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Transversal Whitney topology and singularities of Haefliger foliations

Shyuichi IZUMIYA and Kunihide MARUYAMA

January 5, 2001

Dedicated to the memory of Professor Luiz A. Favaro

Abstract

In order to study singularities of Haefliger foliation, we define the notion of transversally Whitney C^∞-topology modulo a regular foliation on the set of C^∞-mappings into a foliated manifold. We prove a kind of transversality theorem with respect to this new topology. All arguments we use here are analogous to those of the theory for the ordinary Whitney C^∞-topology. However, this is the first attempt for the study of generic properties of Haefliger foliations

1 Introduction

In his paper [2] Prof. Luiz A. Favaro studied *$\mathcal{F}$-stability* of C^∞-mappings into a foliated manifold. The definition of $\mathcal{F}$-stability is given as follows: Let M, N be C^∞-manifolds and N is regularly foliated by $\mathcal{F}$. We consider the space of C^∞-mappings $C^\infty(M,N)$ with the Whitney C^∞ topology. We say that $f \in C^\infty(M,N)$ is *$\mathcal{F}$-stable* if there exists a neighbourhood V_f of f in $C^\infty(M,N)$ such that for any $g \in V_f$, there exist a diffeomorphism $h : M \longrightarrow M$ and a diffeomorphism $k : N \longrightarrow N$ taking leaves of $\mathcal{F}$ to leaves such that $g = k \circ f \circ h^{-1}$.

The infinitesimal version of the above stability is defined as follows: $f \in C^\infty(M,N)$ is *infinitesimally $\mathcal{F}$-stable* if for any $w \in \Gamma^\infty(f^*TN)$, there exist $u \in \Gamma^\infty(TM)$, $v \in \Gamma^\infty(TN)$ with $\pi(v)$ is locally constant along the leaves and $w = df \circ u + v \circ f$, where $\pi : TN \longrightarrow TN/T\mathcal{F}$ is the canonical projection.

One of the purpose in his paper is to consider that the above two definitions are equivalent or not. He has shown that it is true in the local sense. Of course his assertion is rather clear in the present time, because there are a lot of tools and results in singularity theory now (c.f., Damon [1]). Global $\mathcal{F}$-stability is, however, still an open question now. Unfortunately, we can not contribute to the global $\mathcal{F}$-stability in this paper. Let us consider the meaning of the above stability of mappings. We say that a C^∞-mapping $F : M \longrightarrow N$ is $\mathcal{F}^\perp$-nonsingular at $x \in M$ if f is transversal to $\mathcal{F}$ at x. Otherwise, we say that f is $\mathcal{F}^\perp$-singular at x. The above notion of stability corresponds to the stability of $\mathcal{F}^\perp$-singularities. One of the motivations to study

Key Words and Phrases. Whitney topology, Haefliger foliations, singularities, transversality

$\mathcal{F}^{\perp}$-singularities is given in the study of singular Haefliger foliations. It is clear that if f is $\mathcal{F}^{\perp}$-nonsingular at any point, $f^{-1}\mathcal{F}$ is a regular foliation. If f has $\mathcal{F}^{\perp}$-singularities, then $f^{-1}\mathcal{F}$ is a singular Haefliger foliation. We now review the definition of Haefliger foliations. Suppose that there exist an open covering $M = \cup_{i \in I} V_i$ and a local C^∞-mapping $\phi_i : V_i \longrightarrow \mathbb{R}^q$ on each V_i with the following properties:

(1) If $V_i \cap V_j \neq \emptyset$, there exist open neighbourhoods U_i of $\phi_i(V_i \cap V_j)$, U_j of $\phi_j(V_i \cap V_j)$ and a diffeomorphism $\psi_{ij} : U_j \longrightarrow U_i$ such that $\psi_{ij} \circ \phi_i(x) = \phi_j(x)$ for $x \in V_i \cap V_j$.

(2) If $V_i \cap V_j \cap V_k \neq \emptyset$, $\psi_{ij}, \psi_{jk}, \psi_{ik}$ satisfy the cocycle condition: $\psi_{ik}(x) = \psi_{ij} \circ \psi_{jk}(x)$ for any $x \in V_i \cap V_j \cap V_k$.

