}, \tag{29}$$

$$\lim_{t\downarrow 0}\ \bar{v}(t,x') = \min_{\eta\in[0,1]}\ v_0(x',\eta) \quad x' \in \mathbf{R}^{n-1}. \tag{30}$$

The answers to (A) and (B) for (1)-(3) and (4)-(6) follow from the above Theorems.

Theorem 1.3. *Let $u_\varepsilon(\varepsilon > 0)$ be the solutions of (1)-(3). Assume that (7),(8),(20) and (21) hold. Then, we have the following.*
(i) If (22) holds, there exists an unique function $\bar{u}(t,x')$ $(= \bar{v}(t,x')$ in Theorem 1.1) and

$$\lim_{\varepsilon\downarrow 0}\ \sup_{[0,T']\times\{|x'|\leq R\}\times(0,\varepsilon)} |u_\varepsilon(t,x',x_n) - \bar{u}(t,x')| = 0 \quad \textit{for any } T' > 0,\ R > 0 \tag{31}$$

and $\bar{u}$ is the solution of (24).
(ii) If (26) holds, there exists an unique function $\bar{u}(t,x')(= \bar{v}(t,x'))$ and (24),(31) hold by replacing $[0,T']$ to $[0,T\wedge T']$, and $t \geq 0$ to $0 \leq t \leq T$.
(iii) If (27) holds, there exists an unique function $\bar{u}(t,x')(= \bar{v}(t,x'))$ and (24),(31) hold by replacing $[0,T']$ to $[0,T\wedge T']$, and $t \geq 0$ to $0 \leq t \leq T$.

Theorem 1.4. *Let $u_\varepsilon(\varepsilon > 0)$ be the solutions of (4)-(6). Assume that (8),(19) hold. Then there exists an unique function $\bar{u}(t,x')(= \bar{v}(t,x')$ in Theorem 1.2) such that*

$$\lim_{\varepsilon\downarrow 0}\ \sup_{[t_1,T_1]\times\{|x'|\leq R\}\times(0,\varepsilon)} |u_\varepsilon(t,x',x_n) - \bar{u}(t,x')| = 0 \tag{32}$$

$$\textit{for all } t_1,\ T_1\ \in (0,\infty) \textit{ satisfying } t_1 < T_1 \textit{ and all } R > 0,$$

and $\bar{u}$ is the solution of (29) with the initial condition

$$\lim_{t\downarrow 0}\bar{u}(t,x') = \min_{\eta\in[0,1]} v_0(x',\eta) = \min_{x_n\in[0,\varepsilon]} u_\varepsilon(x',x_n) \quad x' \in \mathbf{R}^{n-1}. \tag{33}$$

Remark 1.1. By Theorem 1.3, (24), (25), we see that the limit interfacial energy is $\bar{\gamma}(p') = \gamma(p',0)$ $(p' \in \mathbf{R}^{n-1})$. Clearly, $\bar{\gamma}$ is positively homogeneous of degree one. If γ is C^2, so is $\bar{\gamma}$. For γ and $\bar{\gamma}$ we define their Wulff shapes by

$$\mathrm{W}_\gamma = \{x \in \mathbf{R}^n;\ \langle x, q\rangle\ \leq \gamma(q) \qquad \text{for all } q \in \mathbf{R}^n\},$$
$$\mathrm{W}_{\bar{\gamma}} = \{x' \in \mathbf{R}^{n-1};\ \langle x', q'\rangle \leq \bar{\gamma}(q') \qquad \text{for all } q' \in \mathbf{R}^{n-1}\}.$$

It turns out that

$$\mathrm{W}_{\bar{\gamma}} = \mathrm{Proj}\ \mathrm{W}_\gamma\ , \tag{34}$$

where Proj is the orthogonal projection $\mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}^n$. (Indeed, if γ is convex and positively homogeneous of degree one, then the Wulff shape W_γ is the unique convex compact (nonempty) set whose support function equals to γ, i.e. $\gamma(p) = \sup_{x \in W_\gamma} \langle x, p \rangle$, $p \in \mathbf{R}^n$; see e.g. [26, Theorem 1.7.1].

Thus

$$\bar{\gamma}(p') = \gamma(p', 0) = \sup\{\langle x', p' \rangle \ ; (x', x_n) \in W_\gamma \text{ for some } x_n\}, \ p' \in \mathbf{R}^{n-1}$$

or

$$\bar{\gamma}(p') = \sup\{\langle x', p' \rangle; \ x' \in \text{Proj } \mathrm{W}_\gamma\}.$$

Therefore (34) follows.) We thank Professor Mi-Ho Giga who suggested to consider anisotropy in a thin domain as well as this remark (without proof).

Remark 1.2. The initial condition (33) for the limit $\bar{u}$ the solutions of (4)-(6) shows the existence of the initial layer, which is particular to the first-order case. (The same phenomenon is pointed out in F. Bagagiolo and M. Bardi [7] , for the Hamilton-Jacobi operator.) We do not know if the initial layer occurs to (1)-(3).

Remark 1.3. Our convergence results do not directly imply the convergence of $\Gamma_t^\varepsilon \to \bar{\Gamma}_t$ since there may be a fattening phenomenon [12], [15]. If there is no fattening, our results imply the convergence $\Gamma_t^\varepsilon \to \bar{\Gamma}_t$ in local Hausdorff topology. This is easy to check as proved in M.-H. Giga & Y. Giga [15, Lemma 8.4].

Before ending this long introduction, we shall give the plan of the paper. §2 is devoted to the proofs of the convergences of $v_\varepsilon(u_\varepsilon)$ to $\bar{v}$, (23) and (28) in Theorems 1.1 and 1.2. In §3, we give the proofs of Theorems 1.1, 1.2 and of Theorems 1.3, 1.4. To prove these results, we implicitly use the so-called formal asymptotic expansion based on the averaging principle (or ergodicity) of the diffusion system and the deterministic controlled system. We derive the limit equation (24) by using this formal asymptotic expansion in §4.

Throughout of this paper, we use the theory of viscosity solutions, for which we refer the readers to M. G. Crandall - H. Ishii - P.-L. Lions [9].

§2. Proofs of the convergences of $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$

The goal of this section is to prove the following two results.

