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On a limiting motion and self-intersections of curves moved by the intermediate surface diffusion flow

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We give a rigorous proof that the solution curve of the intermediate surface diffusion flow equation converges to that of the averaged curvature flow equation locally in time as the diffusion coefficient D goes to infinity. As an application of this convergence result, we also prove that a self-intersection of curves can be developed by the intermediate surface diffusion flow for any positive D.

1 Introduction

This is a preliminary version of our project on the intermediate law (1). In this paper we discuss motion of curves in the plane. Although it can be generalized to higher dimensional cases as is done later by the authors [3], we believe the proof given here is more elementary and explicit and is also of some interest.

We study a nonlocal geometric evolution equation of the form:

$$V = -\Delta_{\Gamma(t)}\Big(\frac{1}{D} - \frac{1}{M}\Delta_{\Gamma(t)}\Big)^{-1}\kappa \quad \text{on } \Gamma(t) \text{ for } t > 0. \tag{1}$$

Here $\Gamma(t)$ is an unknown evolving closed curve immersed in $\boldsymbol{R}^2$ depending on time $t > 0$. The operator $\Delta_{\Gamma(t)}$ denotes the Laplace-Beltrami operator on $\Gamma(t)$. For $\Gamma(t)$, V denotes its outward normal velocity and κ stands for its curvature with the sign convention that the curvature of a circle is negative. $D > 0$ is the diffusion coefficient and $M > 0$ is the mobility constant. The equation (1) is called the *intermediate surface diffusion flow equation* which was first proposed by J. W. Cahn and J. E. Taylor [1]. We consider (1) as the initial value problem with the initial condition

$$\Gamma(0) = \Gamma_0, \tag{2}$$

where the initial closed curve Γ_0 which is immersed in $\boldsymbol{R}^2$ is assumed to belong to $C^{2+\alpha}$ with $0 < \alpha < 1$. Our interest here is restricted to large D, so that we restrict ourselves to the case that $D \geq 1$.

The purpose of this paper is to prove that the problem (1)-(2) admits a unique local solution $\Gamma^D(\cdot)$ in $C^{1,2+\alpha}$ on a common time interval $[0,T]$ with a $T > 0$ independent of $D \geq 1$, and the solution $\Gamma^D(\cdot)$ converges to the unique solution $\Gamma(\cdot)$ of the *averaged curvature flow equation*

$$V = M(\kappa - \kappa_{av}) \quad \text{on } \Gamma(t) \text{ for } t > 0 \tag{3}$$

with $\Gamma(0) = \Gamma_0$ as $D \to \infty$ in $C^{1,2+\alpha}$ on $[0,T]$. Here κ_{av} is the averaged curvature defined by

$$\kappa_{av}(t) := \frac{1}{L(t)} \int_0^{L(t)} \kappa(t,s)ds, \tag{4}$$

where $L(t)$ is the length of $\Gamma(t)$ and s denotes the arc-length parameter. Moreover, as an application of the above convergence result, we show a self-intersection of the solution curve of (1).

The equation (1) was first proposed by J. W. Cahn and J. E. Taylor [1, 10] to describe a geometric growth law for a moving surface where surface diffusion is the only transport mechanism and the only driving force for surface motion is the reduction in total surface free energy. The first mathematical analysis for (1) was presented by C. M. Elliott and H. Garcke [2], who established both global existence and stability results when Γ_0 is close to a circle. J. Escher and G. Simonett [4] extended their result to the higher dimensional case. But the D-dependence of the solutions to (1)-(2) was not investigated in [2, 4]. In [1] it was conjectured that the formal limit of (1) as $D \to \infty$ becomes (3). In this paper we carefully investigate the D-dependence of the solutions of (1) in our setting different from those of [2, 4] and then we justify this conjecture.

We here restrict our analysis to local-in-time solutions. We have two reasons to do so. First, the global-in-time solvability of (1) for all kinds of configurations of initial closed curves is not yet known. Secondly, (3) is convexity-preserving [6, 7], whereas (1) does not seem so. Therefore the global-in-time convergence of (1) toward (3) in $C^{1,2+\alpha}$ as $D \to \infty$ may be delicate problem to treat.

Owing to the convergence result as $D \to \infty$, it is expected that we can find a particular behavior of solutions to (1) for *any* $D > 0$ by investigating (3). We note that (3) is easier to treat than (1). Here we study a self-intersection of the solution curve of (1)-(2). For this phenomenon, we remind the reader the works by M. Gage [6], U. F. Mayer and G. Simonett [9] for (3). Their results suggest that a self-intersection for (1) can also occur because (1) and (3) are linked by the limit as $D \to \infty$. In our paper we prove for (1) with any $D > 0$ that this phenomenon actually occurs. We here emphasize an announcement that the above result is recently extended to higher dimensional case by the authors [3].

