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Supersymmetric Methods for Constructing Soliton-type Solutions to Multi-component nonlinear Schrödinger and Klein-Gordon Equations

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Abstract

We consider a multi-component nonlinear partial differential equation in the two-dimensional space-time $\mathbf{R}^2$ which unifies Schrödinger and Klein-Gordon equations in $\mathbf{R}^2$. By supersymmetric methods connected with shape-invariant potentials, we show that a class of soliton-type solutions to the equation can be constructed from solutions to nonlinear equations of some types for superpotentials. Moreover we present new soliton-type solutions which are written in terms of q-deformed hyperbolic functions.

Key words: nonlinear Schrödinger equation, nonliner Klein-Gordon equation, soliton, supersymmetry, superpotential, shape-invariant potential, q-deformed hyperbolic function

1 Introduction

In this paper we present soliton-type solutions to the following nonlinear partial differential equation for a $\mathbf{C}^N$-valued function

$$\Psi(x,t) = (\Psi_1(x,t), \cdots, \Psi_N(x,t))$$

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on the two-dimensional space-time $\mathbf{R}^2 = \{(x,t)|x,t \in \mathbf{R}\}$ $(N \geq 1)$:

$$i\alpha\frac{\partial\Psi(x,t)}{\partial t} + \beta\frac{\partial\Psi(x,t)}{\partial x} + \gamma\frac{\partial^2\Psi(x,t)}{\partial t^2} + \frac{\partial^2\Psi(x,t)}{\partial x^2}$$
$$+\rho\Psi(x,t) + \kappa|\Psi(x,t)|^{2p}\Psi(x,t) = 0, \tag{1.1}$$

where α, β, γ, ρ, $\kappa \in \mathbf{C}$ $(\kappa \neq 0)$, $p \in \mathbf{R} \setminus \{0\}$ (not necessarily an integer) are constants and

$$|\Psi(x,t)|^2 := \sum_{n=1}^{N} |\Psi_n(x,t)|^2.$$

Eq.(1.1) unifies N-component nonlinear Schrödinger and Klein-Gordon equations on $\mathbf{R}^2$. The basic idea of the method taken in the present paper comes from a paper [3] which discusses a use of supersymmetric quantum mechanics in constructing soliton-type solutions to a multi-component nonlinear Schrödinger equation on $\mathbf{R}^2$

$$i\frac{\partial\Phi(x,t)}{\partial t} + \frac{\partial^2\Phi(x,t)}{\partial x^2} + \kappa|\Phi(x,t)|^2\Phi(x,t) = 0, \tag{1.2}$$

the case of (1.1) with $p = 1$, $\alpha = 1, \beta = 0, \gamma = 0, \rho = 0$. Our primary concern is to clarify relations between supersymmetric structures and soliton-type solutions to Eq.(1.1). We show that there exists a correspondence from solutions of a nonlinear differential equation to soliton-type solutions of (1.1) (see Propositions 3.1, 3.3 and 4.2). Then, using this structure and results on supersymmetric quantum mechanics connected with shape-invariant potentials [1], we find new soliton-type solutions to (1.1) in the following cases: (i) $N = 1$, $p \in \mathbf{R} \setminus \{0, -1\}$:arbitrary ; (ii) $N = 2$, $p = 1$; (iii) $N = 3$, $p = 1$. These solutions are written in terms of q-deformed hyperbolic functions with $q > 0$.

2 Basic facts

We seek solutions $\Psi = (\Psi_1, \cdots, \Psi_N)$ to (1.1) in the form

$$\Psi_n(x,t) = e^{i\theta_n(x,t)}\psi_n(x - vt), \quad n = 1, \cdots, N, \tag{2.1}$$

where $v \in \mathbf{R}$ is a constant,

$$\theta_n(x,t) := \mu_n x - \omega_n t \tag{2.2}$$

with μ_n, $\omega_n \in \mathbf{R}$ being constants and ψ_n is a twice continuously differentiable function on $\mathbf{R}$. This is a soliton-type solution. We assume that μ_n and ω_n satisfy

$$\mu_n = \frac{1}{2i}(i\alpha v - 2i\omega_n v\gamma - \beta), \tag{2.3}$$
$$2\omega_n v\Im\gamma = v\Im\alpha + \Re\beta. \tag{2.4}$$

Condition (2.4), which ensures that μ_n is a real number, implies that, if $v\Im\gamma \neq 0$, then $\omega_n = (v\Im\alpha + \Re\beta)/(2v\Im\gamma)$ independently of n, but, if $v\Im\gamma = 0$, then only $v\Im\alpha + \Re\beta = 0$.

We set

$$
\begin{aligned}
a &:= 1 + \gamma v^2, && (2.5)\\
b_n &:= \alpha\omega_n - \gamma\omega_n^2 - \mu_n^2 + \rho + i\beta\mu_n, && (2.6)\\
& && (2.7)
\end{aligned}
$$

The following fact is the starting point of our analysis here:

Lemma 2.1 *The function $\Psi := (\Psi_1, \cdots, \Psi_N)$ with Ψ_n given by (2.1) $(n = 1, \cdots, N)$ is a solution to (1.1) if and only if, for each $n = 1, \cdots, N$,*

$$
a\frac{d^2\psi_n(x)}{dx^2} + \kappa\left(\sum_{j=1}^{N} |\psi_j(x)|^2\right)^p \psi_n(x) + b_n\psi_n(x) = 0. \tag{2.8}
$$

Proof. Direct computation. □

By Lemma 2.1, we try to construct solutions $\psi = (\psi_1, \cdots, \psi_N)$ to Eq.(2.8). A basic general idea for that is stated in the following lemma.

Lemma 2.2 *Let N be fixed and V be a function on $\mathbf{R}$ such that there exist N twice differentiable functions ψ_n^V on $\mathbf{R}$ and constants E_n $(n = 1, \cdots, N)$ satisfying*

$$
-\frac{d^2\psi_n^V(x)}{dx^2} + V(x)\psi_n^V(x) = E_n\psi_n^V(x). \tag{2.9}
$$

and

$$
aV(x) + \kappa\left(\sum_{j=1}^{N} |\psi_j^V(x)|^2\right)^p + K = 0 \tag{2.10}
$$

with $K := b_n - aE_n$ independently of $n = 1, \cdots, N$. Then

$$
\Psi^V(x) := \left(e^{i\theta_1(x,t)}\psi_1^V(x - vt), \cdots, e^{i\theta_N(x,t)}\psi_N^V(x - vt)\right) \tag{2.11}
$$

is a solution to (1.1).

Proof. Eq.(2.9) and (2.10) imply (2.8) with $\psi_n = \psi_n^V$. □

Lemma 2.2 relates (1.1) to the eigenvalue problem of the Schrödinger operator $-d^2/dx^2 + V$.

In general it would be difficult to find a potential V having the properties described in the assumption of Lemma 2.2. But we show below that a use of an idea of supersymmetry makes it possible to find such potentials V.

We shall discuss the case $N = 1$ and the case $N \geq 2$ separately.

Remark 2.1 We can consider also "local" soliton solutions to Eq.(1.1) which are of the form Ψ with (2.1), but, defined only in $\{(x,t) \in \mathbf{R}^2 | L_1 < x - vt < L_2\}$ with L_1 and L_2 being finite real constants. Such solutions can be constructed from local solutions of (2.8) on the interval (L_1, L_2). Indeed, we show that, in the case $N = 1$, local solutions to Eq.(1.1) exist (see Proposition 3.4).

