Title	Instability in the spectral and the Fredholm properties of an infinite dimensional Dirac operator on the abstract Boson-Fermion Fock space
Author(s)	Arai, A.
Citation	Hokkaido University Preprint Series in Mathematics, 507, 1-6
Issue Date	2000-12-1
DOI	10.14943/83653
Doc URL	http://hdl.handle.net/2115/69257
Type	bulletin (article)
File Information	pre507.pdf

[Instructions for use](#)

**Instability in the Spectral and the Fredholm
Properties of an Infinite Dimensional Dirac
Operator on the Abstract Boson-Fermion Fock
Space**

Asao Arai

Series #507. December 2000

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

- #482 N. H. Bingham and A. Inoue, Tauberian and Mercerian theorems for systems of kernels, 16 pages. 2000.
- #483 N. H. Bingham and A. Inoue, Abelian, Tauberian and Mercerian theorems for arithmetic sums, 29 pages. 2000.
- #484 I. A. Bogaevski and G. Ishikawa, Lagrange mappings of the first open Whitney umbrella, 22 pages. 2000.
- #485 A. Arai and H. Kawano, A class of deformations of the Schrödinger representation of the Heisenberg commutation relation and exact solution to a Heisenberg equation and a Schrödinger equation, 22 pages. 2000.
- #486 T. Nakazi, Functions in N_+ with the positive real parts on the boundary, 21 pages. 2000.
- #487 Y. Shibukawa, Classification of the R -operator, 36 pages. 2000.
- #488 A. Inoue, Asymptotic behaviour for partial autocorrelation functions of fractional ARIMA processes, 20 pages. 2000.
- #489 S. Ohtani, Construction of unramified Galois extensions over maximal abelian extensions of algebraic number fields, 14 pages. 2000.
- #490 T. Nakazi and T. Yamamoto, The real part of an outer function and a Helson-Szegö weight, 13 pages. 2000.
- #491 A. Yamagami, On Gouvêas conjecture on controlling the conductor, 11 pages. 2000.
- #492 I. Tsuda and M. Hatakeyama, Making sense of internal logic: Theory and a case study, 10 pages. 2000.
- #493 I. Tsuda, Towards an interpretation of dynamic neural activity in terms of chaotic dynamical systems, 73 pages. 2000.
- #494 T. Mikami, Optimal control for absolutely continuous stochastic processes and the mass transportation problem, 17 pages. 2000.
- #495 M. Arisawa and Y. Giga, Anisotropic curvature flow in a very thin domain, 21 pages. 2000.
- #496 T. Nakazi, Backward shift invariant subspaces in the bidisc, 9 pages. 2000.
- #497 Y. Giga, K. Inui, J. Kato and S. Matsui, Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations, 4 pages. 2000.
- #498 Y. Giga, M. Paolini and P. Rybka, On the motion by singular interfacial energy, 21 pages. 2000.
- #499 J. Escher and Y. Giga, On a limiting motion and self-interactions of curves moved by the intermediate surface diffusion flow, 12 pages. 2000.
- #500 I. Tsuda and S. Kuroda, Cantor coding in the hippocampus, 20 pages. 2000.
- #501 M. Tsujii, Fat solenoidal attractors, 20 pages. 2000.
- #502 A. Arai, Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, 19 pages. 2000.
- #503 Y. Giga, S. Matsui and O. Sawada, Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity, 19 pages. 2000.
- #504 A. Inoue and Y. Kasahara, Partial autocorrelation functions of the fractional ARIMA processes with negative degree of differencing, 14 pages. 2000.
- #505 T. Nakazi, Interpolation problem for ℓ^1 and a uniform algebra, 12 pages. 2000.
- #506 R. Kobayashi and Y. Giga, On anisotropy and curvature effects for growing crystals, 38 pages. 2000.

