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A Uniform Algebra**

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Interpolation Problem For  $\ell^1$  And A Uniform Algebra

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Abstract. Let  $A$  be a uniform algebra and  $M(A)$  the maximal ideal space of  $A$ . A sequence  $\{a_n\}_n$  in  $M(A)$  is called  $\ell^1$ -interpolating if for every sequence  $(\alpha_n)$  in  $\ell^1$  there exists a function  $f$  in  $A$  such that  $f(a_n) = \alpha_n$  for all  $n$ . In this paper, an  $\ell^1$ -interpolating sequence is studied for an arbitrary uniform algebra. For some special uniform algebras, an  $\ell^1$ -interpolating sequence is equivalent to an  $\ell^\infty$ -interpolating sequence which is familiar for us. However, in general these two interpolating sequences may be different from each other.

## 1. Introduction

Let  $A$  be a uniform algebra on a compact Hausdorff space  $X$  and  $M(A)$  the maximal ideal space of  $A$ . Throughout this paper we assume that  $\{a_n\}_n$  is an infinite sequence of distinct points in  $M(A)$ . For  $1 \leq p \leq \infty$ , a sequence  $\{a_n\}_n$  is called  $\ell^p$ -interpolating if for every sequence  $(\alpha_n)$  in  $\ell^p$  there exists a function  $f$  in  $A$  such that  $f(a_n) = \alpha_n$  for all  $n$ . An  $\ell^\infty$ -interpolating sequence is called usually interpolating.

If  $A = H^\infty(D)$ , that is, the set of all the bounded analytic functions on the unit disc  $D$  in  $\mathbb{C}$ , then an  $\ell^\infty$ -interpolating sequence was studied by Carleson [2]. Carleson [2] determined an  $\ell^\infty$ -interpolating sequence when  $\{a_n\}$  in  $D$ . In the general situation, that is, for  $\{a_n\}$  in  $M(H^\infty(D))$ , Izuchi [5] studied an  $\ell^\infty$ -interpolating sequence. Recently, Hatori [3] showed that an  $\ell^1$ -interpolating sequence is equivalent to an  $\ell^\infty$ -interpolating sequence when  $\{a_n\}$  in  $D$ . In this paper we study an  $\ell^1$ -interpolating sequence for an arbitrary uniform algebra  $A$  when  $\{a_n\}_n$  in  $M(A)$ . For  $\{a_n\}_n$  in  $M(A)$  put

$$J = \{f \in A ; f = 0 \text{ on } \{a_n\}_n\},$$

$$J_k = \{f \in A ; f = 0 \text{ on } \{a_n\}_{n \neq k}\}$$

and

$$\rho_k = \sup\{|f(a_k)| ; f \in J_k, \|f\| \leq 1\}.$$

For  $a, b$  in  $M(A)$

$$\sigma(a, b) = \sup\{|f(a)| ; f(b) = 0, \|f\| \leq 1\}.$$

When  $A = H^\infty(D)$  and  $\{a_n\}_n$  is in  $D$ ,

$$\sigma(a_k, a_n) = \left| \frac{a_k - a_n}{1 - \bar{a}_k a_n} \right| \quad \text{and} \quad \rho_k = \prod_{n \neq k} \left| \frac{a_k - a_n}{1 - \bar{a}_k a_n} \right|.$$

In general, we don't know that

$$\rho_k = \prod_{n \neq k} \sigma(a_k, a_n).$$

However under some mild condition (Hypothesis I in Section 4), we can show that

$$\rho_k \geq \prod_{n \neq k} \sigma(a_k, a_n).$$

In general,  $\rho_k > 0$  if and only if  $J_k \not\supseteq J$ . Hence  $\rho_k > 0$  if and only if there exists a function  $f_k$  in  $A$  such that  $f_k(a_n) = \delta_{nk}$ . In this paper, for  $\{a_n\}_n$  in  $M(A)$  we assume that  $\rho_k > 0$  for all  $k$ .