In this case, $\mathcal{H} = \{(\phi_i, V_i) \mid i \in I\}$ is called *a Haefliger foliation* and q is called *the codimension of* $\mathcal{H}$. We can extend each level set $\phi_i^{-1}(c) \subset V_i$ to V_j if $\phi_i^{-1}(c) \cap V_j \neq \emptyset$. The maximal connected component consisting of such level sets is called *a leaf of* $\mathcal{H}$. We say that $x \in M$ is *a singular point* of $\mathcal{H}$ if x is a singular point of ϕ_i for $x \in V_i$. It has been known (and easy to prove) the following proposition.

Proposition 1.1 *Let $\mathcal{H}$ be a Haefliger foliation on M. Then there exist a smooth manifold N, a regular foliation $\mathcal{F}$ and a C^∞-mapping $f : M \longrightarrow N$ such that $f^{-1}\mathcal{F} = \mathcal{H}$.*

This proposition gives a strong motivation for the study of $\mathcal{F}^{\perp}$-singularities of C^∞-mapping $f : M \longrightarrow N$. In the definition of $\mathcal{F}$-stability and infinitesimal $\mathcal{F}$-stability, the directions for perturbations of $f \in C^\infty(M, N)$ contains too many information for the study of $\mathcal{F}^{\perp}$-singularities. It can be perturbed into the tangent direction of leaves of $\mathcal{F}$. In this paper, we study the stability of singularities of the Haefliger foliation $f^{-1}\mathcal{F}$. For the purpose, we do not need the perturbation corresponding to the tangent direction of leaves. Our main purpose is summarised that we introduce a certain topology on $C^\infty(M, N)$ describing the perturbation of $f \in C^\infty(M, N)$ with respect to only the transversal direction of $\mathcal{F}$.

All arguments in this note are analogous to those for the jet-transversality theory on the ordinary Whitney C^∞-topology. Nevertheless, we write down the results because these contain some new concepts and these are the fundamental results for the study of singularities of Haefliger foliations.

Apologies and Acknowledgments It was 1994 August, the first author proposed collaboration to Prof. L. A. Favaro on the study of smooth mappings along the above direction. We promised to proceed this project in that time. However, we have not been able to realise it because of his sorrowful death. The first author really would like to apologise to him that we did not start the project earlier.

On the other hand, the authors would like to thank the organisers of "The 6th Workshop On Real and Complex Singularities" for giving us an opportunity to talk about this incomplete project. The first author is also grateful to people at USP and USFsc for their kind and warm-hearted hospitality during his stay in Sao Carlos.

2 Transversal Whitney topology modulo $\mathcal{F}$

In this section we introduce a new topology on $C^\infty(M, N)$ along the line of the description of Whitney C^∞-topology in [3]. First, we borrow the notion of jet spaces modulo $\mathcal{F}$ from Ikegami[4] for our purpose. Let $f, g : (M, x) \longrightarrow (N, y)$ be C^∞-map germs. Since $\mathcal{F}$ is a regular foliation on N, there exists a local chart $\eta_1 \times \eta_2 : U \longrightarrow D^q \times D^r \subset \mathbb{R}^q \times \mathbb{R}^r$ around $y \in N$

such that $U \cap \mathcal{F} = \{(\eta_1 \times \eta_2)^{-1}(\{c\} \times D^r) \mid c \in D^q\}$. For any $k \geq 1$, we say that f and g have *k-th order contact at x modulo $\mathcal{F}$* if $\eta_1 \circ f$ and $\eta_1 \circ g$ have k-th order contact at x in the ordinary sense (cf., [3]). We can easily show that the above relation is an equivalence relation among the set of all C^∞-map germs $C^\infty(M,N)_{(x,y)}$. We denote that the set of equivalence classes by $J^k(M,N;\mathcal{F})_{(x,y)}$ and we call it *a k-jet space modulo $\mathcal{F}$*. For $k = 0$, we consider that $J^0(M,N;\mathcal{F})_{(x,y)} = M \times N$. The element in $J^k(M,N;\mathcal{F})_{(x,y)}$ represented by a map-germ $f : (M,x) \longrightarrow (N,y)$ is denoted $j^k_{\mathcal{F}}f(x)$. We can define *the k-jet bundle modulo $\mathcal{F}$* by

$$J^k(M,N;\mathcal{F}) = \cup_{(x,y)\in M\times N} J^k(M,N;\mathcal{F})_{(x,y)}.$$

The topology on $J^k(M,N;\mathcal{F})$ is defined like as the ordinary k-jet bundle. We also define a mapping $\alpha \times \beta : J^k(M,N;\mathcal{F}) \longrightarrow M \times N$ by $\alpha \times \beta(j^k_{\mathcal{F}}f(x)) = (x, f(x))$. We can summarise basic properties of $J^k(M,N;\mathcal{F})$. For the detailed descriptions, see Ikegami[4].