3 Solutions in the case $N = 1$

3.1 General aspects

We first review a model of supersymmetric quantum mechanics introduced by Witten [5]. In the present paper, we do not discuss the model in Hilbert space framework. For the purpose here, it is enough to work in the algebraic vector space

$$\mathcal{V} := C^\infty(\mathbf{R}), \tag{3.1}$$

the vector space of infinitely differentiable functions on $\mathbf{R}$.

For a real-valued function $W \in \mathcal{V}$, we introduce linear operators

$$A_W := -\frac{d}{dx} + W, \quad A_W^+ := \frac{d}{dx} + W. \tag{3.2}$$

acting on $\mathcal{V}$ and define

$$H_W^+ := A_W^+ A_W, \quad H_W^- := A_W A_W^+. \tag{3.3}$$

We have

$$H_W^\pm = -\frac{d^2}{dx^2} + V_W^\pm, \tag{3.4}$$

where

$$V_W^\pm := W^2 \pm W' \tag{3.5}$$

($W' := dW/dx$). The pair (H_W^+, H_W^-) defines a supersymmetric Hamiltonian [5]. In supersymmetric quantum mechanics, the function W is called a *superpotential.*

Let $c \neq 0$ be a non-zero complex constant and

$$\Omega_{W,c}(x) := c e^{\int_0^x W(y)dy}. \tag{3.6}$$

Then $\Omega_W \in \mathcal{V}$ and

$$A_W \Omega_{W,c} = 0. \tag{3.7}$$

Hence

$$H_W^+ \Omega_{W,c} = 0, \tag{3.8}$$

which implies that

$$-\frac{d^2\Omega_{W,c}(x)}{dx^2} + V_W^+(x)\Omega_{W,c}(x) = 0. \tag{3.9}$$

Hence we can apply Lemma 2.2 with $N = 1$ to obtain the following proposition. In the case $N = 1$, we set

$$\omega := \omega_1, \ \mu := \mu_1, \ b := b_1, \ \theta := \theta_1. \tag{3.10}$$

Proposition 3.1 *The function*

$$\Psi_W(x,t) := e^{i\theta(x,t)}\Omega_{W,c}(x - vt) \tag{3.11}$$

is a solution to Eq.(1.1) with $N = 1$ if and only if W satisfies

$$a[W(x)^2 + W'(x)] + \kappa |c|^{2p} e^{2p\int_0^x W(y)dy} + b = 0. \tag{3.12}$$

Proof. In Lemma 2.2, we need only consider the case where $N = 1$, $V = V_W^+$, $\psi_1^V = \Omega_{W,c}$ and $E_1 = 0$. $\square$

Proposition 3.1 gives an interesting correspondence between a solution W to the nonlinear integro-differential equation (3.12) and a solution to (1.1) with $N = 1$.

In view of Proposition 3.1, it is important to find all the solutions to Eq.(3.12).

If we put

$$\phi(x) := \int_0^x W(y)dy + \Lambda$$

with Λ being an arbitrary constant, then Eq.(3.12) takes the form

$$\phi''(x) + \phi'(x)^2 + Ae^{2p\phi(x)} + B = 0 \tag{3.13}$$

with $A = \kappa|c|^{2p}e^{-2p\Lambda}/a$ and $B = b/a$, provided that $a \neq 0$. We shall give a detailed analysis of Eq.(3.13) in Appendix of the present paper and classify solutions to Eq.(3.13).

We may also make use of the supersymmerric partner H_W^- to construct solutions to (1.1). A key fact for that is the following characteristic structure of a supersymmetric Hamiltonian (cf. [1, Lemma 2.2]).

Lemma 3.2 *Let $H_W^- f = Ef$ with $f \in \mathcal{V}$ and E a constant. Let $g := A_W^+ f$. Then $H_W^+ g = Eg$.*

Proof. Direct computation using definition (3.3). $\square$

We take another superpotential U on $\mathbf{R}$ such that

$$V_U^+ + E = V_W^- \tag{3.14}$$

with E a constant.

Proposition 3.3 *Assume (3.14) . Let*

$$\psi_E := A_W^+ \Omega_{U,c} \tag{3.15}$$

with $c \in \mathbf{C}$ a non-zero constant. Suppose that

$$a(W(x)^2 + W'(x)) + \kappa|\psi_E(x)|^{2p} + b - aE = 0. \tag{3.16}$$

Then the fucntion $\Psi(x,t) := e^{i\theta(x,t)}\psi_E(x - vt)$ is a solution to Eq.(1.1) with $N = 1$.

Proof. By (3.14), we have $H_W^- = H_U^+ + E$. Hence

$$H_W^- \Omega_{U,c} = (H_U^+ + E)\Omega_{U,c} = E\Omega_{U,c}. \tag{3.17}$$

By (3.15) and Lemma 3.2, we have $H_W^+ \psi_E = E\psi_E$. Thus an application of Lemma 2.2 with $N = 1$ gives the desired result. $\square$

Remark 3.1 For a given $W \in \mathcal{V}$, Eq.(3.14) for U is a differential equation of Riccati type. Hence general solutions to Eq.(3.14) can be constructed from its special solutions. For example, let U_1 be a special solution to Eq.(3.14) and $F_1(x) := \int U_1(x)dx$. Then

$$U := U_1 + \frac{e^{-2F_1}}{\int e^{-2F_1}dx + c}$$

is a general solution to Eq.(3.14) with c being an arbitrary constant.

3.2 Exact solutions

Let $c \neq 0$ be a complex constant, and D and Λ be real constants. In the case $a \neq 0$, we define a function g_p on $\mathbf{R}$ by

$$g_p(y) := \begin{cases} -\dfrac{\kappa|c|^{2p}e^{-2p\Lambda}}{a(1+p)}e^{2py} + De^{-2y} - \dfrac{b}{a}, & p \neq -1, \\ \left(D - \dfrac{2\kappa e^{2\Lambda}}{|c|^2 a}y\right) e^{-2y} - \dfrac{b}{a}, & p = -1 \end{cases} \tag{3.18}$$

$(y \in \mathbf{R})$.

The following proposition describes general structures of solutions to Eq.(1.1).

Proposition 3.4 *Assume that $a \neq 0$. Let G_p be a primitive function of either $1/\sqrt{g_p}$ or $-1/\sqrt{g_p}$ on*

$$D_+ := \{y \in \mathbf{R} | g_p(y) > 0\}, \tag{3.19}$$

and G_p^{-1} denote the inverse function of G_p.

(i) *For every open interval $J \subset D_+$,*

$$\Psi(x,t) := ce^{i\theta(x,t)}e^{-\Lambda}e^{G_p^{-1}(x-vt)} \tag{3.20}$$

is a solution to Eq.(1.1) with $N = 1$ on $\{(x,t) | x - vt \in G_p(J)\}$.

(ii) *Suppose that there exists an open interval $J \subset D_+$ such that $G_p(J) = (0, L)$ with $L > 0$ or $L = \infty$ and*

$$G_p^{-1}(0) := \lim_{x \downarrow 0} G_p^{-1}(x) \text{ exists and } g_p(G_p^{-1}(0)) = 0. \tag{3.21}$$

Let

$$\phi_p(x) := \begin{cases} G_p^{-1}(x) & x \in [0, L) \\ G_p^{-1}(-x) & x \in (-L, 0) \end{cases}. \tag{3.22}$$

Then

$$\Psi(x,t) := ce^{i\theta(x,t)}e^{-\Lambda}e^{\phi_p(x-vt)} \tag{3.23}$$

is a solution to Eq.(1.1) with $N = 1$ on $\{(x,t) | x - vt \in (-L, L)\}$.