In Section 2, for an arbitrary uniform algebra we show that  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence if and only if  $\inf_k \rho_k > 0$ . In Section 3, we define a finite  $\ell^1$ -interpolating sequence and we give a necessary and sufficient condition for this. In Section 4, we show that if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ , then  $\{a_n\}_n$  is always a finite  $\ell^1$ -interpolating sequence and under some mild condition it is an  $\ell^1$ -interpolating sequence. In some sense,

this type theorem for an  $\ell^\infty$ -interpolating sequence was conjectured in [1]. In Section 5, we apply the results in the previous sections to concrete uniform algebras. In Section 6, we give a remark about an  $\ell^\infty$ -interpolating sequence.

## 2. $\ell^1$ -interpolating sequence

In this section, we show that  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence if and only if  $\inf_k \rho_k > 0$ . The argument in 'only if' part of Lemma 1 is similar to one which was used by Hatori [3] when  $A = H^\infty(D)$ .

**Lemma 1.**  *$\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence if and only if there exists a sequence  $\{f_n\}_n$  in  $A$  such that  $f_n(a_k) = \delta_{nk}$  ( $n \geq 1, k \geq 1$ ) and  $\sup_n \|f_n + J\| < \infty$ .*

*Proof.* Suppose  $M = \sup_n \|f_n + J\| < \infty$  and  $f_n(a_k) = \delta_{nk}$ . Let  $\varepsilon$  be arbitrary positive constant. For each  $n$  there exists  $g_n$  in  $J$  such that  $\|f_n + g_n\| \leq M + \varepsilon$ . If  $(\alpha_n) \in \ell^1$ , put

$$f = \sum_{n=1}^{\infty} \alpha_n (f_n + g_n).$$

Then  $f$  belongs to  $A$  and  $f(a_n) = \alpha_n$  for  $n = 1, 2, \dots$ . Suppose  $S = \{a_n\}_n$  is an  $\ell^1$ -interpolating sequence. Then there exists a sequence  $\{f_n\}_n$  in  $A$  such that  $f_n(a_k) = \delta_{nk}$ . For  $(\alpha_n) \in \ell^1$ , put

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n f_n|_S,$$

then by hypothesis there exists a function  $f$  such that  $T(\alpha_n) = f|_S$ . Since  $A|_S$  is algebraically isomorphic to the quotient algebra  $A/J$ , we put the quotient norm of  $A/J$  on  $A|_S$ . By the closed graph theorem,  $T$  is bounded from  $\ell^1$  to  $A|_S$  and so

$$\|f_k + J\| = \|f_k|_S\| \leq \|T\|$$

because  $T(\{\delta_{nk}\}_n) = f_k|_S$ . □

**Lemma 2.** *Suppose  $\{f_n\}_n$  is a sequence in  $A$  such that  $f_n(a_k) = \delta_{nk}$ . Then*

*$\|f_n + J\| = 1/\rho_n$  for  $n = 1, 2, \dots$*

*Proof.* Since  $(\rho_n f_n)(a_k) = \rho_n \delta_{nk}$ ,  $\|\rho_n f_n + J\| \geq 1$ . By definition of  $\rho_n$ , for each  $\ell \geq 1$  there exists  $g_\ell \in A$  such that  $\|g_\ell\| = 1$ ,  $g_\ell(a_n) = 0$  for  $n \neq k$  and

$$\rho_k - \frac{1}{\ell} \leq g_\ell(a_k) \leq \rho_k.$$

Put  $G_\ell = g_\ell/g_\ell(a_k)$ , then  $G_\ell \in A$  and

$$\frac{1}{\rho_k} \leq \|G_\ell\| = \frac{1}{|g_\ell(a_k)|} \leq \frac{1}{\rho_k - 1/\ell}.$$

Moreover,  $G_\ell(a_k) = 1$ ,  $G_\ell(a_n) = 0$  for  $n \neq k$  and so  $G_\ell \in f_k + J$ . Since  $\|f_k + J\| \leq (\rho_k - 1/\ell)^{-1}$  for any  $\ell \geq 1$ ,  $\|\rho_k f_k + J\| \leq 1$ . □

**Theorem 1.** Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ .  $\{a_n\}_n$  is a  $\ell^1$ -interpolating sequence if and only if  $\inf_k \rho_k > 0$ .

Proof. It is clear by Lemmas 1 and 2. □

### 3. Finite $\ell^1$ -interpolating sequence

We say that  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence if there exists a finite positive constant  $\gamma$  which satisfies the following : For any finite  $\ell \geq 1$  and for any  $(\alpha_n)$  in the unit ball of  $\ell^1$ , there exists a function  $F_\ell$  in  $A$  such that

$$F_\ell(a_n) = \alpha_n \text{ for } 1 \leq n \leq \ell$$

and  $\|F_\ell\| \leq \gamma$ .

