Interpolation Problem For ℓ^1 And A Uniform Algebra

Takahiko Nakazi

Series #505. December 2000

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#481 T. Nakazi, Two dimensional Q-algebras, 11 pages. 2000.

#482 N. H. Bingham and A. Inoue, Tauberian and Mercerian theorems for systems of kernels, 16 pages. 2000.

#483 N. H. Bingham and A. Inoue, Abelian, Tauberian and Mercerian theorems for arithmetic sums, 29 pages. 2000.

#484 I. A. Bogaevski and G. Ishikawa, Lagrange mappings of the first open Whitney umbrella, 22 pages. 2000.

#485 A. Arai and H. Kawano, A class of deformations of the Schrödinger representation of the Heisenberg commutation relation and exact solution to a Heisenberg equation and a Schrödinger equation, 22 pages. 2000.

#486 T. Nakazi, Functions in N_+ with the positive real parts on the boundary, 21 pages. 2000.

#487 Y. Shibukawa, Classification of the R-operator, 36 pages. 2000.

#488 A. Inoue, Asymptotic behaviour for partial autocorrelation functions of fractional ARIMA processes, 20 pages. 2000.

#489 S. Ohtani, Construction of unramified Galois extensions over maximal abelian extensions of algebraic number fields, 14 pages. 2000.

#490 T. Nakazi and T. Yamamoto, The real part of an outer function and a Helson-Szegö weight, 13 pages. 2000.

#491 A. Yamagami, On Gouvêás conjecture on controlling the conductor, 11 pages. 2000.

#492 I. Tsuda and M. Hatakeyama, Making sense of internal logic: Theory and a case study, 10 pages. 2000.

#493 I. Tsuda, Towards an interpretation of dynamic neural activity in terms of chaotic dynamical systems, 73 pages. 2000.

#494 T. Mikami, Optimal control for absolutely continuous stochastic processes and the mass transportation problem, 17 pages. 2000.

#495 M. Arisawa and Y. Giga, Anisotropic curvature flow in a very thin domain, 21 pages. 2000.

#496 T. Nakazi, Backward shift invariant subspaces in the bidisc, 9 pages. 2000.

#497 Y. Giga, K. Inui, J. Kato and S. Matsui, Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations, 4 pages. 2000.

#498 Y. Giga, M. Paolini and P. Rybka, On the motion by singular interfacial energy, 21 pages. 2000.

#499 J. Escher and Y. Giga, On a limiting motion and self-interactions of curves moved by the intermediate surface diffusion flow, 12 pages. 2000.

#500 I. Tsuda and S. Kuroda, Cantor coding in the hippocampus, 20 pages. 2000.

#501 M. Tsujii, Fat solenoidal attractors, 20 pages. 2000.

#502 A. Arai, Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, 19 pages. 2000.

#503 Y. Giga, S. Matsui and O. Sawada, Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity, 19 pages. 2000.

#504 A. Inoue and Y. Kasahara, Partial autocorrelation functions of the fractional ARIMA processes with negative degree of differencing, 14 pages. 2000.

Interpolation Problem For ℓ^1 And A Uniform Algebra

Takahiko Nakazi

This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

2000 Mathematics subject classification : 46J10, 46J15.

Keywords and phrases : uniform algebra, ℓ^1, interpolation, maximal ideal space, pseudo-hyperbolic distance

Abstract. Let A be a uniform algebra and $M(A)$ the maximal ideal space of A. A sequence $\{a_n\}_n$ in $M(A)$ is called ℓ^1-interpolating if for every sequence (α_n) in ℓ^1 there exists a function f in A such that $f(a_n) = \alpha_n$ for all n. In this paper, an ℓ^1-interpolating sequence is studied for an arbitrary uniform algebra. For some special uniform algebras, an ℓ^1-interpolating sequence is equivalent to an ℓ^∞-interpolating sequence which is familiar for us. However, in general these two interpolating sequences may be different from each other.

1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X and $M(A)$ the maximal ideal space of A. Throuhout this paper we assume that $\{a_n\}_n$ is an infinite sequence of distinct points in $M(A)$. For $1 \leq p \leq \infty$, a sequence $\{a_n\}_n$ is called ℓ^p-interpolating if for every sequence (α_n) in ℓ^p there exists a function f in A such that $f(a_n) = \alpha_n$ for all n. An ℓ^∞-interpolating sequence is called usually interpolating.