Proposition 2.1 *Let M,N be smooth manifolds such that N is regularly foliated by $\mathcal{F}$ with* codim$\mathcal{F} = q$.
a) $J^k(M,N;\mathcal{F})$ *is a smooth manifold.*
b) *For any smooth mapping $f : M \longrightarrow N$, we can define a smooth mapping $j^k_{\mathcal{F}}f : M \longrightarrow J^k(M,N;\mathcal{F})$ by $(j^k_{\mathcal{F}}f)(x) = j^k_{\mathcal{F}}f(x)$. We call $j^k_{\mathcal{F}}f$ a k-jet extension modulo $\mathcal{F}$.*
c) $\alpha \times \beta : J^k(M,N;\mathcal{F}) \longrightarrow M \times N$ *is a smooth fibre bundle with the fibre*

$$B^k_{m,q} = \{P : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^q, 0) \mid P \ : \ \text{polynomial map with degree} P \leq k\ \}.$$

We now define the notion of Whitney C^∞-topology modulo $\mathcal{F}$. The idea is very simple. We adopt $J^k(M,N;\mathcal{F})$ for the new topology instead of $J^k(M,N)$ for the ordinary Whitney C^∞-topology. For any non-negative integer k and a subset $U \subset J^k(M,N;\mathcal{F})$, we denote that

$$M^k_{\mathcal{F}}(U) = \{f \in C^\infty(M,N) \mid j^k_{\mathcal{F}}f(M) \subset U\ \}.$$

It is clear that $M^k_{\mathcal{F}}(U) \cap M^k_{\mathcal{F}}(V) = M^k_{\mathcal{F}}(U \cap V)$. It follows from these facts that $W^{\mathcal{F}}_k = \{M^k_{\mathcal{F}}(U) \mid U : \text{open subset}\}$ form a basis of a topology on $C^\infty(M,N)$. We can induce a topology on $C^\infty(M,N)$ such that the basis of the topology is given by $W^{\mathcal{F}}_\infty = \cup_{k=0}^\infty W^{\mathcal{F}}_k$. We denote $C^\infty(M,N;\mathcal{F})$ the space of smooth mappings with this new topology.

We call this topology *Whitney C^∞-topology modulo $\mathcal{F}$.* We can prove several basic facts on Whitney C^∞-topology modulo $\mathcal{F}$ like as the ordinary Whitney C^∞-topology.

Proposition 2.2 1) *$C^\infty(M,N;\mathcal{F})$ is a Baire space.*
2) *The mapping*

$$j^k_{\mathcal{F}} : C^\infty(M,N;\mathcal{F}) \longrightarrow C^\infty(M, J^k(M,N;\mathcal{F}))$$

defined by $j^k_{\mathcal{F}}(f) = j^k_{\mathcal{F}}f$ is continuous.

Here we adopt the ordinary Whitney C^∞-topology on $C^\infty(M, J^k(M,N;\mathcal{F}))$.

Proof. Let d_k be a complete metric on $J^k(M,N;\mathcal{F})$. Let $U_1, U_2, \dots$ be countably many open dense subsets of $C^\infty(M,N;\mathcal{F})$. For any open subset $V \subset C^\infty(M,N;\mathcal{F})$, we have to show that $V \cap \cap_{i=1}^\infty U_i \neq \emptyset$. Since $V \subset C^\infty(M,N;\mathcal{F})$ is an open subset with respect to Whitney C^∞ topology modulo $\mathcal{F}$, there exist a natural number k_0 and an open subset $W \subset J^{k_0}(M,N;\mathcal{F})$ such that $M_{\mathcal{F}}(\bar{W}) \subset V$ and $M_{\mathcal{F}}(W) \neq \emptyset$. It is enough to show that $M_{\mathcal{F}}(\bar{W}) \cap \cap_{i=1}^\infty U_i \neq \emptyset$.