Proposition 2.1. *Let u_ε ($\varepsilon > 0$) be the solutions of (1)-(3), and assume that (7),(8),(20) and (21) hold. Then, there exists a constant $C > 0$ such that*

$$\sup_{\substack{x' \in \mathbf{R}^{n-1} \\ x_n \in (0,\varepsilon) \\ t \geq 0}} |\nabla_x u_\varepsilon| \leq C \qquad \textit{uniformly in} \quad \varepsilon \in (0, 1]. \tag{35}$$

Furthermore, if (22) holds, there exists a constant $C > 0$ such that

$$\sup_{\substack{t\geq 0\\ x'\in\mathbf{R}^{n-1}\\ x_n\in(0,\varepsilon)}} \left|\frac{\partial u_\varepsilon}{\partial t}\right| \leq C \qquad \textit{uniformly in} \quad \varepsilon\in(0,1], \tag{36}$$

and if (26) or (27) holds, there exists a constant $C > 0$ such that

$$\sup_{\substack{t\in[0,T]\\ x'\in\mathbf{R}^{n-1}\\ x_n\in(0,\varepsilon)}} \left|\frac{\partial u_\varepsilon}{\partial t}\right| \leq C \qquad \textit{uniformly in} \quad \varepsilon\in(0,1]. \tag{37}$$

Proposition 2.2. *Let $v_\varepsilon(\varepsilon > 0)$ be the solutions of (16)-(18), and assume that (8) and (19) hold. Then,*

$$\lim_{\substack{\varepsilon\to 0\\ t\to 0}} v_\varepsilon(t,x',\eta) \leq \min_{\zeta\in[0,1]} v_0(x',\zeta) \quad \textit{uniformly in } x'\in\mathbf{R}^n, \eta\in[0,1] \tag{38}$$

in the sense that

$$\lim_{\substack{\varepsilon\to 0\\ t\to 0}} \sup_{(x',\eta)\in\mathbf{R}^{n-1}\times[0,1]} [v_\varepsilon(t,x',\eta) - \min_{\zeta\in[0,1]} v_0(x',\zeta)] \leq 0$$

and

$$\lim_{\substack{\varepsilon\to 0\\ t\to 0}} v_\varepsilon(t,x',\eta) = \min_{\zeta\in[0,1]} v_0(x',\zeta) \quad x'\in\mathbf{R}^{n-1},\ \eta\in[0,1]. \tag{39}$$

In the case of the mean curvature flow equation ($M\equiv 1$, $\gamma = |p|$, $C\equiv 0$ in (1)), the estimate (35) and (36) are shown in Y. Giga, M. Ohnuma and M. -H. Sato [17]. We extend their proof to our case.

Proof of Proposition 2.1. First, we regularize (1) in the similar way to Y. -G. Chen, Y. Giga and S. Goto [13], [14]. Let ρ_δ $(\delta > 0)$ be the mollifier ($\rho\in C_c^\infty(\mathbf{R}^n)$, $\rho\geq 0$, Supp $\rho\in B(0,1)$ (the ball centered at origin with radius 1), $\int_{\mathbf{R}^n}\rho\, dx = 1$, $\rho_\delta(\cdot) = \frac{1}{\delta^n}\rho\left(\frac{\cdot}{\delta}\right)$). We set

$$\gamma_\delta(p) = \rho_\delta * \gamma(p) \ + \ \delta\sqrt{|p|^2+\delta^2} \qquad p\in\mathbf{R}^n,\ 0<\delta<1, \tag{40}$$

so that γ_δ is smooth on $\mathbf{R}^n$ and strictly convex,

$$M_\delta(p) = \rho_\delta * M\left(\frac{p}{|p|}\right),$$

$$\left\{\begin{array}{l} C_\delta(x,t,p) \text{ is a smooth approximation of } C\left(x,t,\frac{p}{|p|}\right) \\ \text{such that for any } \ t_1>0 \\ \displaystyle\sup_{0<\delta<1}\ \sup_{\mathbf{R}^n\times[0,t_1]\times\mathbf{R}^n} \left(|\nabla_x C_\delta| + \left|\frac{\partial C_\delta}{\partial t}\right| + |C_\delta|\right) < \infty \\ \displaystyle\text{and } C_\delta\to C,\ \frac{\partial C_\delta}{\partial t}\to\frac{\partial C}{\partial t},\ \frac{\partial C_\delta}{\partial x}\to\frac{\partial C}{\partial x} \text{ uniformly} \\ \text{in every compact set in } \mathbf{R}^n\times[0,t_1]\times S^{n-1}. \\ C_\delta \text{ satisfies (21).} \end{array}\right. \tag{41}$$

Consider

$$(42)\ \frac{\partial u_\varepsilon^\delta}{\partial t} = \sqrt{|\nabla u_\varepsilon^\delta|^2+\delta^2}\ M_\delta\ (\nabla u_\varepsilon^\delta)\ \left(\sum_{i=1} \frac{\partial}{\partial x_i}\frac{\partial \gamma_\delta}{\partial p_i}(\nabla u_\varepsilon^\delta)\right) - \sqrt{|\nabla u_\varepsilon^\delta|^2+\delta^2}\ C_\delta\ (x,t,\nabla u_\varepsilon^\delta)$$

$$\equiv\ F_\delta\ (\nabla\ u_\varepsilon^\delta,\ \nabla^2\ u_\varepsilon^\delta) - \sqrt{|\nabla u_\varepsilon^\delta|^2+\delta^2}\ C_\delta\ (x,t,\nabla u_\varepsilon^\delta)$$

$$t\ge 0,\ (x',x_n)\in \mathbf{R}^{n-1}\times(0,\varepsilon),$$

$$(43)\ \frac{\partial u_\varepsilon^\delta}{\partial x_n} = 0 \qquad t\ge 0,\quad (x',x_n)\in \mathbf{R}^{n-1}\times(0,\varepsilon),$$

$$(44)\ u_\varepsilon^\delta(0,x',x_n) = u_0(x') \qquad x'\in\mathbf{R}^{n-1},\ x_n\in(0,\varepsilon).$$

For the simplicity, we shall write $u(t,x',x_n) = u_\varepsilon^\delta(t,x',x_n)$ until the end of the proof. From (40) and the regularity results of the uniformly quasilinear parabolic equations (see e.g. Y. G. Chen, Y. Giga and S. Goto [14], O. A. Ladyzhenskya, V. A. Solonnikov and N. N. Ural'tseva, [20], G. M. Lieberman [21]), $u(t,x',x_n)$ exists globally and is C^1 in $t\ge 0$ and C^2 in $(x',x_n)\ \in \mathbf{R}^{n-1}\times(0,\varepsilon)$. To see (35), put $V=|\nabla u|^2$ and differentiate it in t to have

$$\begin{aligned}\frac{\partial V}{\partial t} &= 2\sum_i \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial t\partial x_i}\\ &= 2\sum_i \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\{F_\delta(\nabla u,\nabla^2 u) - \sqrt{|\nabla u|^2+\delta^2}\ C_\delta\ (x,t,\nabla u)\}.\end{aligned}\tag{45}$$

