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Backward Shift Invariant Subspaces In The Bidisc

by

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Abstract. Suppose T_ϕ is a Toeplitz operator with a symbol ϕ on the Hardy space on the bidisc, H^2 . Let N be a backward shift invariant subspace of H^2 , that is, N is an invariant subspace under T_z^* and T_w^* . Let P be the orthogonal projection from H^2 onto N . For ϕ in H^∞ , put $S_\phi = PT_\phi|N$. In this paper, we are interested in backward shift invariant subspaces which satisfy $S_z S_w^* = S_w^* S_z$. In particular, we show that $S_z S_w^* = S_w^* S_z$ if N is of finite dimension.

§1. Introduction

Let T^2 be the torus that is the Cartesian product of two unit circles T in \mathbf{C} . Let $1 \leq p \leq \infty$. The usual Lebesgue spaces, with respect to the Haar measure m of T^2 , are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{T^2} f(z, w) \bar{z}^j \bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative. H^p is called the Hardy space. As $T^2 = (T, z) \times (T, w)$, $H^p(T, z)$ and $H^p(T, w)$ denote the one variable Hardy spaces.

Let P_{H^2} be the orthogonal projection from L^2 onto H^2 . For ϕ in L^∞ , the Toeplitz operator T_ϕ is defined by

$$T_\phi f = P_{H^2}(\phi f) \quad (f \in H^2).$$

A closed subspace N of H^2 is said to be backward shift invariant if

$$T_z^* N \subset N \quad \text{and} \quad T_w^* N \subset N.$$

A closed subspace M of H^2 is said to be shift invariant if $T_z M \subset M$ and $T_w M \subset M$. The orthogonal complement of N is shift invariant. In this paper, we are interested in backward shift invariant subspaces. Let P_N and P_M be the orthogonal projection from H^2 onto N and M , respectively. For ϕ in H^∞ , put

$$S_\phi = P_N T_\phi P_N|_N \quad \text{and} \quad V_\phi = P_M T_\phi P_M|M.$$

It is known in [4] that $V_z V_w^* = V_w^* V_z$ if and only if $M = qH^2$ for some inner function q in H^∞ . In this paper, we are interested in backward shift invariant subspaces which satisfy $S_z S_w^* = S_w^* S_z$. We will write $P = P_N$ and $Q = I - P_N$ where I is an identity operator on H^2 and N is a backward shift invariant subspace. In this paper, we study two operators

$$A = QT_z P \quad \text{and} \quad B = PT_w^* Q$$

when $AB = 0$ or $BA = 0$. $AB = 0$ is equivalent to $V_z V_w^* = V_w^* V_z$, and $BA = 0$ is equivalent to $S_z S_w^* = S_w^* S_z$. In §2, we show that $AB|M = V_w^* V_z - V_z V_w^*$ and $BA|N = S_z S_w^* - S_w^* S_z$. Moreover we determine invariant subspaces when $A = 0$ or $B = 0$. In §3, we show the following : If $BA = 0$, then $N \subseteq H^2 \ominus qH^2$ where q is a one variable inner function. Conversely if $N = H^2 \ominus qH^2$ and q is one variable, then $BA = 0$. As a result, $AB = BA = 0$ if and only if $A = 0$ or $B = 0$. Moreover we show that if N is of finite dimension, then $BA = 0$.

For an operator C , $\ker C$ denotes the kernel of C and $\text{ran } C$ denotes the range of C .

All results in this paper can be generalized to backward shift invariant subspaces in the polydisc by essentially same method. Such invariant subspaces have been studied by Ahern and Clark [1], and Cotlar and Sadosky [3] from different points of view.

§2. $A = 0$ or $B = 0$

Let N be a backward shift invariant subspace and M be the orthogonal complement of N in H^2 . Put $P = P_N$ and $Q = I - P_N$, then Q is the orthogonal projection from H^2 onto M .

Lemma 1

(1) $AB = QT_w^*QT_zQ - QT_zQT_w^*Q$ and so $AB|M = V_w^*V_z - V_zV_w^*$.

(2) $BA = PT_zPT_w^*P - PT_w^*PT_zP$ and so $BA|N = S_zS_w^* - S_w^*S_z$.

(3) $\ker A = \{f \in N : T_zf \in N\} \oplus M$.

(4) $\ker B = \{f \in M : T_w^*f \in M\} \oplus N$.