2 Parametrization

For simplicity of descriptions we set $\delta := 1/D$ with $0 < \delta \leq 1$ and $M := 1$ and we consider the equation

$$V = -\Delta_{\Gamma(t)}(\delta - \Delta_{\Gamma(t)})^{-1}\kappa \quad \text{on } \Gamma(t) \text{ for } t > 0 \tag{5}$$

with (2). We note that (5) can be written as

$$V = \kappa - \kappa_{av} - \delta(\delta - \Delta_{\Gamma(t)})^{-1}(\kappa - \kappa_{av}). \tag{6}$$

This can be seen from the facts that $\Delta_{\Gamma(t)}$ and $(\delta - \Delta_{\Gamma(t)})^{-1}$ commute and $\Delta_{\Gamma(t)}\kappa_{av} \equiv 0$. We also note that (6) is valid also for $\delta = 0$. In fact, the fact $\int_{\Gamma(t)}(\kappa - \kappa_{av})ds = 0$ shows that $\kappa - \kappa_{av}$ is orthogonal to the constants which are the 0-eigenvectors of $\Delta_{\Gamma(t)}$. Thus we see that $(-\Delta_{\Gamma(t)})^{-1}$ makes sense for the operand $\kappa - \kappa_{av}$. From this observation we are allowed to consider (6) for $\delta \in [0, 1]$. In particular, we note that (1) with $\delta = 0$ corresponds to the averaged curvature flow equation (3).

We parametrize (6) on a fixed closed reference curve near Γ_0. Let Σ be the reference curve which is assumed to be smooth (at least C^3) and parametrized as

$$\Sigma = \{\sigma(\eta) \in \boldsymbol{R}^2; \eta \in \boldsymbol{T} := \boldsymbol{R}/l\boldsymbol{Z}\}.$$

Here l and η are the total length and the arc-length parameter of Σ, respectively; $\sigma(\eta)$ runs clockwise as η increases. Assuming that $\Gamma(t)$ is close to Σ for $t \in [0, T]$ with a $T > 0$ small enough, we parametrize $\Gamma(t)$ as a graph over Σ. More precisely, we set

$$\Gamma(t) = \{(\Theta_{\rho(t)}(\sigma))(\eta) \in \boldsymbol{R}^2; \eta \in \boldsymbol{T}\},$$

where $(\Theta_{\rho(t)}(\sigma))(\eta) := \sigma(\eta) + \rho(t,\eta)\nu(\eta)$ for $\eta \in \boldsymbol{T}$, and $\rho(t,\eta)$ is the signed distance function of $\Gamma(t)$ from Σ at η and $\nu(\eta)$ is the outward unit normal vector field of Σ at η. Following the idea of Escher and Simonett [4], we pull (6) back to Σ by $\Theta_{\rho(t)}$. Let $\Theta^*_{\rho(t)}$ be the pull-back operator from $\Gamma(t)$ to Σ induced by $\Theta_{\rho(t)}$. As in [2] we obtain

$$\Theta^*_\rho V = \frac{1 - \lambda\rho}{g[\rho]^{1/2}}\rho_t,$$

$$\Theta^*_\rho \kappa = \frac{1}{g[\rho]^{3/2}}\Big((1 - \lambda\rho)\rho_{\eta\eta} + 2\lambda\rho_\eta^2 + \lambda'\rho\rho_\eta + \lambda\cdot(1 - \lambda\rho)^2\Big) =: K[\rho],$$

$$\Theta^*_\rho \kappa_{av} = \frac{\int_0^l K[\rho]\sqrt{g[\rho]}d\eta}{\int_0^l \sqrt{g[\rho]}d\eta} =: K_{av}[\rho].$$

Here $\lambda(\eta)$ denotes the curvature of Σ at η and $g[\rho]$ is the parametrization of the Euclidean metric of $\Gamma(t)$ defined by

$$g[\rho] := \left|\frac{\partial\Theta_\rho(\sigma)}{\partial\eta}\right|^2 = (1 - \lambda\rho)^2 + \rho_\eta^2.$$

Moreover, it follows from [4] that $\Theta^*_{\rho(t)}(\delta - \Delta_{\Gamma(t)})^{-1} = (\delta - \Delta_{\rho(t)})^{-1}\Theta^*_{\rho(t)}$, where $\Delta_{\rho(t)}$ is the parametrization of $\Delta_{\Gamma(t)}$ given by

$$\Delta_{\rho(t)} = \frac{1}{g[\rho(t)]}\left(\partial_\eta^2 - \frac{\partial_\eta(g[\rho(t)])}{2g[\rho(t)]}\partial_\eta\right). \tag{7}$$

For notational simplicity we put $G[\rho] := (1-\lambda\rho)^{-1}\sqrt{g[\rho]}$. Then, pulling back (6)-(2) to Σ by Θ_ρ, we have the following nonlinear nonlocal partial differential equation

$$\rho_t = F^\delta[\rho] \quad \text{in } \Sigma_T, \quad \rho(0) = \rho_0 \quad \text{in } \Sigma. \tag{8}$$

Here we set $\Sigma_T := [0,T]\times\Sigma$, $F^\delta[\rho] := F_1[\rho] - F_2[\rho] + F_3^\delta[\rho]$ with

$$F_1[\rho] := G[\rho]K[\rho], \quad F_2[\rho] := G[\rho]K_{av}[\rho],$$

$$F_3^\delta[\rho] := -\delta G[\rho](\delta - \Delta_\rho)^{-1}(K[\rho] - K_{av}[\rho]).$$

For later convenience, we also write $F_1[\rho]$, $G[\rho]$, $K[\rho]\sqrt{g[\rho]}$, and $\sqrt{g[\rho]}$ by means of functions $f(\eta, p_0, p_1, p_2)$, $\varphi(\eta, p_0, p_1)$, $\psi(\eta, p_0, p_1, p_2)$, and $\omega(\eta, p_0, p_1)$ with $\eta \in \boldsymbol{T}$, $|p_0| \le 1/(2\|\lambda\|_{C(\Sigma)})$, and $(p_1, p_2) \in \boldsymbol{R}^2$ such as

$$F_1[\rho] = f(\eta, \rho, \rho_\eta, \rho_{\eta\eta}), \quad G[\rho] = \varphi(\eta, \rho, \rho_\eta),$$

$$K[\rho]\sqrt{g[\rho]} = \psi(\eta, \rho, \rho_\eta, \rho_{\eta\eta}), \quad \sqrt{g[\rho]} = \omega(\eta, \rho, \rho_\eta).$$

3 Uniform local existence result

In this section we establish a unique local existence result of (8) which ensures a uniformity of solutions with respect to $\delta \in [0,1]$ in the following sense.