If $A = H^\infty(D)$, that is, the set of all the bounded analytic functions on the unit disc D in $\mathbf{C}$, then an ℓ^∞-interpolating sequence was studied by Carleson [2]. Carleson [2] determined an ℓ^∞-inlerpolating sequence when $\{a_n\}$ in D. In the general situation, that is, for $\{a_n\}$ in $M(H^\infty(D))$, Izuchi [5] studied an ℓ^∞-interpolating sequence. Recently, Hatori [3] showed that an ℓ^1-interpolating sequence is equivalent to an ℓ^∞-interpolating sequence when $\{a_n\}$ in D. In this paper we study an ℓ^1-interpolating sequence for an arbitrary uniform algebra A when $\{a_n\}_n$ in $M(A)$. For $\{a_n\}_n$ in $M(A)$ put

$$J = \{f \in A \ ; \ f = 0 \ \text{ on } \ \{a_n\}_n\},$$
$$J_k = \{f \in A \ ; \ f = 0 \ \text{ on } \ \{a_n\}_{n \neq k}\}$$

and

$$\rho_k = \sup\{|f(a_k)| \ ; \ f \in J_k, \ \|f\| \leq 1\}.$$

For a, b in $M(A)$

$$\sigma(a, b) = \sup\{|f(a)| \ ; \ f(b) = 0, \ \|f\| \leq 1\}.$$

When $A = H^\infty(D)$ and $\{a_n\}_n$ is in D,

$$\sigma(a_k, a_n) = \left|\frac{a_k - a_n}{1 - \bar{a}_k a_n}\right| \quad \text{and} \quad \rho_k = \textstyle\prod_{n \neq k} \left|\frac{a_k - a_n}{1 - \bar{a}_k a_n}\right|.$$

In general, we don't know that

$$\rho_k = \textstyle\prod_{n \neq k} \sigma(a_k, a_n).$$

However under some mild condition (Hypothesis I in Section 4), we can show that

$$\rho_k \geq \textstyle\prod_{n \neq k} \sigma(a_k, a_n).$$

In general, $\rho_k > 0$ if and only if $J_k \underset{\neq}{\supset} J$. Hence $\rho_k > 0$ if and only if there exists a function f_k in A such that $f_k(a_n) = \delta_{nk}$. In this paper, for $\{a_n\}_n$ in $M(A)$ we assume that $\rho_k > 0$ for all k.

In Section 2, for an arbitrary uniform algebra we show that $\{a_n\}_n$ is an ℓ^1-interpolating sequence if and only if $\inf_k \ \rho_k > 0$. In Section 3, we define a finite ℓ^1-interpolating sequence and we give a necessary and sufficient condition for this. In Section 4, we show that if $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$, then $\{a_n\}_n$ is always a finite ℓ^1-interpolating sequence and under some mild condition it is an ℓ^1-interpolating sequence. In some sence, this type theorem for an ℓ^∞-interpolating sequence was conjectured in [1]. In Section 5, we apply the results in the previous sections to concrete uniform algebras. In Section 6, we give a remark about an ℓ^∞-interpolating sequence.

2. ℓ^1-interpolating sequence

In this section, we show that $\{a_n\}_n$ is an ℓ^1-interpolating sequence if and only if $\inf_k \rho_k > 0$. The argument in 'only if' part of Lemma 1 is similar to one which was used by Hatori [3] when $A = H^\infty(D)$.

Lemma 1. *$\{a_n\}_n$ is an ℓ^1-interpolating sequence if and only if there exists a sequence $\{f_n\}_n$ in A such that $f_n(a_k) = \delta_{nk}$ $(n \geq 1, k \geq 1)$ and $\sup_n \|f_n + J\| < \infty$.*
Proof. Suppose $M = \sup_n \|f_n + J\| < \infty$ and $f_n(a_k) = \delta_{nk}$. Let ε be arbitrary positive constant. For each n there exists g_n in J such that $\|f_n + g_n\| \leq M + \varepsilon$. If $(\alpha_n) \in \ell^1$, put

$$f = \textstyle\sum_{n=1}^{\infty} \alpha_n (f_n + g_n).$$

Then f belongs to A and $f(a_n) = \alpha_n$ for $n = 1, 2, \ldots$. Suppose $S = \{a_n\}_n$ is an ℓ^1-interpolating sequence. Then there exists a sequence $\{f_n\}_n$ in A such that $f_n(a_k) = \delta_{nk}$. For $(\alpha_n) \in \ell^1$, put

$$T(\alpha_n) = \textstyle\sum_{n=1}^{\infty} \alpha_n f_n | S,$$

then by hypothesis there exists a function f such that $T(\alpha_n) = f|S$. Since $A|S$ is algebraically isomorphic to the quotient algebra A/J, we put the quotient norm of A/J on $A|S$. By the closed graph theorem, T is bounded from ℓ^1 to $A|S$ and so