For the purpose, we choose functions $f_1, f_2, \cdots \in C^\infty(M,N;\mathcal{F})$, non-negative integers $k_1, k_2, \cdots$ and open sets $W_i \subset J^{k_i}(M,N;\mathcal{F})$ with the following properties:

(A_i) $f_i \in M_{\mathcal{F}}(W) \cap \cap_{j=1}^{i-1} M_{\mathcal{F}}(W_j) \cap U_i$,

(B_i) $M_{\mathcal{F}}(\bar{W}) \subset U_i$ and $f_i \in M_{\mathcal{F}}(W_i)$,

(C_i) $d_s(j^s_{\mathcal{F}} f_i(x), j^s_{\mathcal{F}} f_{i-1}(x)) < \dfrac{1}{2^i}$ for $i > 1$ and $1 \leq s \leq i$.

(D_i) For any $x \in M$, there exists a local chart $\eta_1 \times \eta_2 : V \longrightarrow D^q \times D^r$ of N around $f_1(x)$ such that $V \cap \mathcal{F} = \{\eta^{-1}(c) \mid c \in D^q\}$, $f_i(x) \in V$ and $\eta_2 \circ f_i = \eta_2 \circ f_1$ on $(f_i)^{-1}(V) \cap (f_1)^{-1}(V)$ for any i.

The condition D_i is independent on the choice of local charts $(\eta_1 \times \eta_2, V)$ and it is the essential difference from the arguments on the ordinary Whitney C^∞-topology. We can choose the above subjects by the almost same arguments as those on the ordinary Whitney C^∞-topology. By the condition (C_i), $j^s_{\mathcal{F}} f_i(x)$ $(i = 1, 2, \dots)$ is a Cauchy sequence in $J^s(M, N; \mathcal{F})$, so that we have the limit $g_s(x) = \lim_{i\to\infty} j^s_{\mathcal{F}} f_i(x)$. Since $J^0(M, N; \mathcal{F}) = M \times N$, we have $j^0_{\mathcal{F}} f_i(x) = (x, f_i(x))$. Therefore, we define a mapping $g : M \longrightarrow N$ by $g_0(x) = (x, g(x))$. In order to prove that g is a smooth mapping, we need the condition (D_i). By exactly the same arguments as those on the ordinary Whitney C^∞-topology, we can show that $\eta_1 \circ g$ is smooth for a local chart $\eta_1 \times \eta_2 : V \longrightarrow D^q \times D^r$ of N around $g(x)$. By the condition (D_i), $\eta_2 \circ g = \eta_2 \circ f_1$. This means that g is smooth.

By the conditions (A_i), (B_i) and exactly the same arguments as those on the ordinary Whitney C^∞-topology, we can show that $g \in M_{\mathcal{F}}(\bar{W}) \cap \cap_{i=1}^{\infty} U_i$. This completes the proof of the assertion 1).

Let $U \subset J^\ell(M, J^k(M, N; \mathcal{F})$ be an open subset. Since

$$M(U) = \{f \in C^\infty(M, J^k(M, N; \mathcal{F})) \mid j^\ell f(M) \subset U\}$$

is an element of the open basis for the ordinary Whitney C^∞-topology, it is enough to show that $(j^k_{\mathcal{F}})^{-1}(M(U))$ is an open set in $C^\infty(M, N; \mathcal{F})$. We define a map

$$\alpha_{k,\ell} : J^{k+\ell}(M, N; \mathcal{F}) \longrightarrow J^\ell(M, J^k(M, J^k(M, N; \mathcal{F}))$$

by $\alpha_{k,\ell}(\sigma) = j^\ell(j^k_{\mathcal{F}} f)(x)$, where $\sigma = j^{k+\ell}_{\mathcal{F}} f(x)$. It is clear that $\alpha_{k,\ell}$ is a well-defined smooth map. Therefore, $\alpha^{-1}_{k,\ell}(U)$ is an open submanifold of $J^{k+\ell}(M, N; \mathcal{F})$. Since $\alpha_{k,\ell} \circ j^{k+\ell}_{\mathcal{F}} f = j^\ell \circ j^k_{\mathcal{F}} f$, we have $M_{\mathcal{F}}(\alpha^{-1}_{k,\ell}(U)) = (j^k_{\mathcal{F}})^{-1}(M(U))$, so that it is open. This completes the proof. □

3 A jet transversality theorem modulo $\mathcal{F}$

In this section we formulate the transversality theorem modulo $\mathcal{F}$ and give a proof. The proof of the theorem is a direct analogy of the proof of the original jet transversality theorem[3], so that we only give the sketch of the proof here.