Theorem 3.1 *(Uniform local existence). Let $\alpha \in (0,1)$ and let $\rho_0 \in C^{2+\alpha}(\Sigma)$ with*

$$\|\rho_0\|_{C(\Sigma)} \le \frac{1}{4\|\lambda\|_{C(\Sigma)}} =: \frac{\gamma}{2}. \tag{9}$$

Then there are positive constants $T_0 = T_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$ and $N_0 = N_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$, which are independent of $\delta \in [0,1]$, such that (8) admits a unique solution ρ^δ in $C^{1,2+\alpha}(\Sigma_{T_0})$ satisfying

$$\|\rho^\delta\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \le N_0 \quad \text{for } \delta \in [0,1]. \tag{10}$$

Remark 3.2 *(i) The condition (9) is imposed to keep the factor $1 - \lambda\rho^\delta$ positive and away from 0 in Σ_{T_0} and for $\delta \in [0,1]$.*

(ii) Local existence results of (6) for each $\delta \ge 0$ are previously obtained in [2, 4] but δ-dependence of solutions is not rigorously studied there.

We prove Theorem 3.1 by a contraction argument. To do so, we linearize (8) around ρ_0. Let $\mathcal{A}$ be a linear operator defined by $\mathcal{A} := \mathcal{A}_1 - \mathcal{A}_2$ with

$$\mathcal{A}_1 r := \sum_{i=0}^{2} f_{p_i} \partial_\eta^i r, \tag{11}$$

$$\begin{aligned}\mathcal{A}_2 r \ := \ & \frac{1}{\int_0^l \omega d\eta}\Big(\varphi \cdot \int_0^l \sum_{i=0}^{2} \psi_{p_i} \partial_\eta^i r d\eta + \int_0^l \psi d\eta \sum_{i=0}^{1} \varphi_{p_i} \partial_\eta^i r \\ & - \frac{\int_0^l \psi d\eta}{\int_0^l \omega d\eta} \int_0^l \sum_{i=0}^{1} \omega_{p_i} \partial_\eta^i r d\eta\Big),\end{aligned} \tag{12}$$

for $r \in C^2(\Sigma)$, where f, φ, ψ, ω, and their first derivatives are evaluated at $(\eta, \rho_o, \rho_0', \rho_0'')$. (For each $k = 1, 2$ the expression of $\mathcal{A}_k$ can be obtained by deriving the first variation of F_k at ρ_0 in the r-direction.) Let $\rho(t, \eta)$ be a given function belonging to $C^{1,2+\alpha}(\Sigma_T)$ with $\|\rho\|_{C(\Sigma_T)} \leq \gamma$. Then we consider the inhomogeneous linear equation for unknown fuction $r(t, \eta)$ of the form

$$r_t = \mathcal{A} r + \hat{F}^\delta[\rho] \quad \text{in } \Sigma_T, \quad r(0) = \rho_0 \quad \text{in } \Sigma \tag{13}$$

with a $T > 0$. Here $\hat{F}^\delta[\rho]$ is defined by $\hat{F}^\delta[\rho] := \hat{F}_1[\rho] - \hat{F}_2[\rho] + F_3^\delta[\rho]$ with $\hat{F}_k[\rho] := F_k[\rho] - \mathcal{A}_k \rho$ $(k = 1, 2)$. We note that (13) with $\rho = r$ is equivalent to (8).

For (13) we have the following unique existence result.

Lemma 3.3 *Let $\delta \in [0, 1]$ and let $\alpha \in (0, 1)$. Assume that $\rho_0 \in C^{2+\alpha}(\Sigma)$ and $\rho \in C^{1,2+\alpha}(\Sigma_T)$ with $\|\rho_0\|_{C(\Sigma)} \leq \gamma$, $\|\rho\|_{C(\Sigma_T)} \leq \gamma$ where γ is in (9). Then (13) admits a unique solution r^δ in $C^{1,2+\alpha}(\Sigma_T)$ satisfying*

$$\|r^\delta\|_{C^{1,2+\alpha}(\Sigma_T)} \leq a(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})\big(\|\rho_0\|_{C^{2+\alpha}(\Sigma)} + \|\hat{F}^\delta[\rho]\|_{C^{0,\alpha}(\Sigma_T)}\big), \tag{14}$$

where the constant a is independent of $\delta \in [0, 1]$.

