GENERIC SPECIAL CURVES

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GENERIC SPECIAL CURVES

Shyuichi IZUMIYA and Nobuko TAKEUCHI

We study generic properties of cylindrical helices and Bertrand curves as applications of singularity theory for plane curves and spherical curves.

1 Introduction

In [7] we have studied cylindrical helices and Bertrand curves from the view point as curves on ruled surfaces. These curves are different generalisation of circular helices which are called special curves in classical treatises [3]. Although these curves are called special curves, these curves form large classes of space curves. We can show that cylindrical helices can be constructed from plane curves and Bertrand curves can be constructed from spherical curves. Moreover, all these curves can be constructed by such a way (cf., Theorems 3.1, 4.1). These facts mean that we can study generic properties of these special curves as applications of singularity theory to plane curves and spherical curves. In generic differential geometry, we first consider model submanifolds which have standard properties like as lines, planes and spheres in $\mathbb{R}^3$ (cf., [1]). After that we estimate how "generic" submanifolds are different from or similar like as model submanifolds. Usually, we estimate the contact between "generic" submanifold and model submanifolds. Since both notions of cylindrical helices and Bertrand curves are generalisation of the notion of circular helices, we adopt circular helices as model manifolds here.

The basic notions and properties of space curves are reviewed in §2. In §3, we study generic properties of cylindrical helices as an application of Singularity theory to plane curves. We also study Bertrand curves in §4. We give examples of cylindrical helices and Bertrand curves in §5.

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All manifolds and maps considered here are of class C^∞ unless otherwise stated.

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2 Basic notions and properties

We now review some basic concepts on classical differential geometry of space curves in Euclidean space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, we denote $\boldsymbol{x} \cdot \boldsymbol{y}$ as the standard inner product. Let $\tilde{\gamma} : I \longrightarrow \mathbb{R}^3$ be a curve with $\dot{\tilde{\gamma}}(t) \neq 0$, where $\dot{\tilde{\gamma}}(t) = d\gamma/dt(t)$. We also denote the norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\|$. The *arc-length* of a curve $\tilde{\gamma}$, measured from $\tilde{\gamma}(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\tilde{\gamma}}(t)\| \, dt.$$

Then a parameter s is determined such that $\|\tilde{\gamma}'(s)\| = 1$, where $\tilde{\gamma}'(s) = d\tilde{\gamma}/ds(s)$. So we say that a curve $\tilde{\gamma}$ is *parameterised by the arc-length* if it satisfies $\|\tilde{\gamma}'(s)\| = 1$. Throughout in this paper, we denote s the arc-length of space curves. Let us denote $\boldsymbol{T}(s) = \tilde{\gamma}'(s)$ and we call $\boldsymbol{T}(s)$ a *unit tangent vector* of $\tilde{\gamma}$ at s. We define the *curvature* of $\tilde{\gamma}$ by $\kappa(s) = \|\tilde{\gamma}''(s)\|$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $\boldsymbol{N}(s)$ of the curve $\tilde{\gamma}$ at s is given by $\tilde{\gamma}''(s) = \kappa(s)\boldsymbol{N}(s)$. The unit vector $\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$ is called a *unit binormal vector* of the curve $\tilde{\gamma}$ at s. Then the following Frenet-Serret formula holds:

$$\left\{ \begin{array}{lcl} \boldsymbol{T}'(s) & = & \kappa(s)\boldsymbol{N}(s) \\ \boldsymbol{N}'(s) & = & -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s) \\ \boldsymbol{B}'(s) & = & -\tau(s)\boldsymbol{N}(s), \end{array} \right.$$

where $\tau(s)$ is the torsion of the curve $\tilde{\gamma}$ at s. For any unit speed curve $\tilde{\gamma} : I \longrightarrow \mathbb{R}^3$, we call $\boldsymbol{D}(s) = \tau(s)\boldsymbol{T}(s) + \kappa(s)\boldsymbol{B}(s)$ *the Darboux vector* of $\tilde{\gamma}$ (cf., [8], Section 5.2). We consider the normalisation of the Darboux vector $\boldsymbol{d}(s) = \boldsymbol{D}(s)/\|\boldsymbol{D}(s)\|$, which is called *the spherical Darboux image* or *the Darboux indicatrix* of $\tilde{\gamma}$.

For a general parameter t of a space curve $\tilde{\gamma}$, we can calculate the curvature and the torsion as follows:

$$\kappa(t) = \frac{\|\dot{\tilde{\gamma}}(t) \times \ddot{\tilde{\gamma}}(t)\|}{\langle \dot{\tilde{\gamma}}(t), \dot{\tilde{\gamma}}(t) \rangle^{\frac{3}{2}}}, \quad \tau(t) = \frac{\det\left(\dot{\tilde{\gamma}}(t)\ \ddot{\tilde{\gamma}}(t)\ \tilde{\gamma}^{(3)}(t)\right)}{\|\dot{\tilde{\gamma}}(t) \times \ddot{\tilde{\gamma}}(t)\|^2}.$$

A curve $\tilde{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called *a cylindrical helix* if the tangent lines of $\tilde{\gamma}$ make a constant angle with a fixed direction. It has been known that the curve $\tilde{\gamma}(s)$ is a cylindrical helix if and only if $(\tau/\kappa)(s) = \text{constant}$. If both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant, it is, of course, a cylindrical helix. We call it *a circular helix*. On the other hand, a curve $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *Bertrand curve* if there exists a curve $\bar{\gamma} : I \longrightarrow \mathbb{R}^3$ such that the principal normal lines of $\widetilde{\gamma}$ and $\bar{\gamma}$ at $s \in I$ are equal. In this case $\bar{\gamma}$ is called a *Bertrand mate* of $\widetilde{\gamma}$. Any plane curve γ is a Bertrand curve whose Bertrand mates are parallel curves of γ. For a Bertrand curve $\widetilde{\gamma}$, it has been known that if there exists a point $s_0 \in I$ such that $\tau(s_0) = 0$, then $\widetilde{\gamma}$ is a plane curve. Therefore, the torsion of a spatial Bertrand curve does not vanish. For a space curve $\widetilde{\gamma}(s)$ with $\tau(s) \neq 0$, the curve $\widetilde{\gamma}$ is a Bertrand curve if and only if there exist nonzero real numbers A, B such that $A\kappa(s) + B\tau(s) = 1$ for any $s \in I$ (cf., [3, 7]). So a circular helix is a Bertrand curve.