For  $\{a_n\}_n$  in  $M(A)$  and  $1 \leq k \leq \ell < \infty$ , put

$$J^\ell = \{f \in A ; f(a_n) = 0 \text{ if } 1 \leq n \leq \ell\},$$

$$J_k^\ell = \{f \in A ; f(a_n) = 0 \text{ if } 1 \leq n \leq \ell, n \neq k\}$$

and

$$\rho_{k,\ell} = \sup\{|f(a_k)| ; f \in J_k^\ell, \|f\| \leq 1\}.$$

Then  $\rho_{k,\ell} \geq \rho_{k,\ell+1}$  and  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} \geq \rho_k$ .

**Lemma 3.**  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence if and only if for each  $\ell \geq 1$  there exists a sequence  $\{f_{\ell,n}\}_{n=1}^\ell$  in  $A$  such that  $f_{\ell,n}(a_k) = \delta_{nk}$  for  $1 \leq k \leq \ell$  and  $\sup_\ell \sup_{1 \leq n \leq \ell} \|f_{\ell,n} + J^\ell\| < \infty$ .

Proof.  $(\alpha_n)$  denotes an element in the unit ball of  $\ell^1$ . Suppose

$M = \sup_\ell \sup_{1 \leq n \leq \ell} \|f_{\ell,n} + J^\ell\| < \infty$  and  $f_{\ell,n}(a_k) = \delta_{nk}$  for  $1 \leq k \leq \ell$ , then for any finite  $\ell \geq 1$

$$\|\sum_{n=1}^\ell \alpha_n f_{\ell,n} + J^\ell\| \leq (\sum_{n=1}^\ell |\alpha_n|)M.$$

If  $\gamma = M + 1$ , then for any  $\ell \geq 1$  there exists a  $g_\ell \in J^\ell$  such that  $\|\sum_{n=1}^\ell \alpha_n f_{\ell,n} + g_\ell\| \leq \gamma$ . Set  $F_\ell = \sum_{n=1}^\ell \alpha_n f_{\ell,n} + g_\ell$ , then  $F_\ell(a_n) = \alpha_n$  for  $1 \leq n \leq \ell$  and  $\|F_\ell\| \leq \gamma$ . Suppose  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence. Since  $\{a_n\}_n$  is an infinite sequence of distinct points in  $M(A)$ , for each  $\ell \geq 1$  there exists a sequence  $\{f_{\ell,n}\}_{n=1}^\ell$  in  $A$  such that  $f_{\ell,n}(a_k) = \delta_{nk}$  for  $1 \leq k \leq n$ . Put



$$T^\ell(\alpha_n) = \sum_{n=1}^{\ell} \alpha_n f_{\ell,n} + J^\ell$$

then  $\|T^\ell(\alpha_n)\| \leq \|T^\ell\|(\sum_{n=1}^{\ell} |\alpha_n|)$ . If  $\|T^\ell\| \rightarrow \infty$  as  $\ell \rightarrow \infty$ , then there exists  $(\alpha_n)$  in the unit ball of  $\ell^1$  such that  $\|T^\ell(\alpha_n)\| \rightarrow \infty$  as  $\ell \rightarrow \infty$ . On the other hand, by hypothesis  $\|T^\ell(\alpha_n)\| \leq \gamma < \infty$  for all  $\ell$ . This contradiction implies that  $M = \sup_{\ell} \|T^\ell\| < \infty$ . This shows that for any  $\ell \geq 1$  and any  $k \geq 1$  with  $k \leq \ell$ ,

$$\|f_{\ell,k} + J^\ell\| = \|T^\ell(\{\delta_{kn}\})\| \leq M.$$

□

**Lemma 4.** For  $\ell = 1, 2, \dots$  and  $1 \leq k \leq \ell$ ,  $\|f_k + J^\ell\| = 1/\rho_{k,\ell}$ .

Proof is almost the same as the proof of Lemma 2.

□

**Theorem 2.** Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ .  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence if and only if  $\inf_k \lim_{\ell \rightarrow \infty} \rho_{k,\ell} > 0$ .