Now,

$$\begin{aligned}&2\sum_i \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i} F_\delta\ (\nabla u,\nabla^2 u)\\
=\ &2\sum_{i,\ell}\frac{\partial F_\delta}{\partial p_\ell}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i\partial x_\ell} + 2\sum_{k\ell}\frac{\partial F_\delta}{\partial r_{k\ell}}\sum_i \frac{\partial u}{\partial x_i}\left(\frac{\partial^3 u}{\partial x_k\partial x_\ell\partial x_i}\right)\\
=\ &\sum_\ell \frac{\partial F_\delta}{\partial p_\ell}\frac{\partial V}{\partial x_\ell} + \sum_{k\ell}\frac{\partial F_\delta}{\partial r_{k\ell}}\frac{\partial^2 V}{\partial x_k\partial x_\ell} - 2\sum_{k,\ell}\sum_i \frac{\partial F_\delta}{\partial r_{k\ell}}\frac{\partial^2 u}{\partial x_k\partial x_i}\frac{\partial^2 u}{\partial x_\ell\partial x_i}\\
\le\ &\sum_\ell \frac{\partial F_\delta}{\partial p_\ell}\frac{\partial V}{\partial x_\ell} + \sum_{k\ell}\frac{\partial F_\delta}{\partial r_{k\ell}}\frac{\partial^2 V}{\partial x_k\partial x_\ell},\end{aligned}\tag{46}$$

for from (40) the matrix $\left(\frac{\partial F_\delta}{\partial r_{k\ell}}\right)_{1\le k,\ell\le n}$ is positively definite. It follows that

$$\begin{aligned}&2\sum_i \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left\{\sqrt{|\nabla u|^2+\delta^2}\ C_\delta\ (x,t,\nabla u)\right\}\\
=\ &\sum_i \frac{C_\delta}{\sqrt{V+\delta^2}}\frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i} + 2\sqrt{V+\delta^2}\sum_i \frac{\partial u}{\partial x_i}\frac{\partial C_\delta}{\partial x_i} + \sqrt{V+\delta^2}\sum_\ell \frac{\partial C_\delta}{\partial p_\ell}\frac{\partial V}{\partial x_\ell}\\
\ge\ &\sum_i \frac{C_\delta}{\sqrt{V+\delta^2}}\frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i} + \sqrt{V+\delta^2}\sum_\ell \frac{\partial C_\delta}{\partial p_\ell}\frac{\partial V}{\partial x_\ell},\end{aligned}\tag{47}$$

from (21), (41).

By introducing (46), (47) into (45) we obtain

$$(48) \qquad \frac{\partial V}{\partial t} \leq \sum_{k\ell} \frac{\partial F_\delta}{\partial r_{k\ell}} \frac{\partial^2 V}{\partial x_k \partial x_\ell} + \sum_{\ell} \frac{\partial F_\delta}{\partial p_\ell} \frac{\partial V}{\partial x_\ell} - \sum_i \frac{C_\delta}{\sqrt{V+\delta^2}} \frac{\partial u}{\partial x_i} \frac{\partial V}{\partial x_i}$$
$$- \sqrt{V+\delta^2} \sum_\ell \frac{\partial C_\delta}{\partial p_\ell} \frac{\partial V}{\partial x_\ell} \qquad t \geq 0,\ (x', x_n) \in \mathbf{R}^{n-1} \times (0, \varepsilon).$$

From (43) it now follows that

$$(49) \quad \frac{\partial V}{\partial x_n} = 2\sum_i \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_n} = 2\sum_{i \neq n} \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_n} + 2 \frac{\partial u}{\partial x_n} \frac{\partial^2 u}{\partial x_n \partial x_n} = 0$$
$$t \geq 0,\ x' \in \mathbf{R}^{n-1},\ x_n = 0,\ \varepsilon.$$

Therefore, from the maximum principle, V attains its maximum at $t = 0$, and from (20) we get (35).

Next, to see (36) or (37), differentiate the both hands sides of (42) in t, and set $W = \frac{\partial u}{\partial t}$. Then,

$$(50) \quad \frac{\partial W}{\partial t} = \sum_i \frac{\partial F_\delta}{\partial p_i} \frac{\partial W}{\partial x_i} + \sum_{ij} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 W}{\partial x_i \partial x_j} - C_\delta \sum_i \frac{1}{\sqrt{|\nabla u|^2 + \delta^2}} \frac{\partial u}{\partial x_i} \frac{\partial W}{\partial x_i}$$
$$-\sqrt{|\nabla u|^2 + \delta^2}\, \frac{\partial C_\delta}{\partial t} - \sqrt{|\nabla u|^2 + \delta^2} \sum_i \frac{\partial C_\delta}{\partial p_i} \frac{\partial W}{\partial x_i}$$

with

$$(51) \qquad \frac{\partial W}{\partial x_n} = 0 \qquad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ x_n = 0, \varepsilon,$$

$$(52) \qquad W(0,x) = F_\delta\,(\nabla u_0, \nabla^2 u_0) - \sqrt{|\nabla u_0|^2 + \delta^2}\, C_\delta\,(x,\ 0,\ \nabla u_0).$$

Thus, if (22) holds, from the maximum principle and (20),

$$\sup_{\substack{t \geq 0 \\ x' \in \mathbf{R}^{n-1} \\ x_n \in (0,\varepsilon)}} |W(t, x', x_n)| \leq \sup_{\substack{x' \in \mathbf{R}^{n-1} \\ x_n \in (0,\varepsilon)}} W(0,x) \leq C \qquad \text{for all } \varepsilon > 0,$$

and (36) is proved. On the other hand, if (26) holds,

$$\frac{\partial W}{\partial t} \leq \sum_i \frac{\partial F_\delta}{\partial p_i} \frac{\partial W}{\partial x_i} + \sum_{ij} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 W}{\partial x_i \partial x_j} - C_\delta \sum_i \frac{1}{\sqrt{|\nabla u|^2 + \delta^2}} \frac{\partial u}{\partial x_i} \frac{\partial W}{\partial x_i}$$
$$-\sqrt{|\nabla u|^2 + \delta^2} \sum_i \frac{\partial C_\delta}{\partial p_i} \frac{\partial W}{\partial x_i}$$

from which and (51), we know that W takes its maximum at $t = 0$, that is from (52)

$$(53) \qquad \sup_{\substack{0 \leq t \leq T \\ x' \in \mathbf{R}^{n-1} \\ x_n \in (0,\varepsilon)}} W(t, x', x_n) \leq \sup_{\substack{x'a \in \mathbf{R}^{n-1} \\ x_n \in (0,\varepsilon)}} W(0,x) \leq C \qquad \text{for all } \varepsilon > 0.$$

Moreover, since u_ε are convex ((26)) and since we have shown that $|V|_\infty$ is bounded, from (42) there is a bound C such that

$$W \geq C \qquad \text{for all } \varepsilon > 0. \tag{54}$$

Thus (53) and (54) yield (37). The similar argument leads (37), if we assume (27).