Instability in the Spectral and the Fredholm Properties of an Infinite Dimensional Dirac Operator on the Abstract Boson-Fermion Fock Space

Asao Arai *

Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan

Email: arai@math.sci.hokudai.ac.jp

November 23, 2000

to be published in the Proceedings of the Conference PARTIAL DIFFERENTIAL EQUATIONS 2000 held in Clausthal, Germany (July 24–July 28, 2000)

Abstract

A perturbed Dirac operator $Q(\alpha)$ on the abstract Boson-Fermion Fock space is considered, where $\alpha \in \mathbb{C}$ is a perturbation (coupling) parameter and the unperturbed operator $Q(0)$ is taken to be a free infinite dimensional Dirac operator introduced by the author (A. Arai, J. Funct. Anal. **105**(1992), 342–408). The following results are reported: (i) Under some conditions, the kernel of $Q(\alpha)$ is one dimensional for all $\alpha \neq \alpha_0$ with some $\alpha_0 \neq 0$ and degenerate at $\alpha = \alpha_0$, while, under another condition, the kernel of $Q(\alpha)$ is one dimensional for all $\alpha \in \mathbb{C}$. (ii) There are cases where, for all sufficiently large $|\alpha|$ with $\alpha < 0$, $Q(\alpha)$ has infinitely many non-zero eigenvalues even if $Q(0)$ has no non-zero eigenvalues. This is a strong coupling effect. (iii) Fredholm property of $Q(\alpha)$ also depends on the coupling parameter α .

Key words: Boson-Fermion Fock space, supersymmetric quantum field, infinite dimensional Dirac operator, non-regular perturbation, kernel, spectrum, Fredholm property, strong coupling effect

*Supported by the Grant-In-Aid No.11440036 for Scientific Research from the Ministry of Education, Science, Sports and Culture, Japan.

1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces. Then the abstract Boson-Fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with the pair $(\mathcal{H}, \mathcal{K})$ is defined by

$$\mathcal{F}(\mathcal{H}, \mathcal{K}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{K}), \quad (1.1)$$

where $\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{H}$ is the Boson Fock space over $\mathcal{H}$ ($\bigotimes_s^n \mathcal{H}$ denotes the n -fold symmetric tensor product Hilbert space of $\mathcal{H}$; $\bigotimes_s^0 \mathcal{H} := \mathbb{C}$) and $\mathcal{F}_f(\mathcal{K}) := \bigoplus_{p=0}^{\infty} \bigwedge^p \mathcal{K}$ is the Fermion Fock space over $\mathcal{K}$ ($\bigwedge^p \mathcal{K}$ denotes the p -fold anti-symmetric tensor product Hilbert space of $\mathcal{K}$; $\bigwedge^0 \mathcal{K} := \mathbb{C}$).

Let $\mathcal{C}(\mathcal{H}, \mathcal{K})$ be the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$. Then, for each $A \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, one can define a Dirac-type operator Q_A on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ [Ar92] (for the definition of Q_A , Section 2 below). The operator Q_A is an infinite dimensional version of free Dirac operators on finite dimensional spaces.

In [Ar92] some fundamental properties of Q_A were established. Moreover, a perturbed Dirac operator of the form $Q_A(V) := Q_A + V$ was considered in view of index theory, where V is a symmetric operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$, and a functional integral representation for the index of $Q_A(V)$ restricted to a subspace, called the “bosonic subspace”, was derived (for related aspects and further developments, see [Ar89], [Ar91], [Ar93], [Ar93], [Ar94], [Ar96], [Ar97], [AM91], [AM93]). It still remains, however, as an important problem, to investigate spectral properties of $Q_A(V)$. In this paper, as a first step towards this direction, we present a perturbation of Q_A which is not of the form V considered in [Ar92] and rather simple, but, gives rise to interesting *nonperturbative instability phenomena* in spectral and Fredholm properties. For proofs on the results reported in this paper, see [Ar00].