$$\|f_k + J\| = \|f_k|S\| \leq \|T\|$$

because $T(\{\delta_{nk}\}_n) = f_k|S$. □

Lemma 2. *Suppose $\{f_n\}_n$ is a sequence in A such that $f_n(a_k) = \delta_{nk}$. Then $\|f_n + J\| = 1/\rho_n$ for $n = 1, 2, \ldots$.*
Proof. Since $(\rho_n f_n)(a_k) = \rho_n \delta_{nk}$, $\|\rho_n f_n + J\| \geq 1$. By definition of ρ_n, for each $\ell \geq 1$ there exists $g_\ell \in A$ such that $\|g_\ell\| = 1$, $g_\ell(a_n) = 0$ for $n \neq k$ and

$$\rho_k - \frac{1}{\ell} \leq g_\ell(a_k) \leq \rho_k.$$

Put $G_\ell = g_\ell / g_\ell(a_k)$, then $G_\ell \in A$ and

$$\frac{1}{\rho_k} \leq \|G_\ell\| = \frac{1}{|g_\ell(a_k)|} \leq \frac{1}{\rho_k - 1/\ell}.$$

Moreover, $G_\ell(a_k) = 1$, $G_\ell(a_n) = 0$ for $n \neq k$ and so $G_\ell \in f_k + J$. Since $\|f_k + J\| \leq (\rho_k - 1/\ell)^{-1}$ for any $\ell \geq 1$, $\|\rho_k f_k + J\| \leq 1$. □

Theorem 1. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$. $\{a_n\}_n$ is a ℓ^1-interpolating sequence if and only if* $\inf_k \rho_k > 0$.
Proof. It is clear by Lemmas 1 and 2.

$\square$

3. Finite ℓ^1-interpolating sequence

We say that $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence if there exists a finite positive constant γ which satisfies the following : For any finite $\ell \geq 1$ and for any (α_n) in the unit ball of ℓ^1, there exists a function F_ℓ in A such that

$$F_\ell(a_n) = \alpha_n \quad \text{for} \quad 1 \leq n \leq \ell$$

and $\|F_\ell\| \leq \gamma$.

For $\{a_n\}_n$ in $M(A)$ and $1 \leq k \leq \ell < \infty$, put

$$J^\ell = \{f \in A \ ; \ f(a_n) = 0 \ \text{ if } \ 1 \leq n \leq \ell\},$$

$$J_k^\ell = \{f \in A \ ; \ f(a_n) = 0 \ \text{ if } \ 1 \leq n \leq \ell, \ n \neq k\}$$

and

$$\rho_{k,\ell} = \sup\{|f(a_k)| \ ; \ f \in J_k^\ell, \ \|f\| \leq 1\}.$$

Then $\rho_{k,\ell} \geq \rho_{k,\ell+1}$ and $\lim_{\ell\to\infty} \rho_{k,\ell} \geq \rho_k$.

Lemma 3. *$\{a_n\}_n$ is a finite ℓ^1-interpolating sequence if and only if for each $\ell \geq 1$ there exists a sequence $\{f_{\ell,n}\}_{n=1}^\ell$ in A such that $f_{\ell,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq \ell$ and* $\sup_\ell \sup_{1\leq n\leq \ell} \|f_{\ell,n} + J^\ell\| < \infty$.
Proof. (α_n) denotes an element in the unit ball of ℓ^1. Suppose $M = \sup_\ell \sup_{1\leq n\leq \ell} \|f_{\ell,n} + J^\ell\| < \infty$ and $f_{\ell,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq \ell$, then for any finite $\ell \geq 1$

$$\| \textstyle\sum_{n=1}^\ell \alpha_n f_{\ell,n} + J^\ell\| \leq (\sum_{n=1}^\ell |\alpha_n|)M.$$

If $\gamma = M+1$, then for any $\ell \geq 1$ there exists a $g_\ell \in J^\ell$ such that $\|\Sigma_{n=1}^\ell \alpha_n f_{\ell,n} + g_\ell\| \leq \gamma$. Set $F_\ell = \Sigma_{n=1}^\ell \alpha_n f_{\ell,n} + g_\ell$, then $F_\ell(a_n) = \alpha_n$ for $1 \leq n \leq \ell$ and $\|F_\ell\| \leq \gamma$. Suppose $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence. Since $\{a_n\}_n$ is an infinite sequence of distinct points in $M(A)$, for each $\ell \geq 1$ there exists a sequence $\{f_{\ell,n}\}_{n=1}^\ell$ in A such that $f_{\ell,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq n$. Put