Proof. It is clear by Lemma 3 and Lemma 4.

□

#### 4. Uniformly separated sequence

When  $A = H^\infty(D)$  and  $\{a_n\}_n$  is in  $D$ , for any  $k \geq 1$

$$\rho_k = \prod_{n \neq k} \sigma(a_k, a_n) = \lim_{\ell \rightarrow \infty} \rho_{k,\ell}.$$

When  $\{a_n\}_n$  is in  $M(A)$ , Izuchi [5] showed essentially that  $\inf_k \rho_k > 0$  implies  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . However, this is not true in general. If  $\sum_{n=1}^{\infty} (1 - \rho_n) < \infty$ , then  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . In fact,  $\rho_n \leq \sigma(a_k, a_n)$  for  $n \neq k$  and so  $\prod_{n=1}^{\infty} \rho_n \leq \prod_{n \neq k} \sigma(a_k, a_n)$  for any  $k \geq 1$ . When  $\sum_{n=1}^{\infty} (1 - \rho_n) < \infty$ ,  $0 < \prod_{n=1}^{\infty} \rho_n$  and so  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . In this section, we study these three quantities.

**Lemma 5.**

(1) For any  $\ell \geq 1$ ,  $\rho_{k,\ell} \geq \prod_{n \neq k}^{\ell} \sigma(a_k, a_n)$ . Hence for any  $k \geq 1$

$$\lim_{\ell \rightarrow \infty} \rho_{k,\ell} \geq \prod_{n \neq k} \sigma(a_k, a_n).$$

(2) For  $1 \leq n \leq \ell$  and  $n \neq k$ ,  $\rho_{k,\ell} \leq \sigma(a_k, a_n)$ . Hence for any  $k \geq 1$

$$\lim_{\ell \rightarrow \infty} \rho_{k,\ell} \leq \inf_{n \neq k} \sigma(a_k, a_n).$$

Proof. (1) Fix any positive constant  $\varepsilon > 0$ . For each  $n$  with  $\ell \geq n \geq 1$  and  $n \neq k$ , there exists  $F_n^\varepsilon \in A$  such that  $\|F_n^\varepsilon\| \leq 1$ ,  $F_n^\varepsilon(a_n) = 0$  and

$$\sigma(a_k, a_n) \geq |F_n^\varepsilon(a_k)| \geq \sigma(a_k, a_n) - \varepsilon.$$

Then  $F^\varepsilon = \prod_{n \neq k}^\ell F_n^\varepsilon$  belongs to  $J_{\ell,k}$ ,  $\|F^\varepsilon\| \leq 1$  and

$$\rho_{\ell,k} \geq |F^\varepsilon(a_k)| \geq \prod_{n \neq k}^\ell \{\sigma(a_k, a_n) - \varepsilon\}.$$

As  $\varepsilon \rightarrow 0$   $\rho_{\ell,k} \geq \prod_{n \neq k}^\ell \sigma(a_k, a_n)$  for any  $\ell \geq 1$  and hence  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} \geq \prod_{n \neq k} \sigma(a_k, a_n)$ . (2) is clear by the definitions of  $\rho_{k,\ell}$  and  $\sigma(a_k, a_n)$  for  $1 \leq n \leq \ell$  and  $n \neq k$ .  $\square$

**Theorem 3.** *Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ .*

- (1) *If  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$  then  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence.*
- (2) *If  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence then  $\inf_{n \neq k} \sigma(a_k, a_n) > 0$ .*

Proof. (1) By (1) of Lemma 5,  $\inf_k \lim_{\ell \rightarrow \infty} \rho_{k,\ell} > 0$  and so by Theorem 2  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence. (2) By Theorem 2  $\inf_k \lim_{\ell \rightarrow \infty} \rho_{k,\ell} > 0$  and so by (2) of Lemma 5  $\inf_{n \neq k} \sigma(a_k, a_n) > 0$ .  $\square$

**Hypothesis I.** *Let  $A$  be a uniform algebra and  $\{a_n\}_n$  in  $M(A)$ . If  $g_\ell$  is a function in  $A$  and  $\|g_\ell\| \leq 1$  for  $\ell = 1, 2, \dots$ , then there exist a subsequence  $\{g_{\ell(j)}\}_j$  of  $\{g_\ell\}_\ell$  and a function  $g$  in  $A$  such that  $\|g\| \leq 1$  and  $\lim_{j \rightarrow \infty} g_{\ell(j)}(a_n) = g(a_n)$  for any  $n \geq 1$*

