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Toeplitz Operators And Weighted Norm Inequalities On The Bidisc

By

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To The Memory Of Professor K.Seddighi

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

2000 Mathematics Subject Classification. Primary 47 B 35, 42 B 20

Abstract. Let H^p be the Hardy space on the bidisc and $1 < p < \infty$. For a function ϕ in L^∞, we study the norm of the Hankel operator H_ϕ on H^p and the invertibility of the Toeplitz operator T_ϕ on H^p. The latter is strongly related with a weighted norm inequalities on the bidisc.

§1. Introduction

Let m be the normalized Lebesgue measure on the torus T^2. For $1 \le p \le \infty$, $L^p = L^p(T^2, m)$ denotes the Lebesgue space and $H^p = H^p(T^2, m) = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0$ if $\ell < 0$ or $n < 0\}$, that is, H^p denotes the usual Hardy space on T^2. Let $K^p = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0$ if $\ell \le 0$ and $n \le 0\}$. Then $K^p = \{f \in L^q \ ; \ \int fgdm = 0$ if $g \in H^q\}$ where $1/p + 1/q = 1$. Put $H = H^p \cap L$ and $K = L^p \cap L$ where L denotes the set of all trigonometric polynomials on T^2. Suppose m_z and m_w denote the normalized Lebesgue measures on the torus $T = T_z$ and $T = T_w$. Then $T^2 = T_z \times T_w$ and $m = m_z \times m_w$.

Let P be a projection from L onto H with $P = 0$ on K. Then P can be extended boundedly to L^p for $1 < p < \infty$ and P is an orthogonal projection when $p = 2$. For a function ϕ in L^∞, the Hankel operator determined by ϕ is

$$H_\phi f = (I - P)(\phi f) \quad (f \in H^p)$$

and the Toeplitz operator determined by ϕ is

$$T_\phi f = P(\phi f) \quad (f \in H^p).$$

For the bounded linear operator A on H^p or L^p, $\|A\|_p$ denotes the norm of A. When $p = 2$, and H^p is the Hardy space on the disc, Z.Nehari [10] proved that $\|H_\phi\|_2 = \|\phi + H^\infty\|$. It is not difficult to generalize this for $p \neq 2$, that is,

$$\|\phi + H^\infty\| \le \|H_\phi\|_p \le \|1 - P\|_q \|\phi + H^\infty\|.$$

This is different from a formula in [1, Theorem 2.11] because the Hankel operator is different from ours. In Section 2, we generalize the formula above for $\|H_\phi\|_p$ to the Hardy space on the bidisc. When $p = 2$, this is known (see [2],[4],[8] and [9]). When H^p is the Hardy space on the disc, R.Rochberg [13] showed that T_ϕ is invertible on H^p if and only if $\phi = k\phi_0$ and $\phi_0 = \bar{h}_0/h_0$ where k is an invertible function in H^∞ and h_0 is an outer function in H^p with $|h_0|^p$ satisfying the (A_p)-condition (see [10, Theorem 1]). If $p = 2$, this reduces to a theorem of H.Widom and A.Devinatz. In Section 4, for some special symbol ϕ we generalize the above theorem of R.Rochberg to the Hardy space on the bidisc. In Section 3, we define (A_p)-condition on T^2 and give a theorem of Hunt, Muckenhoupt and Wheeden on T^2. This is used in Section 4.

An order relation can be inrtoduced in Z^2. Let L_r be a line with rational slope r in the plane. S_r denotes all lattice points on one side of L_r, together with those on the right side ray of L_r from the origin. When L is a real axis, that is, $L = L_0$, then $S_0 = \{(m, 0) \ ; \ m > 0\} \cup \{(m, n) \ ; \ n > 0\}$. When L is an imaginary axis, that is, $L = L_{-\infty}$, then $S_{-\infty} = \{(0, n) \ ; \ n > 0\} \cup \{(m.n) \ ; \ m > 0\}$. This order is non-archimedean, and Z^2 has the smallest positive element (m_0, n_0) in S_r. We assume that S_r contains Z^2_+, that is, $-\infty \le r \le 0$. When $-\infty < r < 0$, $|m_0|$ and $|n_0|$ have no common factor except 1 and $r = n_0/m_0$, and let $(m_1, n_1) = (0, 1)$. When $r = 0$, $(m_0, n_0) = (1, 0)$ and let