Proof. (1) Since $T_zQ = QT_zQ$ and $T_zT_w^* = T_w^*T_z$,

$$\begin{aligned} AB &= QT_zPT_w^*Q \\ &= QT_zT_w^*Q - QT_zQT_w^*Q \\ &= QT_w^*QT_zQ - QT_zQT_w^*Q. \end{aligned}$$

Since $T_w^*P = PT_w^*P$ and $T_w^*T_z = T_zT_w^*$,

$$\begin{aligned} BA &= PT_w^*QT_zP \\ &= PT_w^*T_zP - PT_w^*PT_zP \\ &= PT_zPT_w^*P - PT_w^*PT_zP. \end{aligned}$$

(3) and (4) are clear.

Theorem 2.

(1) $A = 0$ if and only if $N = H^2 \ominus qH^2$ where q is a one variable inner function with $q = q(w)$.

(2) $B = 0$ if and only if $M = qH^2$ where q is a one variable inner function with $q = q(z)$.

(3) $A = B = 0$ if and only if $N = \{0\}$ or $N = H^2$.

Proof. (2) follows from (1). We will show (1). $H^2 = N \oplus M$ and $T_zM \subset M$. Suppose $A = 0$. By (3) of Lemma 1, $T_zN \subset N$. Put $N_0 = N \ominus T_zN$ and $M_0 = M \ominus T_zM$. Then

$$H^2 = \sum_{n=0}^{\infty} \oplus (N_0 \oplus M_0)z^n = \sum_{n=0}^{\infty} \oplus H^2(T, w)z^n.$$

By (1) of Lemma 1, $V_w^*V_z = V_zV_w^*$ and so $V_z^*V_w = V_wV_z^*$ because $AB = 0$. Hence $V_w(\ker V_z^*) \subseteq \ker V_z^*$ and $\ker V_z^* = M_0$. Therefore by the Beurling theorem [2], $M_0 = qH^2(T, w)$ and q is a one variable inner function with $q = q(w)$. Hence $M = qH^2$ and so $N = H^2 \ominus qH^2$.

Corollary 1

(1) $AB = BA = 0$ if and only if $A = 0$ or $B = 0$.

(2) If $N = H^2 \ominus qH^2$ and q is an inner function and $S_z S_w^* = S_w^* S_z$, then q is a one variable inner function.

§3. $AB = 0$ or $BA = 0$

Suppose that N is a backward shift invariant subspace and M is a shift invariant subspace. By Lemma 1, $AB = 0$ if and only if $V_w^* V_z = V_z V_w^*$, and $BA = 0$ if and only if $S_z S_w^* = S_w^* S_z$. Hence we know (see [4],[5],[6]) that $AB = 0$ if and only if $M = qH^2$ for some inner function q . In this section, we study N when $BA = 0$, that is, $S_z S_w^* = S_w^* S_z$.

Lemma 2.

$$\text{ran } A = \{M \ominus zM\} \ominus \{H^2(T, w) \cap M\}$$

and

$$\text{ker } B = H^2(T, z) \cap M \oplus wM \oplus N$$

Proof. Since $(T_w^* f, g) = (f, wg)$ if $f, g \in H^2$,

$$\begin{aligned} & \{f \in M ; T_w^* f \in M\} \\ &= M \cap \{H^2 \ominus wN\} = H^2(T, z) \cap M \oplus wM \end{aligned}$$

because $H^2 \ominus wN = (H^2 \ominus wH^2) \oplus w(H^2 \ominus N)$ and $N = H^2 \ominus M$. Hence by (4) of Lemma 1, $\text{ker } B = \{f \in M ; T_w^* f \in M\} \oplus N = H^2(T, z) \cap M \oplus wM \oplus N$. By the same argument, $\text{ker } A^* = H^2(T, w) \cap M \oplus zM \oplus N$ and so

$$\begin{aligned} \text{ran } A &= H^2 \ominus \text{ker } A^* \\ &= \{M \ominus zM\} \ominus \{H^2(T, w) \cap M\}. \end{aligned}$$

Lemma 3

(1) Let q be a one variable inner function with $q = q(w)$. $N \subseteq H^2 \ominus qH^2$ if and only if $\text{ran } A \supseteq \{M \ominus zM\} \ominus qH^2(T, w)$.

(2) Let q be a one variable inner function with $q = q(z)$. $N \subseteq H^2 \ominus qH^2$ if and only if $\text{ker } B \supseteq qH^2(T, z) \oplus wM \oplus N$.