**Hypothesis II.** *Let  $A$  be a uniform algebra and  $\{a_n\}_n$  in  $M(A)$ . For any  $a, b$  in  $\{a_n\}_n$  with  $a \neq b$ , if the function  $f$  in  $A$  satisfies  $f(a) = f(b) = 0$  and  $\|f\| \leq 1$ , then for any  $\varepsilon > 0$  there exist two functions  $g$  and  $h$  in  $A$  such that  $\|g\| \leq 1 + \varepsilon$ ,  $\|h\| \leq 1 + \varepsilon$ ,  $g(a) = 0$ ,  $h(b) = 0$  and  $f = gh$ .*

**Lemma 6.** *Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ . If  $\{a_n\}_n$  satisfies Hypothesis I then  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} = \rho_k$  for any  $k \geq 1$  and so a finite  $\ell^1$ -interpolating sequence is an  $\ell^1$ -interpolating sequence.*

Proof.  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} \geq \rho_k$  for any  $k \geq 1$ . If  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} > \varepsilon > 0$ , then for each  $\ell$  there exists  $g_\ell \in J_k^\ell$  such that  $\|g_\ell\| \leq 1$  and  $|g_\ell(a_k)| \geq \varepsilon > 0$ . By hypothesis, there exists  $g \in J_k$  such that  $\|g\| \leq 1$  and  $|g(a_k)| \geq \varepsilon > 0$ . Thus  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} \leq \rho_k$  and so  $\lim_{\ell \rightarrow \infty} \rho_{k,\ell} = \rho_k$ . This and Theorem 1 and Theorem 2 also imply that a finite  $\ell^1$ -interpolating sequence is an  $\ell^1$ -interpolating sequence.  $\square$

**Lemma 7.** *Assume Hypothesis II. If  $f$  is a function in  $J_{k,\ell}$  with  $\|f\| \leq 1$ , then for any  $\varepsilon > 0$   $f = \prod_{n \neq k}^\ell f_n$ ,  $f_n(a_n) = 0$  ( $n \neq k$ ) and  $\|f_n\| \leq (1 + \varepsilon)^{\ell-1}$ .*

*Proof.* We may assume  $k = 1$ . Fix any  $\varepsilon > 0$ . By Hypothesis II,  $f = g_2 g_3$ ,  $\|g_j\| \leq 1 + \varepsilon$  ( $j = 2, 3$ ) and  $g_2(a_2) = g_3(a_3) = 0$ . Since  $f(a_4) = 0$ ,  $g_2(a_4) = 0$  or  $g_3(a_4) = 0$ . We may assume  $g_2(a_4) = 0$ . By Hypothesis II,  $g_2 = g_{22} g_{24}$ ,  $\|g_{2j}\| \leq (1 + \varepsilon)^2$  ( $j = 2, 4$ ), and  $g_{22}(a_2) = g_{24}(a_4) = 0$ . Hence there exist  $h_2, h_3, h_4$  such that  $f = h_2 h_3 h_4$ ,  $\|h_j\| \leq (1 + \varepsilon)^2$  ( $j = 2, 3, 4$ )  $h_2(a_2) = h_3(a_3) = h_4(a_4) = 0$ . This argument implies the proof.  $\square$

**Lemma 8.** *Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ . If  $\{a_n\}_n$  satisfies Hypothesis II, then for  $1 \leq k \leq \ell$   $\rho_{k,\ell} = \prod_{k \neq n}^\ell \sigma(a_k, a_n)$ . Moreover, if  $\{a_n\}_n$  satisfies Hypothesis I then  $\rho_k = \prod_{k \neq n} \sigma(a_k, a_n)$ .*