The proof of Lemma 3.3 is postponed in Section 6. Lemma 3.3 shows us that the problem (13) induces the mapping $S^\delta : C^{1,2+\alpha} \to C^{1,2+\alpha}$; $\rho \mapsto r$.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let N be positive parameter and set

$$\mathcal{B}_{T,N,\gamma} = \{\rho \in C^{1,2+\alpha}(\Sigma_T); \|\rho\|_{C^{1,2+\alpha}(\Sigma_T)} \leq N, \quad \|\rho\|_{C(\Sigma_T)} \leq \gamma, \quad \rho(0) = \rho_0\}.$$

Then $\mathcal{B}_{T,N,\gamma}$ is a complete metric space endowed with the norm $\|\cdot\|_{C^{1,2+\alpha}(\Sigma_T)}$. For $\rho \in \mathcal{B}_{T,N,\gamma}$, we denote by $S^\delta[\rho]$ the unique solution r^δ of (13) given by Lemma 3.3. We shall show the estimates:

$$\|S^\delta[\rho]\|_{C^{1,2+\alpha}(\Sigma_T)} \leq C_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C_1(N) T^{\alpha/2}, \tag{15}$$

$$\|S^\delta[\rho_1] - S^\delta[\rho_2]\|_{C^{1,2+\alpha}(\Sigma_T)} \leq C_2(N) T^{\alpha/2} \|\rho_1 - \rho_2\|_{C^{1,2+\alpha}(\Sigma_T)} \tag{16}$$

for ρ, ρ_1, $\rho_2 \in \mathcal{B}_{T,N,\gamma}$, where the constants C_i $(i = 0, 1, 2)$ are independent of $\delta \in [0, 1]$.

Here we once admit (15) and (16) and continue the proof. Choose N_0 and T_0 as

$$N_0 := 2C_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}), \quad C_1(N_0)T_0^{\alpha/2} \le \frac{N_0}{2}, \quad C_2(N_0)T_0^{\alpha/2} \le \frac{1}{2}, \quad N_0 T_0 \le \frac{\gamma}{2}. \tag{17}$$

Note that N_0 and T_0 are independent of $\delta \in [0,1]$. Then we see that S^δ is a contraction mapping from $\mathcal{B}_{T_0,N_0,\gamma}$ into itself and it admits a unique fixed point ρ^δ in $\mathcal{B}_{T_0,N_0,\gamma}$. This ρ^δ is obviously the desired solution we are going to seek.

We only prove (15), since the proof of (16) can be done identically. Throughout this proof, we denote by $C(q)$ universal constants depending on q but independent of $\delta \in [0,1]$; their values may be different in each occasion. In view of (14), if we establish the estimate

$$\|\hat{F}^\delta[\rho]\|_{C^{0,\alpha}(\Sigma_T)} \le C(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C(N)T^{\alpha/2}, \tag{18}$$

then (18) immediately yields (15).

We shall show (18). By a standard method utilyzing its particular structure of $\hat{F}_1$, we can obtain

$$\|\hat{F}_1[\rho]\|_{C^{0,\alpha}(\Sigma_T)} \le C(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C(N)T^{\alpha/2} \tag{19}$$

(see [8, Theorem 8.5.4]). For $\hat{F}_2[\rho]$, we use the formula:

$$\begin{aligned} \hat{F}_2[\rho] &= F_2[\rho_0] - \mathcal{A}_2\rho_0 \\ &\quad + \int_0^1 \int_0^\theta F_2''[(1-\sigma)\rho_0 + \sigma\rho](\rho - \rho_0, \rho - \rho_0)\, d\sigma d\theta. \end{aligned}$$

Here $F_2''[w](u,v)$ for u, v, $w \in \mathcal{B}_{T,N,\gamma}$ is the second variation of F_2 to the (u,v)-direction and its explicit form is given by

$$\begin{aligned} F_2''[w](u,v) &= \frac{1}{\int_0^l \omega d\eta}\Big(\varphi \cdot \int_0^l \sum_{i,j=0}^{2} \psi_{p_i p_j} \partial_\eta^i u \partial_\eta^j v d\eta + \int_0^l \sum_{i=0}^{2} \psi_{p_i} \partial_\eta^i u d\eta \sum_{j=0}^{1} \varphi_{p_j} \partial_\eta^j v \\ &\quad + \sum_{i=0}^{1} \varphi_{p_i} \partial_\eta^i u \int_0^l \sum_{j=0}^{2} \psi_{p_j} \partial_\eta^j v d\eta\Big) + Q[w](u,v), \end{aligned}$$

where φ, ψ, ω, and their derivatives are evaluated at $(\eta, w, w_\eta, w_{\eta\eta})$ and the symmetric bilinear form $Q[w](u,v)$ stands for the term consisting of the up to first order derivatives of u and v. Then, after a straightforward but rather long caluculation, we obtain

$$\begin{aligned} \|\hat{F}_2[\rho]\|_{C^{0,\alpha}(\Sigma_T)} &\le \|F_2[\rho_0] - \mathcal{A}_2\rho_0\|_{C^\alpha(\Sigma)} \\ &\quad + \int_0^1 \int_0^\theta \|F_2''[(1-\sigma)\rho_0 + \theta\rho](\rho - \rho_0, \rho - \rho_0)\|_{C^{0,\alpha}(\Sigma_T)} d\sigma d\theta \\ &\le C(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C(N)T^{\alpha/2}. \end{aligned} \tag{20}$$

We are going to estimate $F_3^\delta[\rho]$. We have nothing to prove when $\delta = 0$, for $F_3^0[\rho] = 0$ due to its definiton. So we investigate the case $\delta \in (0,1]$. The key in this case is the following lemma, which ensures the existence of uniform bounds for the operator $\delta(\delta - \Delta_\rho)^{-1}$ with respect to $\delta \in (0,1]$.