$(m_1, n_1) = (0,1)$. When $r = -\infty$, $(m_0, n_0) = (0,1)$ and let $(m_1, n_1) = (1,0)$. For each half plane S_r, put

$$Z = Z_r = z^{m_0} w^{n_0}$$

and

$$W = W_r = z^{m_1} w^{n_1}.$$

Hence $Z_0 = W_{-\infty} = z$ and $W_0 = Z_{-\infty} = w$, and if $-\infty < r < 0$ then $W_r = w$.

For each r with $-\infty \leq r \leq 0$, put H_r^p = the norm closed linear span of $\cup_{j=-\infty}^{\infty} Z_r^j H^p$ in L^p if $1 \leq p < \infty$ and $\mathbf{H}_r^\infty$ = the weak* closed linear span of $\cup_{j=-\infty}^{\infty} Z_r^j H^\infty$ in L^∞. $\mathcal{L}_r^p$ and $\mathcal{H}_r^p$ denote the norm closure of the set of trigonometric polynomials and analytic polynomials, respectively, of Z_r in L^p if $1 \leq p < \infty$. $\mathcal{L}_r^\infty$ and $\mathcal{H}_r^\infty$ denote the weak* closure. Then

$$\mathbf{H}_r^p = \mathcal{L}_r^p + \mathcal{L}_r^p W + \cdots + \mathcal{L}_r^p W^{n-1} + W^n \mathbf{H}_r^p.$$

Let $\mathcal{E}$ be a conditional expectation from $\mathbf{H}_r^\infty$ onto $\mathcal{L}_r^\infty$. Then $\mathcal{E}$ is multiplicative on $\mathbf{H}_r^\infty$ and $\mathbf{H}_r^\infty + \overline{W\mathbf{H}_r^\infty}$ is weak* dense in L^∞. Hence $\mathbf{H}_r^\infty$ is an extended weak* Dirichlet algebra with respect to $\mathcal{E}$ (see [7]). For $r = 0$ and $r = -\infty$, we will write that $\mathbf{H}_0^p = \mathbf{H}_w^p$, $\mathbf{H}_{-\infty}^p = \mathbf{H}_z^p$, $W\mathbf{H}_0^p = w\mathbf{H}_w^p$, $W\mathbf{H}_{-\infty}^p = z\mathbf{H}_z^p$, $\mathcal{L}_0^\infty = \mathcal{L}_z^\infty$ and $\mathcal{L}_{-\infty}^\infty = \mathcal{L}_w^\infty$ where $\mathcal{L}_z^\infty = L^\infty(T_z, dm_z)$ and $\mathcal{L}_w^\infty = L^\infty(T_w, dm_w)$. $H^p(T_z, dm_z)$ and $H^p(T_w, dm_w)$ denote one variable Hardy spaces. Let P^w be a projection from L onto $(\mathbf{H}_w^\infty) \cap L$ with $P^w = 0$ on $\overline{w\mathbf{H}_w^\infty} \cap L$, and put $P_0^w f = wP^w(\bar{w}f)$. P^z and P_0^z can be defined similarly.

All results in this paper can be generalized easily to the Hardy space on the polydisc.

§2. Hankel operator

Theorem 1 for $p = 2$ is known in [2],[3],[8] and [9]. Proposition 2 for $p = 2$ is written in [8]. However its proof had some gap except $r = 0$ and $r = -\infty$ (see[9]).