Proof of Proposition 2.2. Let M_i $(i = 1, 2)$ be the constants in (8), and let $V_\varepsilon^i (i = 1, 2)$ are respectively the solutions of

$$\frac{\partial V_\varepsilon^i}{\partial t} = M_i \sqrt{|\nabla_{x'} V_\varepsilon^i|^2 + \left(\varepsilon^{-1}\frac{\partial V_\varepsilon^i}{\partial \eta}\right)^2} \quad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta \in (0,1), \tag{55}$$

$$\frac{\partial V_\varepsilon^i}{\partial \eta} = 0 \qquad t \geq 0,\ x' \in \mathbf{R}^{n-1},\ \eta = 0, 1, \tag{56}$$

$$V_\varepsilon^i(0, x', \eta) = v_0(x', \eta) \quad x' \in \mathbf{R}^{n-1},\ \eta \in (0,1). \tag{57}$$

By writing (55) as

$$\frac{\partial V_\varepsilon^i}{\partial t} = M_i \sup_{\substack{(\alpha', \alpha_n) \in \mathbf{R}^n \\ |\alpha'|^2 + \alpha_n{}^2 = 1 \text{ or } 0}} \left\langle \left(\nabla_{x'} V_\varepsilon^i, \frac{\partial V_i^\varepsilon}{\partial \eta}\right), (\alpha', \varepsilon^{-1}\alpha_n) \right\rangle, \tag{55$'$}$$

we can write the solutions V_ε^i explicitely as follows. (See P. -L. Lions [22], [23].)

$$V_\varepsilon^i(t, x', \eta) = \inf_{\alpha = (\alpha', \alpha_n) \in \mathcal{A}} v_0(X'_\alpha(t), X_{n\alpha}(t)) \tag{58}$$

where $\mathcal{A}$ is the set of measurable functions such that

$$\alpha(t) : [0, \infty) \to \{p \in \mathbf{R}^n \,|\, |p| = 0 \text{ or } 1\}$$

and $X_\alpha(t) = (X'_\alpha(t), X_{n\alpha}(t)) \in \mathbf{R}^{n-1} \times (0,1)$ is the solution of the following ordinary differential equation with reflection on the boundary : for $\alpha(t) = (\alpha'(t), \alpha_n(t)) \in \mathcal{A}$,

$$\begin{cases} X'_\alpha(t) = x' + \int_0^t \alpha'(s)\mathrm{ds} & t \geq 0 \\ X_{n\alpha}(t) = \eta + \int_0^t \frac{1}{\varepsilon}\alpha_n(s)\mathrm{ds} - \int_0^t n(X'_\alpha(s), X_{n\alpha}(s))\mathrm{d}A_s & t \geq 0 \\ \text{with } (X'_\alpha(t), X_{n\alpha}(t)) \in \mathbf{R}^{n-1} \times (0,1) & \text{for all } t \geq 0, \\ A_t = \int_0^1 1_{\partial\Omega}(X'_\alpha(s), X_{n\alpha}(s))\mathrm{d}A_s & \text{for } t \geq 0 \text{ is continuous} \\ \text{and nondecreasing,} \end{cases} \tag{59}$$

where $n(\cdot)$ denotes the outward unit normal on the boundary.

Therefore, by using the comparison principle to (55)-(57) $(i = 1, 2)$ and to (16)-(18), by using (58), (19) we have

$$-||v_0||_\infty \leq V_\varepsilon^1(t, x', \eta) \leq v_\varepsilon(t, x', \eta) \leq V_\varepsilon^2(t, x', \eta) \leq ||v_0||_\infty \qquad \text{for all } \varepsilon > 0. \tag{60}$$

Now, let $t_0 > 0$ be fixed and put $\varepsilon_0 = M_2 t_0$.

For an arbitrary $(x_0', \eta_0) \in \mathbf{R}^{n-1} \times (0,1)$, let $\bar{\eta} \in [0,1]$ be

$$\inf_{\eta \in (0,1)} v_0(x', \eta) = v_0(x', \bar{\eta}) \tag{61}$$

Since $|\eta_0 - \bar{\eta}| \leq 1$, $\varepsilon_0 = M_2 t_0 \leq \dfrac{M_2 t_0}{|\bar{\eta} - \eta_0|}$. For any $0 < \varepsilon < \varepsilon_0$, define the control $\alpha_\varepsilon(t) \in \mathcal{A}$ as follows $\Big(\alpha_\varepsilon(t) = (\alpha_\varepsilon'(t), \alpha_{n\varepsilon}(t))\Big)$

$$\alpha_\varepsilon'(t) \equiv 0 (\in \mathbf{R}^{n-1}) \quad \text{for all } t \geq 0,$$
$$\alpha_{n\varepsilon}(t) \begin{cases} = \operatorname{sgn}(\bar{\eta} - \eta_0) 1 & 0 < t < \frac{\varepsilon|\bar{\eta}-\eta_0|}{M_2} \\ = 0 & t \geq \frac{\varepsilon|\bar{\eta}-\eta_0|}{M_2} \end{cases},$$

and solve (59) with $\alpha_\varepsilon(t)$, and the initial condition $(x', \eta) = (x_0', \eta_0)$ to have

$$X_{\alpha_\varepsilon}'(t) = x_0', \; X_{n\alpha_\varepsilon}(t) = \bar{\eta} \quad \text{for all } t \geq \frac{\varepsilon|\bar{\eta} - \eta_0|}{M_2}.$$

By introducing this into (58) for $i = 2$, we get

$$V_\varepsilon^2(t, x_0', \eta_0) \leq v_0(x_0', \bar{\eta}) = \inf_{\eta \in [0,1]} v_0(x_0', \eta) \quad \text{for all } t \geq \frac{\varepsilon}{M_2}, \tag{62}$$

and by combining this inequality with (60), we have

$$\begin{cases} \limsup\limits_{\substack{\varepsilon \to 0 \\ t \to 0}} v_\varepsilon(t, x', \eta) \leq \inf\limits_{\eta \in [0,1]} v_0(x_0', \eta) \\ \qquad\qquad \text{where the convergence is uniform in } x' \in \mathbf{R}^{n-1} \end{cases} \tag{63}$$

Clearly, for an arbitrary $(x_0', \eta_0) \in \mathbf{R}^{n-1} \times (0,1)$ and $\bar{\eta}$ defined in (61),

$$V_\varepsilon^1(0, x_0', \bar{\eta}) \leq V_\varepsilon^1(0, x_0', \eta_0),$$

and from the continuity to the initial value, there exists $t_\varepsilon > 0$ depending on (x_0', η_0) such that

$$v_0(x_0', \bar{\eta}) \leq V_\varepsilon^1(t, x_0', \bar{\eta}) \leq V_\varepsilon^1(t, x_0', \eta_0).$$

Thus

$$\inf_{\eta \in [0,1]} v_0(x_0', \eta) \leq \liminf_{\substack{\varepsilon \to 0 \\ t \to 0}} V_\varepsilon^1(t, x_0', \eta_0). \tag{64}$$

From (60), (63) and (64), (38) and (39) were proved.