Proof. (i) By Proposition B.3-(i), $\phi(x) = G_p^{-1}(x)$ is a solution to Eq.(3.13). In this case, we have $\Omega_{W,c}(x) = ce^{-\Lambda}e^{G_p^{-1}(x)}$. Thus the desired result follows.

(ii) Similar to part (i) (an application of Proposition B.3-(ii)). □

Remark 3.2 More concrete forms of $G_p(y)$ can be found by applying the results in Sections C–E in Appendix. But we omit the details here.

Some of solutions given in Proposition 3.4 may be global in (x, t) and have explicit representations. To write down one of them, we recall *q-deformed hyperbolic functions* which were introduced in [1]:

$$\sinh_q x := \frac{e^x - qe^{-x}}{2}, \quad \cosh_q x := \frac{e^x + qe^{-x}}{2}, \tag{3.24}$$

$$\tanh_q x := \frac{\sinh_q x}{\cosh_q x}, \quad \mathrm{sech}_q x := \frac{1}{\cosh_q x}, \quad x \in \mathbf{R}, \tag{3.25}$$

where $q > 0$ is a deformation parameter. Note that, if $q \neq 1$, then $\sinh_q x$ is not odd and and $\cosh_q x$ is not even:

$$\sinh_q(-x) = -q \sinh_{1/q} x, \quad \cosh_q(-x) = q \cosh_{1/q} x, \quad x \in \mathbf{R}. \tag{3.26}$$

The following formulas can be easily proven:

$$(\sinh_q x)' = \cosh_q x, \tag{3.27}$$

$$(\cosh_q x)' = \sinh_q x, \tag{3.28}$$

$$\cosh_q^2 x - \sinh_q^2 x = q, \tag{3.29}$$

$$(\tanh_q x)' = q\, \mathrm{sech}_q^2 x, \tag{3.30}$$

$$(\mathrm{sech}_q x)' = -(\tanh_q x)(\mathrm{sech}_q x), \tag{3.31}$$

$$\tanh_q^2 x = 1 - q\mathrm{sech}_q^2 x. \tag{3.32}$$

Lemma 3.5 *For all $s \in \mathbf{R}$, the function*

$$W_s(x) := -s \tanh_q(spx) \tag{3.33}$$

on $\mathbf{R}$ *satisfies the equation*

$$W_s(x)^2 + W_s'(x) + \frac{4s^2(1+p)q}{(1+q)^2} e^{2p \int_0^x W_s(y)dy} - s^2 = 0. \tag{3.34}$$

Proof. Let $p \neq 0$. Then, by (3.28), we have

$$\int_0^x W_s(y)dy = -\frac{1}{p} \log \cosh_q(spx) + \frac{1}{p} \log \frac{1+q}{2}. \tag{3.35}$$

This fact together with (3.30) and (3.32) implies (3.34). By the fact that

$$\tanh_q 0 = \frac{1-q}{1+q},$$

one can easily see that Eq.(3.34) with $p = 0$ holds. □

Lemma 3.5 implies that W_s is a solution to Eq.(3.12) with $b = -as^2$, $\kappa |c|^{2p} = 4as^2(1 + p)q/(1+q)^2$. By (3.35), we have for $p \neq 0$

$$e^{\int_0^x W_s(y)dy} = \left(\frac{1+q}{2}\right)^{1/p} \mathrm{sech}_q^{1/p}(spx). \tag{3.36}$$

Thus we have the following result:

Theorem 3.6 *Let $p \neq 0$. Suppose that $\kappa \in \mathbf{R}$ and*

$$\frac{a(1+p)}{\kappa} > 0, \quad b + as^2 = 0. \tag{3.37}$$

Then the function

$$\Psi_{s,q,p}(x,t) := \left(\frac{a(1+p)qs^2}{\kappa} \right)^{1/2p} e^{i\theta(x,t)} \operatorname{sech}_q^{1/p}[sp(x-vt)] \tag{3.38}$$

is a solution to Eq.(1.1).

Remark 3.3 The function $\Psi_{s,1,1}$ (the case $q = p = 1$) is a well-known soliton solution (up to constant multiples) to the nonlinear Schrödinger equation (1.2)(e.g., [3]). Hence Theorem 3.6 gives a generalization of a soliton of this type.

We next give a simple application of Proposition 3.3 with $W = W_s$. Let us consider the case $p = 1$ and set

$$Y(s) := -s \tanh_q(sx).$$

Then we have

$$Y^2 - Y' = s^2.$$

Hence (3.14) hold with $W = Y$, $U = 0$ and $E = s^2$. In this case,

$$\psi_E = cY.$$

Hence, if we take

$$b = 2as^2, \quad |c|^2 = -\frac{2a}{\kappa}$$

then (3.16) is satisfied. Thus, by Proposition 3.1, we obtain the following result.

Theorem 3.7 *Let $b = 2as^2$ and $a/\kappa < 0$. Then*

$$\Psi(x,t) := -s\sqrt{-2a/\kappa}\, e^{i\theta(x,t)} \tanh_q s(x - vt) \tag{3.39}$$

is a solution to Eq.(1.1) with $N = 1$ and $p = 1$.

4 Solutions in the Case $N \geq 2$

4.1 Supersymmetry with shape-invariant potentials

For the case $N \geq 2$, we consider a family of supersymmetric Hamiltonians, generalizing an idea suggested in [3]. Let Λ be a subset of $\mathbf{R}$ and $\{W_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{V}$. As in the case $N = 1$, we introduce linear operators

$$A(\lambda) := -\frac{d}{dx} + W_\lambda, \quad A(\lambda)^+ := \frac{d}{dx} + W_\lambda \tag{4.1}$$

acting on $\mathcal{V}$ and define

$$H_+(\lambda) := A(\lambda)^+ A(\lambda), \quad H_-(\lambda) := A(\lambda)A(\lambda)^+ \tag{4.2}$$

We have

$$H_\pm(\lambda) = -\frac{d^2}{dx^2} + V_\lambda^\pm, \tag{4.3}$$

where

$$V_\lambda^\pm := W_\lambda^2 \pm W_\lambda'. \tag{4.4}$$

We assume the following:

Hypothesis (**W**) There exist mappings $f : \Lambda \to \Lambda$ and $F : f(\Lambda) \to \mathbf{R}$ such that for all $\lambda \in \Lambda$

$$V_{f(\lambda)}^+ + F(f(\lambda)) = V_\lambda^-. \tag{4.5}$$

Remark 4.1 In the physics literature, the functions $V_\lambda^\pm$ satisfying (4.5) are called *shape-invariant potentials.* This notion was first introduced by Gendenshteȋn [2] and developed by many theoretical physicists (e.g., see [4]). The abstract mathematical formulation extending the idea of shape-invariant potentials was given in [1].

We write as $f^0(\lambda) := \lambda$, $f^n(\lambda) := f(f^{n-1}(\lambda))$, $n \geq 1$.