$$T^\ell(\alpha_n) = \Sigma_{n=1}^{\ell} \alpha_n f_{\ell,n} + J^\ell$$

then $\|T^\ell(\alpha_n)\| \leq \|T^\ell\|(\Sigma_{n=1}^{\ell}|\alpha_n|)$. If $\|T^\ell\| \to \infty$ as $\ell \to \infty$, then there exists (α_n) in the unit ball of ℓ^1 such that $\|T^\ell(\alpha_n)\| \to \infty$ as $\ell \to \infty$. On the other hand, by hypothesis $\|T^\ell(\alpha_n)\| \leq \gamma < \infty$ for all ℓ. This contradiction implies that $M = \sup_\ell \|T^\ell\| < \infty$. This shows that for any $\ell \geq 1$ and any $k \geq 1$ with $k \leq \ell$,

$$\|f_{\ell,k} + J^\ell\| = \|T^\ell(\{\delta_{kn}\})\| \leq M.$$

□

Lemma 4. *For $\ell = 1, 2, \ldots$ and $1 \leq k \leq \ell$, $\|f_k + J^\ell\| = 1/\rho_{k,\ell}$.*
Proof is almost the same as the proof of Lemma 2. □

Theorem 2. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$. $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence if and only if* $\inf_k \lim_{\ell\to\infty} \rho_{k,\ell} > 0$.
Proof. It is clear by Lemma 3 and Lemma 4. □

4. Uniformly separated sequence

When $A = H^\infty(D)$ and $\{a_n\}_n$ is in D, for any $k \geq 1$

$$\rho_k = \Pi_{n\neq k}\sigma(a_k, a_n) = \lim_{\ell\to\infty} \rho_{k,\ell}.$$

When $\{a_n\}_n$ is in $M(A)$, Izuchi [5] showed essentially that $\inf_k \rho_k > 0$ implies $\inf_k \Pi_{n\neq k}\sigma(a_k, a_n) > 0$. However, this is not true in general. If $\Sigma_{n=1}^{\infty}(1-\rho_n) < \infty$, then $\inf_k \Pi_{n\neq k}\sigma(a_k, a_n) > 0$. In fact, $\rho_n \leq \sigma(a_k, a_n)$ for $n \neq k$ and so $\Pi_{n=1}^{\infty}\rho_n \leq \Pi_{n\neq k}\sigma(a_k, a_n)$ for any $k \geq 1$. When $\Sigma_{n=1}^{\infty}(1-\rho_n) < \infty$, $0 < \Pi_{n=1}^{\infty}\rho_n$ and so $\inf_k \Pi_{n\neq k}\sigma(a_k, a_n) > 0$. In this section , we study these three quantities.

Lemma 5.

(1) For any $\ell \geq 1$, $\rho_{k,\ell} \geq \Pi_{n\neq k}^{\ell} \sigma(a_k, a_n)$. Hence for any $k \geq 1$

$$\lim_{\ell\to\infty} \rho_{k,\ell} \geq \Pi_{n\neq k} \sigma(a_k, a_n).$$

(2) For $1 \leq n \leq \ell$ and $n \neq k$, $\rho_{k,\ell} \leq \sigma(a_k, a_n)$. Hence for any $k \geq 1$

$$\lim_{\ell\to\infty} \rho_{k,\ell} \leq \inf_{n\neq k} \sigma(a_k, a_n).$$

Proof. (1) Fix any positive constant $\varepsilon > 0$. For each n with $\ell \geq n \geq 1$ and $n \neq k$, there exists $F_n^\varepsilon \in A$ such that $\|F_n^\varepsilon\| \leq 1$, $F_n^\varepsilon(a_n) = 0$ and

$$\sigma(a_k, a_n) \geq |F_n^\varepsilon(a_k)| \geq \sigma(a_k, a_n) - \varepsilon.$$

Then $F^\varepsilon = \prod_{n\neq k}^{\ell} F_n^\varepsilon$ belongs to $J_{\ell,k}$, $\|F^\varepsilon\| \leq 1$ and

$$\rho_{\ell,k} \geq |F^\varepsilon(a_k)| \geq \prod_{n\neq k}^{\ell} \{\sigma(a_k, a_n) - \varepsilon\}.$$

As $\varepsilon \to 0$ $\rho_{\ell,k} \geq \prod_{n\neq k}^{\ell} \sigma(a_k, a_n)$ for any $\ell \geq 1$ and hence $\lim_{\ell\to\infty} \rho_{k,\ell} \geq \prod_{n\neq k} \sigma(a_k, a_n)$. (2) is clear by the definitions of $\rho_{k,\ell}$ and $\sigma(a_k, a_n)$ for $1 \leq n \leq \ell$ and $n \neq k$. □

Theorem 3. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$.*

(1) If $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$ then $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence.
(2) If $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence then $\inf_{n\neq k} \sigma(a_k, a_n) > 0$.