§3. Effective limit equations

This section is devoted to the proofs of Theorems 1.1, 1.2, 1.3 and 1.4.

Proof of Theorem1.1. We shall prove only (i), for (ii) and (iii) can be proved similarly. From (35), (36) in Proposition 2.1,

$$|\nabla_{x'}v_\varepsilon|_{L^\infty(\mathbf{R}^+\times\mathbf{R}^{n-1}\times[0,1])},\ |\frac{\partial v_\varepsilon}{\partial t}|_{L^\infty(\mathbf{R}^+\times\mathbf{R}^{n-1}\times[0,1])} < C \quad \text{for all } \varepsilon\in(0,1),$$
$$|\frac{\partial v_\varepsilon}{\partial \eta}|_{L^\infty(\mathbf{R}^+\times\mathbf{R}^{n-1}\times[0,1])} < C\varepsilon \qquad \text{for all } \varepsilon\in(0,1).$$

Thus, we can extract a subsequence $\varepsilon'\downarrow 0$ such that there is $\bar{v}(t,x')$

$$\lim_{\varepsilon'\downarrow 0} v_{\varepsilon'}(t,x',\eta)=\bar{v}(t,x') \quad \text{locally uniformly in} \quad t\geq 0,\ x'\in\mathbf{R}^{n-1},\ \eta\in[0,1]. \tag{65}$$

Hereafter we shall show that $\bar{v}$ is the unique solution of (24), which also proves (23) that $\bar{v}$ does not depend on the choise of the subsequence $\varepsilon'\downarrow 0$. For this purpose, we must first show the following : if for some $\phi(t,x')\in C^\infty((\mathbf{R}^+\cup\{0\})\times\mathbf{R}^{n-1})$, $\bar{v}-\phi$ takes a strict maximum at $(t_0,x_0)\in\mathbf{R}^+\times\mathbf{R}^{n-1}$, $\bar{v}(t_0,x_0)=\phi(t_0,x_0)$ and $\nabla'_x\ \phi\ (t_0,x_0)\neq 0$.

$$\frac{\partial\phi}{\partial t}(t_0,x_0)\leq\ |\nabla_{x'}\phi(t_0,x_0)|\ M\left(\frac{\left(\nabla_{x'}\phi(t_0,x_0),0\right)}{|\nabla_{x'}\phi(t_0,x_0)|}\right)\ \mathrm{div}_{x'}\left(\mathrm{D}_{p'}\bar{\gamma}\ (\nabla_{x'}\phi(t_0,x_0))\right)$$
$$-|\nabla_{x'}\phi(t_0,x_0)|\ C\ \left(x_0,0,t_0,\frac{(\nabla_{x'}\phi(t_0,x_0),0)}{|\nabla_{x'}\phi(t_0,x_0)|}\right) \tag{66}$$

holds, where $\bar{\gamma}(p')=\gamma(p',0)$. We assume that (66) does not hold, and shall look for the contradiction. Assume that there exists $\theta>0$ such that

$$\frac{\partial\phi}{\partial t}(t_0,x_0)\geq\ |\nabla_{x'}\phi(t_0,x_0)|\ M\left(\frac{(\nabla_{x'}\phi(t_0,x_0),0)}{|\nabla_{x'}\phi(t_0,x_0)|}\right)\mathrm{div}_{x'}\left(\mathrm{D}_{p'}\bar{\gamma}(\nabla_{x'}\phi(t_0,x_0))\right)$$
$$-|\nabla_{x'}\phi(t_0,x_0)|\ C\ \left(x_0,0,t_0,\frac{\left(\nabla_{x'}\phi(t_0,x_0),0\right)}{|\nabla_{x'}\phi(t_0,x_0)|}\right)+2\theta \tag{67}$$

Put for $\varepsilon>0$

$$\tilde{\phi}(t,x',\eta)=\phi(t,x') \qquad t\geq 0,\ x'\in\mathbf{R}^{n-1},\ \eta\in(0,1).$$

From (67), We can easily check that $\tilde{\phi}$ satisfies

$$\frac{\partial\tilde{\phi}}{\partial t}(t,x)\geq\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)^2}\ M\ \left(\frac{\left(\nabla_{x'}\tilde{\phi}(t,x),\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)}{\sqrt{|\nabla_{x'}\tilde{\phi}|^2+(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta})^2}}\right)\left\{\mathrm{div}_{x'}\left(D_{p'}\bar{\gamma}(\nabla_{x'}\tilde{\phi}(t,x))\right)\right.$$
$$+\varepsilon^{-1}\sum_{i=1}^{n}\frac{\partial^2\gamma}{\partial p_i\partial p_n}\left(\left(\nabla_{x'}\tilde{\phi}(t,x),\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)\right)\frac{\partial^2\tilde{\phi}}{\partial x_i\partial\eta}$$

$$+\varepsilon^{-2}\frac{\partial^2\gamma}{\partial p_{n^2}}\left(\left(\nabla_{x'}\tilde{\phi}(t,x),\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)\right)\frac{\partial^2\tilde{\phi}}{\partial\eta^2}\Bigg\}$$

$$-\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}(t,x)\right)^2}\,C\left(x',\varepsilon\eta,t,\frac{\left(\nabla_{x'}\tilde{\phi}(t,x),\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}(t,x)\right)}{\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)^2}}\right)+\theta$$

in $B_r(t_0)\times B_r(x_0)\times(0,1)$,

provided that $\varepsilon>0,\ r>0$ are small enough. The above inequality is equivalent to

$$(68)\quad \frac{\partial\tilde{\phi}}{\partial t}(t,x)\geq\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)^2}\,M\left(\frac{\left(\nabla_{x'}\tilde{\phi},\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)}{\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)^2}}\right)TrG_\varepsilon(\nabla^2_{(x,\eta)}\tilde{\phi})$$

$$-\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)^2}\,C\left(x',\varepsilon\eta,t,\frac{\left(\nabla_{x'}\tilde{\phi},\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)}{\sqrt{|\nabla_{x'}\tilde{\phi}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{\phi}}{\partial\eta}\right)^2}}\right)+\theta$$

in $B_r(t_0)\times B_r(x_0)\times(0,1)$.

