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Efficient Hedging with Coherent Risk Measure

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Abstract

The idea of efficient hedging has been introduced by Föllmer and Leukert (2000). They defined the shortfall risk as the expectation of the shortfall weighted by a loss function, and looked for strategies that minimize the shortfall risk under a capital constraint. In this paper, to measure the shortfall risk, we use the coherent risk measures introduced by Artzner, Delbaen, Eber and Heath (1999). We show that, for a given contingent claim H, the optimal strategy consists in hedging a modified claim φH for some randomized test φ. This is an analogou of the results by Föllmer and Leukert (2000).

KEY WORDS: hedging, shortfall risk, efficient hedging, coherent risk measure, randomized test, Neyman-Pearson lemma, worst conditional expectation

1 Introduction

In a complete financial market, we can replicate a given contingent claim by a self-financing strategy. In an incomplete market, by using a "super-hedging" strategy, we can generate a final wealth that dominates the payoff of the contingent claim. If the seller of a contingent claim hopes to hedge the claim with a smaller initial amount of capital than that required by a perfect (or super-) hedging strategy, then the seller has to accept some risk. In such a situation, the seller seeks the optimal "partial" hedge that can be achieved with his initial amount. In Föllmer and Leukert (2000), they introduced the strategy of "efficient hedging" that minimizes the *shortfall risk* under a capital constraint. They described the investor's attitude towards the shortfall in terms of a loss function, and defined the shortfall risk as the expectation of the shortfall weighted by the loss function. In other words, they used the expected loss functions as risk measures.

*I should like to thank Professor Akihiko Inoue for helpful comments.

In this paper, we use *coherent mesures of risk* , introduced by Artzner, Delbaen, Eber and Heath (1999), as risk measures. They are defined axiomatically by four desirable properties, that is, monotonicity, subadditivity, positive homogeneity, and translation invariance. In Artzner et al. (1999), they restrict themselves to finite probability spaces. Delbaen (2000) extended the definition of coherent risk measures to general probability spaces (see also Kusuoka (2000)). In Delbaen (2000), as the space of random variables, the space L^∞ of all essentially bounded random variables or the space L^0 of all random variables is adopted. We use the intermediate space L^1 instead here. The space L^1 is large enough to be used in our hedging problem yet sufficiently small for nice properties to hold.

We show that, for a given contingent claim H, the optimal strategy consists in hedging a modified claim φH for some randomized test φ. This is an analogou of the results by Föllmer and Leukert (2000).

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{Q}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to P. We write L^1 and L^∞ for $L^1(\Omega, \mathcal{F}, P)$ and $L^\infty(\Omega, \mathcal{F}, P)$, respectively. For $Q \in \mathcal{Q}$, we denote expectation with respect to Q by E^Q and the Radon-Nykodim derivative dQ/dP by Z_Q. Following Artzner et al. (1999) and Delbaen (2000), we give the following definition.

Definition 1.1 We say that a map $\rho : L^1 \to \mathbf{R}$ is a coherent risk measure if the following are satisfied:

(1) For all $X \in L^1$ with $X \geq 0$, we have $\rho(X) \leq 0$.

(2) For all X and $Y \in L^1$, we have $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

(3) If $X \in L^1$ and $\lambda > 0$, then $\rho(\lambda X) = \lambda \rho(X)$.

(4) If $X \in L^1$ and $c \in \mathbf{R}$, then $\rho(X+c) = \rho(X) - c$.

We consider the coherent risk measures that are lower semi-continuous in the L^1-norm. We estabilish a representation theorem for them, which is an analogue of Proposition 4.1 in Artzner et al. (1999) and Theorem 2.3 in Delbaen (2000).

Theorem 1.2 *For a mapping $\rho : L^1 \to \mathbf{R}$, the following are equivalent:*

(1) *The mapping ρ is a lower semi-continuous coherent risk measure.*

(2) *There is a subset $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ such that*

$$\{Z_Q \mid Q \in \tilde{\mathcal{Q}}\} \text{ is a weak*-closed convex subset of } L^\infty \,, \tag{1.1}$$

$$\rho(X) = \sup_{Q \in \tilde{\mathcal{Q}}} E^Q[-X] \qquad (X \in L^1). \tag{1.2}$$

The element of $\tilde{\mathcal{Q}}$ can be interpreted as a *"scenario"* (see Artzner et al. (1999) and Delbaen (2000)). We notice that, for ρ as in Theorem 1.2, the restriction of ρ on L^∞ satisfies the "Fatou property" defined in Delbaen (2000).

Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration on $(\Omega, \mathcal{F})$. For simplicity, we assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T$ is equal to $\mathcal{F}$. The discounted price process of the underlying asset is described as a semimartingale $X = (X_t)_{0 \le t \le T}$ on the filterd probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$. Let $\mathcal{P}$ denote the set of all equivalent martingale measures. We assume absence of arbitrage in the sense that $\mathcal{P} \neq \emptyset$.

A self-financing strategy is described as a pair (V_0, ξ), where V_0 is an initial capital, and ξ is a predictable process such that the resulting value process

$$V_t = V_0 + \int_0^t \xi_s dX_s \qquad (t \in [0,T])$$

is well defined (see Föllmer and Leukert (2000)). A self-financing strategy (V_0, ξ) is said to be *admissible* if the corresponding value process V satisfies

$$V_t \ge 0 \qquad \forall t \in [0,T] \quad P-\text{a.s.}$$

We consider a contingent claim that is defined by a nonnegative random variable $H \in L^1$. We assume that

$$U_0 := \sup_{P^* \in \mathcal{P}} E^{P^*}[H] < \infty.$$

Let ρ be a coherent risk measure on L^1. The shortfall risk we consider here is given by $\rho((V_T - H) \wedge 0)$. For a given amount of initial capital $\tilde{V_0}$ which is smaller than U_0, we want to find an admissible strategy (V_0, ξ) that minimizes the shortfall risk $\rho((V_T - H) \wedge 0)$. Thus we consider the optimization problem

$$\rho((V_T - H) \wedge 0) = \rho\left(\left(V_0 + \int_0^T \xi_s dX_s - H\right) \wedge 0\right) = \min \tag{1.3}$$

under the constraint

$$V_0 \le \tilde{V_0}. \tag{1.4}$$

We take ρ from the class of lower semi-continuous coherent risk measures, and follow the method of Föllmer and Leukert (2000). We define the set

$$\mathcal{R} = \{\varphi : \Omega \to [0,1] \mid \varphi \text{ is } \mathcal{F}\text{–measurable}\}$$

of "randomized tests" φ. We also define the constrained set

$$\mathcal{R}_0 = \left\{\varphi \in \mathcal{R} \mid \sup_{P^* \in \mathcal{P}} E[\varphi H] \le \tilde{V_0}\right\}.$$

We reduce our problem to the following proposition, which corresponds to Proposition 3.1 in Föllmer and Leukert (2000).

Proposition 1.3 *There exists $\tilde{\varphi} \in \mathcal{R}_0$ such that*

$$\inf_{\varphi \in \mathcal{R}_0} \rho(-(1-\varphi)H) = \rho(-(1-\tilde{\varphi})H). \tag{1.5}$$

Let $\tilde{\varphi}$ be the solution to the minimization problem defined by (1.5), and let $\tilde{U}$ be a right-continuous version of the process

$$\tilde{U}_t = \operatorname*{ess.\,sup}_{P^* \in \mathcal{P}} E^{P^*}[\tilde{\varphi} H \mid \mathcal{F}_t].$$

The process $\tilde{U}$ is a $\mathcal{P}$–supermartingale, i.e., a supermartingale under any $P^* \in \mathcal{P}$. By the optional decomposition theorem (see Föllmer and Leukert (2000)), there exists an admissible strategy $(\tilde{V}_0, \tilde{\xi})$ and an increasing optional process $\tilde{C}$ with $\tilde{C}_0 = 0$ such that

$$\tilde{U}_t = \tilde{V}_0 + \int_0^t \tilde{\xi}_s dX_s - \tilde{C}_t.$$

Following Föllmer and Leukert (2000), we give the following definition.

Definition 1.4 For any admissible strategy (V_0, ξ) we define the corresponding success ratio as

$$\varphi_{(V_0,\xi)} = \mathbf{1}_{\{V_T \geq H\}} + \frac{V_T}{H} \mathbf{1}_{\{V_T < H\}}.$$

The next theorem corresponds to Theorem 3.2 in Föllmer and Leukert (2000).

Theorem 1.5 *Let $\tilde{\varphi}$ be a solution to the minimization problem* (1.5) *and let $(\tilde{V}_0, \tilde{\xi})$ be the admissible strategy determined by the optional decomposition of the claim $\tilde{\varphi}H$. Then the strategy $(\tilde{V}_0, \tilde{\xi})$ solves the optimization problem* (1.3), (1.4).