Proof. (1) By (1) of Lemma 5, $\inf_k \lim_{\ell\to\infty} \rho_{k,\ell} > 0$ and so by Theorem 2 $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence. (2) By Theorem 2 $\inf_k \lim_{\ell\to\infty} \rho_{k,\ell} > 0$ and so by (2) of Lemma 5 $\inf_{n\neq k} \sigma(a_k, a_n) > 0$. □

Hypothesis I. *Let A be a uniform algebra and $\{a_n\}_n$ in $M(A)$. If g_ℓ is a function in A and $\|g_\ell\| \leq 1$ for $\ell = 1, 2, \cdots$, then there exist a subsequence $\{g_{\ell(j)}\}_j$ of $\{g_\ell\}_\ell$ and a function g in A such that $\|g\| \leq 1$ and $\lim_{j\to\infty} g_{\ell(j)}(a_n) = g(a_n)$ for any $n \geq 1$*

Hypothesis II. *Let A be a uniform algebra and $\{a_n\}_n$ in $M(A)$. For any a, b in $\{a_n\}_n$ with $a \neq b$, if the function f in A satisfies $f(a) = f(b) = 0$ and $\|f\| \leq 1$, then for any $\varepsilon > 0$ there exist two functions g and h in A such that $\|g\| \leq 1 + \varepsilon$, $\|h\| \leq 1 + \varepsilon$, $g(a) = 0$, $h(b) = 0$ and $f = gh$.*

Lemma 6. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$. If $\{a_n\}_n$ satisfies Hypothesis I then $\lim_{\ell\to\infty} \rho_{k,\ell} = \rho_k$ for any $k \geq 1$ and so a finite ℓ^1-interpolating sequence is an ℓ^1-interpolating sequence.*
Proof. $\lim_{\ell\to\infty} \rho_{k,\ell} \geq \rho_k$ for any $k \geq 1$. If $\lim_{\ell\to\infty} \rho_{k,\ell} > \varepsilon > 0$, then for each ℓ there exists $g_\ell \in J_k^\ell$ such that $\|g_\ell\| \leq 1$ and $|g_\ell(a_k)| \geq \varepsilon > 0$. By hypothesis, there exists $g \in J_k$ such that $\|g\| \leq 1$ and $|g(a_k)| \geq \varepsilon > 0$. Thus $\lim_{\ell\to\infty} \rho_{k,\ell} \leq \rho_k$ and so $\lim_{\ell\to\infty} \rho_{k,\ell} = \rho_k$. This and Theorem 1 and Theorem 2 also imply that a finite ℓ^1-interpolating sequence is an ℓ^1-interpolating sequence. □

Lemma 7. *Assume Hypothesis II. If f is a function in $J_{k,\ell}$ with $\|f\| \leq 1$, then for any $\varepsilon > 0$ $f = \prod_{n \neq k}^{\ell} f_n$, $f_n(a_n) = 0$ $(n \neq k)$ and $\|f_n\| \leq (1+\varepsilon)^{\ell-1}$.*
Proof. We may assume $k = 1$. Fix any $\varepsilon > 0$. By Hypothesis II, $f = g_2 g_3$, $\|g_j\| \leq 1+\varepsilon$ $(j = 2,3)$ and $g_2(a_2) = g_3(a_3) = 0$. Since $f(a_4) = 0$, $g_2(a_4) = 0$ or $g_3(a_4) = 0$. We may assume $g_2(a_4) = 0$. By Hypothesis II, $g_2 = g_{22} g_{24}$, $\|g_{2j}\| \leq (1+\varepsilon)^2$ $(j = 2,4)$, and $g_{22}(a_2) = g_{24}(a_4) = 0$. Hence there exist h_2, h_3, h_4 such that $f = h_2 h_3 h_4$, $\|h_j\| \leq (1+\varepsilon)^2$ $(j = 2,3,4)$ $h_2(a_2) = h_3(a_3) = h_4(a_4) = 0$. This argument implies the proof. □