Theorem 3.1 *Let M, N be smooth manifolds and $\mathcal{F}$ be a regular foliation on N. For any submanifold $W \subset J^k(M, N; \mathcal{F})$, the set*

$$T^{\mathcal{F}}_W = \{f \in C^\infty(M, N; \mathcal{F}) \mid j^k_{\mathcal{F}} f \ : \ \text{transversal to } W\}$$

is a residual subset of $C^\infty(M, N; \mathcal{F})$. If W is a closed set, then $T^{\mathcal{F}}_W$ is an open set.

Proof. The proof is analogous to the proof of the ordinary jet transversality theorem. so that we only describe the different part from the ordinary one. We show that $T^{\mathcal{F}}_W$ is a intersection of countable open dense subsets of $C^\infty(M,N;\mathcal{F})$. Let $W_1, W_2, \cdots \subset W$ be an open covering of W with the following properties:

1) $\bar{W}_i \subset W$.
2) $\bar{W}_i$ is compact.
3) There exist coordinate neighbourhoods U_i of M and V_i of N such that $\alpha\times\beta(W_i) \subset U_i\times V_i$.
4) $\bar{U}_i$ is compact.

Since $\mathcal{F}$ is a regular foliation on N, we can choose the coordinate system $\eta_1 \times \eta_2 : V_i \longrightarrow D^q \times D^r \subset \mathbb{R}^q \times \mathbb{R}^r$ with $V_i \cap \mathcal{F} = \{(\eta_1\times\eta_2)^{-1}(\{c\}\times D^r) \mid c \in D^q\}$. We define a set

$$T^{\mathcal{F}}_{W_i} = \{f \in C^\infty(M,N;\mathcal{F}) \mid j^k_{\mathcal{F}} f \ : \ \text{transversal to } W \text{ on } \bar{W}_i\}.$$

Since $T^{\mathcal{F}}_W = \cap_{i=1}^\infty T^{\mathcal{F}}_{W_i}$, it is enough to show that $T^{\mathcal{F}}_{W_i}$ is an open dense subset.

We denote that

$$T_i = \{g \in C^\infty(M, J^k(M,N;\mathcal{F})) \mid g \ : \ \text{transversal to } W \text{ on } \bar{W}_i\}.$$

Since we adopt the ordinary Whitney C^∞-topology on $C^\infty(M, J^k(M,N;\mathcal{F}))$, it is known that T_i is an open set (cf., [3]). It follows from the fact that $j^k_{\mathcal{F}}$ is continuous, we have an open subset $T_{W_i} = (j^k_{\mathcal{F}})^{-1}(T_i)$. In order to prove the density, let $\psi : U_i \longrightarrow \mathbb{R}^n$ and $\eta_1\times\eta_2 : V_i \longrightarrow D^q \times D^r \subset \mathbb{R}^q\times\mathbb{R}^r$ be coordinate charts. Let

$$\rho : \mathbb{R}^n \longrightarrow [0,1], \quad \rho' : \mathbb{R}^r \longrightarrow [0,1]$$

be continuous functions respectively defined by

$$\rho = \begin{cases} 1 & \text{on a neighbourhood of } \psi\circ\alpha(\bar{W}_i) \\ 0 & \text{off } \psi(U_i) \end{cases}$$

$$\rho' = \begin{cases} 1 & \text{on a neighbourhood of } \eta_1\circ\alpha(\bar{W}_i) \\ 0 & \text{off } \eta_1(V_i). \end{cases}$$

Let $B(m,r:k)$ be a set of polynomial maps $p : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ with degree $p \leq k$ and $p(0)=0$. Let $f : M \longrightarrow N$ be a smooth mapping. For any $b \in B(m,r:k)$, we define a smooth mapping $g_b : M \longrightarrow N$ by

$$g_b(x) = \begin{cases} f(x) \text{ if } x \notin U_i \text{ or } f(x) \notin V_i \\ (\eta_1\times\eta_2)^{-1}(\rho(\psi(x))\rho'(\eta_1(f(x)))b(\psi(x)) + \eta_1(f(x)), \eta_2(f(x))) \text{ otherwise.} \end{cases}$$