Since v_ε satisfies (12) in $B_r(t_0)\times\ B_r(x_0)\times(0,1)$, we have

$$(69)\qquad \max_{B_r(t_0)\times\ B_r(x_0')\times\overline{(0,1)}}(v_\varepsilon-\tilde{\phi})\ \leq\ \max_{\partial\left(B_r(t_0)\times\ B_r(x_0')\times\overline{(0,1)}\right)}(v_\varepsilon-\tilde{\phi})$$

and by letting $\varepsilon\downarrow 0$, we have

$$\max_{B_r(t_0)\times\ B_r(x_0')}(\bar{v}-\phi)\ \leq\ \max_{\partial\left(B_r(t_0)\times\ B_r(x_0')\right)}(\bar{v}-\phi)$$

which contradicts to the fact that (t_0,x_0') is the local strict maximum of $\bar{v}-\phi$. According to definition of the viscosity solution of (24), it remains to consider the case when $\nabla_{x'}\phi\ (t_0,x_0)=0$ and $\nabla_{x'}\nabla_{x'}\phi\ (t_0,x_0)=0$; see [12], [13], [16].

For this purpose, assume that

$$(66)'\qquad \frac{\partial\phi}{\partial t}\ (t_0,x_0)\geq 2\ \theta$$

for some $\theta>0$, and shall look for a contradiction. By using the same notations as before, this yields (68) when $\nabla_{x'}\tilde{\phi}\ (t,x',\eta)\neq 0$, and $\frac{\partial\tilde{\phi}}{\partial t}\ (t,x',\eta)\geq 2\ \theta$ if $\nabla_{x'}\tilde{\phi}\ (t,x',\eta)=0$ and $\nabla_{x'}\nabla_{x'}\tilde{\phi}\ (t,x',\eta)=0$. The same argument leads (69), from which we get a contradiction.

Proof of Theorem 1.2. Set

$$\bar{v}_0(x')=\inf_{\eta\in(0,1)}v_0(x',\eta)\qquad x'\in\mathbf{R}^{n-1}.$$

Let $\bar{v}(t,x')$ be the unique solutions of

$$\frac{\partial \bar{v}}{\partial t} = M\left(\frac{(\nabla_{x'}\bar{v},0)}{|\nabla_{x'}\bar{v}|}\right)|\nabla_{x'}\bar{v}| \qquad t\geq 0,\ x'\in \mathbf{R}^{n-1},$$
$$\bar{v}(0,x')=\bar{v}_0(x').$$

Put

$$\tilde{v}(t,x',\eta)=\bar{v}(t,x') \qquad t\geq 0,\ x'\in\mathbf{R}^{n-1},\ \eta\in(0,1).$$

It is easy to check that for any $\varepsilon>0$, $\tilde{v}$ is the unique solution of

$$\text{(70)}\quad \frac{\partial\tilde{v}}{\partial t} = M\left(\frac{\left(\nabla_{x'}\tilde{v},\varepsilon^{-1}\frac{\partial\tilde{v}}{\partial\eta}\right)}{\sqrt{|\nabla_{x'}\tilde{v}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{v}}{\partial\eta}\right)^2}}\right)\sqrt{|\nabla_{x'}\tilde{v}|^2+\left(\varepsilon^{-1}\frac{\partial\tilde{v}}{\partial\eta}\right)^2} \quad t\geq 0,\ x'\in\mathbf{R}^{n-1},$$
$$\eta\in(0,1),$$
$$\text{(71)}\quad \frac{\partial\tilde{v}}{\partial\eta}=0 \qquad t\geq 0,\ x'\in\mathbf{R}^{n-1},\ \eta=0,1,$$
$$\text{(72)}\quad \tilde{v}(0,x',\eta)=\bar{v}_0(x').$$

The following estimate holds.

Lemma 3.1. *There exists a constant $M_3>0$ such that*

$$\text{(73)}\qquad |\tilde{v}\ (t,x',\eta)-\tilde{v}(0,x',\eta)|\leq M_3 t \qquad t\geq 0,\ x'\in\mathbf{R}^{n-1},\ \eta\in(0,1).$$

Proof of Lemma 3.1. Let M_i $(i=1,2)$ be the constants in (8). Consider for i=1, 2

$$\frac{\partial V_i}{\partial t} = M_i\,|\nabla_{x'}V_i\,| \qquad t\geq 0,\ x'\in\mathbf{R}^{n-1}$$
$$V_i(0,x')=\bar{v}_0(x').$$

From the comparison,

$$\text{(74)}\qquad V_1(t,x')\leq\bar{v}(t,x')\leq V_2(t,x') \qquad t\geq 0,\quad x'\in\mathbf{R}^{n-1}.$$

On the other hand, as explained in the proof of Proposition 2.2,

$$\text{(75)}\qquad V_i(t,x')=\inf_{\alpha\in\mathcal{A}_i}\bar{v}_0\ (X'_\alpha(t)) \qquad i=1,\ 2,$$

where $\mathcal{A}_i=\{\alpha(t):[0,\infty)\to\{0,M_i\}$ measurable function $\}$, and for $\alpha(\cdot)\in\mathcal{A}_i$

$$\text{(76)}\qquad X'_\alpha(t)=x'+\int_0^t\alpha(s)\ \mathrm{ds}.$$

By taking account of (76), (75) can be rewritten as

$$\text{(75)}'\qquad V_i(t,x')=\inf_{\substack{|y'-x'|\leq M_i t\\ y'\in\mathbf{R}^{n-1}}}\bar{v}_0(y').$$

Since $\bar{v}_0(x')$ is Lipschitz $\big((19)\big)$, from (75)$'$

$$V_1(t,x') \geq \bar{v}_0(x') - \mathrm{M}_1 t \qquad (77)$$
$$V_2(t,x') \leq \bar{v}_0(x') + M_2 t.$$

Thus, by (74) and (75),

$$-M_1 t \leq \tilde{v}(t,x',\eta) - \tilde{v}(0,x',\eta) = \bar{v}(t,x') - \bar{v}(0,x') \leq M_2 t\ ,$$

and (73) was proved.