Lemma 8. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$. If $\{a_n\}_n$ satisfies Hypothesis II, then for $1 \leq k \leq \ell$ $\rho_{k,\ell} = \prod_{k \neq n}^{\ell} \sigma(a_k, a_n)$. Moreover, if $\{a_n\}_n$ satisfies Hypothesis, I then $\rho_k = \prod_{k \neq n} \sigma(a_k, a_n)$.*
Proof. By (1) of Lemma 5 it is sufficient to show that $\rho_{k,\ell} \leq \prod_{k \neq n}^{\ell} \sigma(a_k, a_n)$. If $0 < \delta < \rho_{k,\ell}$, then there exists $f \in J_{k,\ell}$ with $\|f\| \leq 1$ such that

$$\rho_{k,\ell} - \delta \leq |f(a_k)| \leq \rho_{k,\ell}.$$

For any $\varepsilon > 0$, by Lemma 7, f can be factorized as $f = \prod_{n \neq k}^{\ell} f_n$, $\|f_n\| \leq (1+\varepsilon)^{\ell-1}$ and $f_n(a_n) = 0$ for $n \neq k$. Hence

$$\prod_{n \neq k}^{\ell} |f_n(a_k)| \leq (1+\varepsilon)^{(\ell-1)(\ell-1)} \prod_{n \neq k}^{\ell} \sigma(a_k, a_n).$$

As $\varepsilon \to 0$ $\rho_{k,\ell} - \delta \leq \prod_{n \neq k}^{\ell} \sigma(a_k, a_n)$. Since δ is arbitrary, $\rho_{k,\ell} \leq \prod_{n \neq k}^{\ell} \sigma(a_k, a_n)$. □

Theorem 4. *Let A be an arbitrary uniform and $\{a_n\}_n$ in $M(A)$*

(1) Under Hypothesis II, $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$.
(2) Under Hypothesis I and II, $\{a_n\}_n$ is an ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$.

Proof. Theorem 1, Theorem 2 and Lemma 8 imply the theorem. □

When $A = H^\infty(D)$ and $\{a_n\}_n$ is in D, $\{a_n\}_n$ satifies Hypotheses I and II. Let A be a disc algebra. Then if $\{a_n\}_n$ is in D then $\{a_n\}_n$ satisfies Hypothesis II (see Section 5). On the other hand, it is easy to see that there exists a sequence $\{a_n\}_n$ in D which does not satisfy Hypothesis I.

5. Special uniform algebras

When $A = H^\infty(D)$ and $\{a_n\}_n$ in D, Hatori [3] showed that $\{a_n\}_n$ is an ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$. Since it is clear that $\{a_n\}_n$ in D satisfies Hypothesis I and II, this is a corollary of (2) of Theorem 4. Corollary 3 is also a result of Hatori [3]. We give another proof of it. Hatori [3] also shows this type theorem for a Hardy space $H^p(1 \leq p < \infty)$ on a finite open Riemann surbace and generalizes a theorem of Shapiro and Shields [7].

Corollary 1. *Let A be a uniform closed algebra between the disc algebra $\mathcal{A}$ and $H^\infty(D)$, and $\{a_n\}_n$ in D. Suppose that f/z belongs to A for f in A with $f(0) = 0$. Then $\{a_n\}_n$ is a finite ℓ^1-interpolating sequence if and only if* $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$.
Proof. If $f \in A$ and $f(a) = 0$ for some $a \in D$, then $f(z)/(z-a)$ belongs to A (see [4]). Hence

$$\frac{1-\bar{a}z}{z-a} f(z) \text{ belongs to } A$$

and $(z-a)/(1-\bar{a}z)$ is a unimodular function in $\mathcal{A}$. Therefore, $\{a_n\}_n$ satisfies Hypothesis II and so (1) of Theorem 4 implies the corollary. □

Corollary 2. *Let $A = H^\infty(D^m)$ and $\{a_n\}_n$ in D^m. Suppose $a_n = (a_n^1, a_n^2, \cdots, a_n^m)$ and $\sum_{n=1}^\infty (1 - |a_n^\ell|) < \infty$ for $1 \leq \ell \leq m$. $\{a_n\}_n$ is an ℓ^1-interpolating sequence if and only if* $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$. Proof. By Theorem 2 and Lemma 6, the 'if' part is showed. We will show the 'only if' part. Put

$$B_k = B_k(z_1, \ldots, z_m) = \textstyle\prod_{\ell=1}^m \prod_{n\neq k}^\infty \dfrac{-a_n^\ell}{|a_n^\ell|} \dfrac{z_\ell - a_n^\ell}{1 - \bar{a}_n^\ell z_\ell},$$

then B_k belngs to $H^\infty(D^m)$ because $\sum_{n=1}^\infty (1 - |a_n^\ell|) < \infty$ for $1 \leq \ell \leq m$. Put $F_k = B_k/B_k(a_k)$, then $F_k(a_n) = \delta_{nk}$ and

$$\|F_k + J\| = |B_k(a_k)|^{-1} \|B_k + J\| = |B_k(a_k)|^{-1}$$

and so $\rho_k = |B_k(a_k)|$. Theorem 1 implies that $\inf_k |B_k(a_k)| = \inf_k \rho_k > 0$. Since

$$\sigma(a_k, a_n) = \max\left(\left|\frac{a_k^1 - a_n^1}{1 - \bar{a}_n^1 a_k^1}\right|, \cdots, \left|\frac{a_k^m - a_n^m}{1 - \bar{a}_n^m a_k^m}\right|\right)$$