We now define a smooth mapping $G : M \times B(m,r:k) \longrightarrow N$ by $G(x,b) = g_b(x)$. We also define a mapping $\Phi : M\times B(m,r:k) \longrightarrow J^k(M,N;\mathcal{F})$ by $\Phi(x,b) = j^k_{\mathcal{F}} g_b(x)$. By the similar arguments as those of in the proof of ordinary jet transversality theorem, we can easily show that there exists an open neighbourhood B of $0 \in B(m,r:k)$ such that $\Phi|M\times B$ is a local diffeomorphism at the point $(x,b) \in M\times B$ with $\Phi(x,b) \in \bar{W}_i$, so that $\Phi|M\times B$ is transversal to W on $\bar{W}_i$. By the fundamental transversality theorem of Thom (cf., [3]), we have a sequence $\{b_j\}_{j=1}^\infty \subset B$ which is uniformly convergent to $0 \in B$ such that $j^k_{\mathcal{F}} g_{b_j}$ is transversal to W on $\bar{W}_i$. Since $g_0 = f$ and $g_b = f$ outside of U_i, we have $\lim_{j\longrightarrow\infty} g_{b_j} = f$ in $C^\infty(M,N;\mathcal{F})$.

On the other hand, suppose that W is a closed set. Since we adopt the ordinary Whitney C^∞-topology on $C^\infty(M, J^k(M,N;\mathcal{F}))$, it has been known that the set

$$T^k_W = \{F \in C^\infty(M, J^k(M,N;\mathcal{F})) \mid F \ : \ \text{transversal to } W\}$$

is an open set (cf., [3]). By the assertion 2) of Proposition 2.2,

$$j^k_{\mathcal{F}} : C^\infty(M,N;\mathcal{F}) \longrightarrow C^\infty(M, J^k(M,N;\mathcal{F}))$$

is continuous.

Hence, $T^{\mathcal{F}}_W = (j^k_{\mathcal{F}})^{-1}(T^k_W)$ is an open subset of $C^\infty(M,N;\mathcal{F})$. □

We remark that the multi-jet version of the above theorem also holds. Since we do not consider global situation here, we omit to describe the assertion and the proof.

4 Applications

In this section we give some applications of the jet transversality theorem modulo $\mathcal{F}$.

1) Thom-Boardman Haefliger foliations Let M, N be smooth manifolds and $\mathcal{F}$ be a regular foliation on N with codim$\mathcal{F} = q$. We now consider *the (ordinary) Thom-Boardman singular set* $\Sigma^I(m,q) \subset J^k(m,q)$. Since $J^k(m,q)$ is considered to be the fibre of $J^k(M,N;\mathcal{F})$, we can define *a Thom-Boardman subbundle modulo* $\mathcal{F}$ $\Sigma^I(M,N;\mathcal{F}) \subset J^k(M,N;\mathcal{F})$ as usual. By Theorem 3.1, the set

$$T_{\Sigma^I(M,N;\mathcal{F})} = \{ f \in C^\infty(M,N;\mathcal{F}) \mid j^k_{\mathcal{F}} f \ : \ \text{transversal to } \Sigma^I(M,N;\mathcal{F}) \ \}$$

is a residual subset of $C^\infty(M,N;\mathcal{F})$. For any $f \in T_{\Sigma^I(M,N;\mathcal{F})}$, we call $\mathcal{H} = f^{-1}\mathcal{F}$ *a Thom-Boardman Haefliger foliation* with I.

Suppose that the Boardman symbol is given by $I = (i_1, i_2, \dots, i_k)$. For any $f \in T_{\Sigma^I(M,N;\mathcal{F})}$, we can choose local covering $\{U_j\}_{j\in J}$ of M and local C^∞-mappings $\phi_j : U_j \longrightarrow D^q \subset \mathbb{R}^q$ which define the Haefliger foliation $\mathcal{H} = f^{-1}\mathcal{F}$. By the construction of the Thom-Boardman singular set, the set

$$\Sigma^{i_1}(\mathcal{H}) = \{ x \in M \mid \operatorname{rank} d(\phi_j)_x = q - i_1 \text{ for } x \in U_j \}$$