3 Plane curves and cylindrical helices

For a space curve, if the torsion always vanishes, then the curve is contained in a plane. In this case we denote the curve γ instead $\widetilde{\gamma}$ and the curvature κ_p instead of κ. For a plane curve $\gamma(t)$, we define a space curve

$$\widetilde{\gamma}(t) = \gamma(t) \pm \left(\cot\theta \int \|\dot{\gamma}(t)\| dt\right) \boldsymbol{a},$$

where θ is a constant number and $\boldsymbol{a}$ is a constant vector with $\langle \dot{\gamma}(t), \boldsymbol{a} \rangle = 0$ and $\|\boldsymbol{a}\| = 1$.

Theorem 3.1 *Under the above notation, $\widetilde{\gamma}$ is a cylindrical helix. Moreover, all cylindrical helices can be constructed by the above method.*

Proof. We now calculate the curvature and the torsion of $\widetilde{\gamma}(t)$. By definition, we have

$$\dot{\widetilde{\gamma}}(t) = \gamma(t) + \cot\theta \boldsymbol{a}, \ddot{\widetilde{\gamma}}(t) = \kappa_p(t)\boldsymbol{n}(t), \widetilde{\gamma}^{(3)}(t) = \dot{\kappa_p}(t)\boldsymbol{n}(t) - (\kappa_p(t))^2 \boldsymbol{t}(t).$$

Therefore, by the formulae of the curvature and the torsion for a general parameter, we can calculate that $\kappa(t) = |\kappa_p(t)| \sin^2\theta$ and the torsion is $\tau(t) = \kappa_p(t) \cot\theta \sin^2\theta$. Therefore, $\widetilde{\gamma}(t)$ is a cylindrical helix.

For the converse, let $\widetilde{\gamma}(s)$ be a cylindrical helix. In this case, the spherical Darboux image $\boldsymbol{d}(s)$ is constant, then we denote that $\boldsymbol{a} = \boldsymbol{d}(s)$. There exists a real number c such that $\tau(s) = c\kappa(s)$, so we choose θ that $\cot\theta = c$. Without loss of generality, we assume that $\sin\theta > 0$. Consider the curve $\gamma(s) = \widetilde{\gamma}(s) - \langle \widetilde{\gamma}(s), \boldsymbol{a} \rangle \boldsymbol{a}$, then $\langle \gamma(s), \boldsymbol{a} \rangle = \langle \gamma'(s), \boldsymbol{a} \rangle = 0$. This means that it is a plane curve. Since $\boldsymbol{a} = \dfrac{\tau(s)\boldsymbol{T}(s) + \kappa(s)\boldsymbol{B}(s)}{\sqrt{\tau(s)^2 + \kappa(s)^2}}$, we have $\langle \widetilde{\gamma}'(s), \boldsymbol{a} \rangle = \cos\theta$. Therefore, we have $\|\widetilde{\gamma}'(s) - \langle \widetilde{\gamma}'(s), \boldsymbol{a} \rangle \boldsymbol{a}\| = \sin\theta$. It follows that

$$\gamma(s) + \left(\cot\theta \int \|\gamma'(s)\| ds\right) \boldsymbol{a} = \widetilde{\gamma}(s) - \cos\theta \boldsymbol{a} + \cot\theta \sin\theta \boldsymbol{a} = \widetilde{\gamma}(s).$$

$\square$

We have the following corollary.

Corollary 3.2 *A plane curve γ is a circle if and only if the corresponding cylindrical helices are circular helices.*

Proof. By the calculation in the proof of the theorem, we have $\kappa(t) = \kappa_p(t)\sin^2\theta$ and $\tau(t) = \kappa_p(t)\cot\theta\sin^2\theta$. Both of these are constant if κ_p is constant. $\Box$

For space curves $\widetilde{\gamma}_i : I \longrightarrow \mathbb{R}^3$ $(i = 1, 2)$, we say that $\widetilde{\gamma}_1(t_0)$ and $\widetilde{\gamma}_2(t_0)$ *have at least* $(k+1)$*-point contact* if $\widetilde{\gamma}_1^{(p)}(t_0) = \widetilde{\gamma}_2^{(p)}(t_0)$ for $0 \leq p \leq k$. We also say that $\widetilde{\gamma}_1(t_0)$ and $\widetilde{\gamma}_2(t_0)$ *have* $(k+1)$*-point contact* if they have at least $(k+1)$-point contact and satisfy the relation $\widetilde{\gamma}_1^{(k+1)}(t_0) \neq \widetilde{\gamma}_2^{(k+1)}(t_0)$. Then we have the following fundamental proposition.

