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The Composition Operators On Weighted Bloch Space

Rikio Yoneda

Abstract

We will characterize the boundedness and compactness of the composition operators on weighted Bloch space $B_{\log} = \{f \in H(D) : \sup_{z \in D} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |f'(z)| < +\infty\}$, where $H(D)$ be the class of all analytic functions on D .

Key Words and Phrases : composition operator, weighted Bloch space, compactness, boundedness.

§1. Introduction

Let D denote the open unit disk in complex plane C and ∂D denote the unit circle in C . For $1 \leq p < +\infty$, the Lebesgue space $L^p(D, dA)$ is defined to be the Banach space of Lebesgue measurable functions on D with

$$\|f\|_{L^p(dA)} := \left(\int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}} < +\infty,$$

where $dA(z)$ is the normalized area measure on D . The Bergman space $L_a^p(D)$ is defined to be the subspace of $L^p(D, dA)$ consisting of analytic functions. The orthogonal projection from $L^2(D, dA)$ onto $L_a^2(D)$ is called the Bergman projection and is denoted by P . Then given a function $f \in L^2(D, dA)$, an operator $H_f : L_a^2(D) \rightarrow (L_a^2(D))^\perp$ is defined by $H_f g = (I - P)(fg)$, $g \in L_a^2(D)$. The operator H_f is called the Hankel operator on the Bergman space with symbol f .

For $0 < p < +\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\|f\|_p := \left(\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty.$$

The orthogonal projection from $L^2(\partial D)$ onto H^2 is called the Szegő projection and is also denoted by P . Then given a function $f \in L^2(\partial D)$, an operator $H_f : H^2 \rightarrow (H^2)^\perp$ is defined by $H_f g = (I - P)(fg)$, $g \in H^2$. The operator H_f is called the Hankel operator on the Hardy space with symbol f .

The Bloch space B of D is defined to be the space of analytic functions f on D such that

$$\|f\|_B := \sup_{z \in D} (1 - |z|^2) |f'(z)| < +\infty.$$

The space of analytic functions on D of bounded mean oscillation , denoted by $BMOA$, consists of functions f in H^2 for which

$$\| f \|_{BMOA} := \sup_I \frac{2}{|I|} \int_{S(I)} |f'(z)|^2 \log \frac{1}{|z|} dA(z) < +\infty,$$

where I denotes a subarc of ∂D , $|I|$ denotes the arclength measure of I , and $S(I) = \{re^{i\theta} : 1-r \leq |I|, e^{i\theta} \in I\}$ (see [6]).

And the following are the classical results for the spaces B and $BMOA$ (see [9]) : For $f \in L_a^2$, Hankel operator $H_{\bar{f}} : L_a^2 \rightarrow (L_a^2)^\perp$ is bounded if and only if $f \in B$. And for $f \in H^2$, Hankel operator $H_{\bar{f}} : H^2 \rightarrow (H^2)^\perp$ is bounded if and only if $f \in BMOA$.

It is well-known that $BMOA \subset B$ (see [6] or [9]). It is called that a holomorphic self map φ of D has Bloch-to- $BMOA$ pullback property if $f \circ \varphi \in BMOA$ for all $f \in B$. In [3], B.R.Choe, W.Ramey, and D.Ullrich studied Bloch-to- $BMOA$ pullbacks.

On the other hand , in [1], K.R.M. Attle proved that for $f \in L_a^2(D)$, the Hankel operator $L_a^1 \rightarrow L^1$ is bounded if and only if

$$\| f \|_{B_{\log}} := \sup_{z \in D} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |f'(z)| < +\infty,$$

and in [4], Cima and Stegenga proved that the Hankel operator $H_f : H^1 \rightarrow H^1$, with an analytic symbol f (see their paper for the definition of this Hankel operator) is bounded if and only if

$$\| f \|_{BMOA_{\log}} := \sup_I \frac{\left(\log \frac{2}{|I|} \right)^2}{|I|} \int_{S(I)} |f'(z)|^2 \log \frac{1}{|z|} dA(z) < +\infty.$$

Accordingly, we define the weighted Bloch space B_{\log} of D to be the space of analytic functions f on D such that $\| f \|_{B_{\log}} < +\infty$. And we define the space of analytic functions on D of weighted bounded mean oscillation , denoted by $BMOA_{\log}$, to be the space of functions f in H^2 for which $\| f \|_{BMOA_{\log}} < +\infty$.