is a smooth submanifold of M. Moreover,

$$\Sigma^{i_1,i_2}(\mathcal{H}) = \{ x \in \Sigma^{i_1}(\mathcal{H}) \mid \operatorname{rank} d(\phi_j|\Sigma^{i_1}(\mathcal{H}))_x = \min(\dim \Sigma^{i_1}(\mathcal{H}), q - i_2) \text{ for } x \in U_j \}$$

is also a smooth submanifold of M. This procedure can be continued to k times. As a special case, we assume that $q = 1$, $k = 1$ and $I = (1)$. For any $f \in T_{\Sigma^1(M,N;\mathcal{F})}$, $\mathcal{H} = f^{-1}\mathcal{F}$ is called *a Morse type Haefliger foliation.* We have the following problem associated to Morse type Haefliger foliations:

Problem Suppose that $\mathcal{F}$ is a regular foliation on N with codim $\mathcal{F} = 1$ and $\mathcal{H} = f^{-1}\mathcal{F}$ is a Morse type Haefliger foliation. Can we have a generalised Morse inequality? It might be related to the topology of the leaf space $N/\mathcal{F}$.

2) Stability of mappings modulo $\mathcal{F}$ For any $f, g \in C^\infty(M,N;\mathcal{F})$, we say that f, g are $\mathcal{A}_{\mathcal{F}}$-*equivalent* if there exist a diffeomorphism $h : M \longrightarrow M$ and a smooth mapping $k : N \longrightarrow N$ taking leaves of $\mathcal{F}$ to leaves such that the induced mapping $\tilde{k} : N/\mathcal{F} \longrightarrow N/\mathcal{F}$ is a homeomorphism and $\tilde{g} = \tilde{k} \circ \tilde{f} \circ h^{-1}$, where $\tilde{f} = \pi \circ f$ and $\pi : N \longrightarrow N/\mathcal{F}$ is the canonical projection. The local version of the above definition is as follows: For smooth map-germs $f : (M,x) \longrightarrow (N,y)$ and $g : (M,x') \longrightarrow (N,y')$, f, g are $\mathcal{A}_{\mathcal{F}}$-*equivalent* if there exist diffeomorphism-germs $h : (M,x) \longrightarrow (M,x')$ and $k_1 : (D^q,0) \longrightarrow (D^q,0)$ such that $k_1 \circ \pi_1 \circ (\eta_1 \times \eta_2) \circ f = \pi_1 \circ (\eta_1' \times \eta_2') \circ g \circ h$, where $\eta_1 \times \eta_2 : (N,y) \longrightarrow (D^q \times D^r, 0)$, $\eta_1' \times \eta_2' : (N,y') \longrightarrow (D^q \times D^r, 0)$ are local charts and

$\pi_1 : (D^q \times D^r, 0) \longrightarrow (D^q, 0)$ is the canonical projection. We remark that the above definition does not depend on the choice of local charts $\eta_1 \times \eta_2$, $\eta_1' \times \eta_2'$.

We now define the notion of stability corresponding to each definition. We say that $f \in C^\infty(M, N; \mathcal{F})$ is *$\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$* if there exists a neighbourhood V_f of f in $C^\infty(M, N; \mathcal{F})$ such that f and g are $\mathcal{A}_\mathcal{F}$-equivalent for any $g \in V_f$. We also say that a smooth map-germ $f : (M, x) \longrightarrow (N, y)$ is *$\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$* if for any representative $\tilde{f} : U \longrightarrow V$ of f with a local chart $\eta_1 \times \eta_2 : V \longrightarrow D^q \times D^r$, there exists a neighbourhood $V_{\tilde{f}}$ of $\tilde{f}$ in $C^\infty(U, V; \mathcal{F}|V)$ such that for any $g \in V_{\tilde{f}}$ there exists a point $x' \in U$ with f and the germ $g : (M, x') \longrightarrow (N, g(x'))$ are $\mathcal{A}_\mathcal{F}$-equivalent. The notion of the $\mathcal{A}_\mathcal{F}$-stability modulo $\mathcal{F}$ for smooth map-germs is rather trivial because it is almost the same as the ordinary definition of the stability of smooth map-germs. However, the local stability of global mappings is important for the study of Haefliger foliations. We say that $f \in C^\infty(M, N; \mathcal{F})$ is *locally $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$* if the map-germ $f : (M, x) \longrightarrow (N, f(x))$ at any point $x \in M$ represented by f is $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$ as a smooth map-germ.