Lemma 3.4 *Let $\delta \in (0,1]$. Assume that $\rho \in C^{1,2+\alpha}(\Sigma_T)$ and $h \in C^{0,\alpha}(\Sigma_T)$. Then,*

(i) $(\delta - \Delta_\rho)^{-1}h$ belongs to $C^{0,2+\alpha}(\Sigma_T)$. In particular,

$$\|\delta(\delta - \Delta_\rho)^{-1}h\|_{C^{0,2}(\Sigma_T)} \leq B_1(\|\rho\|_{C^{0,2}(\Sigma_T)})\|h\|_{C(\Sigma_T)} \quad \text{for } \delta \in (0,1], \tag{21}$$

where the constant B_1 is independent of $\delta \in (0,1]$.

(ii) Moreover, if h also fulfills

$$\int_T h(t,\eta)\sqrt{g[\rho](t,\eta)}d\eta = 0 \quad \text{for } t \in [0,T], \tag{22}$$

then

$$\|(\delta - \Delta_\rho)^{-1}h\|_{C^{0,2}(\Sigma_T)} \leq B_2(\|\rho\|_{C^{0,2}(\Sigma_T)})\|h\|_{C(\Sigma_T)} \quad \text{for } \delta \in (0,1], \tag{23}$$

where the constant B_2 is independent of $\delta \in (0,1]$.

The proof of Lemma 3.4 is postponed in Section 6. We resume to estimate $F_3^\delta[\rho]$. We decompose $F_3^\delta[\rho(t)]$ as

$$\begin{aligned}
F_3^\delta[\rho(t)] &= -\delta(G[\rho(t)] - G[\rho_0])(\delta - \Delta_{\rho(t)})^{-1}(K[\rho(t)] - K_{av}[\rho(t)]) \\
&\quad -\delta G[\rho_0]\big((\delta - \Delta_{\rho(t)})^{-1} - (\delta - \Delta_{\rho_0})^{-1}\big)(K[\rho(t)] - K_{av}[\rho(t)]) \\
&\quad -\delta G[\rho_0](\delta - \Delta_{\rho_0})^{-1}\Big((K[\rho(t)] - K_{av}[\rho(t)]) - (K[\rho_0] - K_{av}[\rho_0])\Big) \\
&\quad -\delta G[\rho_0](\delta - \Delta_{\rho_0})^{-1}(K[\rho_0] - K_{av}[\rho_0]) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Observe that

$$\|\rho - \rho_0\|_{C^{0,2}(\Sigma_T)} \leq C(N)T^{\alpha/2}. \tag{24}$$

This can be checked by means of [8, Lemma 5.1.1]. Using Lemma 3.4 (ii) (and also (24) for (25)), it is not difficult to show

$$\|I_1\|_{C^{0,\alpha}(\Sigma_T)} \leq C(N)T^{\alpha/2}, \tag{25}$$

$$\|I_4\|_{C^{0,\alpha}(\Sigma_T)} \leq C(\|\rho_0\|_{C^2(\Sigma)}). \tag{26}$$

In order to estimate I_2, we use the resolvent equation:

$$(\delta - \Delta_{\rho_0})^{-1} - (\delta - \Delta_{\rho(t)})^{-1} = (\delta - \Delta_{\rho_0})^{-1}(\Delta_{\rho_0} - \Delta_{\rho(t)})(\delta - \Delta_{\rho(t)})^{-1}.$$

Then

$$\begin{aligned}
&\|I_2\|_{C^{0,\alpha}(\Sigma_T)} \\
&\leq 2\|G[\rho_0]\|_{C^\alpha(\Sigma)}\|\delta(\delta - \Delta_{\rho_0})^{-1}(\Delta_{\rho_0} - \Delta_\rho)(\delta - \Delta_\rho)^{-1}(K[\rho] - K_{av}[\rho])\|_{C^{0,\alpha}(\Sigma_T)} \\
&\leq C(\|\rho_0\|_{C^2(\Sigma)})\|(\Delta_{\rho_0} - \Delta_\rho)(\delta - \Delta_\rho)^{-1}(K[\rho] - K_{av}[\rho])\|_{C(\Sigma_T)},
\end{aligned}$$

where we have used Lemma 3.4 (i). Moreover, (24) and Lemma 3.4 (ii) yield

$$\begin{aligned}
&\|(\Delta_{\rho_0} - \Delta_\rho)(\delta - \Delta_\rho)^{-1}(K[\rho] - K_{av}[\rho])\|_{C(\Sigma_T)} \\
&\leq C(N)T^{\alpha/2}\|(\delta - \Delta_\rho)^{-1}(K[\rho] - K_{av}[\rho])\|_{C^{0,2}(\Sigma_T)} \\
&\leq C(N)T^{\alpha/2}\|K[\rho] - K_{av}[\rho]\|_{C(\Sigma_T)} \leq C(N)T^{\alpha/2}.
\end{aligned}$$