We go back to the proof of Theorem 1.2. From now on, we shall prove

$$\lim_{\varepsilon\downarrow 0} v_\varepsilon(t,x',\eta) = \tilde{v}(t,x',\eta) = \bar{v}(t,x'). \qquad (78)$$

From Proposition 2.2,

$$\varlimsup_{\substack{\varepsilon\to 0\\ t\to 0}} v_\varepsilon(t,x',\eta) \leq \bar{v}_0(x') \qquad \text{uniformly in } \ x' \in \mathbf{R}^{n-1}.$$

Thus, by putting.

$$\delta_{\varepsilon,t} = \sup\{(v_\varepsilon(t,x',\eta) - \bar{v}_0(x')) \vee 0\ ;\ x' \in \mathbf{R}^{n-1}, \eta \in (0,1)\},$$

$$\lim_{\substack{\varepsilon\to 0\\ t\to 0}}\ \delta_{\varepsilon,t}\ \downarrow 0\ . \qquad (79)$$

Fix $t_0 > 0$, and consider

$$\frac{\partial \omega_\varepsilon^\pm}{\partial t} = M\left(\frac{\left(\nabla_{x'}\omega_\varepsilon^\pm, \varepsilon^{-1}\frac{\partial\omega_\varepsilon^\pm}{\partial\eta}\right)}{\sqrt{|\nabla_{x'}\omega_\varepsilon^\pm|^2 + \left(\varepsilon^{-1}\frac{\partial\omega_\varepsilon^\pm}{\partial\eta}\right)^2}}\right)\sqrt{|\nabla_{x'}\omega_\varepsilon^\pm|^2 + \left(\varepsilon^{-1}\frac{\partial\omega_\varepsilon^\pm}{\partial\eta}\right)^2},$$
$$t \geq t_0, \quad x' \in \mathbf{R}^{n-1}, \quad \eta \in (0,1),$$
$$\frac{\partial\omega_\varepsilon^\pm}{\partial\eta} = 0 \qquad t \geq t_0, \quad x' \in \mathbf{R}^{n-1}, \quad \eta = 0 \text{ and } 1,$$
$$\omega_\varepsilon^\pm(t_0,x',\eta) = \bar{v}_0(x') \pm \delta_{\varepsilon,t_0} \qquad x' \in \mathbf{R}^{n-1}, \quad \eta \in (0,1)$$

where $\varepsilon > 0$ is an arbitray number in $\varepsilon \in (0, M_2 t_0)$. Of course from (70)-(72),

$$\omega_\varepsilon^\pm(t,x',\eta) = \tilde{v}(t-t_0,x',\eta) \pm\ \delta_{\varepsilon,t_0}. \qquad (80)$$

From (79) and (80),

$$v_\varepsilon(t,x',\eta) \leq \omega_\varepsilon^+(t,x',\eta) = \tilde{v}(t-t_0,x',\eta) + \delta_{\varepsilon,t_0} \qquad (81)$$

Since

$$\bar{v}_0(x') - \delta_{\varepsilon,t_0} \leq \bar{v}_0(x') \leq v_\varepsilon(0,x',\eta) \qquad x' \in \mathbf{R}^{n-1}, \quad \eta \in (0,1),$$

by the comparison,

$$\tilde{v}(t-t_0,x',\eta)-\delta_{\varepsilon,t_0}=\omega_\varepsilon^-(t,x',\eta)\le v_\varepsilon(t,x',\eta). \tag{82}$$

By combining (81) and (82), letting $t_0\to 0$ $(\varepsilon\to 0)$, and by using (73) and (79), we get (78). Since $\bar{v}$ is the solution of (29) and (30), we proved Theorem 1.2.

Proofs of Theorems 1.3 and 1.4. Theorems 1.3 and 1.4 are derived easily from Theorems 1.1 and 1.2 by using (11).

§4. Relationship with the ergodic problem

The effective equation (24) for (1)-(3) can be derived formally as follows. We formally expand the solution $v_\varepsilon(t,x',\eta)$ of (12)-(14)

$$v_\varepsilon(t,x',\eta)=\bar{v}(t,x')+\varepsilon^2 v_1(t,x',\eta)+0(\varepsilon^2)\qquad t\ge 0,\ x'\in\mathbf{R}^{n-1},\ \eta\in(0,1). \tag{83}$$

By introducing (83) into (12), we have

$$\frac{\partial\bar{v}}{\partial t}=|\nabla_{x'}\bar{v}|\ M\ \left(\frac{(\nabla_{x'}\bar{v},0)}{|\nabla_{x'}\bar{v}|}\right)T_\gamma G'Y'-C\left(x',0,t,\frac{(\nabla_{x'}\bar{v},0)}{|\nabla_{x'}\bar{v}|}\right)+O(\varepsilon),\qquad t>0,\ x'\in\mathbf{R}^{n-1},\ \eta\in(0,1), \tag{84}$$

$$\frac{\partial\bar{v}}{\partial\eta}=0\qquad t>0,\ x'\in\mathbf{R}^{n-1},\ \eta=0,\ 1, \tag{85}$$

$$\bar{v}(0,x')=u_0(x')\qquad x'\in\mathbf{R}^{n-1},$$

where $G'=(\gamma'_{ij})$ is the $n\times n$ matrix such that

$$\begin{cases}\gamma'_{ij}=\gamma_{ij}(\nabla_{x'}\bar{v},O(\varepsilon)) & 1\le i,j\le n-1,\\ \gamma'_{ni}=\gamma'_{in}=\varepsilon^{-1}\gamma_{in}(\nabla_{x'}\bar{v},O(\varepsilon)) & 1\le i\le n-1,\\ \gamma'_{nn}=\varepsilon^{-2}\gamma_{nn}(\nabla_{x'}\bar{v},O(\varepsilon)),\end{cases} \tag{86}$$

where $\gamma_{ij}=\frac{\partial^2\gamma}{\partial p_i\partial p_j}$ $(1\le i,\ j\le n)$ and $Y'=(y'_{ij})$ is the $n\times n$ matrix such that

$$\begin{array}{ll} y'_{ij}=\frac{\partial^2\bar{v}}{\partial x_i\partial x_j} & 1\le i,j\le n-1,\\ y'_{ni}=y'_{in}=\varepsilon^2\frac{\partial^2 v_1}{\partial x_i\partial\eta} & 1\le i\le n-1,\\ y'_{nn}=\varepsilon^2\frac{\partial^2 v_1}{\partial\eta^2} & \end{array}$$