Lemma 1. *Suppose $1/p + 1/q = 1$. If h is a function in $\mathbf{H}_r^1$, then there exists an f in $\mathbf{H}_r^p$ and a g in $\mathbf{H}_r^q$ such that $h = fg$, $\|f\|_p \leq \|h\|_1$ and $\|g\|_q \leq \|h\|_1$.*

Proof. In Section 1, we noted that $\mathbf{H}_r^\infty$ is an extended weak* Dirichlet algebra. If h is in $\mathbf{H}_r^1$ then $\chi_{E(h)}\mathcal{E}(\log|h|) > -\infty$ *a.e.* where $E(h)$ is the support set of h. Hence Theorem 4′ in [7] implies the lemma for $p = 2$. For $p \neq 2$, we can prove it similarly because Theorem 4′ can be shown for arbitrary p (see[7, Section 5]).

Lemma 2. *Suppose $1/p + 1/q = 1$. If ϕ is a function in L^∞, then*

$$\begin{aligned}
&\|\phi + \mathbf{H}_r^\infty\| \\
&= \sup\{|\int \phi h dm| \ ; \ h \in W\mathbf{H}_r^1 \ \text{ and } \ \|h\|_1 \le 1\} \\
&= \sup\{|\int \phi f g dm| \ ; \ f \in \mathbf{H}_r^p, \ g \in W\mathbf{H}_r^q, \ \|f\|_p \le 1 \ \text{ and } \ \|g\|_q \le 1\}
\end{aligned}$$

Proof. This is a result of Lemma 1 and Hahn-Banach theorem because the annihilator of $W\mathbf{H}_r^1$ in L^∞ is $\mathbf{H}_r^\infty$.

Theorem 1. *Suppose $1 < p < \infty$, $1/p + 1/q = 1$ and ϕ is function in L^∞. Then*

$$\begin{aligned}
&\max(\|\phi + \mathbf{H}_z^\infty\|, \ \|\phi + \mathbf{H}_w^\infty\|) \le \|H_\phi\| \\
&\le \|1 - P\|_q\{\|P_0^w\|_q\|\phi + \mathbf{H}_w^\infty\| + \|P_0^z\|_q\|\phi + \mathbf{H}_z^\infty\|\}.
\end{aligned}$$

Proof. We will give the lower estimate of $\|H_\phi\|$. Since $\bar{z}^n\mathbf{H}_w^q = \mathbf{H}_w^q$ for any positive integer n, $H^p \times w\mathbf{H}_w^q = \bar{z}^n H^p \times w\mathbf{H}_w^q$. Hence

$$H^p \times K^q \supset \{\bigcup_{n=0}^\infty \bar{z}^n H^p\} \times w\mathbf{H}_w^q$$

Using Lemma 2 in the second equality,

$$\begin{aligned}
&\|H_\phi\| = \sup\{|\int (1-P)\phi f g dm| \ ; \ f \in H^p, \ g \in L^q, \ \|f\|_p \le 1 \ \text{ and } \ \|g\|_q \le 1\} \\
&\ge \sup\{|\int \phi f g dm| \ ; \ f \in H^p, \ g \in K^q, \ \|f\|_p \le 1 \ \text{ and } \ \|g\|_q \le 1\} \\
&\ge \sup\{|\int \phi f g dm| \ ; \ f \in \mathbf{H}_w^p, \ g \in w\mathbf{H}_w^q, \ \|f\|_p \le 1 \ \text{ and } \ \|g\|_q \le 1\} \\
&= \|\phi + \mathbf{H}_w^\infty\|.
\end{aligned}$$

Similarly we can show that $\|H_\phi\| \ge \|\phi + \mathbf{H}_z^\infty\|$.

Now we will give the upper estimate of $\|H_\phi\|$. Here for $F \in L^p$ and $G \in L^q$, put $\langle F, G\rangle = \int F\bar{G} dm$.

$$\begin{aligned}
&\|H_\phi\| = \sup\{|\langle H_\phi f, g\rangle| \ ; \ f \in H^p, \ g \in L^q, \ \|f\|_p \le 1 \ \text{ and } \ \|g\|_q \le 1\} \\
&= \sup\{|\langle \phi f, (I-P)g\rangle| \ ; \ f \in H^p, \ g \in L^q, \ \|f\|_p \le 1 \ \text{ and } \ \|g\|_q \le 1\} \\
&\le \|I - P\|_q \sup\{|\langle \phi f, \bar{h}\rangle| \ ; \ f \in H^p, \ h \in K^q, \ \|f\|_p \le 1 \ \text{ and } \ \|h\|_q \le 1\}.
\end{aligned}$$