We prove Theorem 1.2 in Section 2, and Proposition 1.3 and Theorem 1.5 in Section 3. In Section 4, we consider our hedging problem with some special coherent risk measures.

2 Proof of Theorem 1.2

Proof of Theorem 1.2. It is easy to prove the implication (2) $\Rightarrow$ (1). To prove the converse one (1) $\Rightarrow$ (2), we follow the method of proof of Theorem 2.3 in Delbaen (2000). We put $\phi(X) = -\rho(X)$ and define the set $C = \{X \in L^1 \mid \phi(X) \geq 0\}$. Then since ϕ is upper semi-continuous, the set C is a convex and norm closed cone in L^1. We regard L^∞ and L^1 as a duality pair associated with the nondegenerate bilinear form

$$L^1 \times L^\infty \ni \{X, Y\} \mapsto \langle X, Y \rangle = E[XY] \in \mathbf{R}.$$

Recall that the polar set C° of C is defined by

$$C^\circ = \{Y \in L^\infty \mid E[XY] \geq -1 \ (\forall X \in C)\}$$

(see Aubin (1982, p. 30)). However, since C is cone, we have

$$C^\circ = \{Y \in L^\infty \mid E[XY] \geq 0 \ (\forall X \in C)\}.$$

This implies that C° is also a weak*-closed, convex cone in L^∞.

We put $\Phi = \{Y \in C^\circ \mid E[Y] = 1\}$. Then, it holds that

$$C^\circ = \bigcup_{\lambda \geq 0} \lambda \Phi. \tag{2.1}$$

Indeed, if $Y \in C^\circ$ with $E[Y] > 0$, then we have $Y = \lambda \tilde{Y}$, where $\tilde{Y} = Y/E[Y]$ and $\lambda = E[Y]$. Hence $Y \in \cup_{\lambda \geq 0} \lambda \Phi$. On the other hand, if $Y \in C^\circ$ with $E[Y] = 0$, then $Y = 0$ since $L^1_+ \subset C$. Hence $Y \in \cup_{\lambda \geq 0} \lambda \Phi$. Thus (2.1) follows. The bipolar theorem (see Aubin (1982, p. 32)) then implies that

$$C = \{X \in L^1 \mid E[XY] \geq 0 \ (\forall Y \in \Phi)\}.$$

From this, we find that $\phi(X) \geq 0$ if and only if $E[XY] \geq 0$ for all $Y \in \Phi$. Since $\phi(X - \phi(X)) = 0$, we have that $E[(X - \phi(X))Y] \geq 0$ for all $Y \in \Phi$. Thus

$$\inf_{Y \in \Phi} E[XY] \geq \phi(X).$$

Now, for $\varepsilon > 0$, we have $\phi(X - \phi(X)) = -\varepsilon < 0$, so that there exists a $Y \in \Phi$ such that $E[(X - \phi(X) - \varepsilon)Y] < 0$ or $E[XY] \leq \phi(X) + \varepsilon$. Since ε is arbitrary, we obtain

$$\inf_{Y \in \Phi} E[XY] \leq \phi(X),$$

hence

$$\inf_{Y \in \Phi} E[XY] = \phi(X). \tag{2.2}$$

If we put

$$\tilde{\mathcal{Q}} = \{Q \in \mathcal{Q} \mid Z_Q = Y \text{ for some } Y \in \Phi\},$$

then (2.2) implies (1.1). Since $\{Z_Q \mid Q \in \tilde{\mathcal{Q}}\} = \Phi$, we find that this is the desired representation for ρ.

3 Proofs of Proposition 1.3 and Theorem 1.5

Let $\rho : L^1 \to \mathbf{R}$ be a lower semi-continuous coherent risk measure on L^1. Then, by Theorem 1.2, there exists a subset $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ such that (1.1) and (1.2) hold. We use the representation (1.2) in the proofs of Proposition 1.3 and Theorem 1.5.

Proof of Proposition 1.3. First, we notice that $\mathcal{R}$ is weak*-compact, i.e., $\sigma(L^\infty, L^1)$-compact, in L^∞. Since the map

$$L^\infty \ni \varphi \mapsto \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \in \mathbf{R}$$

is lower semi-continuous in the weak*-topology, the constrained set $\mathcal{R}_0$ is weak*-closed and so is weak*-compact. Since the map

$$L^\infty \ni \varphi \mapsto \sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\varphi)H] \in \mathbf{R}$$

is also lower semi-continuous in the weak*-topology, there exists $\tilde{\varphi} \in \mathcal{R}_0$ satisfying (1.5).