Proposition 3.3 *Let $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3$ be a regular curve with $\kappa \neq 0$. Then there exists an open interval $s_0 \in J \subset I$ and a unique cylindrical helix $\widetilde{\boldsymbol{\delta}} : J \longrightarrow \mathbb{R}^3$ such that $\widetilde{\boldsymbol{\delta}}(t_0) = \widetilde{\gamma}(t_0)$, the curvature of $\widetilde{\boldsymbol{\delta}}$ is $\kappa(t_0)$ and $\widetilde{\gamma}, \widetilde{\boldsymbol{\delta}}$ have at least* 3*-point contact at t_0.*

The proof of Proposition 3.3 is given by solving the natural equation $\kappa_\delta(t) = \kappa(t_0)$ and $\tau_\delta(t) = \tau(t_0)$ under the initial conditions $\widetilde{\boldsymbol{\delta}}(t_0) = \widetilde{\gamma}(t_0)$, $\dot{\widetilde{\boldsymbol{\delta}}}(t_0) = \dot{\widetilde{\gamma}}(t_0)$ and $\ddot{\widetilde{\boldsymbol{\delta}}}(t_0) = \ddot{\widetilde{\gamma}}(t_0)$. We call the circular helix $\widetilde{\boldsymbol{\delta}}$ *the osculating circular helix* of $\widetilde{\gamma}$ at s_0. We remark that if $\widetilde{\gamma}$ is a cylindrical helix, then the spherical Darboux image of the both curves is a constant vector $\boldsymbol{a}$. By the proof of Theorem 3.1, the curve $\boldsymbol{\delta}(s) = \widetilde{\boldsymbol{\delta}}(s) - \langle\widetilde{\boldsymbol{\delta}}(s), \boldsymbol{a}\rangle\boldsymbol{a}$ is the corresponding plane curve. Since $\widetilde{\boldsymbol{\delta}}$ is a circular helix, $\boldsymbol{\delta}$ is a circle. We can easily show that this circle has at least 3-point contact with $\gamma(s) = \widetilde{\gamma}(s) - \langle\widetilde{\gamma}(s), \boldsymbol{a}\rangle\boldsymbol{a}$ at $s_0 = s(t_0)$. This means that $\boldsymbol{\delta}$ is *the circle of the curvature* of γ at s_0. The centre of $\boldsymbol{\delta}$ is called it the centre of the curvature of γ at s_0. The locus of the centre of the curvature of a plane curve γ is called *the evolute of* the curve γ. The evolute of γ is given by $\boldsymbol{e}_\gamma(\sigma) = \gamma(\sigma) + (1/\kappa_p(\sigma))\boldsymbol{n}(\sigma)$, where $\boldsymbol{n}(\sigma)$ is the unit normal vector which is given by the anticlockwise $\pi/2$ rotation of the unit tangent vector $\boldsymbol{t}(\sigma)$ in the plane. For a cylindrical helix $\widetilde{\gamma}(s)$, we consider a curve

$$\boldsymbol{E}_{\widetilde{\gamma}}(s) = \widetilde{\gamma}(s) - \langle\widetilde{\gamma}(s), \boldsymbol{a}\rangle\boldsymbol{a} + \frac{\kappa(s)}{\tau(s)^2 + \kappa(s)^2}\boldsymbol{N}(s).$$

We can show that $\langle\boldsymbol{E}_{\widetilde{\gamma}}(s), \boldsymbol{a}\rangle = \langle\boldsymbol{E}'_{\widetilde{\gamma}}(s), \boldsymbol{a}\rangle = 0$. This means that $\boldsymbol{E}_{\widetilde{\gamma}}(s)$ is a plane curve. We can also show that $\boldsymbol{E}_{\widetilde{\gamma}}(s)$ is the evolute of the curve $\gamma(s) = \widetilde{\gamma}(s) - \langle\widetilde{\gamma}(s), \boldsymbol{a}\rangle\boldsymbol{a}$. We call $\boldsymbol{E}_{\widetilde{\gamma}}(s)$ *the planer evolute of* the cylindrical helix $\widetilde{\gamma}(s)$. We also consider a ruled surface defined by $\boldsymbol{PN}_{\widetilde{\gamma}}(s, u) = \widetilde{\gamma}(s) + u\boldsymbol{N}(s)$ which is called *the principal normal surface* of $\widetilde{\gamma}$. This surface is deeply related to Bertrand curves as we have shown in [7]. Here, we consider the relation to cylindrical helices. For $\boldsymbol{PN}_{\widetilde{\gamma}}(s, u)$, we define the orthogonal projection to a plane by

$$\pi_{\widetilde{\gamma},N}(s, u) = \widetilde{\gamma}(s) - \langle\boldsymbol{a}, \widetilde{\gamma}(s)\rangle\boldsymbol{a} + u\boldsymbol{N}(s).$$

By our construction, we have the relation $\pi_{\widetilde{\gamma},N}(s, u) = \gamma(s) + u\varepsilon\boldsymbol{n}(s)$, where $\varepsilon = \pm 1$.

On the other hand, there is a beautiful application of singularity theory to plane curves (cf., [1]). One of the results is to classify the singularities of the evolute of a generic plane curve and study the geometric meaning of such a point. We summarise the result as follows: Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed plane curve with the curvature $\kappa_p(\sigma)$. We denote σ as the arc-length of the plane curve here.

Theorem 3.4 [1] *Under the same notations as in the above paragraph, we have the following:*

(1) *The evolute* $\mathbf{e}_\gamma$ *is the critical value set of the mapping* $\gamma(s) + u\varepsilon \boldsymbol{n}(s)$.

(2) *The evolute* $\mathbf{e}_\gamma$ *is regular at* σ_0 *if* $\kappa_p'(\sigma_0) \neq 0$.

(3) *The evolute* $\mathbf{e}_\gamma$ *is locally diffeomorphic to the ordinary cusp if and only if* $\kappa_p'(\sigma_0) = 0$ *and* $\kappa_p''(\sigma_0) \neq 0$.

Here, the ordinary cusp is the plane curve given by $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$.