2 A Class of Perturbed Dirac Operators

We denote by $a(f)$ ($f \in \mathcal{H}$) and $b(u)$ ($u \in \mathcal{K}$) the annihilation operators on $\mathcal{F}_b(\mathcal{H})$ and on $\mathcal{F}_f(\mathcal{K})$ respectively (e.g., [BR97, §5.2]). Let $\Omega_b := \{1, 0, 0, \dots\} \in \mathcal{F}_b(\mathcal{H})$ (resp. $\Omega_f := \{1, 0, 0, \dots\} \in \mathcal{F}_f(\mathcal{K})$) be the Fock vacuum in $\mathcal{F}_b(\mathcal{H})$ (resp. $\mathcal{F}_f(\mathcal{K})$). Let $A \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and

$$\begin{aligned} \mathcal{D}_A^\infty := \mathcal{L} \Big\{ & a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \cdots b(u_p)^* \Omega_f \Big| n, p \geq 0, f_j \in C^\infty(A^*A), \\ & j = 1, \dots, n, u_k \in C^\infty(AA^*), k = 1, \dots, p \Big\}, \end{aligned} \quad (2.1)$$

where $\mathcal{L}\{\dots\}$ means the subspace algebraically spanned by the vectors in the set $\{\dots\}$ and $C^\infty(T) := \bigcap_{n=1}^{\infty} D(T^n)$ for a linear operator T on a Hilbert space ($D(T^n)$ denotes the domain of T^n). It follows that $\mathcal{D}_A^\infty$ is dense in $\mathcal{F}(\mathcal{H}, \mathcal{K})$.

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system of $\mathcal{K}$ such that $e_n \in D(A^*)$, $n \in \mathbb{N}$. Then it is shown that there is a unique densely defined closed linear operator d_A on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ such that (i) for all $\Psi \in \mathcal{D}_A^\infty$

$$d_A \Psi = \sum_{n=1}^{\infty} a(A^* e_n) b(e_n)^* \Psi \quad (2.2)$$

independently of the choice of $\{e_n\}_{n=1}^\infty$ and (ii) $\mathcal{D}_A^\infty$ is a core of d_A . It follows that $d_A^2 = 0$. The operator may be regarded as an infinite dimensional version of finite dimensional exterior differential operators. A free Dirac operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by

$$Q_A = d_A + d_A^*. \quad (2.3)$$

We have an orthogonal decomposition

$$\mathcal{F}(\mathcal{H}, \mathcal{K}) = \mathcal{F}_+(\mathcal{H}, \mathcal{K}) \oplus \mathcal{F}_-(\mathcal{H}, \mathcal{K}) \quad (2.4)$$

with

$$\mathcal{F}_+(\mathcal{H}, \mathcal{K}) := \bigoplus_{p=0}^\infty \mathcal{F}_b(\mathcal{H}) \otimes \wedge^{2p}(\mathcal{K}), \quad \mathcal{F}_-(\mathcal{H}, \mathcal{K}) := \bigoplus_{p=0}^\infty \mathcal{F}_b(\mathcal{H}) \otimes \wedge^{2p+1}(\mathcal{K}). \quad (2.5)$$

Let P_+ and P_- be the orthogonal projections onto $\mathcal{F}_+(\mathcal{H}, \mathcal{K})$ and $\mathcal{F}_-(\mathcal{H}, \mathcal{K})$ respectively and define

$$\Gamma := P_+ - P_-. \quad (2.6)$$

For a self-adjoint operator S on $\mathcal{H}$ (resp. $\mathcal{K}$), $d\Gamma_b(S)$ (resp. $d\Gamma_f(S)$) denotes the second quantization of S in the Boson Fock space $\mathcal{F}_b(\mathcal{H})$ (resp. the Fermion Fock space $\mathcal{F}_f(\mathcal{H})$) [BR97, §5.2].