Proof of Theorem 1.5. We consider an admissible strategy (V_0, ξ) with (1.4) and the corresponding success ratio φ. We have from $\varphi H = V_T \wedge H$ that

$$(V_T - H) \wedge 0 = -(H - V_T)_+ = -(H - V_T \wedge H) = -(1-\varphi)H.$$

Since the corresponding value process $(V_t)_{0\le t\le T}$ is a $\mathcal{P}$-supermartingale, we obtain

$$E^{P^*}[\varphi H] \le E^{P^*}[V_T] \le V_0 \le \tilde{V}_0.$$

Thus the success ratio φ belongs to the constrained set $\mathcal{R}_0$ and so we have

$$\rho((V_T - H)\wedge 0) = \sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\varphi)H] \ge \sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\tilde{\varphi})H].$$

In particular, the success ratio $\varphi_{(\tilde{V}_0,\tilde{\xi})}$ satisfies

$$\sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\varphi_{(\tilde{V}_0,\tilde{\xi})})H] \ge \sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\tilde{\varphi})H]. \tag{3.1}$$

On the other hand, we have

$$\varphi_{(\tilde{V}_0,\tilde{\xi})}H = \tilde{V}_T \wedge H \ge \tilde{\varphi}H \qquad \mathrm{P}-\text{a.s.}$$

and so, for all $Q \in \tilde{\mathcal{Q}}$,

$$E^Q[(1-\varphi_{(\tilde{V}_0,\tilde{\xi})})H] \le E^Q[(1-\tilde{\varphi})H].$$

Hence we obtain from (3.1) that

$$\rho((\tilde{V}_T - H)\wedge 0) = \sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\varphi_{(\tilde{V}_0,\tilde{\xi})})H] = \sup_{Q\in\tilde{\mathcal{Q}}} E^Q[(1-\tilde{\varphi})H],$$

which proves the theorem.

4 Optimal hedging

In this section, we study our problem with two special coherent risk measures. The first one is the case of $\tilde{\mathcal{Q}}$ being a singleton, and second one is the worst conditional expectation.

First we take a singleton $\tilde{\mathcal{Q}} = \{Q\}$ with $Z_Q \in L^\infty$ as a scenario set. Then, the corresponding risk measure is

$$\rho(X) = E^Q[-X].$$

Thus we want to minimize the coherent risk measure

$$\rho((V_T - H) \wedge 0) = E^Q[-(V_T - H) \wedge 0] \tag{4.1}$$

under the constraint

$$V_0 \le \tilde{V}_0. \tag{4.2}$$

Theorem 1.5 shows that this is reduced to the optimization problem

$$E^Q[\varphi H] = \max \tag{4.3}$$

under the constraint that $\varphi \in \mathcal{R}$ satisfies

$$\sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \le \tilde{V}_0. \tag{4.4}$$

We assume that H is not trivial, i.e.,

$$E^Q[H] > 0.$$

Then the problem (4.3), (4.4) can be reformulated as

$$E^R[\varphi] = \max \tag{4.5}$$

under the constraint

$$E^{R^*}[\varphi] \le \frac{\tilde{V}_0}{E^{P^*}[H]} \qquad \forall P^* \in \mathcal{P}, \tag{4.6}$$

where the probability measures R and R^* are defined by

$$\frac{dR}{dQ} = \frac{H}{E^Q[H]} \quad , \quad \frac{dR^*}{dP^*} = \frac{H}{E^{P^*}[H]}.$$

In the terminology of the theory of hypothesis testing, the solution $\tilde{\varphi}_Q$ is identified as the most powerful test for the problem in which the null hypothesis is composite but the alternative simple.

In the complete case, by the fundamental lemma of Neyman and Pearson, we can solve the problem explictily.