Lemma 4.1 *Assume* (W). *Let*

$$E_1(\lambda) := 0, \tag{4.6}$$

$$E_n(\lambda) := \sum_{j=1}^{n-1} F(f^j(\lambda)), \qquad n \geq 2, \tag{4.7}$$

$$\psi_{1,\lambda}(x) := e^{\int_0^x W_\lambda(y)dy}, \tag{4.8}$$

$$\psi_{n,\lambda} := A(\lambda)^+ A(f(\lambda))^+ \cdots A(f^{n-2}(\lambda))^+ \psi_{1,f^{n-1}(\lambda)}, \qquad n \geq 2. \tag{4.9}$$

Then, for all $\lambda \in \Lambda$,

$$H_+(\lambda)\psi_{n,\lambda} = E_n(\lambda)\psi_{n,\lambda}, \quad n \geq 1. \tag{4.10}$$

Proof. We prove (4.10) by induction. As in the case of (3.8), (4.10) holds for $n = 1$. Suppose that (4.10) holds for some n. We have

$$\begin{aligned} H_+(\lambda)\psi_{n+1,\lambda} &= A(\lambda)^+ H_-(\lambda) A(f(\lambda))^+ \cdots A(f^{n-1}(\lambda))^+ \psi_{1,f^n(\lambda)} \\ &= A(\lambda)^+ H_-(\lambda)\psi_{n,f(\lambda)}. \end{aligned} \tag{4.11}$$

By Hypothesis (W), we have

$$H_-(\lambda) = H_+(f(\lambda)) + F(f(\lambda)). \tag{4.12}$$

Putting this equation into (4.11) and using the induction hypothesis (4.10), we have

$$H_+(\lambda)\psi_{n+1,\lambda} = [E_n(f(\lambda)) + F(f(\lambda))]\psi_{n+1,\lambda} = E_{n+1}(\lambda)\psi_{n+1,\lambda}.$$

Hence (4.10) holds also for $n + 1$. □

Lemma 4.1 implies that, under Hypothesis (W), for all $n \geq 1$,

$$-\frac{d^2\psi_{n,\lambda}}{dx^2} + V_\lambda^+ \psi_{n,\lambda} = E_n(\lambda)\psi_{n,\lambda}. \tag{4.13}$$

By this fact and Lemma 2.2, we obtain the following fact:

Proposition 4.2 *Let $N \geq 2$ be fixed. Assume (W). Suppose that*

$$aV_\lambda^+(x) + \kappa \left(\sum_{j=1}^{N} |c_j|^2 |\psi_{j,\lambda}(x)|^2 \right)^p + K = 0, \quad x \in \mathbf{R}, \tag{4.14}$$

with c_j's being complex constants,

$$K := b_n - aE_n(\lambda) \tag{4.15}$$

independently of $n = 1, \cdots, N$. Then

$$\Psi_\lambda(x,t) := \left(c_1 e^{i\theta_1(x,t)} \psi_{1,\lambda}(x - vt), \cdots, c_N e^{i\theta_N(x,t)} \psi_{N,\lambda}(x - vt) \right) \tag{4.16}$$

is a solution to (1.1).

Let $\phi_\lambda(x) := \int_0^x W_\lambda(y)dy + \eta_\lambda$ with $\eta_\lambda \in \mathbf{R}$ being a constant. Then Eq.(4.14) is a nonlinear differential equation for $\{\phi_\lambda\}_{\lambda\in\Lambda}$. Hence Proposition 4.2 gives a relation of a nonlinear differential equation to soliton-type solutions of (1.1).

Remark 4.2 As in Lemma 3.2, we have

$$H_-(\lambda)A(\lambda)\psi_{n,\lambda} = E_n(\lambda)A(\lambda)\psi_{n,\lambda}. \tag{4.17}$$

On the other hand, we can show that

$$A(\lambda)\psi_{n,\lambda} = E_n(\lambda)\psi_{n-1,f(\lambda)}. \tag{4.18}$$

Hence, if $E_n(\lambda) \neq 0$, then

$$H_-(\lambda)\psi_{n-1,f(\lambda)} = E_n(\lambda)\psi_{n-1,f(\lambda)}. \tag{4.19}$$

4.2 Exact solutions in the case $N=2$ and $p=1$

Let $s \in \mathbf{R}$ and consider the case where the superpotential W_λ is given by

$$W_\lambda(x) := -\lambda \tanh_q(sx), \quad \lambda \in \mathbf{R}. \tag{4.20}$$

Then the functions $V_\lambda^\pm$ defined by (4.4) take the form

$$V_\lambda^\pm = -\lambda(\lambda \pm s) q \operatorname{sech}_q^2(sx) + \lambda^2. \tag{4.21}$$

Let

$$f_s(\lambda) = \lambda - s, \quad F_s(\lambda) := (\lambda+s)^2 - \lambda^2 = 2\lambda s + s^2. \tag{4.22}$$

Then it is easy to see that

$$V_{f_s(\lambda)}^+ + F_s(f_s(\lambda)) = V_\lambda^-. \tag{4.23}$$

Hence, for each s, W_λ satisfies Hypothesis (W) with $\Lambda = \mathbf{R}$, $F = F_s$ and $f = f_s$. Thus we can apply Lemma 4.1 and Proposition 4.2. To do that, however, we need compute the left hand side of (4.14) in the present case.

We consider only the simplest case $p = 1$ in nonlinearity. Let

$$L_{\lambda,s}^{(N)}(x) := aV_\lambda^+(x) + \kappa\left(\sum_{j=1}^N |c_j|^2 |\psi_{j,\lambda}(x)|^2\right) + K \tag{4.24}$$

and

$$h := \frac{1+q}{2}. \tag{4.25}$$

In the present case, we see that

$$\begin{aligned} E_n(\lambda) &= \sum_{j=1}^{n-1} F_s(f_s^j(\lambda)) = (n-1)s[2\lambda - (n-1)s], & (4.26)\\ \psi_{1,\lambda}(x) &= h^{\lambda/s}\operatorname{sech}_q^{\lambda/s}(sx), & (4.27)\\ \psi_{2,\lambda}(x) &= (s-2\lambda)h^{(\lambda-s)/s}\tanh_q(sx)\operatorname{sech}_q^{(\lambda-s)/s}(sx). & (4.28) \end{aligned}$$

Using this expression, we can show that

$$\begin{aligned} L_{\lambda,s}^{(2)}(x) &= -a\lambda(\lambda+s)q\operatorname{sech}_q^2(sx) + a\lambda^2 + K \\ &\quad +\kappa\left(|c_1|^2 h^{2\lambda/s} - |c_2|^2(2\lambda-s)^2 q h^{2(\lambda-s)/s}\right)\operatorname{sech}_q^{2\lambda/s}(sx) \\ &\quad +\kappa|c_2|^2(2\lambda-s)^2 h^{2(\lambda-s)/s}\operatorname{sech}_q^{2(\lambda-s)/s}(sx). \end{aligned} \tag{4.29}$$

There are two ways to have $L_{\lambda,s}^{(2)} = 0$. One of them is to take $s = \lambda$. Then $L_{\lambda,\lambda}^{(2)} = 0$ if and only if

$$\begin{aligned} K &= -\lambda^2(a+\kappa|c_2|^2), & (4.30)\\ \kappa(|c_1|^2h^2 - |c_2|^2\lambda^2 q) &= 2a\lambda^2 q. & (4.31) \end{aligned}$$

Hence we need only take b_n $(n = 1, 2)$ as

$$\begin{aligned} b_1 &= -\lambda^2(a+\kappa|c_2|^2), & (4.32)\\ b_2 &= -\kappa|c_2|^2\lambda^2 & (4.33) \end{aligned}$$

to have (4.15) for $N = 2$. Thus we obtain the following result.

Theorem 4.3 *Suppose that (4.31)–(4.33) hold. Then*

$$\Psi(x,t) := \left(c_1 e^{i\theta_1(x,t)} h \,\mathrm{sech}_q \lambda(x-vt),\ -c_2\lambda e^{i\theta_2(x,t)} \tanh_q \lambda(x-vt)\right) \tag{4.34}$$

is a solution to Eq.(1.1) with $N=2$ and $p=1$.