*Proof.* By (1) of Lemma 5 it is sufficient to show that  $\rho_{k,\ell} \leq \prod_{k \neq n}^\ell \sigma(a_k, a_n)$ . If  $0 < \delta < \rho_{k,\ell}$ , then there exists  $f \in J_{k,\ell}$  with  $\|f\| \leq 1$  such that

$$\rho_{k,\ell} - \delta \leq |f(a_k)| \leq \rho_{k,\ell}.$$

For any  $\varepsilon > 0$ , by Lemma 7,  $f$  can be factorized as  $f = \prod_{n \neq k}^\ell f_n$ ,  $\|f_n\| \leq (1 + \varepsilon)^{\ell-1}$  and  $f_n(a_n) = 0$  for  $n \neq k$ . Hence

$$\prod_{n \neq k}^\ell |f_n(a_k)| \leq (1 + \varepsilon)^{(\ell-1)(\ell-1)} \prod_{n \neq k}^\ell \sigma(a_k, a_n).$$

As  $\varepsilon \rightarrow 0$   $\rho_{k,\ell} - \delta \leq \prod_{n \neq k}^\ell \sigma(a_k, a_n)$ . Since  $\delta$  is arbitrary,  $\rho_{k,\ell} \leq \prod_{n \neq k}^\ell \sigma(a_k, a_n)$ .  $\square$

**Theorem 4.** *Let  $A$  be an arbitrary uniform and  $\{a_n\}_n$  in  $M(A)$*

(1) *Under Hypothesis II,  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ .*

(2) *Under Hypothesis I and II,  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ .*

*Proof.* Theorem 1, Theorem 2 and Lemma 8 imply the theorem.  $\square$

When  $A = H^\infty(D)$  and  $\{a_n\}_n$  is in  $D$ ,  $\{a_n\}_n$  satisfies Hypotheses I and II. Let  $A$  be a disc algebra. Then if  $\{a_n\}_n$  is in  $D$  then  $\{a_n\}_n$  satisfies Hypothesis II (see Section 5). On the other hand, it is easy to see that there exists a sequence  $\{a_n\}_n$  in  $D$  which does not satisfy Hypothesis I.

## 5. Special uniform algebras

When  $A = H^\infty(D)$  and  $\{a_n\}_n$  in  $D$ , Hatori [3] showed that  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . Since it is clear that  $\{a_n\}_n$  in  $D$  satisfies Hypothesis I and II, this is a corollary of (2) of Theorem 4. Corollary 3 is also a result of Hatori [3]. We give another proof of it. Hatori [3] also shows this type theorem for a Hardy space  $H^p(1 \leq p < \infty)$  on a finite open Riemann surface and generalizes a theorem of Shapiro and Shields [7].

**Corollary 1.** *Let  $A$  be a uniform closed algebra between the disc algebra  $\mathcal{A}$  and  $H^\infty(D)$ , and  $\{a_n\}_n$  in  $D$ . Suppose that  $f/z$  belongs to  $A$  for  $f$  in  $A$  with  $f(0) = 0$ . Then  $\{a_n\}_n$  is a finite  $\ell^1$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ .*

Proof. If  $f \in A$  and  $f(a) = 0$  for some  $a \in D$ , then  $f(z)/(z - a)$  belongs to  $A$  (see [4]). Hence

$$\frac{1 - \bar{a}z}{z - a} f(z) \text{ belongs to } A$$

and  $(z - a)/(1 - \bar{a}z)$  is a unimodular function in  $\mathcal{A}$ . Therefore,  $\{a_n\}_n$  satisfies Hypothesis II and so (1) of Theorem 4 implies the corollary.  $\square$

**Corollary 2.** *Let  $A = H^\infty(D^m)$  and  $\{a_n\}_n$  in  $D^m$ . Suppose  $a_n = (a_n^1, a_n^2, \dots, a_n^m)$  and  $\sum_{n=1}^\infty (1 - |a_n^\ell|) < \infty$  for  $1 \leq \ell \leq m$ .  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . Proof. By Theorem 2 and Lemma 6, the 'if' part is showed. We will show the 'only if' part. Put*

$$B_k = B_k(z_1, \dots, z_m) = \prod_{\ell=1}^m \prod_{n \neq k} \frac{-a_n^\ell}{|a_n^\ell|} \frac{z_\ell - a_n^\ell}{1 - \bar{a}_n^\ell z_\ell},$$

then  $B_k$  belongs to  $H^\infty(D^m)$  because  $\sum_{n=1}^\infty (1 - |a_n^\ell|) < \infty$  for  $1 \leq \ell \leq m$ . Put  $F_k = B_k/B_k(a_k)$ , then  $F_k(a_n) = \delta_{nk}$  and