We remark that the condition $\kappa_p'(\sigma_0) = 0$ and $\kappa_p''(\sigma_0) \neq 0$ is equivalent to the condition that the curve $\boldsymbol{\gamma}$ and the circle of the curvature at $\boldsymbol{\gamma}(\sigma_0)$ have 4-point contact. This point is called *the vertex of* $\boldsymbol{\gamma}$. Since we have $\kappa(s) = |\kappa_p(s)| \sin^2\theta$ and $\tau(s) = \kappa_p(s)\cot\theta\sin^2\theta$, we conclude the following proposition as a corollary of Theorem 3.4.

Proposition 3.5 *Let* $\widetilde{\gamma}(s)$ *be a cylindrical helix. Then we have the following:*

(1) *The planer evolute* $\boldsymbol{E}_{\widetilde{\gamma}}(s)$ *is the critical value set of the orthogonal projection* $\pi_{\widetilde{\gamma},N}$ *of the principal normal surface* $\boldsymbol{PN}_{\widetilde{\gamma}}$ *of* $\widetilde{\gamma}$.

(2) *The planer evolute* $\boldsymbol{E}_{\widetilde{\gamma}}(s)$ *is locally diffeomorphic to the ordinary cusp if and only if* $\kappa'(s_0) = \tau'(s_0) = 0$, $\kappa''(s_0) \neq 0$ *and* $\tau''(s_0) \neq 0$.

The last condition is equivalent to the condition that the cylindrical helix $\widetilde{\gamma}$ *and the osculating circular helix at* $\widetilde{\gamma}(s_0)$ *have 4-point contact.*

In [1] it has been shown that number of points with the condition $\dot{\kappa}_p(\sigma_0) = 0$ and $\ddot{\kappa}_p(\sigma_0) \neq 0$ is finite and there are no points with $\dot{\kappa}_p(\sigma_0) = \ddot{\kappa}_p(\sigma_0) = 0$ for a "generic" plane curve. By Theorem 3.1, we might say that singularities of the planer evolute of a "generic" cylindrical helix are ordinary cusps.

4 Spherical curves and Bertrand curves

In this section we describe the method to construct Bertrand curves from spherical curves. Let $\boldsymbol{\gamma} : I \longrightarrow S^2$ be a unit speed spherical curve. In this section we denote σ as the arc-length parameter of $\boldsymbol{\gamma}$. Let us denote $\boldsymbol{t}(\sigma) = \dot{\boldsymbol{\gamma}}(\sigma)$, and we call $\boldsymbol{t}(\sigma)$ a *unit tangent vector* of $\boldsymbol{\gamma}$ at σ, where $\dot{\boldsymbol{\gamma}} = d\boldsymbol{\gamma}/d\sigma$. We now set a vector $\boldsymbol{s}(\sigma) = \boldsymbol{\gamma}(\sigma) \times \boldsymbol{t}(\sigma)$. By definition, we have an orthonormal frame $\{\boldsymbol{\gamma}(\sigma), \boldsymbol{t}(\sigma), \boldsymbol{s}(\sigma)\}$ along γ. This frame is called the *Sabban frame of* $\boldsymbol{\gamma}$ [8]. Then we have the following *spherical Frenet-Serret formula* of $\boldsymbol{\gamma}$:

$$\begin{cases} \dot{\boldsymbol{\gamma}}(\sigma) &= \boldsymbol{t}(\sigma) \\ \dot{\boldsymbol{t}}(\sigma) &= \boldsymbol{\gamma}(\sigma) + \kappa_g(\sigma)\boldsymbol{s}(\sigma) \\ \dot{\boldsymbol{s}}(\sigma) &= -\kappa_g(s)\boldsymbol{t}(\sigma), \end{cases}$$

where $\kappa_g(\sigma)$ is *the geodesic curvature* of the curve γ in S^2 which is given by $\kappa_g(\sigma) = \det(\boldsymbol{\gamma}(\sigma)\ \boldsymbol{t}(\sigma)\ \dot{\boldsymbol{t}}(\sigma))$. We now define a space curve

$$\widetilde{\gamma}(\sigma) = a\int \gamma(\sigma)d\sigma + a\cot\theta \int \boldsymbol{s}(\sigma)d\sigma,$$

where a, θ are constant numbers. The following theorem is the key in this section:

Theorem 4.1 *Under the above notation, $\tilde{\gamma}$ is a Bertrand curve. Moreover, all Bertrand curves can be constructed by the above method.*

Proof. We now calculate the curvature and the torsion of $\tilde{\gamma}(\sigma)$. By definition, we have

$$\dot{\tilde{\gamma}}(\sigma) = a(\boldsymbol{\gamma}(\sigma) + \cot\theta \boldsymbol{s}(\sigma)), \quad \ddot{\tilde{\gamma}}(\sigma) = a(1 - \cot\theta\kappa_g(\sigma))\boldsymbol{t}(\sigma),$$

and

$$\tilde{\gamma}^{(3)}(\sigma) = -a\cot\theta\dot{\kappa}_g(\sigma)\boldsymbol{t}(\sigma) + a(1 - \cot\theta\kappa_g(\sigma))(-\boldsymbol{\gamma}(\sigma) + \kappa_g(\sigma)\boldsymbol{s}(\sigma)).$$

Therefore, by the formulae of the curvature and the torsion for a general parameter, we can calculated as follows:

$$\kappa(\sigma) = \varepsilon\frac{\sin^2\theta(1 - \kappa_g(\sigma)\cot\theta)}{a} \quad \tau(\sigma) = \frac{\sin^2\theta(\kappa_g(\sigma) + \cot\theta)}{a},$$

where $\varepsilon = \pm 1$. It follows from these formulae that $a(\varepsilon\kappa(\sigma) + \cot\theta\tau(\sigma)) = 1$, so that $\tilde{\gamma}(s)$ is a Bertrand curve.