We now define the infinitesimal version of the above notions of stability. We have the normal bundle $T\mathcal{F}^\perp = TN/T\mathcal{F}$ of the foliation $\mathcal{F}$. We say that $f \in C^\infty(M, N; \mathcal{F})$ is *infinitesimally $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$* if for any $w \in \Gamma^\infty(f^*\mathcal{F}^\perp)$, there exist $u \in \Gamma^\infty(TM)$, $v \in \Gamma^\infty(T\mathcal{F}^\perp)$ such that $w = \pi_\mathcal{F} \circ df \circ u + v \circ f$, where $\pi_\mathcal{F} : TN \longrightarrow T\mathcal{F}^\perp$ is the canonical projection. We can also define the germ version of the infinitesimally $\mathcal{A}_\mathcal{F}$-stability modulo $\mathcal{F}$ as usual. A smooth map-germ $f : (M, x) \longrightarrow (N, y)$ is *infinitesimally $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$ as a germ* if for any $w \in \Gamma^\infty(f^*\mathcal{F}^\perp)_x$, there exist $u \in \Gamma^\infty(TM)_x$, $v \in \Gamma^\infty(T\mathcal{F}^\perp)_{f(x)}$ such that $w = \pi_\mathcal{F} \circ df \circ u + v \circ f$ for any $x \in M$, where we denote that $\Gamma^\infty(f^*\mathcal{F}^\perp)_x, u \in \Gamma^\infty(TM)_x, \Gamma^\infty(T\mathcal{F}^\perp)_{f(x)}$ are sets of germs of smooth sections of each vector bundle. Since the local $\mathcal{A}_\mathcal{F}$-group is the geometric subgroup of $\mathcal{A}$ and $\mathcal{K}$ in the sense of Damon[1], the infinitesimally $\mathcal{A}_\mathcal{F}$-stability modulo $\mathcal{F}$ implies the local $\mathcal{A}_\mathcal{F}$-stability modulo $\mathcal{F}$. The converse is also true by the jet transversality theorem modulo $\mathcal{F}$.

Proposition 4.1 *For $f \in C^\infty(M, N; \mathcal{F})$, f is locally $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$ if and only if f is infinitesimally $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$ in the local sense.*

Moreover, we have the following characterisation of local stability: Since the fibre of the jest bundle modulo $\mathcal{F}$ $J^k(M, N; \mathcal{F})$ can be identified with the ordinary jet space $J^k(m, q)$, we can consider the $\mathcal{K}$-orbit in $J^k(m, q)$. Here $\mathcal{K}$ is the group corresponding to the contact equivalence in the sense of Mather (c.f., [3, 5]). For any $z = j^k_\mathcal{F} f(x) \in J^k(M, N; \mathcal{F})$, we denote the $\mathcal{K}$-orbit through z by $\mathcal{K}^k(z)$. By the characterisation theorem of Mather[5], we have the following proposition:

Proposition 4.2 *Let $f : (M, x) \longrightarrow (N, y)$ be a smooth map-germ. Then f is $\mathcal{A}_\mathcal{F}$-stable modulo $\mathcal{F}$ if and only if $j^k_\mathcal{F} f$ is transversal to $\mathcal{K}^k(z)$ for $k \geq q + 1$.*

It follows from the transversality theorem modulo $\mathcal{F}$ and the celebrated results of Mather[5] that we have the following theorem.

Theorem 4.3 *If* $(\dim M, \operatorname{codim} \mathcal{F})$ *is in the nice dimensions in the sense of* Mather[5], *the set of locally $\mathcal{A}_\mathcal{F}$-stable mappings modulo $\mathcal{F}$ is open and dense in $C^\infty(M, N; \mathcal{F})$.*

Final remark We remark that Prof. J. P. Dufour pointed out that if we consider the foliation $\mathcal{F}$ given by the irrational flow on the torus $N = T^2$, there might be no global $\mathcal{A}_\mathcal{F}$-stable mappings modulo $\mathcal{F}$ from any compact manifold into N. This means that we need strong assumption on the foliation for the existence of global $\mathcal{A}_\mathcal{F}$-stable mappings modulo $\mathcal{F}$.

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