Proposition 4.1 *Assume that $\mathcal{P} = \{P^*\}$. Then the most powerful test $\tilde{\varphi}_Q \in \mathcal{R}$ is given by*

$$\tilde{\varphi}_Q = \mathbf{1}_{\{Z_Q > \tilde{a} Z_{P^*}\}} + \gamma \mathbf{1}_{\{Z_Q = \tilde{a} Z_{P^*}\}},$$

where

$$\tilde{a} = \inf\left\{a \mid E^{P^*}\left[H\mathbf{1}_{\{Z_Q > a Z_{P^*}\}}\right] \le \tilde{V}_0\right\}$$

and

$$\gamma = \begin{cases} \dfrac{\tilde{V}_0 - E^{P^*}\left[H\mathbf{1}_{\{Z_Q > \tilde{a} Z_{P^*}\}}\right]}{E^{P^*}\left[H\mathbf{1}_{\{Z_Q = \tilde{a} Z_{P^*}\}}\right]} & \text{if } P^*(\{Z_Q = \tilde{a} Z_{P^*}\} \cap \{H > 0\}) > 0, \\ \text{an arbitrary value from } [0,1] & \text{if } P^*(\{Z_Q = \tilde{a} Z_{P^*}\} \cap \{H > 0\}) = 0. \end{cases}$$

Remark 4.2 When Q is equal to P, this proposition coincides with Proposition 4.1 in Föllmer and Leukert (2000).

Proof of Proposition 4.1. From the Neyman-Pearson lemma (see Schmetterer (1974, Chapter III, Theorem 3.3)) in terms of R and R^*, we obtain that

$$\tilde{\varphi}_Q = \mathbf{1}_{\{Z_R > \tilde{b} Z_{R^*}\}} + \beta \mathbf{1}_{\{Z_R = \tilde{b} Z_{R^*}\}},$$

where

$$\tilde{b} = \inf \left\{ b \mid P^*(Z_R > b Z_{R^*}) \leq \frac{\tilde{V}_0}{E^{P^*}[H]} \right\}$$

and

$$\beta = \begin{cases} \dfrac{\tilde{V}_0/E^{P^*}[H] - R^*(Z_R > \tilde{b} Z_{R^*})}{R^*(Z_R = \tilde{b} Z_{R^*})} & \text{if } R(Z_R = \tilde{b} Z_{R^*}) > 0, \\ \text{an arbitrary value from } [0,1] & \text{if } R(Z_R = \tilde{b} Z_{R^*}) = 0. \end{cases}$$

We have

$$\{Z_R > b Z_{R^*}\} = \left\{ Z_Q = b Z_{P^*} \frac{E^Q[H]}{E^{P^*}[H]} \right\} \cap \{H > 0\},$$

$$\frac{\tilde{b} E^Q[H]}{E^{P^*}[H]} = \tilde{a},$$

and $\gamma = \beta$. So

$$\tilde{\varphi}_Q \mathbf{1}_{\{H>0\}} = \mathbf{1}_{\{Z_Q > \tilde{a} Z_{P^*}\}} + \gamma \mathbf{1}_{\{Z_Q = \tilde{a} Z_{P^*}\}}.$$

Since

$$E^Q[\tilde{\varphi}_Q H] = E^Q[\tilde{\varphi}_Q H \mathbf{1}_{\{H>0\}}],$$

the proposition follows.

Next we take the *worst conditional expectation* introduced by Artzner et al. (1999). In our setting, this measure is given by

$$\mathrm{WCE}_\alpha(X) = \sup \left\{ E\left[(-X) \frac{\mathbf{1}_A}{P(A)} \right] \mid A \in \mathcal{F},\ P(A) > \alpha \right\} \qquad (X \in L^1),$$

where $\alpha \in (0,1)$. Now, for $\alpha \in (0,1]$, we define another coherent risk measure on L^1 as

$$\rho_\alpha(X) = \sup \{ E[(-X)f] \mid f \in \Phi_\alpha \},$$

where

$$\Phi_\alpha = \{ f \mid f \text{ is } \mathcal{F}\text{-measurable},\ 0 \leq f \leq \alpha^{-1} \text{ P - a.s.},\ E[f] = 1 \}.$$