Remark that

$$T_rG'Y'=T_r\bar{G}\ \bar{Y}+O(\varepsilon)+\gamma_{nn}(\nabla_{x'}\bar{v},0)\ \frac{\partial^2 v_1}{\partial\eta^2}\ (t,x',\eta) \tag{87}$$

where $\bar{G} = (\gamma_{ij}(\nabla_x \bar{v}, 0))_{1\leq i,j\leq n-1}$ and $\bar{Y} = (\nabla^2_{x'}\bar{v}(t,x'))$ are the $(n-1)\times(n-1)$ matrices. Thus, if for an arbitray fixed $(t_0, x'_0, p'_0) \in \mathbf{R}^+ \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$, there exists an unique constant number $\bar{H}(t_0, x'_0, p'_0)$ such that the problem

$$(88)\qquad \begin{array}{ll} \bar{H}(t_0, x'_0, p'_0) = \gamma_{nn}(p'_0, 0)\, \frac{\partial^2 v_1}{\partial \eta^2} & \eta \in (0,1), \\ \frac{\partial v_1}{\partial \eta} = 0 & \eta = 0,\ 1, \end{array}$$

has at least a solution $v_1(\eta)$, this $v_1(\eta)$ may serve as the corrector $v_1(t, x', \eta)$ in the expansion (83), and from (84) and (87), we get

$$(89)\qquad \frac{\partial \bar{v}}{\partial t} = |\nabla'_x \bar{v}|\ M\ \left(\frac{(\nabla_{x'}\bar{v}, 0)}{|\nabla_{x'}\bar{v}|}\right) T_r\bar{G}\ \bar{Y} - C\left(x', 0, t, \frac{(\nabla_{x'}\bar{v}, 0)}{|\nabla_{x'}\bar{v}|}\right) + \bar{H}(t, x', \nabla_{x'}\bar{v}),$$

the effective limit equation for $\bar{v}(t,x')$. The problem (88) is called the ergodic problem for the Hamilton-Jacobi-Bellman equation, and we refer the readers to M. Arisawa [2] and [3], M. Arisawa - P. -L.Lions [6] etc. The following is a typical result of the ergodic problem with the Neumann boundary condition.

Proposition 4.1. *Let $a(\eta) >^{\exists} a_0 > 0$ be a function in $\eta \in (0,1)$, $f(\eta)$ be a Lipschitz continuous function in $\eta \in (0,1)$. Then there exists an unique number d_f such that*

$$\begin{cases} d_f - a(\eta)\frac{\partial^2 v_1}{\partial \eta^2}(\eta) - f(\eta) = 0 & \eta \in (0,\ 1), \\ \frac{\partial v_1}{\partial \eta} = 0 & \eta = 0,\ 1. \end{cases}$$

admits a viscosity solution $v_1(\eta)$.

By using Proposition 4.1, the above unique number in (88) $\bar{H}(t_0, x'_0, p'_0) \equiv 0$ $(v_1(\eta) \equiv$ constant), and introducing this into (89), we get the effective anisotropic mean curvature equation (24). In the preceding sections 2 and 3, we rigorously confirmed this formal argument.

References

[1] O. Alvarez and M. Bardi, *Viscosity solutions methods for singular perturbations in deterministic and stochastic control,* preprint.

[2] M. Arisawa, *Some ergodic problems for Hamilton-Jcobi equations in Hilbert space,* Differential Integral Equations, **9** (1996) 59-70.

[3] M. Arisawa, *Ergodic problem for the Hamilton-Jacobi-Bellman equation I. Existence of the ergodic attractor,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **14** (1997) 415-438.

[4] M. Arisawa, *Multiscale homogenizations for first-order Hamilton-Jacobi equations,* To appear in Advances in Differential Equations.

[5] M. Arisawa, *Quasi-periodic homogenizations for second-order Hamilton-Jacobi-Bellman equations,* To appear in J. Math. Anal. Appl.

[6] M. Arisawa and P.-L. Lions, *On ergodic stochastic control,* Comm. Partial Differential Equations **23** (1998) 2187-2217.

[7] F. Bagagiolo and M. Bardi, *Singular perturbation of a finite horizon problem with state-space constraints,* SIAM J. Control Optim. **36** (1998) 2040-2060.

[8] A. Bensoussan, J. L. Lions and G. Pananicolaou, *Asymptotic analysis for periodic structures,* Stud. Math. Appl. **5** North-Holland, Amsterdam, (1978).

[9] M. G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Math. Soc. **27** (1992) 1-67.

[10] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE,* Proc. Roy. Soc. Edinburgh **111A** (1989) 359-375.

[11] L. C. Evans, *Periodic homogenization of certain fully nonlinear partial differential equations,* Proc. Roy. Soc. Edinburgh **120A** (1992) 245-265.

[12] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. I,* J. Differential Geom. **33** (1991) 635-681.

[13] Y. G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* J. Differential Geom. **33** (1991), 749-786.

[14] Y. G. Chen, Y. Giga and S. Goto, *Analysis toward snow crystal growth, International Symposium on Functional Analysis and Related topics. (ed. S. Koshi),* World Scientific (1991) 43-57.

[15] M. -H. Giga and Y. Giga, *Generalized motion by nonlocal curvature in the plane,* Hokkaido University Preprint Series in Math. **#478** (2000).

[16] Y. Giga, *A level set method for surface evolution equations,* Sugaku **47** (1995), 321-340, English translation, Sugaku Expositions 10 (1997), 217-241.

[17] Y. Giga, M. Ohnuma and M.-H. Sato, *On the strong maximum principle and large time behavior of generalized mean curvature flow with the Neumann boundary condition,* J. Differential Equations **154** (1999) 107-131.

[18] M. E. Gurtin, *Thermomechanics of evolving Phase Boundaries in the Plane,* Oxford Mathematical Monographs, Clarendon Press, (1993).

[19] K. Horie and H. Ishii, *Homogenization of Hamilton-Jacobi equations on domains with small scale periodic structure.* Indiana Univ. Math. J. **47** (1998) 1011-1058.

[20] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quarilinear Equations of Parabolic Type,* American Math. Soc. Providence. R.I., (1968).

[21] G. M. Lieberman, *Second Order Parabolic Differential Equations,* World Scientific, Singapore, (1996).

[22] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations,* Research Notes in Mathematics **69**. Pitman, Boston MA (1982).

[23] P.-L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations,* Duke Math. J. **52** (1985) 793-820.

[24] P.-L. Lions and R. Jensen, *Some asymptotic peoblems in fully nonlinear elliptic equations and stochastic control,* Ann. Scuola Norm. Pisa Cl. Sci. (4) **11** (1989) 129-176.

[25] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan, *Homogenizations of Hamilton-Jacobi equations,* preprint.

[26] R. Schneider, *Convex Bodies: the Brunn-Minkowski theory,* Encyclopedia Math. Appl. **44** Cambridge University Press (1993)

[27] J. Taylor, J. Cahn and C. Handwerker, *Geometric models of crystal growth,* Acta Metall. **40** (1992), 1443-1474.