Basic properties of Q_A are as follows [Ar92]:

- (i) The operator Q_A is self-adjoint and essentially self-adjoint on $\mathcal{D}_A^\infty$.
- (ii) The operator Γ leaves $D(Q_A)$ invariant and $\Gamma Q_A + Q_A \Gamma = 0$ on $D(Q_A)$.
- (iii) The following operator equations hold :

$$Q_A^2 = d_A^* d_A + d_A d_A^* = d\Gamma_b(A^* A) \otimes I + I \otimes d\Gamma_f(AA^*), \quad (2.7)$$

where I denotes identity.

We perturb Q_A through a perturbation of d_A . Let $g \in D(A)$, $g \neq 0$, and $v \in D(A^*)$, $v \neq 0$ and define

$$d(\alpha) := d_A + \alpha a(g) \otimes b(v)^*. \quad (2.8)$$

where $\alpha \in \mathbb{C}$ is a coupling constant. It is easy to see that $D(d(\alpha)) \supset \mathcal{D}_A^\infty$ and $d(\alpha)|_{\mathcal{D}_A^\infty}$ is closable. Let

$$\bar{d}(\alpha) := \overline{d(\alpha)|_{\mathcal{D}_A^\infty}}, \quad (2.9)$$

the closure of $d(\alpha)|_{\mathcal{D}_A^\infty}$, and

$$Q(\alpha) := \bar{d}(\alpha) + \bar{d}(\alpha)^*. \quad (2.10)$$

This is the perturbed Dirac operator considered in this paper.

3 Results

- Theorem 3.1** (i) For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is self-adjoint, and essentially self-adjoint on $\mathcal{D}_A^\infty$ with $Q(\alpha) = \overline{Q_A + V_{g,v}}$, where $V_{g,v} := \alpha a(g) \otimes b(v)^* + \alpha^* a(g)^* \otimes b(v)$. Moreover, Γ leaves $D(Q(\alpha))$ invariant and $\Gamma Q(\alpha) + Q(\alpha) \Gamma = 0$ on $D(Q(\alpha))$.
- (ii) Suppose that A is injective and $g \in D(|A|^{-1})$. Then, for all $|\alpha| < (\|v\| \| |A|^{-1} g \|)^{-1}$, $D(Q(\alpha)) = D(Q_A)$.
- (iii) For all $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q(\alpha) - z)^{-1}$ is strongly continuous in $\alpha \in \mathbb{C}$.

To describe the spectral properties of $Q(\alpha)$, we introduce a bounded linear operator $T_{g,v}$ from $\mathcal{H}$ to $\mathcal{K}$ by

$$T_{g,v}f := (g, f)v, \quad f \in \mathcal{H}, \quad (3.1)$$

where $(\cdot, \cdot)$ denotes inner product, and define

$$A(\alpha) := A + \alpha T_{g,v}. \quad (3.2)$$

For a linear operator T on a Hilbert space, we denote by $\sigma(T)$ (resp. $\sigma_p(T)$) the (resp. point) spectrum of T . A general feature of the spectra of $Q(\alpha)$ is given in the following theorem:

Theorem 3.2 For all $\alpha \in \mathbb{C}$, $\sigma(Q(\alpha))$ and $\sigma_p(Q(\alpha))$ are symmetric with respect to the origin and

$$\sigma(Q(\alpha)) = \{0\} \cup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma(A(\alpha)^* A(\alpha)), j = 1, \dots, n \right\} \right)}, \quad (3.3)$$

$$\sigma_p(Q(\alpha)) = \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma_p(A(\alpha)^* A(\alpha)), j = 1, \dots, n \right\} \right), \quad (3.4)$$

with

$$\dim \ker(Q(\alpha) - \lambda) = \dim \ker(Q(\alpha) + \lambda), \quad \lambda \in \sigma_p(Q(\alpha)). \quad (3.5)$$

This theorem shows that the spectrum and the point spectrum of $Q(\alpha)$ are completely determined from those of $A(\alpha)^* A(\alpha)$.