The other way to have $L^{(2)}_{\lambda,s}=0$ is to take $s=\lambda/2$. Let $\lambda \neq 0$. Then $L^{(2)}_{\lambda,\lambda/2}=0$ if and only if

$$K = -a\lambda^2, \tag{4.35}$$

$$aq = \frac{3}{2}\kappa|c_2|^2h^2. \tag{4.36}$$

$$|c_1|^2h^2 - |c_2|^2\left(\frac{3\lambda}{2}\right)^2 q = 0, \tag{4.37}$$

In this case we need only to take b_n as

$$b_1 = -a\lambda^2, \quad b_2 = -\frac{a\lambda^2}{4} \tag{4.38}$$

to have (4.15) for $N=2$. Thus we obtain the following result.

Theorem 4.4 *Let $\lambda \neq 0$ and suppose that (4.36)–(4.38) hold. Then*

$$\begin{aligned}\Psi(x,t) \ :=\ & \left(c_1 e^{i\theta_1(x,t)} h^2 \mathrm{sech}_q^2 \frac{\lambda(x-vt)}{2},\right. \\ & \left. -\frac{3}{2}\lambda h c_2 e^{i\theta_2(x,t)} \tanh_q \frac{\lambda(x-vt)}{2} \mathrm{sech}_q \frac{\lambda(x-vt)}{2}\right)\end{aligned} \tag{4.39}$$

is a solution to Eq.(1.1) with $N=2$ and $p=1$.

4.3 Exact solutions in the case $N=3$ and $p=1$

We next consider the case $N=3$ and $p=1$. We have

$$\begin{aligned}\psi_{3,\lambda}(x) = {}& (3s-2\lambda)h^{(\lambda-2s)/s} \\ & \times\left\{q(2\lambda-s)\mathrm{sech}_q^{\lambda/s}(sx) - 2(\lambda-s)\mathrm{sech}_q^{(\lambda-2s)/s}(sx)\right\}.\end{aligned} \tag{4.40}$$

Hence we obtain

$$\begin{aligned} L^{(3)}_{\lambda,s}(x) \ =\ & a\lambda^2 + K - a\lambda(\lambda+s)q\,\mathrm{sech}_q^2(sx) \\ & +\kappa\Big\{|c_1|^2h^{2\lambda}/s - |c_2|^2(2\lambda-s)^2qh^{2(\lambda-s)/s} \\ & +|c_3|^2h^{2(\lambda-2s)/s}q^2(2\lambda-s)^2\Big\}\mathrm{sech}_q^{2\lambda/s}(sx) \\ & +\kappa\left\{|c_2|^2(2\lambda-s)^2h^{2(\lambda-s)/s} - 4|c_3|^2h^{2(\lambda-2s)/s}(2\lambda-s)(\lambda-s)\right\} \\ & \times \mathrm{sech}_q^{2(\lambda-s)/s}(sx) \\ & +4\kappa|c_3|^2(\lambda-s)^2h^{2(\lambda-2s)/s}\mathrm{sech}_q^{2(\lambda-2s)/s}(sx). \end{aligned} \tag{4.41}$$

As in the preceeding case $N=2$, there are two choices for s that gives $L^{(3)}_{\lambda,s}=0$. The one is to take $s=\lambda$. In this case we obtain the following result.

Theorem 4.5 *Suppose that b_1 and b_2 are given by (4.32) and (4.33) respectively, $b_3 = b_1$ and*

$$\kappa(|c_1|^2 h^2 - |c_2|^2 \lambda^2 q + |c_3|^2 h^{-2} q^2 \lambda^2) = 2a\lambda^2 q. \tag{4.42}$$

Then

$$\begin{aligned}\Psi(x,t) \;&:=\; \Big(c_1 e^{i\theta_1(x,t)} h \operatorname{sech}_q \lambda(x-vt), -c_2\lambda e^{i\theta_2(x,t)} \tanh_q \lambda(x-vt), \\ &\quad c_3 e^{i\theta_3(x,t)} \lambda^2 h^{-1} q \operatorname{sech}_q \lambda(x-vt)\Big)\end{aligned} \tag{4.43}$$

is a solution to Eq.(1.1) with $N = 3$ and $p = 1$.

The other choice is to take $s = \lambda/3$. In this case we obtain the following result.

Theorem 4.6 *Suppose that*

$$b_1 = -a\lambda^2, \tag{4.44}$$

$$b_2 = -\frac{4}{9}a\lambda^2, \tag{4.45}$$

$$b_3 = -\frac{1}{9}a\lambda^2, \tag{4.46}$$

$$|c_1|^2 h^4 - |c_2|^2 \left(\frac{5}{3}\lambda\right)^2 q h^2 + |c_3|^2 q^2 \left(\frac{5}{3}\lambda\right)^2 = 0, \tag{4.47}$$

$$|c_2|^2 h^2 = \frac{8}{3}|c_3|^2 q\lambda, \tag{4.48}$$

$$aq = \frac{4}{3}\kappa |c_3|^2 h^2. \tag{4.49}$$

Then

$$\begin{aligned}\Psi(x,t) \;&:=\; \Big(c_1 e^{i\theta_1(x,t)} h^3 \operatorname{sech}_q^3 \frac{\lambda(x-vt)}{3}, \\ &\quad -\frac{5}{3} c_2 e^{i\theta_2(x,t)} h^2 \lambda \tanh_q \frac{\lambda(x-vt)}{3} \operatorname{sech}_q^2 \frac{\lambda(x-vt)}{3}, \\ &\quad \frac{1}{3}\lambda^2 h c_3 e^{i\theta_3(x,t)} [5q \operatorname{sech}_q^3 \frac{\lambda(x-vt)}{3} - 4\operatorname{sech}_q \frac{\lambda(x-vt)}{3}]\Big)\end{aligned} \tag{4.50}$$

is a solution to Eq.(1.1) with $N = 3$ and $p = 1$.

In the same manner as above, one may continue to calculate $L_{\lambda,s}^{(N)}$ for $N \geq 4$ and check if there exist constants $c_j, j = 1, \cdots, N$ such that $L_{\lambda,s}^{(N)} = 0$. It is an interesting problem to show whether or not, for *all* $N \geq 4$, there exist constants $c_j, j = 1, \cdots, N$ such that $L_{\lambda,s}^{(N)} = 0$. But this problem is left open.

Appendix: Solutions to a nonlinear ordinary differential equation

A The equation considered and basic properties

In this appendix we consider the following nonlinear ordinary differential equation for a real-valued function f on an open interval $I \subset \mathbf{R}$:

$$f''(x) + f'(x)^2 + Ae^{kf(x)} + B = 0, \quad x \in I, \tag{A.1}$$

where $f' = df/dx, f'' = d^2f/dx^2$, and A, B and k are real constants satisfying

$$A \neq 0, \quad k \neq 0. \tag{A.2}$$

Remark A.1 If $A = 0$, then a general solution to Eq.(A.1) is given by

$$f(x) = \begin{cases} d + \log|\cos\sqrt{B}(x-c)|, & B > 0 \\ d + \log\cosh\sqrt{|B|}(x-c), & B < 0 \end{cases}, \tag{A.3}$$

where c and d are arbitrary real constants. If $k = 0$, then then a general solution to Eq.(A.1) is given by (A.3) with B replaced by $A + B$.