By a direct calculation, we see that $BMOA_{\log} \subset B_{\log}$ (see [6] or [9]). We will also say that a holomorphic self map φ of D has B_{\log} -to- $BMOA_{\log}$ pullback property if $f \circ \varphi \in BMOA_{\log}$ for all $f \in B_{\log}$.

Let φ denote a holomorphic function taking D into D and C_φ denote the composition operator with φ . Then it is trivial that C_φ is bounded on B (see [7]). In [7], K.Madigan and A.Matheson showed the following result about the compactness of C_φ on B :

Theorem A. Let φ be a holomorphic function taking D into D . Then C_φ is compact on B if and only if for every $\epsilon > 0$, there exists $0 < r < 1$ such that

$$\sup_{|\varphi(z)| > r} \left(\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| \right) < \epsilon.$$

In this paper, we study the boundedness and compactness of the composition operator on B_{\log} . Moreover by using B_{\log} -to- $BMOA_{\log}$ pullback property, we also study the composition operator on $BMOA_{\log}$. Throughout this paper, C_i, K_i for $i = 0, 1, 2$, C, K will denote positive constant whose value is not necessary the same at each occurrence.

§2. The composition operator on the weighted Bloch space B_{\log}

In this section, we study the boundedness and compactness of the composition operator on B_{\log} .

Theorem 1. Let φ be a holomorphic function taking D into D . Then C_φ is bounded on B_{\log} if and only if
$$\sup_{z \in D} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) < +\infty.$$

proof. Suppose that
$$\sup_{z \in D} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) < +\infty.$$
 Let f be in the space B_{\log} . Then we see that

$$\begin{aligned} & \sup_{z \in D} |(f \circ \varphi)'(z)| (1 - |z|^2) \log \frac{2}{1 - |z|^2} \\ &= \sup_{z \in D} |(f'(\varphi(z)))| |\varphi'(z)| (1 - |z|^2) \log \frac{2}{1 - |z|^2} \\ &\leq \sup_{z \in D} |(f'(\varphi(z)))| (1 - |\varphi(z)|^2) \left(\log \frac{2}{1 - |\varphi(z)|^2} \right) \sup_{z \in D} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) \\ &\leq C \|f\|_{B_{\log}}. \end{aligned}$$

To prove the converse, suppose that C_φ is bounded on B_{\log} . Then $\|f \circ \varphi\|_{B_{\log}} \leq C \|f\|_{B_{\log}}$ for all $f \in B_{\log}$. For $w \neq 0$, let f_w be the anti-derivative of

$$\left(1 - \frac{\bar{w}^2}{|w|^2} z^2\right)^{-1} \left(\log \frac{2}{1 - \frac{\bar{w}^2}{|w|^2} z^2}\right)^{-1}$$

with $f_w(0) = 0$. Since $\sup_{z_1 \in D} (1 - |z_1|^2) \left(\log \frac{2}{1 - |z_1|^2}\right) |1 - z_1^2|^{-1} \left|\log \frac{2}{1 - z_1^2}\right|^{-1} < +\infty$, applying $z_1 = \frac{\bar{w}}{|w|} z$, we have

$$\sup_{z \in D} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2}\right) \left|1 - \frac{\bar{w}^2}{|w|^2} z^2\right|^{-1} \left|\log \frac{2}{1 - \frac{\bar{w}^2}{|w|^2} z^2}\right|^{-1} < +\infty.$$