For the converse, let $\tilde{\gamma}(s)$ be a Bertrand curve. There exist non zero real number A, B such that $A\kappa(s) + B\tau(s) = 1$. We put $a = A$, $\cot\theta = B/a$. We assume that $a > 0$ and choose $\varepsilon = \pm 1$ with $\varepsilon\sin\theta/a > 0$. We now define a spherical curve

$$\boldsymbol{\gamma}(s) = \varepsilon(\sin\theta\boldsymbol{T}(s) - \cos\theta\boldsymbol{B}(s)).$$

Then we have

$$\boldsymbol{\gamma}'(s) = \varepsilon(\kappa(s) + \cot\theta\tau(s))\boldsymbol{N}(s) = \frac{\varepsilon}{a}\sin\theta\boldsymbol{N}(s).$$

Let σ be the arc-length parameter of $\boldsymbol{\gamma}$, then we have $d\sigma/ds = (\varepsilon/a)\sin\theta$. We also have

$$a\boldsymbol{\gamma}(s)\frac{d\sigma}{ds} = a\varepsilon(\sin\theta\boldsymbol{T}(s) - \cos\theta\boldsymbol{B}(s))\frac{\varepsilon}{a}\sin\theta = \sin\theta(\sin\theta\boldsymbol{T}(s) - \cos\theta\boldsymbol{B}(s))$$

and

$$\begin{aligned} a\cot\theta\boldsymbol{\gamma}(s) \times \frac{d\boldsymbol{\gamma}}{d\sigma}\frac{d\sigma}{ds} &= a\cot\theta\varepsilon(\sin\theta\boldsymbol{T}(s) - \cos\theta\boldsymbol{B}(s)) \times \frac{\varepsilon}{a}\sin\theta\boldsymbol{N}(s) \\ &= \cos\theta(\sin\theta\boldsymbol{B}(s) + \cos\theta\boldsymbol{T}(s)). \end{aligned}$$

Since $\boldsymbol{s} = \boldsymbol{\gamma} \times \dfrac{d\boldsymbol{\gamma}}{d\sigma}$, we have

$$\begin{aligned} & a\int\boldsymbol{\gamma}(\sigma)d\sigma + a\cot\theta\int\boldsymbol{s}(\sigma)d\sigma \\ &= \int\sin\theta(\sin\theta\boldsymbol{T}(s) - \cos\theta\boldsymbol{B}(s))ds + \int\cos\theta(\sin\theta\boldsymbol{B}(s) + \cos\theta\boldsymbol{T}(s))ds \\ &= \int\boldsymbol{T}(s) = \boldsymbol{\gamma}(s). \end{aligned}$$

$\square$

We now consider the contact between Bertrand curves and circular helices. We have the following corollary of Theorem 4.1.

Corollary 4.2 *The spherical curve γ is a circle if and only if the corresponding Bertrand curves are circular helices.*

Proof. By the proof of Theorem 4.1, we have

$$\dot{\kappa}(\sigma) = \frac{-\varepsilon\dot{\kappa}_g(\sigma)\cos\theta}{a}, \quad \dot{\tau}(\sigma) = \frac{\sin^2\theta\dot{\kappa}_g(\sigma)}{a}.$$

The spherical curve γ is a circle if and only if $\dot{\kappa}_g(\sigma) \equiv 0$. This condition is equivalent to the condition that $\dot{\kappa}(\sigma) = \dot{\tau}(\sigma) \equiv 0$. $\square$

Let $\gamma : I \longrightarrow S^2$ be a spherical curve with $\kappa_g \neq 0$. For any $\sigma_0 \in I$, we consider a unit vector

$$\boldsymbol{e}_0 = \frac{1}{\sqrt{\kappa_g(\sigma_0)^2 + 1}}(\kappa_g(\sigma_0)\boldsymbol{\gamma}(\sigma_0) + \boldsymbol{s}(\sigma_0)).$$

and a circle $S^1(\boldsymbol{e}_0, c_0) = \{\boldsymbol{x} \in S^2 \mid \langle \boldsymbol{x}, \boldsymbol{e}_0 \rangle = c_0\}$, where $c_0 = \dfrac{\kappa_g(\sigma_0)}{\sqrt{\kappa_g(\sigma_0)^2 + 1}}$. Then we define the height function $h_{\boldsymbol{e}_0} : S^2 \longrightarrow \mathbb{R}$ by $h_{\boldsymbol{e}_0}(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{e}_0 \rangle - c_0$. We can easily calculate that

$$h_{\boldsymbol{e}_0} \circ \gamma(\sigma_0) = (h_{\boldsymbol{e}_0} \dot{\circ} \gamma)(\sigma_0) = (h_{\boldsymbol{e}_0} \ddot{\circ} \gamma)(\sigma_0) = 0.$$

This means that $S^1(\boldsymbol{e}_0, c_0)$ is the circle of curvature of γ at $\gamma(\sigma_0)$. Moreover, if we calculate $(h_{\boldsymbol{e}_0} \circ \gamma)^{(3)}(\sigma_0)$ and $(h_{\boldsymbol{e}_0} \circ \gamma)^{(4)}(\sigma_0)$, then we can show that $S^1(\boldsymbol{e}_0, c_0)$ and γ have 4-point contact at $\gamma(\sigma_0)$ if and only if $\dot{\kappa}_g(\sigma_0) = 0$ and $\ddot{\kappa}_g(\sigma_0) \neq 0$. These arguments induce that the centre of curvature of γ at $\gamma(\sigma_0)$ is given by $\boldsymbol{e}_0$. We call the locus of the centre of curvature of γ *the spherical evolute* of γ, which is given by

$$\boldsymbol{e}_\gamma(\sigma) = \frac{1}{\sqrt{\kappa_g(\sigma)^2 + 1}}(\kappa_g(\sigma)\boldsymbol{\gamma}(\sigma) + \boldsymbol{s}(\sigma)).$$

Then we have the following proposition.