(see [1, p 162]),

$$|B_k(a_k)| \leq \textstyle\prod_{k\neq n} \sigma(a_k, a_n).$$

This shows the corollary. $\square$

Corollary 3. *Let R be a finite open Riemann surface and $A = H^\infty(R)$ the set of all bounded analytic functions on R. When $\{a_n\}_n$ in R, $\{a_n\}_n$ is a ℓ^1-interpolating sequence if and only if* $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$.
Proof. It is known [8] that $\{a_n\}_n$ is an ℓ^∞-interpolating sequence if and only if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$. If $\{a_n\}_n$ is an ℓ^1-interpolating sequence then $\inf_k \rho_k > 0$ by Theorem 1 and so by [8, Theorem 5.9] $\{a_n\}_n$ is a ℓ^∞-interpolating sequence.

Let $D_n = \{z \in \mathbf{C} \; ; \; |z - c_n| < r_n\}$, $c_n > 0$ as $D_n \cap D_m = \emptyset \;\; (n \neq m)$, $D_n \subset D \backslash \{0\} \;\; (n = 1, 2, 3, \cdots)$ and $\sum_{n=1}^\infty r_n / c_n < \infty$. $U = D \backslash \overline{\cup_n D_n}$ is called a Zalcman domain [9]. When $A = H^\infty(U)$ and $\{a_n\}_n$ is in U, if $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$, then $\{a_n\}_n$ is an ℓ^1-interpolating sequence by (1) of Theorem 3 and Lemma 6 because $\{a_n\}_n$ satisfies Hypothesis I but $\{a_n\}_n$ is not necessarily an ℓ^∞-interpolating sequence by [6].

6. ℓ^∞-interpolating sequence

When $\{a_n\}_n$ in $M(A)$ satisfies Hypothesis I, it is very nice to give a sufficient condition or a necessary condition for an ℓ^∞-interpolating sequence. Berndtsson, Chang and Lin [1] give the following problem : Let $A = H^\infty(Y)$ and $\{a_n\}_n \subset Y$ is a bounded domain $Y \subset \mathbf{C}^n$. Suppose $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$. Is $\{a_n\}_n$ a ℓ^∞-interpolating sequence ? In Proposition 1, $\sum_{n=1}^\infty (1 - \rho_n) < \infty$ and so by the remark above Lemma 5 $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$.

Proposition 1. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$. Suppose $\{a_n\}_n$ satisfies Hypothesis I. If $\rho_n \geq 2(n^t+1)(n^t+2)/\{(n^t+1)^2 + (n^t+2)^2\}$ for $n = 1, 2, 3, \cdots$ and some $t > 1$, then $\{a_n\}_n$ is an ℓ^∞-interpolating sequence.*
Proof. By Hypothesis I there exists a sequence $\{F_n\}_n$ in A such that $\|F_n\| \leq 1$, $F_n(a_k) = 0$ if $k \neq n$ and $|F_n(a_n)| = \rho_n$ for $n = 1, 2, \ldots$. Now Izuchi [5, Theorem 1] has proved essentially the theorem. We use the notations in [5, Theorem 1]. Put $\rho_n = 2(1-\delta_n)/\{1 + (1-\delta_n)^2\}$, $0 < \delta_n \leq 1/(n^t+2)$ because $\rho_n \geq 2(n^t+1)(n^t+2)/\{(n^t+2)^2 + (n^t+1)^2\}$. If $\varepsilon_n = 1/n^\rho$, then $\sum_{n=1}^\infty \varepsilon_n < \infty$ and so $\prod_{n=1}^\infty (1 + \varepsilon_n) < \infty$. Then

$$\delta_n < 1 - \frac{1}{\sqrt{1 + 2\varepsilon_n}}.$$

By the proof of [5, Theorem 1], there exists a sequence $G_n \in A$ such that

$$\sum_{n=1}^{\infty}|G_n| \leq \sum_{n=1}^{\infty}(1+\varepsilon_n) < \infty \text{ on } X.$$

Hypothesis I implies that $\{a_n\}_n$ is an ℓ^∞-interpolating sequence. $\square$