Proof. (1) If $\text{ran } A \supseteq \{M \ominus zM\} \ominus qH^2(T, w)$, then $qH^2(T, w) \subseteq M$ and so $N \subseteq H^2 \ominus qH^2$. If $N \subseteq H^2 \ominus qH^2$, then $M \supset qH^2(T, w)$ and $qH^2(T, w)$ is orthogonal to zM . By Lemma 2, $\text{ran } A \supseteq \{M \ominus zM\} \ominus qH^2(T, w)$. (2) If $\text{ker } B \supseteq qH^2(T, z) \oplus wM \oplus N$, then $qH^2(T, z) \subseteq M$ and so $N \subseteq H^2 \ominus qH^2$. If $N \subseteq H^2 \ominus qH^2$, then $M \supset qH^2(T, z)$ and $qH^2(T, z)$ is orthogonal to $wM \oplus N$. By Lemma 2, $\text{ker } B \supseteq qH^2(T, z) \oplus wM \oplus N$.

Lemma 4. When M is an invariant subspace of H^2 , if $wM \supseteq M \ominus zM$, then $M = \{0\}$.

Proof. Put $M_0 = M \ominus zM$, then $M = \sum_{j=0}^{\infty} \oplus M_0 z^j$. If $wM \supset M_0$, then $wM \supseteq \sum_{j=0}^{\infty} \oplus M_0 z^j = M$ because $zM \subseteq M$. This implies $M = \{0\}$.

Lemma 2 implies that $A = 0$ if and only if $M \ominus zM = H^2(T, w) \cap M$, and $B = 0$ if and only if $M = H^2(T, z) \cap M \oplus wM$. Hence they imply trivially the condition in (1) of Theorem 3.

Theorem 3.

(1) $S_z S_w^* = S_w^* S_z$, if and only if $H^2(T, z) \cap M \oplus wM \supseteq \{M \ominus zM\} \ominus \{H^2(T, w) \cap M\}$.

(2) If $S_z S_w^* = S_w^* S_z$, then $N \subseteq H^2 \ominus qH^2$ for some one variable inner function q or $N = H^2$.

(3) If $N = H^2 \ominus qH^2$ for some one variable inner function q , then $S_z S_w^* = S_w^* S_z$.

(4) If $S_z S_w^* = S_w^* S_z$, $A \neq 0$ and $B \neq 0$, then $N \subseteq (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ where $q_1 = q_1(w)$ and $q_2 = q_2(z)$ are one variable inner functions.

Proof. (1) $S_z S_w^* = S_w^* S_z$ if and only if $\ker B \supseteq \text{ran } A$. Hence Lemma 2 implies (1). (2) If $H^2(T, z) \cap M \neq \{0\}$, then by Lemma 2 and a theorem of Beurling $N \subseteq H^2 \ominus qH^2$ for some one variable inner function $q = q(z)$. If $H^2(T, z) \cap M = \{0\}$, then by (1) $wM \supseteq \{M \ominus zM\} \ominus \{H^2(T, w) \cap M\}$. If $H^2(T, w) \cap M \neq \{0\}$, then by Lemma 2 and a theorem of Beurling $N \subseteq H^2 \ominus qH^2$ for some one variable inner function $q = q(w)$. If $H^2(T, w) \cap M = \{0\}$, then $wM \supseteq M \ominus zM$ and so by Lemma 4, $M = \{0\}$ and so $N = H^2$. (3) By Theorem 2, $BA = 0$ and so $S_z S_w^* = S_w^* S_z$. (4) follows from the proof of (2).

Lemma 4. Suppose $|\alpha| < 1$ and $|\beta| < 1$.

$$(1) \frac{z}{1 - \bar{\alpha}z} = \frac{\alpha}{1 - \bar{\alpha}z} + \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

$$(2) \frac{z}{(1 - \bar{\alpha}z)^\ell} = \frac{-\frac{1}{\bar{\alpha}}}{(1 - \bar{\alpha}z)^{\ell-1}} + \frac{\frac{1}{\bar{\alpha}}}{(1 - \bar{\alpha}z)^\ell} \quad (\ell \geq 1).$$

$$(3) \frac{\bar{w}}{(1 - \bar{\beta}w)^m} = \sum_{j=1}^m \frac{\bar{\beta}}{(1 - \bar{\beta}w)^j} + \bar{w} \quad (m \geq 1).$$