Proposition 4.3 *Let $\gamma : I \longrightarrow S^2$ be a spherical curve and $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3$ be a Bertrand curve corresponding to γ. Then the spherical Darboux image of $\widetilde{\gamma}$ is equal to the spherical evolute of γ.*

Proof. By the proof of Theorem 4.1, we have

$$\kappa(\sigma) = \varepsilon\frac{\sin^2\theta(1 - \kappa_g(\sigma)\cot\theta)}{a} \quad \tau(\sigma) = \frac{\sin^2\theta(\kappa_g(\sigma) + \cot\theta)}{a}$$

and

$$\boldsymbol{T}(\sigma) = a(\boldsymbol{\gamma}(\sigma) + \cot\theta\boldsymbol{s}(\sigma))\frac{d\sigma}{ds}, \; \boldsymbol{N}(\sigma) = \varepsilon\boldsymbol{t}(\sigma).$$

Then we have $\boldsymbol{B}(\sigma) = \varepsilon a(d\sigma/ds)(\boldsymbol{s}(\sigma) - \cot\theta\boldsymbol{\gamma}(\sigma))$. We can easily show that

$$\boldsymbol{D}(\sigma) = \tau(\sigma)\boldsymbol{T}(\sigma) + \kappa(\sigma)\boldsymbol{B}(\sigma) = \frac{d\sigma}{ds}(\boldsymbol{s}(\sigma) + \kappa_g(\sigma)\boldsymbol{\gamma}(\sigma)).$$

Therefore, we have $\boldsymbol{d}(\sigma) = \boldsymbol{e}_\gamma(\sigma)$. $\square$

Corresponding to the Darboux vector of a general space curve $\widetilde{\gamma}$, we have the following developable surface:

$$RG_{\widetilde{\gamma}}(s,u) = \boldsymbol{B}(s) + u\boldsymbol{T}(s).$$

We have called $RG_{\widetilde{\gamma}}$ *the rectifying Gaussian surface* of $\widetilde{\gamma}$ in [6]. Nevertheless, we call it *the Darboux developable* of $\widetilde{\gamma}$ here. The reason is that the surface has always singularities and the set of singular points are the image of $\widetilde{\boldsymbol{D}}(s) = (\tau/\kappa)(s)\boldsymbol{T}(s) + \boldsymbol{B}(s)$ which is called *the modified Darboux vector* of $\widetilde{\gamma}$. In [6] we have studied singularities of the spherical Darboux image and the Darboux developable of a space curve. One of the main results in [6] is as follows:

Theorem 4.4 *Let $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3$ be a regular space curve with $\kappa \neq 0$ and $\tau \neq 0$. Then we have the following:*

(1) *The spherical Darboux image* $\mathbf{d}$ *of* $\widetilde{\gamma}$ *is locally diffeomorphic to the ordinary cusp C at $\mathbf{d}(s_0)$ if and only if $(\tau/\kappa)'(s_0) = 0$ and $(\tau/\kappa)''(s_0) \neq 0$.*

(2) (a) *The Darboux developable of $\widetilde{\gamma}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at $u_0\boldsymbol{T}(s_0) + \boldsymbol{B}(s_0)$ if and only if $u_0 = (\tau/\kappa)(s_0)$ and $(\tau/\kappa)'(s_0) \neq 0$.*

(b) *The Darboux developable of $\widetilde{\gamma}$ is locally diffeomorphic to the swallowtail SW at $u_0\boldsymbol{T}(s_0) + \boldsymbol{B}(s_0)$ if and only if $u_0 = (\tau/\kappa)(s_0)$, $(\tau/\kappa)'(s_0) = 0$ and $(\tau/\kappa)''(s_0) \neq 0$*

Here, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v, (u,v) \in \mathbb{R}^2 \}$ is the swallowtail.

Remarks The assertion (1) had been known by Fenchel[4]. A point $\widetilde{\gamma}(t_0)$ of a curve $\widetilde{\gamma}$ in $\mathbb{R}^3$ is called *twisting* if $\dot{\overline{\tau/\kappa}}(t_0) = 0$. Such a point is systematically studied by Uribe-Vargas[9] and Romero-Fuster and Sanabria-Codesal[2]. In [5], we have shown that the Darboux developable of $\widetilde{\gamma}$ is locally diffeomorphic to the cuspidal cross cap CCR at $u_0\boldsymbol{T}(s_0) + \boldsymbol{B}(s_0)$ if and only if $u_0 = 0$, $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$. Here, $CCR = \{(x_1, x_2, x_3) \mid x_2^2 = x_1^2 x_3^3 \}$. As we already mentioned that if there exists a point s_0 of a Bertrand curve with $\tau(s_0) = 0$, then the curve must be a plane curve. Therefore, the cuspidal crosscap does not appear as singularities for Darboux developables of Bertrand curves.

In the construction of the Bertrand curve in Theorem 4.1, suppose that we fix real numbers a, θ. Let $\gamma_i : I \longrightarrow S^2$ $(i = 1, 2)$ be spherical curves, then γ_1, γ_2 have $(k+1)$-point contact at σ_0 if and only if $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ have $(k+1)$-contact at σ_0. This means that if $S^1(\boldsymbol{e}_0, c_0)$ is the circle of curvature of γ at $\gamma(\sigma_0)$, then the corresponding circular helix is the osculating circular helix of $\widetilde{\gamma}$ at $\widetilde{\gamma}(\sigma_0)$. Moreover, we have the following lemma:

Lemma 4.5 *Let $\gamma : I \longrightarrow S^2$ be a spherical curve and $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3$ be one of the corresponding Bertrand curves with $\kappa \neq 0$. Then the following three conditions are equivalent:*

(1) $\dot{\kappa}_g(\sigma_0) = 0$ *and* $\ddot{\kappa}_g(\sigma_0) \neq 0$.