Since $K^q = w\mathbf{H}_w^q + z\mathbf{H}_z^q$, by Lemma 2

$$\begin{aligned}
&\sup\{|\langle \phi f, \bar{h}\rangle| \ ; \ f \in H^p, \ h \in K^q, \ \|f\|_p \le 1 \ \text{ and } \ \|h\|_q \le 1\} \\
&\le \sup\{|\langle \phi f, \overline{P_0^w h}\rangle| \ ; \ f \in H^p, \ h \in K^q, \ \|f\|_p \le 1 \ \text{ and } \ \|h\|_q \le 1\} \\
&\quad + \sup\{|\langle \phi f, \overline{P_0^z h}\rangle| \ ; \ f \in H^p, \ h \in K^q, \ \|f\|_p \le 1 \ \text{ and } \ \|h\|_q \le 1\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|P_0^w\|_q \sup\{|\int \phi f k dm| \ ; \ f \in H^p, \ k \in w\mathbf{H}_w^q, \ \|f\|_p \leq 1 \ \text{ and } \ \|k\|_q \leq 1\} \\
&\quad + \|P_0^z\|_q \sup\{|\int \phi f k dm| \ ; \ f \in H^p, \ k \in \mathbf{H}_z^q, \ \|f\|_p \leq 1 \ \text{ and } \ \|k\|_g \leq 1\} \\
&\leq \|P_0^w\|_q \sup\{|\int \phi f k dm| \ ; \ f \in \mathbf{H}_w^p, \ k \in w\mathbf{H}_w^q, \ \|f\|_p \leq 1 \ \text{ and } \ \|k\|_q \leq 1\} \\
&\quad + \|P_0^z\|_q \sup\{|\int \phi f k dm| \ ; \ f \in \mathbf{H}_z^p, \ k \in z\mathbf{H}_z^q, \ \|f\|_p \leq 1 \ \text{ and } \ \|k\|_q \leq 1\} \\
&\leq \|P_0^w\|_q \|\phi + \mathbf{H}_w^\infty\| + \|P_0^z\|_q \|\phi + \mathbf{H}_z^\infty\|.
\end{aligned}
$$

This completes the proof.

Corollary 1. *Suppose $p = 2$ and ϕ is a function in $\mathbf{H}_w^\infty$. Then*

$$\|H_\phi\| = \|\phi + \mathbf{H}_z^\infty\|.$$

Proof. Apply Theorem 1 as $p = 2$.

Corollary 2. *Suppose $p = 2$ and ϕ is a continuous function on T^2. Then* $\lim_{n\to\infty} \|H_{\phi z^n}\| = \|\phi + \mathbf{H}_w^\infty\|$ *and* $\lim_{n\to\infty} \|H_{\phi w^n}\| = \|\phi + \mathbf{H}_z^\infty\|$.

Proof. By Theorem 1 for $p = 2$.

$$
\begin{aligned}
&\|\phi + \mathbf{H}_w^\infty\| = \|\phi z^n + \mathbf{H}_w^\infty\| \\
&\leq \|H_{\phi z^n}\| \leq \|\phi z^n + \mathbf{H}_w^\infty\| + \|\phi z^n + \mathbf{H}_z^\infty\| \\
&= \|\phi + \mathbf{H}_w^\infty\| + \|\phi + \bar{z}^n \mathbf{H}_z^\infty\|.
\end{aligned}
$$

Since ϕ is continuous, $\lim_{n\to\infty} \|\phi + \bar{z}^n \mathbf{H}_z^\infty\| = 0$.