To state properties of the kernel of $Q(\alpha)$, we introduce the following conditions on $\{A, g, v\}$:

- (C.1) A is injective and $v \in D(A^{-1})$ with $(g, A^{-1}v) \neq 0$. In this case we introduce a constant

$$\alpha_0 := -\frac{1}{(g, A^{-1}v)}. \quad (3.6)$$

- (C.2) A^* is injective and $g \in D(A^{*-1})$ with $(A^{*-1}g, v) \neq 0$. In this case we introduce a constant

$$\beta_0 := -\frac{1}{(A^{*-1}g, v)}. \quad (3.7)$$

(C.3) A is injective, and $v \notin D(A^{-1})$ or $v \in D(A^{-1})$ with $(g, A^{-1}v) = 0$.

(C.4) A^* is injective, and $g \notin D(A^{*-1})$ or $g \in D(A^{*-1})$ with $(A^{*-1}g, v) = 0$.

For a linear operator T on a Hilbert space, we set $\text{nul } T := \dim \ker T$.

Theorem 3.3 (i) Suppose that (C.1) and (C.2) hold. Let

$$\Psi_{n,j} := a(A^{-1}v)^{*n} \Omega_b \otimes b(A^{*-1}g)^{*j} \Omega_f, \quad n = 0, 1, 2, \dots, \quad j = 0, 1. \quad (3.8)$$

Then $\text{nul } Q(\alpha_0) = \infty$ with $\ker Q(\alpha_0) = \overline{\mathcal{L}\{\Psi_{n,j} | n \geq 0, j = 0, 1\}}$. Moreover, for all $\alpha \neq \alpha_0$,

$$\text{nul } Q(\alpha) = 1, \quad \ker Q(\alpha) = \{c\Omega_b \otimes \Omega_f | c \in \mathbb{C}\}. \quad (3.9)$$

(ii) Suppose that (C.1) and (C.4) hold. Then $\text{nul } Q(\alpha_0) = \infty$ with

$$\ker Q(\alpha_0) = \overline{\mathcal{L}\{\Psi_{n,0} | n \geq 0\}}.$$

Moreover, for all $\alpha \neq \alpha_0$, (3.9) holds.

(iii) Suppose that (C.2) and (C.3) hold. Then $\text{nul } Q(\beta_0) = 2$ with $\ker Q(\beta_0) = \mathcal{L}\{\Psi_{0,j} | j = 0, 1\}$.

(iv) Suppose that (C.3) and (C.4) hold. Then, for all $\alpha \in \mathbb{C}$, (3.9) holds.

As for non-zero eigenvalues of $Q(\alpha)$, we have the following result:

Theorem 3.4 Consider the case where $\mathcal{H} = \mathcal{K}$, A is a nonnegative self-adjoint operator with $\ker A = \{0\}$ and $g = v \in D(A^{-1})$ (then $\alpha_0 = -1/(v, A^{-1}v) < 0$). Let $\alpha < \alpha_0$. Then, there exists a constant $x_0(\alpha) < 0$ such that $\alpha(v, (x_0(\alpha) - A)^{-1}v) = 1$, and for all $n \in \{0\} \cup \mathbb{N}$,

$$\pm \sqrt{n}|x_0(\alpha)| \in \sigma_p(Q(\alpha)). \quad (3.10)$$

Note that this theorem holds even if Q_A has no non-zero eigenvalues. As the condition that $\alpha < \alpha_0$ shows, this is a *strong coupling effect*.