$$\|F_k + J\| = |B_k(a_k)|^{-1} \|B_k + J\| = |B_k(a_k)|^{-1}$$

and so  $\rho_k = |B_k(a_k)|$ . Theorem 1 implies that  $\inf_k |B_k(a_k)| = \inf_k \rho_k > 0$ . Since

$$\sigma(a_k, a_n) = \max \left( \left| \frac{a_k^1 - a_n^1}{1 - \bar{a}_n^1 a_k^1} \right|, \dots, \left| \frac{a_k^m - a_n^m}{1 - \bar{a}_n^m a_k^m} \right| \right)$$

(see [1, p 162]),

$$|B_k(a_k)| \leq \prod_{k \neq n} \sigma(a_k, a_n).$$

This shows the corollary. □

**Corollary 3.** *Let  $R$  be a finite open Riemann surface and  $A = H^\infty(R)$  the set of all bounded analytic functions on  $R$ . When  $\{a_n\}_n$  in  $R$ ,  $\{a_n\}_n$  is a  $\ell^1$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ .*

*Proof.* It is known [8] that  $\{a_n\}_n$  is an  $\ell^\infty$ -interpolating sequence if and only if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . If  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence then  $\inf_k \rho_k > 0$  by Theorem 1 and so by [8, Theorem 5.9]  $\{a_n\}_n$  is a  $\ell^\infty$ -interpolating sequence.

Let  $D_n = \{z \in \mathbf{C} ; |z - c_n| < r_n\}$ ,  $c_n > 0$  as  $D_n \cap D_m = \emptyset$  ( $n \neq m$ ),  $D_n \subset D \setminus \{0\}$  ( $n = 1, 2, 3, \dots$ ) and  $\sum_{n=1}^{\infty} r_n/c_n < \infty$ .  $U = D \setminus \overline{\bigcup_n D_n}$  is called a Zalcman domain [9]. When  $A = H^\infty(U)$  and  $\{a_n\}_n$  is in  $U$ , if  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ , then  $\{a_n\}_n$  is an  $\ell^1$ -interpolating sequence by (1) of Theorem 3 and Lemma 6 because  $\{a_n\}_n$  satisfies Hypothesis I but  $\{a_n\}_n$  is not necessarily an  $\ell^\infty$ -interpolating sequence by [6].

## 6. $\ell^\infty$ -interpolating sequence

When  $\{a_n\}_n$  in  $M(A)$  satisfies Hypothesis I, it is very nice to give a sufficient condition or a necessary condition for an  $\ell^\infty$ -interpolating sequence. Berndtsson, Chang and Lin [1] give the following problem : Let  $A = H^\infty(Y)$  and  $\{a_n\}_n \subset Y$  is a bounded domain  $Y \subset \mathbf{C}^n$ . Suppose  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ . Is  $\{a_n\}_n$  a  $\ell^\infty$ -interpolating sequence ? In Proposition 1,  $\sum_{n=1}^{\infty} (1 - \rho_n) < \infty$  and so by the remark above Lemma 5  $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$ .

**Proposition 1.** *Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ . Suppose  $\{a_n\}_n$  satisfies Hypothesis I. If  $\rho_n \geq 2(n^t + 1)(n^t + 2)/\{(n^t + 1)^2 + (n^t + 2)^2\}$  for  $n = 1, 2, 3, \dots$  and some  $t > 1$ , then  $\{a_n\}_n$  is an  $\ell^\infty$ -interpolating sequence.*

*Proof.* By Hypothesis I there exists a sequence  $\{F_n\}_n$  in  $A$  such that  $\|F_n\| \leq 1$ ,  $F_n(a_k) = 0$  if  $k \neq n$  and  $|F_n(a_n)| = \rho_n$  for  $n = 1, 2, \dots$ . Now Izuchi [5, Theorem 1] has proved essentially the theorem. We use the notations in [5, Theorem 1]. Put  $\rho_n = 2(1 - \delta_n)/\{1 + (1 - \delta_n)^2\}$ ,  $0 < \delta_n \leq 1/(n^t + 2)$  because  $\rho_n \geq 2(n^t + 1)(n^t + 2)/\{(n^t + 1)^2 + (n^t + 2)^2\}$ . If  $\varepsilon_n = 1/n^p$ , then  $\sum_{n=1}^{\infty} \varepsilon_n < \infty$  and so  $\prod_{n=1}^{\infty} (1 + \varepsilon_n) < \infty$ . Then

$$\delta_n < 1 - \frac{1}{\sqrt{1 + 2\varepsilon_n}}.$$

By the proof of [5, Theorem 1], there exists a sequence  $G_n \in A$  such that

$$\sum_{n=1}^{\infty} |G_n| \leq \sum_{n=1}^{\infty} (1 + \varepsilon_n) < \infty \text{ on } X.$$