Thus we arrive at

$$\|I_2\|_{C^{0,\alpha}(\Sigma_T)} \le C(N)T^{\alpha/2}. \tag{27}$$

For I_3 we use Lemma 3.4 (i) to get

$$\begin{aligned}
&\|I_3\|_{C^{0,\alpha}(\Sigma_T)} \\
&\le 2\|G[\rho_0]\|_{C^\alpha(\Sigma)}\|\delta(\delta-\Delta_{\rho_0})^{-1}\Big((K[\rho]-K_{av}[\rho])-(K[\rho_0]-K_{av}[\rho_0])\Big)\|_{C^{0,\alpha}(\Sigma_T)} \\
&\le C(\|\rho_0\|_{C^2(\Sigma)})\|(K[\rho]-K_{av}[\rho])-(K[\rho_0]-K_{av}[\rho_0])\|_{C(\Sigma_T)} \\
&\le C(N)T^{\alpha/2}.
\end{aligned} \tag{28}$$

From (25)-(28) we obtain

$$\|F_3^\delta[\rho]\|_{C^{0,\alpha}(\Sigma_T)} \le C(\|\rho_0\|_{C^2(\Sigma)}) + C(N)T^{\alpha/2}. \tag{29}$$

Thus we have proved (18) by (19), (20), and (29), and therefore (15) has established. This completes the proof of Theorem 3.1. q.e.d.

4 Convergence result

In this section we prove that the unique solution of (1)-(2) obtained in Theorem 3.1 converges to that of (3)-(2).

Now we state our main result.

Theorem 4.1 *(Convergence result). Let $\delta \in [0,1]$. Let ρ^δ be the unique solutions of (8) in Theorem 3.1. Then it holds*

$$\|\rho^\delta - \rho^0\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \le \delta C_*(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) \quad \textit{for } \delta \in [0,1], \tag{30}$$

where the constant C_* is independent of $\delta \in [0,1]$.

Proof. Set $z^\delta := \rho^\delta - \rho^0$. Then z^δ fulfills

$$\begin{cases} z_t^\delta = \mathcal{A}z^\delta + \hat{F}_1[\rho^\delta] - \hat{F}_1[\rho^0] - (\hat{F}_2[\rho^\delta] - \hat{F}_2[\rho^0]) + F_3^\delta[\rho^\delta] & \text{in } \Sigma_{T_0}, \\ z^\delta(0) = 0 \quad \text{in } \Sigma. \end{cases}$$

It follows from (14) that

$$\|z^\delta\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \le a \cdot \Big(\sum_{k=1}^{2} \|\hat{F}_k[\rho^\delta] - \hat{F}_k[\rho^0]\|_{C^{0,\alpha}(\Sigma_{T_0})} + \|F_3^\delta[\rho^\delta]\|_{C^{0,\alpha}(\Sigma_{T_0})}\Big). \tag{31}$$

The first term in the right hand side of (31) can be estimated as in (16) by using the fact that $z^\delta(0) = 0$ in Σ. So we have for some constant $C(N_0)$ independent of $\delta \in [0,1]$

$$\sum_{k=1}^{2} \|\hat{F}_k[\rho^\delta] - \hat{F}_k[\rho^0]\|_{C^{0,\alpha}(\Sigma_{T_0})} \le C(N_0)T_0^{\alpha/2}\|z^\delta\|_{C^{1,2+\alpha}(\Sigma_{T_0})}.$$

For $F_3^\delta[\rho^\delta]$, we use Lemma 3.4 (ii) and (10) to get for some constant $C'(N_0)$ independent of $\delta \in [0,1]$

$$\begin{aligned} \|F_3^\delta[\rho^\delta]\|_{C^{0,\alpha}(\Sigma_{T_0})} &\leq 2\delta\|G[\rho^\delta]\|_{C^{0,\alpha}(\Sigma_{T_0})}\|(\delta-\Delta_{\rho^\delta})^{-1}\Big(K[\rho^\delta]-K_{av}[\rho^\delta]\Big)\|_{C^{0,\alpha}(\Sigma_{T_0})} \\ &\leq \delta C'(N_0). \end{aligned}$$

Thus we arrive at $\|z^\delta\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \leq C_2(N_0)T_0^{\alpha/2}\|z^\delta\|_{C^{1,2+\alpha}(\Sigma_{T_0})} + \delta C_3(N_0)$. Here we note that $C_2(N_0)$ can be the same as the one in (16). Then, by (17) we obtain (30) with $C_* := 2C_3$ and the proof is complete. q.e.d.

5 Self-intersection

As an application of Theorem 4.1, we show a formation of self-intersection of closed curves moved by (1)-(2) for *any* $D > 0$.

Theorem 5.1 *(Self-intersection). Let $\alpha \in (0,1)$ and let $\delta \in (0,1]$. Then, there is an embedded closed curve Γ_0 in $C^{2+\alpha}$ such that the solution curve of (5) with the initial data Γ_0 established in Theorem 3.1 loses its embeddedness in finite time although it stays $C^{1,2+\alpha}$.*

Proof. (a) Due to U. F. Mayer and G. Simonett [9], there is an embedded closed curve $\Gamma_0 \in C^{2+\alpha}$ such that the solution $\Gamma^0 = \{\Gamma^0(t)\}_{t\in[0,T_0]}$ of (5) with $\delta = 0$ and with $\Gamma^0(0) = \Gamma_0$ established in Theorem 3.1 loses its embeddedness for a $t' \in (0, T_0]$.