By Theorem 3.1 (i)-(ii), the operator $Q_+(\alpha)$ defined by

$$D(Q_+(\alpha)) := D(Q(\alpha)) \cap \mathcal{F}_+(\mathcal{H}, \mathcal{K}), \quad Q_+(\alpha)\Psi := Q(\alpha)\Psi, \quad \Psi \in D(Q_+(\alpha)) \quad (3.11)$$

is a densely defined closed linear operator from $\mathcal{F}_+(\mathcal{H}, \mathcal{K})$ to $\mathcal{F}_-(\mathcal{H}, \mathcal{K})$. We define an index of $Q(\alpha)$ by

$$\text{ind}_\Gamma(Q(\alpha)) := \text{nul } Q_+(\alpha) - \text{nul } Q_+(\alpha)^*, \quad (3.12)$$

the index of $Q_+(\alpha)$, provided that $\text{nul } Q_+(\alpha) < \infty$ or $\text{nul } Q_+(\alpha)^* < \infty$.

Results on the (semi-)Fredholm property and the index $\text{ind}_\Gamma Q(\alpha)$ are as follows:

Theorem 3.5 (i) Suppose that (C.1) and (C.2) hold. Then $Q(\alpha_0)$ is not semi-Fredholm. Moreover, for all $\alpha \neq \alpha_0$, $Q(\alpha)$ is Fredholm with $\text{ind}_\Gamma Q(\alpha) = 1$.

- (ii) Suppose that (C.1) and (C.4) hold. Then $Q(\alpha_0)$ is semi-Fredholm with $\text{ind}_\Gamma Q(\alpha) = \infty$. Moreover, for all $\alpha \neq \alpha_0$, $Q(\alpha)$ is Fredholm with $\text{ind}_\Gamma Q(\alpha) = 1$.
- (iii) Suppose that (C.2) and (C.3) hold. Then $Q(\beta_0)$ is Fredholm with $\text{ind}_\Gamma Q(\beta_0) = 0$. Moreover, for all $\alpha \neq \beta_0$, $Q(\alpha)$ is Fredholm with $\text{ind}_\Gamma Q(\alpha) = 1$.
- (iv) Suppose that (C.3) and (C.4) hold. Then, for all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is Fredholm with $\text{ind}_\Gamma Q(\alpha) = 1$.

References

- [Ar89] A. Arai, Path integral representation of the index of Kähler-Dirac operators on an infinite dimensional manifold, *J. Funct. Anal.* **82**(1989), 330-369.
- [Ar91] A. Arai, A general class of infinite dimensional Dirac operators and related aspects, *Functional Analysis & Related Topics* (Ed. S. Koshi), pages 85-98, World Scientific, Singapore, 1991.
- [Ar92] A. Arai, A general class of infinite dimensional Dirac operators and path integral representation of their index, *J. Funct. Anal.* **105**(1992), 342-408.
- [Ar93] A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, *J. Geom. Phys.* **11**(1993), 465-490.
- [Ar93] A. Arai, Supersymmetric extension of quantum scalar field theories, *Quantum and Non-Commutative Analysis* (Ed. H. Araki et al), pages 73-90, Kluwer Academic Publishers, Dordrecht 1993.
- [Ar94] A. Arai, On self-adjointness of Dirac operators in boson-fermion Fock spaces, *Hokkaido Math. Jour.* **23**(1994), 319-353.
- [Ar96] A. Arai, Supersymmetric quantum field theory and infinite dimensional analysis, *Sugaku Expositions* **9**(1996), 87-98.
- [Ar97] A. Arai, Strong anticommutativity of Dirac operators on Boson-Fermion Fock spaces and representations of a supersymmetry algebra, *Math. Nachr.* **207** (1999), 61-77.
- [Ar00] A. Arai, Spectral properties of a Dirac operator on the abstract Boson-Fermion Fock space, in preparation.
- [AM91] A. Arai and I. Mitoma, De Rham-Hodge-Kodaira decomposition in ∞ - dimensions, *Math. Ann.* **291** (1991), 51-73.
- [AM93] A. Arai and I. Mitoma, Comparison and nuclearity of spaces of differential forms on topological vector spaces, *J. Funct. Anal.* **111**(1993), 278-294.
- [BR97] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, Second Edition, Springer, Berlin, Heidelberg, 1997.