Hence we have $f_w \in B_{\log}$ for $w \neq 0$. Since C_φ is bounded on B_{\log} , we have $\|f_w \circ \varphi\|_{B_{\log}} < +\infty$. Thus for $w \neq 0$,

$$\sup_{z \in D} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2}\right) |(f_w \circ \varphi)'(z)| \leq K < +\infty.$$

For any $z \in D$ and $w \neq 0 \in D$, we have $(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |f'_w(\varphi(z))| |\varphi'(z)| \leq K$. Fix an arbitrary $z \in D$ with $\varphi(z) \neq 0$, applying $w = \varphi(z)$ to the above inequality, we have

$$(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) \left| 1 - \frac{\overline{\varphi(z)}^2}{|\varphi(z)|^2} \varphi(z)^2 \right|^{-1} \left| \log \frac{2}{1 - \frac{\overline{\varphi(z)}^2}{|\varphi(z)|^2} \varphi(z)^2} \right|^{-1} |\varphi'(z)| \leq K < \infty.$$

Hence for an arbitrary $z \in D$ with $\varphi(z) \neq 0$, $\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \leq K < +\infty$. For an arbitrary $z \in D$ with $\varphi(z) = 0$, since $\varphi \in B_{\log}$, we have

$$\left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) = \frac{1}{\log 2} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |\varphi'(z)| < +\infty.$$

This completes the proof of the theorem. \square

For example, for $z, w \in D$, $\varphi(z) = \frac{w - z}{1 - \bar{w}z}$ satisfies the condition of Theorem 1.

Theorem 2. Let φ be a holomorphic function taking D into D . Then C_φ is compact on B_{\log} if and only if $\varphi \in B_{\log}$ and for every $\epsilon > 0$ there exists $0 < r < 1$ such that

$$\sup_{|\varphi(z)| > r} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) < \epsilon.$$

proof. Suppose that $\varphi \in B_{\log}$ and for every $\epsilon > 0$ there exists $0 < r < 1$ such that

$$\sup_{|\varphi(z)| > r} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) < \epsilon.$$

Suppose that f_n is bounded sequence in B_{\log} converging to 0 uniformly on compact subsets of D . Then it is enough to prove that $\|f_n \circ \varphi\|_{B_{\log}} \rightarrow 0$ ($n \rightarrow \infty$) (see [5], [7] and [8]). Let $K := \sup_n \|f_n\|_{B_{\log}}$. For any $\epsilon > 0$, there is $0 < r < 1$ such that

$$\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| < \frac{\epsilon}{2K},$$

whenever $|\varphi(z)| > r$. So we have

$$\begin{aligned} & (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |(f_n \circ \varphi)'(z)| \\ &= \frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| (1 - |\varphi(z)|^2) \left(\log \frac{2}{1 - |\varphi(z)|^2} \right) |f'_n(\varphi(z))| \leq \frac{\epsilon}{2K} K = \frac{\epsilon}{2}. \end{aligned}$$

Hence $(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |(f_n \circ \varphi)'(z)| < \frac{\epsilon}{2}$ whenever $|\varphi(z)| > r$.

On the other hand, $f_n \circ \varphi(0) \rightarrow 0$ and $(1 - |w|^2) \left(\log \frac{2}{1 - |w|^2} \right) |f_n'(w)| \rightarrow 0$ for $|w| \leq r$ ($n \rightarrow +\infty$). By the assumption $\varphi \in B_{\log}$, we have

$$\sup_{|\varphi(z)| \leq r} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) \leq C \|\varphi\|_{B_{\log}} < +\infty.$$

So put $K_1 := \sup_{|\varphi(z)| \leq r} \left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right)$. Then for $\frac{\epsilon}{K_1} > 0$ and for large enough n ,

$$(1 - |w|^2) \left(\log \frac{2}{1 - |w|^2} \right) |f_n'(w)| \leq \frac{\epsilon}{2K_1} \text{ whenever } |w| \leq r \text{ and } |f_n \circ \varphi(0)| < \frac{\epsilon}{2}.$$