(b) First we show the result when $\delta > 0$ is small. Let $\delta \in (0,1]$ and let $\Gamma^\delta = \{\Gamma^\delta(t)\}_{t\in[0,T_0]}$ be the solution of (5) with $\Gamma^\delta(0) = \Gamma_0$ established in Theorem 3.1. Then, by (30) and (a) we can find a small $\delta_0 \in (0,1]$ such that for any $\delta \in (0, \delta_0]$, $\Gamma^\delta(t')$ must not be embedded curve.

(c) We show the result for any $\delta > 0$. For this purpose let us consider the rescaling as $\bar{t} = \lambda^{-2}t$, $(\bar{x},\bar{y}) = \lambda^{-1}(x,y)$ for $\lambda > 0$, where the bar-variables stand for the rescaled ones. Then the normal velocity, the curvature, and the Laplace-Beltrami operator are rescaled as $\bar{V} = \lambda V$, $\bar{\kappa} = \lambda\kappa$, and $\bar{\Delta} = \lambda^2\Delta$. Thus the equation (5) is rescaled to

$$\bar{V} = -\bar{\Delta}(\lambda^2\delta - \bar{\Delta})^{-1}\bar{\kappa} \quad \text{on } \bar{\Gamma}(\bar{t}) \text{ for } \bar{t} > 0. \tag{32}$$

Let $\delta > 0$ be fixed arbitrarily. As in (a)-(b), we can find an embedded closed curve $\bar{\Gamma}_0$ and $\bar{t}_1 > 0$ such that for a sufficiently small $\lambda = \lambda(\delta) > 0$, the solution $\bar{\Gamma}$ of (32) with $\bar{\Gamma}(0) = \bar{\Gamma}_0$ is not embedded at $\bar{t}_1$. This means that the originally scaled solution Γ of (5) with $\Gamma(0) = \Gamma_0$, where Γ_0 is originally scaled curve for $\bar{\Gamma}_0$, is not embedded at $t_1 := \lambda^2\bar{t}_1$, although Γ_0 is embedded. This completes the proof. q.e.d.

6 Proofs of Lemmas 3.3 and 3.4

Proof of Lemma 3.3. Let $X := C(\Sigma)$ and let A be the realization of $\mathcal{A}$ in X which is defined by $A : D(A) \to X; \rho \mapsto \mathcal{A}\rho$, where $D(A) := C^2(\Sigma)$. We shall show that A is sectorial in

X. Once this is verified, then it turs out that A generates an analytic semigroup $\{e^{tA}\}_{t\geq 0}$ in X and the unique solution of (13) is given by the variation of constants formula

$$r^{\delta}(t) = e^{tA}\rho_0 + \int_0^t e^{(t-s)A}F^{\delta}[\rho](s)ds$$

and it satisfies the desired estimate (14) (see [8, Chapter 5]). It also turs out that the constant $a(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$ in (14) arises from the resolvent estimate of A.

In order to verify that A is sectorial in X, we owe a result concerning the perturbation of the generator [8, Proposition 2.4.1 (ii)]. Let A_1 be the realization of $\mathcal{A}_1$ in X which is defined by $A_1 : D(A_1)(:= C^2(\Sigma)) \to X; \rho \mapsto \mathcal{A}_1\rho$. Then, according to its particular form of $\mathcal{A}_1$ in (11), one can show by a standard method that A_1 is sectorial in X (see for example [8, Chapter 3]). Let A_2 be the realization of $\mathcal{A}_2$ in X which is defined by $A_2 : D(A_2)(:= D(A_1)) \to X; \rho \mapsto \mathcal{A}_2\rho$. Then, by means of (12), it is straightforward to show that $A_2 \in L(D(A_2), C^{\alpha}(\Sigma))$ with

$$\|A_2\rho\|_{C^{\alpha}(\Sigma)} \leq C(\|\rho_0\|_{C^{1+\alpha}(\Sigma)})\|\rho\|_{C^2(\Sigma)} \quad \text{for } \rho \in C^2(\Sigma).$$

This means that A_2 is regarded as a perturbation of A_1 in the sense of [8, Proposition 2.4.1 (ii)], and it follows that $A(:= A_1 - A_2)$ is sectorial in X. This completes the proof. q.e.d.

Proof of Lemma 3.4. For simplicity of notation, we write $g(t,\eta) = g[\rho](t,\eta)$. Throughout this proof, we denote by $C(q_1, q_2, \cdots)$ various constants dependeing on q_1, $q_2 \cdots$, but independent of $\delta \in (0,1]$, whose values may be different in each occasion.

First we show (ii). Let us consider the equation

$$\delta u - \Delta_{\rho} u = h. \tag{33}$$

It follows from the Schauder theory that for any $t \in [0,T]$, (33) admits a unique solution $u(t,\cdot) \in C^{2+\alpha}(\Sigma)$.