Remark A.2 Every twice differentiable solution f to Eq.(A.1) on I is in fact infinitely differentiable on I. This is easily seen by writing (A.1) as

$$f''(x) = -f'(x)^2 - Ae^{kf(x)} - B. \tag{A.4}$$

Lemma A.1 *Let f be a solution to Eq.(A.1) and, for each $s \in \mathbf{R}$,*

$$f_s(x) := f(sx), \quad sx \in I, \tag{A.5}$$

a scaling of f. Then

$$f_s'' + f_s'^2 + s^2Ae^{kf_s} + s^2B = 0. \tag{A.6}$$

Proof. Direct computation. □

Lemma A.1 shows a scaling law of Eq.(A.1).

Remark A.3 For each $t \in \mathbf{R}$, let $\tilde{f}_t(x) := f(x-t), x \in I + t$. Then, if f is a solution to Eq.(A.1) on I, then $\tilde{f}_t$ is a solution to Eq.(A.1) on $I + t$. This is a translation invariance of Eq.(A.1).

Let $0 < L < \infty$ or $L = +\infty$. The following lemma concerns a continuation of a solution of Eq.(A.1) on the interval $(0, L)$ to a solution of Eq.(A.1) on $(-L, L)$.

Lemma A.2 *Suppose that f is a solution to Eq.(A.1) on the interval $(0, L)$ such that $f(0) := \lim_{x>0, x\to 0} f(x)$ exists and*

$$\lim_{x>0, x\to 0} f'(x) = 0. \tag{A.7}$$

Let

$$F(x) := \begin{cases} f(x) & x \in [0, L) \\ f(-x) & x \in (-L, 0) \end{cases} \tag{A.8}$$

Then F is a solution to Eq.(A.1) on $(-L, L)$ and $F \in C^\infty(-L, L)$.

Proof. Since $f(-x) = f_{-1}(x)$, it follows from Lemma A.1 that F satisfies Eq.(A.1) on $(-L, 0) \cup (0, L)$. It follows from the definition of F that F is continuous at $x = 0$ with $F(0) = f(0)$. By (A.7), $\lim_{x>0, x\to 0} F'(x) = 0$ and

$$\lim_{x<0, x\to 0} F'(x) = -\lim_{x<0, x\to 0} f'(-x) = 0.$$

Hence F is differentiable at $x = 0$ with $F'(0) = 0$. Using (A.1), we have for all $x \in (-L, 0) \cup (0, L)$

$$\frac{F'(x) - F'(0)}{x} = \frac{1}{x}\int_0^x F''(y)dy = -\frac{1}{x}\int_0^x \left\{F'(y)^2 + Ae^{kF(y)} + B\right\} dy,$$

which implies that F' is differentiable at $x = 0$ with $F''(0) = -Ae^{kf(0)} - B$. Thus F is twice differentiable on $(-L, L)$ satisfying (A.1). By Remark A.2, F is infinitely differentiable on $(-L, L)$. □

For $k \neq -2$, we set

$$c(A, k) := -\frac{2A}{2+k} \tag{A.9}$$

and define a function g on $\mathbf{R}$ by

$$g(y) := \begin{cases} c(A, k)e^{ky} + Ce^{-2y} - B & k \neq -2 \\ (C - 2Ay)e^{-2y} - B & k = -2 \end{cases} \tag{A.10}$$

$y \in \mathbf{R}$, where C is an arbitrary constant.

The following lemma gives a necessary condition for a function f to be a solution to Eq.(A.1) on I:

Lemma A.3 *Suppose that f is a solution to Eq.(A.1) on an open interval I such that f is one-to-one on I. Then*

$$f'(x)^2 = g(f(x)), \quad x \in I. \tag{A.11}$$

Proof. Let $h(x) = f'(x)$, $x \in I$ and $y = f(x), x \in I$. Then h can be regarded as a function of $y \in f(I)$ and

$$f'' = \frac{dh}{dx} = \frac{dh}{dy}h.$$

Hence Eq.(A.1) gives

$$h\frac{dh}{dy} + h^2 + Ae^{ky} + B = 0.$$

Let $u(y) = h^2(x)$ with $y = f(x)$. Then we have

$$\frac{du}{dy} = -2u - 2(Ae^{ky} + B). \tag{A.12}$$

Hence $u(y) = g(y)$ with C being an arbitrary constant. Thus (A.11) holds for all $x \in I$. □

B General forms of solutions

A sufficient condition for Eq.(A.1) to have a solution is given in the following lemma:

Lemma B.1 *Let f be a twice differentiable function on an open interval I such that*

$$f'(x)^2 = g(f(x)), \quad x \in I. \tag{B.1}$$

Suppose that $f'(x) \neq 0$ for all $x \in I$. Then f is a solution to Eq.(A.1) on I.

Proof. By (B.1), $2f''(x)f'(x) = f'(x)g'(f(x))$, $x \in I$. Hence $2f''(x) = g'(f(x))$, which implies that

$$f''(x) + f'(x)^2 = \frac{1}{2}g'(f(x)) + g(f(x)). \tag{B.2}$$

By (A.12) with $u = g$, the right hand side of (B.2) is equal to $-Ae^{kf(x)} - B$. Thus f satisfies (A.1). □

Lemma B.2 *Let f_+ be a twice differentiable function on $(0, L)$ such that*

$$f_+'(x)^2 = g(f_+(x)), \quad x \in (0, L), \tag{B.3}$$

and $f'(x) \neq 0$ for all $x \in (0, L)$. Suppose that $f_+(0) := \lim_{x>0,x\to 0} f_+(x)$ exists and $\lim_{x>0,x\to 0} f_+'(x) = 0$. Let

$$f(x) = \begin{cases} f_+(x) & x \in [0, L) \\ f_+(-x) & x \in (-L, 0) \end{cases} \tag{B.4}$$

Then f is a solution to Eq.(A.1) on $(-L, L)$.

Proof. This follows from Lemma B.1 and Lemma A.2. □

Proposition B.3 *Let $G(y)$ be a primitive function of either $1/\sqrt{g(y)}$ or $-1/\sqrt{g(y)}$ on*

$$D_g^+ := \{y \in \mathbf{R} | g(y) > 0\}. \tag{B.5}$$

(i) *For every open interval $J \subset D_g^+$, $G^{-1}(x)$ is a solution to Eq.(A.1) on $G(J)$.*

(ii) *Suppose that there exists an open interval $J \subset D_g^+$ such that $G(J) = (0, L)$ and*

$$G^{-1}(0) := \lim_{x\downarrow 0} G^{-1}(x) \text{ exists and } g(G^{-1}(0)) = 0. \tag{B.6}$$

Then the function

$$f(x) := \begin{cases} G^{-1}(x) & x \in [0, L) \\ G^{-1}(-x) & x \in (-L, 0) \end{cases} \tag{B.7}$$

is a solution to Eq.(A.1) on $(-L, L)$.

Proof. (i) Let $f(x) = G^{-1}(x),\ x \in G(J)$. Then $G(f(x)) = x$. Hence $G'(f(x))f'(x) = 1$, which implies that $f'(x)^2 = g(f(x)), x \in J$. Hence, by Lemma B.1, f is a solution to Eq.(A.1).

(ii) Let $f_+(x) = G^{-1}(x), x \in (0, L)$. Then, as in part (i), f_+ is a solution to Eq.(A.1) on $(0, L)$. Let (B.6) be satisfied. Then

$$\lim_{x>0, x\to 0} f_+'(x)^2 = g(G^{-1}(0)) = 0.$$

Hence, applying Lemma B.2, we see that f is a solution to Eq.(A.1) on $(-L, L)$. □

C Classification of solutions (I): The case $k > 0$

Using Proposition B.3, we can classify solutions to Eq.(A.1).