For each $X \in L^1$ and $\alpha \in (0,1]$, both the coherent risk measures $\rho_\alpha(X)$ and $\mathrm{WCE}_\alpha(X)$ are bounded by $\alpha^{-1} \|X\|_1$. This implies that these coherent risk measures are continuous in the L^1-norm (see Inoue (2001, Lemma 2.1)). As mentioned in Delbaen (2000, p. 12), if $(\Omega, \mathcal{F}, P)$ is nonatomic, then $\rho_\alpha(X) = \mathrm{WCE}_\alpha(X)$ for $X \in L^\infty$. Since L^∞ is dense in L^1, we have that, for all $X \in L^1$,

$$\rho_\alpha(X) = \mathrm{WCE}_\alpha(X).$$

We consider our hedging problem with ρ_α ($\alpha \in (0, 1]$) as a measure of risk. We do not need to assume that $(\Omega, \mathcal{F}, P)$ is nonatomic. Thus we consider the minimization problem of finding $\tilde{\varphi} \in \mathcal{R}_0$ such that

$$\rho_\alpha(-(1-\tilde{\varphi})H) = \inf_{\varphi \in \mathcal{R}_0} \rho_\alpha(-(1-\varphi)H). \tag{4.7}$$

Lemma 4.3 *Let $X \in L^1$ such that $X \geq 0$. If $P(X > 0) \leq \alpha$, then we have*

$$\rho_\alpha(-X) = \frac{1}{\alpha} E^P[X]. \tag{4.8}$$

Proof. We fix $X \in L^1$ with $X \geq 0$. Then, by Theorem 1.2 in Inoue (2001), we have

$$\rho_\alpha(-X) = \frac{1}{\alpha} E^P[X(\mathbf{1}_{\{X>k\}} + \beta \mathbf{1}_{\{X=k\}})], \tag{4.9}$$

where

$$k = \inf\{a \in \mathbf{R} \mid P(X > a) \leq \alpha\}$$

and

$$\beta = \begin{cases} \dfrac{\alpha - P(X > k)}{P(X = k)} & \text{if } P(X = k) > 0, \\ 0 & \text{if } P(X = k) = 0. \end{cases}$$

If $P(X > 0) \leq \alpha$, then $k = 0$ and hence (4.8) follows.

For special H, our problem with ρ_α reduces to that with ρ_1 which has already been treated in Föllmer and Leukert (2000) as $l(x) = x$.

Proposition 4.4 *Suppose $P(H > 0) \leq \alpha$. Then the solution to the minimization problem (4.7) is the most powerful test $\tilde{\varphi}_P$.*

Proof. By lemma 4.3, we have

$$\rho_\alpha(-(1-\varphi)H) = \frac{1}{\alpha} E^P[(1-\varphi)H].$$

Therefore $\tilde{\varphi}_P$ minimizes $\rho_\alpha(-(1-\varphi)H)$ in $\mathcal{R}_0$.

Example. We consider the standard Black-Scholes model as in Föllmer and Leukert (2000, §6.2). Then, the discounted price process is given by

$$X_t = x_0 \exp\left(\sigma W_t + \left(m - \frac{\sigma^2}{2}\right) t\right),$$

where $m \in \mathbf{R}$, $\sigma > 0$, $x_0 > 0$, and W is a one-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$. The unique equivalent martingale measure P^* is given by

$$\frac{dP^*}{dP} = \exp\left(-\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) = \mathrm{const.} X_T{}^{-m/\sigma^2}.$$

We assume that $m > 0$, and consider a European call $H = (X_T - K)_+$ as in Föllmer and Leukert (2000, §6.2). The cost of replication of this claim is

$$U_0 = E^{P^*}[H] = x_0 N(d_+) - KN(d_-),$$

where

$$d_\pm = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{x_0}{K}\right) \pm \frac{1}{2}\sigma\sqrt{T}$$

and N denotes the standard Gaussian cumulative distribution function. Let $\tilde{V}_0$ be a positive constant such that $\tilde{V}_0 \le E^{P^*}[H]$. We assume

$$P(H > 0) = N(m\sqrt{T} + d_-) \le \alpha.$$

Then by Proposition 4.4, the most powerful test $\tilde{\varphi}_P$ solves the minimization problem (4.7). By Föllmer and Leukert (2000, §6.2), $\tilde{\varphi}_P$ is given by

$$\tilde{\varphi}_P = \mathbf{1}_{\{X_T > c\}},$$

where the constant c is determined by

$$\begin{aligned} \tilde{V}_0 &= E^{P^*}[H\mathbf{1}_{\{X_T>c\}}] \\ &= x_0 N\left(\frac{1}{\sigma\sqrt{T}} \log\left(\frac{x_0}{c}\right) + \frac{1}{2}\sigma\sqrt{T}\right) - KN\left(\frac{1}{\sigma\sqrt{T}} \log\left(\frac{x_0}{c}\right) - \frac{1}{2}\sigma\sqrt{T}\right). \end{aligned}$$

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