Hence for any $\epsilon > 0$ and for large enough n ,

$$\begin{aligned} & (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |(f_n \circ \varphi)'(z)| \\ &= (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |f_n'(\varphi(z))| |\varphi'(z)| \\ &= (1 - |\varphi(z)|^2) \left(\log \frac{2}{1 - |\varphi(z)|^2} \right) |f_n'(\varphi(z))| \frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \leq \frac{\epsilon}{2K_1} K_1 = \frac{\epsilon}{2}. \end{aligned}$$

Hence we have for any $\epsilon > 0$ and for large enough n ,

$$|f_n \circ \varphi(0)| + (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |(f_n \circ \varphi)'(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

To prove the converse, suppose that there exists a sequence $\{z_n\}_n \subset D$ and for some $\epsilon_0 > 0$ such that $|z_n| \rightarrow 1^-$ and $\left(\frac{(1 - |z|^2) \log \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \right) > \epsilon_0$. Then we show that C_φ is not compact. By taking a subsequence, it may be assumed that $w_n = \varphi(z_n) \rightarrow w_0 := \varphi(z_0) \in \partial D$. Let $f_n(z) = \log \left(\log \frac{2}{1 - \overline{w_n} z} \right)$ and $f_{0,n}(z) = \log \left(\log \frac{2}{1 - \overline{w_0} e^{i(\theta_0 - \theta_n)} z} \right)$ where $\theta_n = \arg \varphi(z_n)$ and $\theta_0 = \arg \varphi(z_0)$. Since $\theta_n \rightarrow \theta_0$ ($n \rightarrow \infty$), we see that $\{g_n\}_n = \{f_n - f_{0,n}\}_n$ converges to 0 uniformly on compact subsets of D . Since $\frac{\overline{w_0} e^{i(\theta_0 - \theta_n)} (1 - |w_n|^2) \log \frac{2}{1 - |\overline{w_n}|^2}}{\overline{w_n} (1 - |w_n|) \log \frac{2}{1 - |w_n|}} \rightarrow 2$ ($n \rightarrow +\infty$) and $|\varphi(z_n)| \rightarrow 1$ ($n \rightarrow +\infty$), we have for all n

$$\begin{aligned} & \|C_\varphi g_n\|_{B_{\log}} = \|C_\varphi f_n - C_\varphi f_{0,n}\|_{B_{\log}} \\ & \geq (1 - |z_n|^2) \left(\log \frac{2}{1 - |z_n|^2} \right) |(C_\varphi f_n)'(z_n) - (C_\varphi f_{0,n})'(z_n)| \\ &= (1 - |z_n|^2) \left(\log \frac{2}{1 - |z_n|^2} \right) |f_n'(\varphi(z_n)) - f_{0,n}'(\varphi(z_n))| |\varphi'(z_n)| \\ &= (1 - |z_n|^2) \left(\log \frac{2}{1 - |z_n|^2} \right) |\varphi'(z_n)| \left| \frac{\overline{w_n}}{(1 - |w_n|^2) \log \frac{2}{1 - |w_n|^2}} - \frac{\overline{w_0} e^{i(\theta_0 - \theta_n)}}{(1 - |w_n|) \log \frac{2}{1 - |w_n|}} \right| \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{(1 - |z_n|^2) \log \frac{2}{1 - |z_n|^2}}{(1 - |\varphi(z_n)|^2) \log \frac{2}{1 - |\varphi(z_n)|^2}} |\varphi'(z_n)| \right) |\varphi(z_n)| \left| 1 - \frac{\overline{w_0} e^{i(\theta_0 - \theta_n)} (1 - |w_n|^2) \log \frac{2}{1 - |w_n|^2}}{\overline{w_n} (1 - |w_n|^2) \log \frac{2}{1 - |w_n|^2}} \right| \\
&\geq \epsilon_0 > 0.
\end{aligned}$$

Hence we see that C_φ is not compact. This completes the proof of the theorem. \square

§3. The composition operator on weighted $BMOA_{\log}$

In this section, we study the composition operator on weighted $BMOA_{\log}$. But, by a direct calculation, it is not easy that we do see when $C_\varphi f \in BMOA_{\log}$ for $f \in BMOA_{\log}$. So by the following proposition, we can get the condition of a holomorphic function φ that $C_\varphi f \in BMOA_{\log}$ for $f \in B_{\log}$ holds. In particular, we can get the condition of a holomorphic function φ that $C_\varphi f \in BMOA_{\log}$ for $f \in BMOA_{\log}$ holds because of $BMOA_{\log} \subset B_{\log}$.