We first note the following fact:

Lemma C.1 (i) *Let $k \neq -2$. Then g' has a zero if and only if $kc(A,k)C > 0$. In that case, the zero is unique and given by*

$$a_0 := \frac{1}{k+2} \log \frac{2C}{kc(A,k)}. \tag{C.1}$$

(ii) *Let $k = -2$. Then g' has a unique zero given by*

$$b_0 := \frac{A+C}{2A} \tag{C.2}$$

Proof. We have

$$g'(y) = \begin{cases} kc(A,k)e^{ky} - 2Ce^{-2y} & k \neq -2 \\ -2(A + C - 2Ay)e^{-2y} & k = -2 \end{cases} \tag{C.3}$$

This implies the desired results. □

For a constant $c \in \mathbf{R}$ such that $g(y) \geq 0$ for all $y \geq c$, we set

$$L_c^+ := \int_c^\infty \frac{1}{\sqrt{g(y)}} dy. \tag{C.4}$$

For a constant $d \in \mathbf{R}$ such that $g(y) \geq 0$ for all $y \leq d$, we set

$$L_d^- := \int_{-\infty}^d \frac{1}{\sqrt{g(y)}} dy. \tag{C.5}$$

In what follows in this section, we assume that

$$k > 0. \tag{C.6}$$

C.1 The case $c(A,k) > 0,\ C < 0$

In this case g is strictly monotone increasing with $g(y) \sim c(A,k)e^{ky} \sim \infty$ as $y \to \infty$ and $g(y) \sim Ce^{-y} \sim -\infty$ as $y \to -\infty$. Hence g has a unique zero $y_0 \in \mathbf{R}$ satisfying

$$c(A,k)e^{ky_0} + Ce^{-2y_0} = B. \tag{C.7}$$

Moreover $g(y) > 0$ for all $y > y_0$, and $g(y) < 0$ for all $y < y_0$. Since $kc(A,k)C < 0$ $g'(y_0) \neq 0$ by Lemma C.1-(i). It follows that $L^+_{y_0}$ is finite. Let

$$G_{y_0,+}(y) := \int_{y_0}^{y} \frac{1}{\sqrt{g(t)}}dt, \quad y \geq y_0, \tag{C.8}$$

where, for all $c \in \mathbf{R}$, we set $\int_c^c(\ \cdot\ )dy := 0$. Then $G_{y_0,+}([y_0,\infty)) = [0, L^+_{y_0})$ and $\lim_{x\downarrow 0} G^{-1}_{y_0,+}(x) = G^{-1}_{y_0,+}(0) = y_0$. Hence, by Proposition B.3-(ii), the function

$$f(x) := \begin{cases} G^{-1}_{y_0,+}(x) & x \in [0, L^+_{y_0}) \\ G^{-1}_{y_0,+}(-x) & x \in (-L^+_{y_0}, 0) \end{cases} \tag{C.9}$$

is a solution to Eq.(A.1) on $(-L^+_{y_0}, L^+_{y_0})$.

C.2 The case $c(A,k) < 0,\ C > 0$

In this case g is strictly monotone decreasing with $g(y) \sim c(A,k)e^{ky} \sim -\infty$ as $y \to \infty$ and $g(y) \sim Ce^{-2y} \sim \infty$ as $y \to -\infty$. Hence g has a unique zero $y_0 \in \mathbf{R}$ satisfying (C.7). The function

$$G_{y_0,-}(y) := \int_{y}^{y_0} \frac{1}{\sqrt{g(t)}}dt, \quad y \leq y_0 \tag{C.10}$$

is a primitive function of $-1\sqrt{g(t)}$ on $(-\infty, y_0)$. The integral $L^-_{y_0}$ is finite. As in the preceding case, the function

$$f(x) := \begin{cases} G^{-1}_{y_0,-}(x) & x \in [0, L^-_{y_0}) \\ G^{-1}_{y_0,-}(-x) & x \in (-L^-_{y_0}, 0) \end{cases} \tag{C.11}$$

is a solution to Eq.(A.1) on $(-L^-_{y_0}, L^-_{y_0})$.

C.3 The case $c(A,k) > 0,\ C > 0$

In this case g' has a unique zero a_0 satisfying (C.1). The function g is strictly monotone increasing on $[a_0,\infty)$ and strictly monotone decreasing on $(-\infty, a_0)$ with with $g(y) \sim c(A,k)e^{ky} \sim \infty$ as $y \to \infty$ and $g(y) \sim Ce^{-2y} \sim \infty$ as $y \to -\infty$ We consider the cases $g(a_0) < 0$ and $g(a_0) > 0$ separately.

(a) The case $g(a_0) < 0$

In this case, g has only two zeros. We denote them y_1 and y_2 respectively with $y_1 < y_2$. The integral $L^+_{y_1}$ is finite. Let

$$G_1(y) := \int_{y_1}^{y} \frac{1}{\sqrt{g(t)}}dt, \quad y \geq y_1. \tag{C.12}$$

Then the function

$$f_1(x) := \begin{cases} G_1^{-1}(x) & x \in [0, L_{y_1}^+) \\ G_1^{-1}(-x) & x \in (-L_{y_1}^+, 0) \end{cases} \tag{C.13}$$

is a solution to Eq.(A.1) on $(-L_{y_1}^+, L_{y_1}^+)$.

The integral $L_{y_2}^-$ is finite. Let

$$G_2(y) := \int_y^{y_2} \frac{1}{\sqrt{g(t)}} dt, \quad y \leq y_2. \tag{C.14}$$

Then the function

$$f_2(x) := \begin{cases} G_2^{-1}(x) & x \in [0, L_{y_2}^-) \\ G_2^{-1}(-x) & x \in (-L_{y_2}^-, 0) \end{cases} \tag{C.15}$$

is a solution to Eq.(A.1) on $(-L_{y_2}^-, L_{y_2}^-)$.

(b) The case $g(a_0) > 0$

In this case $g(y) > 0$ for all $y \in \mathbf{R}$. Let

$$G_+(y) := \int_c^y \frac{1}{\sqrt{g(t)}} dt, \tag{C.16}$$

where c is an arbitrary real constant. Then $G_+^{-1}(x)$ is a solution to Eq.(A.1) on $(0, L_c^+)$.

Let

$$G_-(y) := \int_y^c \frac{1}{\sqrt{g(t)}} dt, \tag{C.17}$$

where c is an arbitrary real constant. Then $G_-^{-1}(x)$ is a solution to Eq.(A.1) on $(0, L_c^-)$. Note that $L_c^{\pm} < \infty$.

C.4 The case $c(A, k) < 0$, $C < 0$

In this case, if $g(a_0) < 0$, then $g(y) < 0$ for all $y \in \mathbf{R}$. Hence we consider only the case $g(a_0) > 0$. In this case g has only two zeros. We denote them w_1 and w_2 respectively with $w_1 < w_2$. We have that $g(y) > 0$ if and only if $w_1 < y < w_2$.

Let

$$G_{w_1}(y) := \int_{w_1}^y \frac{1}{\sqrt{g(t)}} dt, \quad w_1 \leq y \leq w_2. \tag{C.18}$$

Then the function

$$f_{w_1}(x) := \begin{cases} G_{w_1}^{-1}(x) & x \in [0, G_{w_1}(w_2)) \\ G_{w_1}^{-1}(-x) & x \in (-G_{w_1}(w_2), 0) \end{cases} \tag{C.19}$$

is a solution to Eq.(A.1) on $(-G_{w_1}(w_2), G_{w_1}(w_2))$.