Proof. (1) and (2) are clear. Since $\frac{\bar{w}}{1 - \bar{\beta}w} = \frac{\bar{\beta}}{1 - \bar{\beta}w} + \bar{w}$,

$$\begin{aligned} \frac{\bar{w}}{(1 - \bar{\beta}w)^m} &= \frac{1}{(1 - \bar{\beta}w)^{m-1}} \left(\frac{\bar{\beta}}{1 - \bar{\beta}w} + \bar{w} \right) \\ &= \frac{\bar{\beta}}{(1 - \bar{\beta}w)^m} + \frac{\bar{w}}{(1 - \bar{\beta}w)^{m-1}} \end{aligned}$$

$$= \frac{\bar{\beta}}{(1-\bar{\beta}w)^m} + \frac{\bar{\beta}}{(1-\bar{\beta}w)^{m-1}} + \frac{\bar{w}}{(1-\bar{\beta}w)^{m-2}}.$$

This implies (3)

Theorem 4. *If N is a backward shift invariant subspace of finite dimension in H^2 , then $S_z S_w^* = S_w^* S_z$.*

Proof. It is known in [1, p969] that N is spanned by partial derivatives of the Cauchy kernels $(1-\bar{\alpha}z)^{-1}(1-\bar{\beta}w)^{-1}$ for (α, β) in $Z(M) = \bigcap_{f \in M} \{(\alpha, \beta) \in D^2; f(\alpha, \beta) = 0\}$.

Here $Z(M) = \bigcup_{j=0}^k \{(\alpha_j, \beta_j)\}$ is a finite set in D^2 . We may assume that $(0, 0) \in Z(M)$ and $(\alpha_0, \beta_0) = (0, 0)$. When $(0, 0) \notin Z(M)$, the proof is easier than that of the case, $(0, 0) \in Z(M)$. We will prove that

$$S_z S_w^* = S_w^* S_z \text{ on } (1-\bar{\alpha}z)^{-\ell}(1-\bar{\beta}w)^{-m}$$

where $\ell \geq 0$, $m \geq 0$, $(\alpha, \beta) \in Z(M)$ and $(1-\bar{\alpha}z)^{-\ell}(1-\bar{\beta}w)^{-m} \in N$. When $(\alpha, \beta) = (0, 0)$, we assume that $(1-\bar{\alpha}z)^{-\ell}(1-\bar{\beta}w)^{-m} = z^\ell w^m$.

Case I. $\ell \geq 2$ and $m \geq 0$: Suppose $m \geq 1$. By Lemma 4, for $0 < j \leq m$

$$\frac{z}{(1-\bar{\alpha}z)^\ell} \frac{1}{(1-\bar{\beta}w)^j} \text{ belongs to } N$$

because $(1-\bar{\alpha}z)^{-\ell}(1-\bar{\beta}w)^{-m} \in N$ and $\ell \geq 2$. Hence since $m \geq 1$, by Lemma 4

$$\begin{aligned} S_z S_w^* (1-\bar{\alpha}z)^{-\ell}(1-\bar{\beta}w)^{-m} &= S_z P \left\{ (1-\bar{\alpha}z)^{-\ell} \frac{\bar{w}}{(1-\bar{\beta}w)^m} \right\} \\ &= \frac{z}{(1-\bar{\alpha}z)^\ell} \sum_{j=1}^m \frac{\bar{\beta}}{(1-\bar{\beta}w)^j} \end{aligned}$$

because $\frac{z}{(1-\bar{\alpha}z)^\ell} \frac{1}{(1-\bar{\beta}w)^j} \in N$ for $0 < j \leq m$. By the same argument, we can show

that $S_w^* S_z (1-\bar{\alpha}z)^{-\ell}(1-\bar{\beta}w)^{-m} = \frac{z}{(1-\bar{\alpha}z)^\ell} \sum_{j=1}^m \frac{\bar{\beta}}{(1-\bar{\beta}w)^j}$. Suppose $m = 0$. $S_w^* S_z (1-\bar{\alpha}z)^{-\ell} = S_w^* P \left\{ \frac{z}{(1-\bar{\alpha}z)^\ell} \right\} = S_w^* \frac{z}{(1-\bar{\alpha}z)^\ell} = 0$ because $\frac{z}{(1-\bar{\alpha}z)^\ell} \in N$. It is clear that $S_z S_w^* (1-\bar{\alpha}z)^{-\ell} = 0$.