(2) $\dot{\overline{(\tau/\kappa)}}(\sigma_0) = 0$ *and* $\ddot{\overline{(\tau/\kappa)}}(\sigma_0) \neq 0$.

(3) $\dot{\kappa}(\sigma_0) = \dot{\tau}(\sigma_0) = 0$, $\ddot{\kappa}(\sigma_0) \neq 0$ *and* $\ddot{\tau}(\sigma_0) \neq 0$.

Proof. By the proof of Theorem 4.1, we have

$$\kappa(\sigma) = \varepsilon\frac{\sin^2\theta(1-\kappa_g(\sigma)\cot\theta)}{a} \quad \tau(\sigma) = \frac{\sin^2\theta(\kappa_g(\sigma)+\cot\theta)}{a}.$$

It follows from these formulae that the condition (1) is equivalent to the condition (3). Since $\kappa \neq 0$, we have $\sin\theta - \kappa_g(\sigma)\cos\theta \neq 0$. Therefore, we have

$$(\tau/\kappa)(\sigma) = \frac{\sin\theta\kappa_g(\sigma)+\cos\theta}{\varepsilon(\sin\theta-\kappa_g(\sigma)\cos\theta)}.$$

Then we can calculate that

$$(\dot{\tau/\kappa})(\sigma) = \frac{\varepsilon\dot{\kappa}_g(\sigma)}{(\sin\theta-\kappa_g(\sigma)\cos\theta)^2},$$

and

$$(\ddot{\tau/\kappa})(\sigma) = \frac{\varepsilon\{\ddot{\kappa}_g(\sigma)(\sin\theta-\kappa_g(\sigma)\cos\theta)^2 + 2\dot{\kappa}_g(\sigma)^2(\sin\theta-\kappa_g(\sigma)\cos\theta)\cos\theta\}}{(\sin\theta-\kappa_g(\sigma)\cos\theta)^4}.$$

Thus, the condition (1) is equivalent to the condition (2). □

Eventually, we have the following theorem:

Theorem 4.6 *Let $\tilde{\gamma}(s)$ be a Bertrand curve with $\kappa \neq 0$ and $\tau \neq 0$. Then we have the following:*

(1) *The spherical Darboux image $\boldsymbol{d}$ of $\tilde{\gamma}$ is locally diffeomorphic to the ordinary cusp C at $\boldsymbol{d}(s_0)$ if and only if $\kappa'(s_0) = \tau'(s_0) = 0$, $\kappa''(s_0) \neq 0$ and $\tau''(s_0) \neq 0$.*

(2) (a) *The Darboux developable of $\tilde{\gamma}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at $u_0\boldsymbol{T}(s_0) + \boldsymbol{B}(s_0)$ if and only if $u_0 = (\tau/\kappa)(s_0)$ and $\kappa'(s_0) \neq 0$ and $\tau'(s_0) \neq 0$.*

(b) *The Darboux developable of $\tilde{\gamma}$ is locally diffeomorphic to the swallowtail SW at $u_0\boldsymbol{T}(s_0) + \boldsymbol{B}(s_0)$ if and only if $u_0 = (\tau/\kappa)(s_0)$, $\kappa'(s_0) = \tau'(s_0) = 0$, $\kappa''(\sigma_0) \neq 0$ and $\tau''(s_0) \neq 0$.*

The conditions (1) *and* (2),(b) *are equivalent to the condition that the Bertrand curve $\tilde{\gamma}$ and the osculating circular helix at $\tilde{\gamma}(s_0)$ have 4-point contact.*

We can prove that the number of points with the condition $\dot{\kappa}_g(\sigma_0) = 0$ and $\ddot{\kappa}_g(\sigma_0) \neq 0$ is finite and there are no points with $\dot{\kappa}_g(\sigma_0) = \ddot{\kappa}_g(\sigma_0) = 0$ for a "generic" spherical curve by the similar method like as the case for plane curves in [1]. By Theorem 4.1, we might say that Theorem 4.6 asserts a "generic" properties for Bertrand curves.

5 Examples

In this section we give examples of cylindrical helices and Bertrand curves by using the method in the previous sections. We also draw pictures of the planer evolute a cylindrical helix and the Darboux developable of a Bertrand curve.

An example of cylindrical helices and it's picture is given as follows.

Example 5.1 Applying a Euclidean motion, we can move any plane to the $(x_1, x_2, 0)$-plane in $\mathbb{R}^3$. We now consider in this situation. If we choose a plane curve $\gamma(t) = (x_1(t), x_2(t), 0)$ and $\boldsymbol{a} = (0, 0, 1)$, the corresponding cylindrical helix is

$$\widetilde{\gamma}(t) = (x_1(t), x_2(t), c\int \sqrt{\dot{x_1}(t)^2 + \dot{x_2}(t)^2 dt}),$$

for a given constant number c.

We now consider a plane curve $\gamma(\tau) = (2\cos\tau, \sin\tau, 0)$ then we have a cylindrical helix $\widetilde{\gamma}(\tau) = (2\cos\tau, \sin\tau, E(\tau, -3))$, where $E(\tau, m) = \int_0^\tau \sqrt{1 - m\sin^2\sigma}\,d\sigma$. The planer evolute of $\widetilde{\gamma}$ is $\boldsymbol{E}_\gamma(\theta) = ((3/2)\cos\theta(1 - 3\sin^2\theta), -3\sin\theta, 0)$. These curves are depicted as in Fig.1.