Let

$$G_{w_2}(y) := \int_y^{w_2} \frac{1}{\sqrt{g(t)}} dt, \quad w_1 \leq y \leq w_2. \tag{C.20}$$

Then the function

$$f_{w_2}(x) := \begin{cases} G_{w_2}^{-1}(x) & x \in [0, G_{w_2}(w_1)) \\ G_{w_2}^{-1}(-x) & x \in (-G_{w_2}(w_1), 0) \end{cases} \tag{C.21}$$

is a solution to Eq.(A.1) on $(-G_{w_2}(w_1), G_{w_2}(w_1))$.

C.5 The case $C = 0$

In this case a primitive function $G(y)$ of $\pm 1/\sqrt{g(y)}$ can be explicitly computed in the form

$$G(y) = \tanh_q^{-1} \alpha\sqrt{|B| - |c(A,k)|e^{ky}},$$

with α being a constant and $q > 0$ is arbitrary. As a result, we see that, for all $t \in \mathbf{R}\setminus\{0\}$, the function

$$f(x) := \frac{1}{k}\log\frac{4t^2 q}{|c(A,k)|k^2} + \frac{2}{k}\log \operatorname{sech}_q tx \tag{C.22}$$

is a solution to Eq.(A.1) on $\mathbf{R}$ with $B = -4t^2/k^2$ if

$$A(k+2) > 0. \tag{C.23}$$

This is a *global* solution.

D Classification of solutions (II): The case $k < 0$, $k \neq -2$

Throughout this section, we assume that

$$k < 0,\ k \neq -2. \tag{D.1}$$

D.1 The case $c(A,k) > 0,\ C < 0$

In this case, by Lemma C.1, g' has a uniuqe zero a_0 given by (C.1). We only present a solution in the case where $-2 < k < 0$ and $B < 0$. In this case, g is strictly monotone increasing in $(-\infty, a_0)$ and strictly monotone decreasing in (a_0, ∞) with $g(y) \to -B > 0$ as $y \to \infty$ and $g(y) \sim -|C|e^{-2y} \sim -\infty$ as $y \to -\infty$ Hence g has a unique zero y_0 satisfying (C.7) and $g(y) > 0$ for all $y > y_0$. Hence we can define

$$G(y) := \int_{y_0}^{y} \frac{1}{\sqrt{g(t)}}dt, \quad y \geq y_0. \tag{D.2}$$

We have $G([y_0,\infty)) = [0,\infty)$ with $g(G^{-1}(0)) = g(y_0) = 0$. Thus, by Proposition B.3-(ii), the function

$$f(x) := \begin{cases} G^{-1}(x) & x \in [0,\infty) \\ G^{-1}(-x) & x \in (-\infty, 0) \end{cases} \tag{D.3}$$

is a solution to Eq.(A.1) on $\mathbf{R}$. This is a global solution.

Other cases for (k, B) produce only local solutions.

D.2 The case $c(A,k) < 0,\ C > 0$

In this case too, by Lemma C.1, g' has a uniuqe zero a_0 given by (C.1).

Let $k > -2$ and $B < 0$. Then g is strictly monotone increasing in (a_0, ∞) and strictly monotone decreasing in $(-\infty, a_0)$ with $g(y) \to -B > 0$ as $y \to \infty$ and $g(y) \sim Ce^{-2y} \sim \infty$

as $y \to -\infty$. Let $g(a_0) < 0$. Then g has only two zeros. We denote them by y_1 and y_2 respectively ($y_1 < y_2$) and define

$$G(y) := \int_{y_2}^{y} \frac{1}{\sqrt{g(t)}} dt, \quad y \geq y_2. \tag{D.4}$$

We have $G([y_2, \infty)) = [0, \infty)$ and the function f given by (D.3) is a solution to Eq.(A.1) on $\mathbf{R}$.

Let $k < -2$ and $B < 0$. Then g is strictly monotone decreasing in (a_0, ∞) and strictly monotone increasing in $(-\infty, a_0)$ with $g(y) \to -B > 0$ as $y \to \infty$ and $g(y) \sim c(A, k)e^{ky} \sim -\infty$ as $y \to -\infty$. Hence g has a unique zero y_0. Let

$$G(y) := \int_{y_0}^{y} \frac{1}{\sqrt{g(t)}} dt, \quad y \geq y_0. \tag{D.5}$$

Then we have $G([y_0, \infty)) = [0, \infty)$ and the function f given by (D.3) is a solution to Eq.(A.1) on $\mathbf{R}$.

D.3 The case $c(A, k) > 0$, $C > 0$

In this case we have $g'(y) < 0$ for all $y \in \mathbf{R}$. Hence g is strictly monotone decreasing with $g(y) \to -B$ as $y \to \infty$ and $g(y) \to \infty$ as $y \to -\infty$. Then we see that the present case produces only local solutions to Eq.(A.1). We omit them writing them down here.

D.4 The case $c(A, k) < 0$, $C < 0$

In this case we have $g'(y) > 0$ for all $y \in \mathbf{R}$. Hence g is strictly monotone increasing with $g(y) \to -B$ as $y \to \infty$ and $g(y) \to -\infty$ as $y \to -\infty$. Hence, if $B < 0$, then g has a unique zero y_0. Let $B < 0$ and define

$$G(y) := \int_{y_0}^{y} \frac{1}{\sqrt{g(t)}} dt, \quad y \geq y_0. \tag{D.6}$$

Then $G([y_0, \infty)) = [0, \infty)$ and the function f given by (D.3) is a solution to Eq.(A.1) on $\mathbf{R}$.

D.5 The case $C = 0$

In this case, if $B < 0$ and $c(A, k) < 0$, then a global solution to Eq.(A.1) is given in the same form as in (C.22)

E Classification of solutions (III): The case $k = -2$

In this case g' has a unique zero b_0 given by (C.2).

E.1 The case $A < 0$

In this case, g is strictly monotone increasing on $(-\infty, b_0)$ and strictly monotone decreasing on (b_0, ∞) with $g(y) \to -B$ as $y \to \infty$ and $g(y) \to -\infty$ as $y \to -\infty$.

Let $B < 0$. Then g has a unique zero z_0 satisfying

$$(C - 2Az_0)e^{-2z_0} = B. \tag{E.1}$$

Let

$$G(y) := \int_{z_0}^{y} \frac{1}{\sqrt{g(t)}} dt, \quad y \geq z_0. \tag{E.2}$$

Then $G([z_0, \infty)) = [0, \infty)$ and the function f given by (D.3) is a solution to Eq.(A.1) on $\mathbf{R}$.

E.2 The case $A > 0$

In this case, g is strictly monotone increasing on (b_0, ∞) and strictly monotone decreasing on $(-\infty, b_0)$ with $g(y) \to -B$ as $y \to \infty$ and $g(y) \to \infty$ as $y \to -\infty$.

Let $B < 0$ and $g(b_0) < 0$. Then g has only two zeros z_1 and z_2 $(z_1 < z_2)$. Let

$$G(y) := \int_{z_2}^{y} \frac{1}{\sqrt{g(t)}} dt, \quad y \geq z_2. \tag{E.3}$$

Then $G([z_2, \infty)) = [0, \infty)$ and the function f given by (D.3) is a solution to Eq.(A.1) on $\mathbf{R}$.

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