Proposition 3. If $\sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} \frac{(1 - |z|^2) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2 \left(\log \frac{2}{1 - |\varphi(z)|^2}\right)^2} dA(z) < +\infty$, then

$\varphi(z)$ has B_{\log} -to- $BMOA_{\log}$ pullback property. Conversely, if there exists a sequence $\{w_n\}_{n=1}^\infty \subset \partial D$ such that $\varphi(D) \subset \bigcup_{n=1}^\infty \{z \in D : |1 - \overline{w_n} z| < \lambda(1 - |z|^2)\}$ where $\lambda > 0$, if $\varphi(z)$ has B_{\log} -to- $BMOA_{\log}$ pullback property, then

$$\sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} \frac{(1 - |z|^2) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2 \left(\log \frac{2}{1 - |\varphi(z)|^2}\right)^2} dA(z) < +\infty.$$

proof. Let $f \in B_{\log}$. Then we have

$$\begin{aligned}
&\sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |(f \circ \varphi)'|^2 (1 - |z|^2) dA(z) \\
&\leq \sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |(f'(\varphi(z)))|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \\
&\leq \sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} \frac{(1 - |z|^2) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2 \left(\log \frac{2}{1 - |\varphi(z)|^2}\right)^2} dA(z) \|f\|_{B_{\log}} < +\infty.
\end{aligned}$$

Since $g \in BMOA_{\log}$ is equivalent to $\sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2) dA(z) < +\infty$, we see $f \circ \varphi \in BMOA_{\log}$ for all $f \in B_{\log}$.

Suppose that there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset \partial D$ such that $\varphi(D) \subset \cup_{n=1}^{\infty} \{z \in D : |1 - \overline{w_n}z| < \lambda(1 - |z|^2)\}$ where $\lambda > 0$ and that $\varphi(z)$ has B_{\log} -to- $BMOA_{\log}$ pullback property. Put $f_n(z) := \log\left(\log\frac{2}{1 - \overline{w_n}z}\right)$. Then it is clear that $f_n \in B_{\log}$. Let $G := \cup_{n=1}^{\infty} \{z \in D : |1 - \overline{w_n}z| < \lambda(1 - |z|^2)\}$. Then for $z \in G$,

$$(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |f'_n(z)| = \frac{(1 - |\overline{w_n}z|^2) \log \frac{2}{1 - |\overline{w_n}z|^2}}{|1 - \overline{w_n}z| \left| \log \frac{2}{1 - \overline{w_n}z} \right|} \geq C > 0.$$

Since $\varphi(D) \subset \cup_{n=1}^{\infty} \{z \in D : |1 - \overline{w_n}z| < \lambda(1 - |z|^2)\}$, we have

$$(1 - |\varphi(z)|^2) \left(\log \frac{2}{1 - |\varphi(z)|^2} \right) |f'_n(\varphi(z))| \geq C > 0.$$

Since $\varphi(z)$ has B_{\log} -to- $BMOA_{\log}$ pullback property and $f_n \in B_{\log}$, we have $f_n \circ \varphi \in BMOA_{\log}$. Hence we have

$$\begin{aligned} & \sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} \frac{(1 - |z|^2)|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2 \left(\log \frac{2}{1 - |\varphi(z)|^2}\right)^2} dA(z) \\ & \leq \frac{1}{C} \sup_I \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |(f_n \circ \varphi)'(z)|^2 (1 - |z|^2) dA(z) < \infty. \quad \square \end{aligned}$$

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