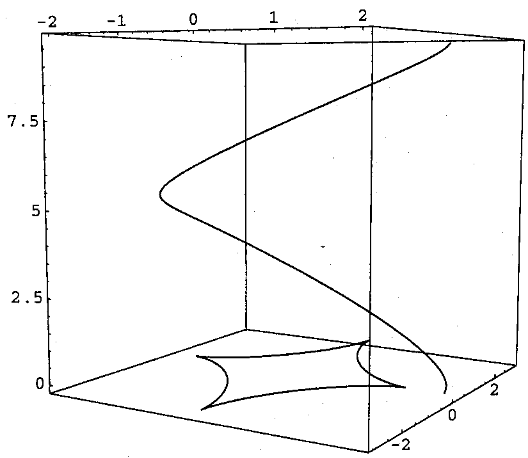

Fig.1.

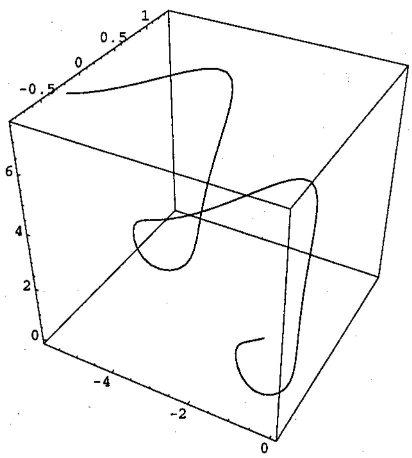

Fig.2.

We also give an example of Bertrand curves.

Example 5.2 We consider a spherical curve $\gamma(\tau) = (\sin\tau, \sin\tau\cos\tau, \cos^2\tau)$. By using the method in §4, we have the following Bertrand curve ($a = 1$ and $\cot\theta = 1$):

$$\begin{aligned}\widetilde{\gamma}(\tau) = \quad & \left(-\cos\tau + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}F(\tau, \frac{1}{2}) + \frac{1}{2}\left(-4E(\tau, \frac{1}{2}) + 3F(\tau, \frac{1}{2})\right)\right),\right. \\ & -2\arctan\left(\frac{\sqrt{2}\sin\tau}{\sqrt{3 + \cos(2\tau)}}\right) - \frac{1}{4}\cos(2\tau) + \frac{\sqrt{3 + \cos(2\tau)}\sin\tau}{2\sqrt{2}}, \\ & \left.\frac{\tau}{2} - \frac{\cos\tau\sqrt{3 + \cos(2\tau)}}{2\sqrt{2}} + \frac{3}{2}\log(\sqrt{2}\cos\tau + \sqrt{3 + \cos(2\tau)}) + \frac{1}{4}\sin(2\tau)\right),\end{aligned}$$

where $F(\tau, m) = \int_0^\tau \frac{d\sigma}{1 - m\sin^2\sigma}$. Pictures of the curve is given in Fig.2. Moreover, we can draw both of pictures of the spherical Darboux image of γ and γ itself (cf., Fig.3). The Fig. 4 is a part of the Darboux developable of $\widetilde{\gamma}$. We can observe the swallowtail.

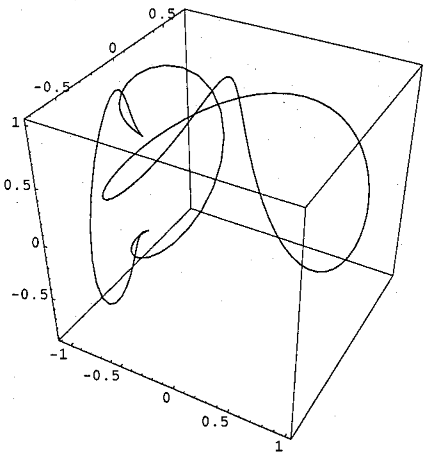

Fig.3.

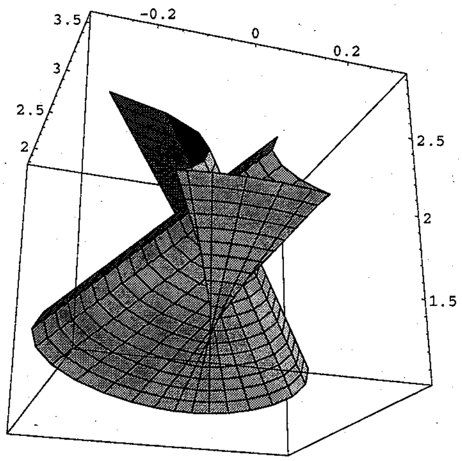

Fig.4.

REFERENCES

[1] J.W. Bruce and P.J.Giblin. *Curves and Singularities, 2nd. ed.* Cambridge Univ. Press, Cambridge, (1992).

[2] M.C. Romero-Fuster and E. Sanabria-Codesal. *Generalized helices, twistings and flattenings of curves in n-space.* preprint (2000).

[3] M. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall, New Jersey (1976).

[4] W. Fenchel. *The differential Geometry of Closed Space Curves.* Bull. Am. Math.Soc. 57 pp.44–54 (1951).

[5] S. Izumiya. *Generating families of developable surfaces in $\mathbb{R}^3$.* preprint Isaac Newton Institute for Mathematical Sciences, Cambridge University, preprint series, NI-00024-SGT (2000)

[6] S. Izumiya, H. Katsumi and T. Yamasaki. *The rectifying developable and the spherical Darboux image of a space curve.* Geometry and topology of caustics-Caustics '98- Banach Center Publications, 50 pp. 137–149 (1999) .

[7] S.Izumiya and N. Takeuchi. *Special curves and ruled surfaces.* preprint Isaac Newton Institute for Mathematical Sciences, Cambridge University, preprint series, NI-00023-SGT (2000).

[8] J. Koenderink. *Solid shape.* MIT Press, Cambridge, MA, (1990).

[9] R. Uribe-Vargas. *On the Twistings of Closed Curves in $\mathbb{R}^n$ and the Focal Curve, the Darboux Vertices and the Spherical Indicatrices of a Closed Curve in the Euclidean Space $\mathbb{R}^3$.*. preprint (2000).

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