Hypothesis I implies that  $\{a_n\}_n$  is an  $\ell^\infty$ -interpolating sequence. □

**Proposition 2.** *Let  $A$  be an arbitrary uniform algebra and  $\{a_n\}_n$  in  $M(A)$ . Suppose  $\{f_k\}_k$  is a sequence in  $A$  such that  $f_k(a_n) = \delta_{nk}$ .  $\{a_n\}_n$  is a  $\ell^p$ -interpolating sequence if and only if*

$$\sup_{\phi \in A^* \cap J^\perp, \|\phi\| \leq 1} (\sum_{n=1}^{\infty} |\phi(f_n)|^q)^{1/q} < \infty,$$

where  $1/p + 1/q = 1$  and  $A^* \cap J^\perp = \{\phi \in A^* ; \phi = 0 \text{ on } J\}$ . For  $p = 1$  and  $q = \infty$  we assume that

$$\sup_{\phi} (\sum_{n=1}^{\infty} |\phi(f_n)|^q)^{1/q} = \sup_{\phi} \sup_n |\phi(f_n)| = \sup_n \|f_n + J\|,$$

Proof. Suppose

$$\sup_{\phi \in A^* \cap J^\perp, \|\phi\| \leq 1} (\sum_{n=1}^{\infty} |\phi(f_n)|^q)^{1/q} = \gamma_q < \infty.$$

For any  $\phi \in A^* \cap J^\perp$  with  $\|\phi\| \leq 1$  and any  $\ell < \infty$ ,

$$|\phi(\sum_{n=1}^{\ell} \alpha_n f_n)| \leq (\sum_{n=1}^{\ell} |\alpha_n|^p)^{1/p} (\sum_{n=1}^{\ell} |\phi(f_n)|^q)^{1/q}$$

and so

$$\|\sum_{n=1}^{\infty} \alpha_n \tilde{f}_n\| \leq \gamma_q (\sum_{n=1}^{\infty} |\alpha_n|^p)^{1/p},$$

where  $\tilde{f}_n = f_n + J$ . Thus if  $\{\alpha_n\}_n \in \ell^p$  then  $\tilde{f} = \sum_{n=1}^{\infty} \alpha_n \tilde{f}_n$  belongs to  $A/J$ . Then  $f(a_n) = \alpha_n$  for  $n = 1, 2, \dots$  and so  $\{a_n\}_n$  is an  $\ell^p$ -interpolating sequence. Conversely, suppose  $S = \{a_n\}_n$  is an  $\ell^p$  interpolating sequence. For  $(\alpha_n) \in \ell^p$ , put

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n f_n |S,$$

then there exists a function  $f$  such that  $T(\alpha_n) = f|S$ . Since  $T$  turns to be bounded from  $\ell^p$  to  $A/J$  (see Lemma 1), for  $\phi \in A^*/J^\perp$  with  $\|\phi\| \leq 1$

$$|\phi(f)| = |\sum_{n=1}^{\infty} \alpha_n \phi(f_n)| \leq \|T\| (\sum_{n=1}^{\infty} |\alpha_n|^p)^{1/p}.$$

Hence  $\sup_{\phi \in A^* \cap J^\perp, \|\phi\| \leq 1} (\sum_{n=1}^{\infty} |\phi(f_n)|^q)^{1/q} < \infty$ . □

Hatori [3] is interested in when an  $\ell^1$ -interpolating sequence shows an  $\ell^\infty$ -interpolating sequence. He showed that if  $A = H^\infty(R)$  and  $\{a_n\}_n$  in  $R$ , then  $\{a_n\}_n$  is such a sequence (see Corollary 3). In general, Proposition 2 gives a necessary and sufficient condition.

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