Case II. $\ell \leq 1$ and $m \geq 0$: Suppose $\ell = 1$ and $m \geq 1$. Since $(\alpha, \beta) \in Z(M)$,

$$\frac{z}{1-\bar{\alpha}z} = \sum_{j=0}^k \frac{a_j}{1-\bar{\alpha}_j z} + \left(\prod_{j=D}^k \frac{z-\alpha_j}{1-\bar{\alpha}_j z} \right) g$$

where $g \in H^2(T, z)$. Hence as in Case I,

$$\begin{aligned}
& S_z^* S_w^* (1 - \bar{\alpha}z)^{-1} (1 - \bar{\beta}w)^{-m} \\
&= S_z \left\{ \frac{1}{1 - \bar{\alpha}z} \sum_{j=1}^m \frac{\bar{\beta}}{(1 - \bar{\beta}w)^j} \right\} \\
&= P \left\{ \frac{z}{1 - \bar{\alpha}z} \sum_{j=1}^m \frac{\bar{\beta}}{(1 - \bar{\beta}w)^j} \right\} \\
&= P \left\{ \left(\sum_{j=0}^k \frac{a_j}{1 - \bar{\alpha}_j z} \right) \left(\sum_{j=1}^m \frac{\bar{\beta}}{(1 - \bar{\beta}w)^j} \right) + \left(\prod_{j=0}^k \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \right) g \left(\sum_{j=0}^k \frac{a_j}{1 - \bar{\alpha}_j z} \right) \right\} \\
&= \left(\sum_{j=0}^k \frac{a_j}{1 - \bar{\alpha}_j z} \right) \left(\sum_{j=1}^m \frac{\bar{\beta}}{(1 - \bar{\beta}w)^j} \right)
\end{aligned}$$

because $\left(\prod_{j=0}^k \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \right) g \left(\sum_{j=0}^k \frac{a_j}{1 - \bar{\alpha}_j z} \right) \in M$. By the same argument, we can show that $S_w^* S_z (1 - \bar{\alpha}z)^{-1} (1 - \bar{\beta}w)^{-m} = \left(\sum_{j=0}^k \frac{a_j}{1 - \bar{\alpha}_j z} \right) \left(\sum_{j=1}^m \frac{\bar{\beta}}{(1 - \bar{\beta}w)^j} \right)$. For $\ell = m = 0$, it is easy to see it as in Case I.

§4. Conjecture

A backward shift invariant subspace of finite codimension in H^2 has a geometric description. As a result, we can give a reasonable conjecture.

Proposition 5. *N is a backward shift invariant subspace of finite dimension in H^2 if and only if*

$$N = (H^2 \ominus B_1 H^2) \cap (H^2 \ominus B_2 H^2)$$

where $B_1 = B_1(z)$ and $B_2 = B_2(w)$ are finite one variable Blaschke products.

Proof. If N is of finite dimension, then by [1, p969]

$$Z(M) = \bigcap_{f \in M} \{(\alpha, \beta) \in D^2 ; f(\alpha, \beta) = 0\} = \bigcup_{j=1}^n \{(a_j, b_j) \in D\}$$

and

$$M \supseteq B_1 H^2 + B_2 H^2$$

where $0 < n < \infty$, $B_1 = \prod_{j=1}^n \left(\frac{z - a_j}{1 - \bar{a}_j z} \right)^{t(j)}$, $B_2 = \prod_{j=1}^n \left(\frac{w - b_j}{1 - \bar{b}_j w} \right)^{s(j)}$, $1 \leq t(j)$, $s(j) < \infty$ and $N = H^2 \ominus M$. Since N is spanned by partial derivatives of the Cauchy kernels

$(1 - \bar{\alpha}z)^{-1}(1 - \bar{\beta}w)^{-1}$ for (α, β) in $Z(M)$ [1, p969], $N = (H^2 \ominus B_1H^2) \cap (H^2 \ominus B_2H^2)$ because

$$H^2 \ominus B_1H^2 = \left\{ \sum_{j=1}^n \sum_{\ell=1}^{t(j)} \frac{F_{\ell,j}(w)}{(1 - \bar{a}_jz)^\ell}; F_{\ell,j} \in H^2(T, w) \right\}$$

and

$$H^2 \ominus B_2H^2 = \left\{ \sum_{j=1}^n \sum_{\ell=1}^{s(j)} \frac{G_{\ell,j}(z)}{(1 - b_jw)^\ell}; G_{\ell,j} \in H^2(T, z) \right\}.$$

Conjecture. Suppose $N \neq H^2$. Then $S_zS_w^* = S_w^*S_z$ if and only if $N = (H^2 \ominus q_1H^2) \cap (H^2 \ominus q_2H^2)$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions.

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