In the following, we fix arbitrary $t \in [0,T]$. We integrate (33) with respect to $\sqrt{g(t,\eta)}d\eta$ from 0 to l, where l is the total length of Σ. Then, it follows from (22) and the fact $\int_0^l \Delta_{\rho}u\sqrt{g}d\eta = 0$ that $\int_0^l u(t,\eta)\sqrt{g(t,\eta)}d\eta = 0$. This implies

$$\|u(t,\cdot)\|_{C(\Sigma)} \leq l\|u_{\eta}(t,\cdot)\|_{C(\Sigma)}. \tag{34}$$

Then (7) and (34) yield

$$\begin{aligned} \|u_{\eta\eta}(t,\cdot)\|_{C(\Sigma)} &= \|\{g\cdot(\delta u + \frac{g_{\eta}}{2g^2}u_{\eta} - h)\}(t,\cdot)\|_{C(\Sigma)} \\ &\leq \|g\|_{C(\Sigma_T)}\Big(\|u(t,\cdot)\|_{C(\Sigma)} + \Big\|\frac{g_{\eta}}{2g^2}\Big\|_{C(\Sigma_T)}\|u_{\eta}(t,\cdot)\|_{C(\Sigma)} + \|h\|_{C(\Sigma_T)}\Big) \\ &\leq C(\|\rho\|_{C^{0,2}(\Sigma_T)})(\|u_{\eta}(t,\cdot)\|_{C(\Sigma)} + \|h\|_{C(\Sigma_T)}). \end{aligned} \tag{35}$$

Thus our task is reduced to estimate $\|u_{\eta}(t,\cdot)\|_{C(\Sigma)}$. For this purpose, we introduce a change of variables as follows:

$$s = \int_0^{\eta}\sqrt{g(t,\zeta)}d\zeta;\ U(t,s) = u(t,\eta),\ H(t,s) = h(t,\eta)\ (\eta \in [0,l]).$$

Set $L(t) := \int_0^l \sqrt{g(t,\eta)}d\eta$. Then the fact that $\int_0^{L(t)} U_s(t,s)ds = 0$ yields

$$\begin{aligned} |u_\eta(t,\eta)| &= |\sqrt{g(t,\eta)}U_s(t,s)| \le C(\|\rho\|_{C^{0,1}(\Sigma_T)}) \int_0^{L(t)} |U_{ss}(t,s)|ds \\ &\le C(\|\rho\|_{C^{0,1}(\Sigma_T)})\|U_{ss}(t,\cdot)\|_{L^2(0,L(t))}. \end{aligned} \tag{36}$$

On the other hand, we see that (33) in s-coordinate reads

$$\delta U - U_{ss} = H. \tag{37}$$

Multiplying (37) by $-U_{ss}$ and integrating it yield

$$\begin{aligned} &\delta\|U_s(t,\cdot)\|^2_{L^2(0,L(t))} + \|U_{ss}(t,\cdot)\|^2_{L^2(0,L(t))} \\ \le\ & \frac{1}{2}\|U_{ss}(t,\cdot)\|^2_{L^2(0,L(t))} + \frac{1}{2}\|H(t,\cdot)\|^2_{L^2(0,L(t))}. \end{aligned}$$

Hence we have

$$\|U_{ss}(t,\cdot)\|_{L^2(0,L(t))} \le \|H(t,\cdot)\|_{L^2(0,L(t))} \le C(\|\rho\|_{C^{0,1}(\Sigma_T)})\|h\|_{C(\Sigma_T)}. \tag{38}$$

Now (36) and (38) give

$$\|u_\eta(t,\cdot)\|_{C(\Sigma)} \le C(\|\rho\|_{C^{0,1}(\Sigma_T)})\|h\|_{C(\Sigma_T)}. \tag{39}$$

Substituting (39) into (35) and using (34), we finally obtain

$$\|u(t,\cdot)\|_{C^2(\Sigma)} \le B_2(\|\rho\|_{C^{0,2}(\Sigma_T)})\|h\|_{C(\Sigma_T)}.$$

In a similar way we can obtain

$$\begin{aligned} &\|u(t,\cdot) - u(\tau,\cdot)\|_{C^2(\Sigma)} \\ &\le C(\|\rho\|_{C^{0,2}(\Sigma_T)}, \|h\|_{C(\Sigma_T)})(\|h(t,\cdot) - h(\tau,\cdot)\|_{C(\Sigma)} + \|\rho(t,\cdot) - \rho(\tau,\cdot)\|_{C^2(\Sigma)}) \end{aligned}$$

for $t, \tau \in [0,T]$. This shows that $u \in C^{0,2}(\Sigma_T)$ and hence we get (23).

We can also show that u belongs to $C^{0,2+\alpha}(\Sigma_T)$ via further straightforward computation, which we leave to the reader.

Next we show (i). We use the decompoition as follows:

$$\delta(\delta - \Delta_\rho)^{-1}h = \delta(\delta - \Delta_\rho)^{-1}(h - h_{av}) + \delta(\delta - \Delta_\rho)^{-1}h_{av}, \tag{40}$$

where $h_{av}(t) := \int_0^l h(t,\eta)\sqrt{g(t,\eta)}d\eta/L(t)$. It is easy to verify from the uniqueness of the solution in (33) that $\delta(\delta - \Delta_\rho)^{-1}h_{av} = h_{av}$. Now, taking the $C^{0,2}(\Sigma_T)$-norm in (40) and using (23), we have

$$\begin{aligned} \|\delta(\delta - \Delta_\rho)^{-1}h\|_{C^{0,2}(\Sigma_T)} &\le \delta B_2(\|\rho\|_{C^{0,2}(\Sigma_T)})\|h - h_{av}\|_{C(\Sigma_T)} + \|h_{av}\|_{C[0,T]} \\ &\le B_1(\|\rho\|_{C^{0,2}(\Sigma_T)})\|h\|_{C(\Sigma_T)} \quad \text{for } \delta \in (0,1]. \end{aligned}$$

This completes the proof. q.e.d.

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