Proposition 2. *Let A be an arbitrary uniform algebra and $\{a_n\}_n$ in $M(A)$. Suppose $\{f_k\}_k$ is a sequence in A such that $f_k(a_n) = \delta_{nk}$. $\{a_n\}_n$ is a ℓ^p-interpolating sequence if and only if*

$$\sup_{\phi\in A^*\cap J^\perp,\ \|\phi\|\leq 1}\left(\sum_{n=1}^{\infty}|\phi(f_n)|^q\right)^{1/q} < \infty,$$

where $1/p + 1/q = 1$ and $A^* \cap J^\perp = \{\phi \in A^*\ ;\ \phi = 0$ on $J\}$. For $p = 1$ and $q = \infty$ we assume that

$$\sup_\phi\left(\sum_{n=1}^{\infty}|\phi(f_n)|^q\right)^{1/q} = \sup_\phi \sup_n |\phi(f_n)| = \sup_n \|f_n + J\|,$$

Proof. Suppose

$$\sup_{\phi\in A^*\cap J^\perp,\ \|\phi\|\leq 1}\left(\sum_{n=1}^{\infty}|\phi(f_n)|^q\right)^{1/q} = \gamma_q < \infty.$$

For any $\phi \in A^* \cap J^\perp$ with $\|\phi\| \leq 1$ and any $\ell < \infty$,

$$\left|\phi\left(\sum_{n=1}^{\ell}\alpha_n f_n\right)\right| \leq \left(\sum_{n=1}^{\ell}|\alpha_n|^p\right)^{1/p}\left(\sum_{n=1}^{\ell}|\phi(f_n)|^q\right)^{1/q}$$

and so

$$\left\|\sum_{n=1}^{\infty}\alpha_n\tilde{f}_n\right\| \leq \gamma_q\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p},$$

where $\tilde{f}_n = f_n + J$. Thus if $\{\alpha_n\}_n \in \ell^p$ then $\tilde{f} = \sum_{n=1}^{\infty}\alpha_n\tilde{f}_n$ belongs to A/J. Then $f(a_n) = \alpha_n$ for $n = 1, 2, \cdots$ and so $\{a_n\}_n$ is an ℓ^p-interpolating sequence. Conversely, suppose $S = \{a_n\}_n$ is an ℓ^p interpolating sequence. For $(\alpha_n) \in \ell^p$, put

$$T(\alpha_n) = \sum_{n=1}^{\infty}\alpha_n f_n|S,$$

then there exists a function f such that $T(\alpha_n) = f|S$. Since T turns to be bounded from ℓ^p to A/J (see Lemma 1), for $\phi \in A^*/J^\perp$ with $\|\phi\| \leq 1$

$$|\phi(f)| = \left|\sum_{n=1}^{\infty}\alpha_n\phi(f_n)\right| \leq \|T\|\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p}.$$

Hence $\sup_{\phi\in A^*\cap J^\perp,\ \|\phi\|\leq 1}\left(\sum_{n=1}^{\infty}|\phi(f_n)|^q\right)^{1/q} < \infty$. $\square$

Hatori [3] is interested in when an ℓ^1-interpolating sequence shows an ℓ^∞-interpolating sequence. He showed that if $A = H^\infty(R)$ and $\{a_n\}_n$ in R, then $\{a_n\}_n$ is such a sequence (see Corollary 3). In general, Proposition 2 gives a necessary and sufficient condition.

References

1. B.Berndtsson, S.-Y.A.Chang and K.-C.Lin, 'Interpolating sequences in the polydisc', *Trans. Amer. Math. Soc.* **302**(1987), 161-169.

2. L.Carleson, 'An interpolation problem for bounded analytic functions', *Amer. J. Math.* **80**(1958), 921-930.

3. O.Hatori, 'A theorem of Shapiro and Shields for finitety connected domains' Japanese, *Sūrikaisekikenkyūsho Kōkyūroku* **1049**

4. K.Izuchi and Y.Izuchi, 'Algebras of bounded analytic functions containing the disk algebra', *Can. J. Math.*, XXXVIII(1986), 87-108.

5. K.Izuchi, 'Interpolating sequences in the maxinal ideal space of H^∞ II', *Operator Theory* : Advances *and Applications*, Vol.59(1992), pp.221-233.

6. J.Narita, 'Interpolating sequences on planer domains', *Kodai, Math.* J. **13**(1990), 311-316.

7. H.S.Shapiro and A.L.Shields, 'On some interpolation problems for analytic functions', *Amer. J. Math.* **83**(1961), 513-532.

8. E.L.Stout, 'Bounded holomorphic functions on finite Riemann surfaces', *Trans. Amer.*
Math. Soc. **120**(1965), 255-285.

9. L.Zalcman, 'Bounded analytic functions on domains of infinite connectivity', *Trans. Amer. Math. Soc.* **144**(1969), 241-270.

Department of Mathematics
Hokkaido University
Sapporo 060-0810,
Japan
e-mail : nakazi @ math.sci.hokudai.ac.jp