Proposition 2. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. Then*

$$\sup_{-\infty \leq r \leq 0} \|\phi + \mathbf{H}_r^\infty\| \leq \|H_\phi\| \leq \|P\|_p \|\phi + H^\infty\|.$$

Proof. For the lower estimate of $\|H_\phi\|$, the proof is almost parallel to the case of $r = 0$ and $-\infty$ which were proved in Theorem 1. When $p = 2$, the proposition was proved in [8, Theorem 1] with a gap (see [9]). The point is to prove that the linear span of $\bigcup_{n=-\infty}^{\infty} Z^n H^p$ is dense in $\mathbf{H}_r^p$ when $r \neq 0$ and $r \neq -\infty$. For $-\infty < r < 0$,

$$ZH^p \supset \{z^s w^t \ ; \ (s,t) \in \mathbf{Z}^2, \ s \geq m_0, \ t \geq n_0\}$$

and

$$Z^{-1}H^p \supset \{z^s w^t \ ; \ (s,t) \in \mathbf{Z}^2, \ s \geq -m_0, \ t \geq -n_0\}.$$

where $Z = Z_r = z^{m_0} w^{n_0}$, $r = n_0/m_0$ and $n_0 < 0$, $m_0 > 0$. Hence $\bigcup_{n=-\infty}^{\infty} Z^n H^p$ contains $\{z^m w^n \ ; \ (m,n) \in S_r \cup$ (the left side ray of L_r from the origin)$\}$ and the linear span of

$\{z^m w^n \ ; \ (m,n) \in S_r \cup$ (the left side ray of L_r from the origin)$\}$ is dense in $\mathbf{H}^p_r$. This implies that the linear span of $\bigcup_{n=-\infty}^{\infty} Z^n H^p$ is dense in $\mathbf{H}^p_r$.

Corollary 3. *Suppose $p = 2$ and $\bar{\phi}$ is a function in $H^\infty_0 = \{f \in H^\infty \ ; \ \int f dm = 0\}$. Then*

$$\left(\int |\phi|^2 dm\right)^{1/2} \leq \|H_\phi\| \leq \|\phi\|_\infty.$$

Proof. If $\bar{\phi}$ is in H^∞_0, then $\bar{\phi}$ is orthogonal to $\mathbf{H}^2_r$ and hence $\|\phi + \mathbf{H}^\infty_r\| \geq \left(\int |\phi|^2 dm\right)^{1/2}$.

Corollary 4. *Suppose $1 < p < \infty$ and $\phi = \phi_w \phi_z$ is a function in L^∞ where ϕ_w is unimodular in $L^\infty(T_w, m_w)$ and ϕ_z is unimodular in $L^\infty(T_z, m_z)$. Then*

$$\max(\|\phi_z + H^\infty\|, \ \|\phi_w + H^\infty\|) \leq \|H_\phi\| \leq \|P\|_p \|\phi + H^\infty\|.$$

Proof. This is a corollary of Proposition 2 by the following equality :

$$\|\phi_z \phi_w + \mathbf{H}^\infty_z\| = \|\phi_z + \mathbf{H}^\infty_z\| = \|\phi_z + H^\infty(T_z, m_z)\| = \|\phi_z + H^\infty\|.$$

§3. Weighted norm inequality

If W is a nonnegative function in L^1, then

$$\left(\frac{1}{m(E \times F)} \int_{E \times F} W^{-\frac{1}{p-1}} dm\right)^{1-p}$$
$$\leq \frac{1}{m_z(E)} \int_E dm_z \left(\frac{1}{m_w(F)} \int_F W^{-\frac{1}{p-1}} dm_w\right)^{1-p} \leq \frac{1}{m(E \times F)} \int_{E \times F} W dm.$$

where $1 < p < \infty$ and $E \times F$ is a Borel set on T^2. If $W(z,w) = W_1(z) W_2(w)$, and W_1 and W_2 are nonnegative functions in $L^1(T_z) = L^1(T, m_z)$ and $L^1(T_w) = L^1(T, m_w)$, respectively, then

$$\left(\frac{1}{m(E \times F)} \int_{E \times F} W^{-\frac{1}{p-1}} dm\right)^{1-p} = \left(\frac{1}{m_z(E)} \int_E W_1^{\frac{1}{p-1}} dm_z\right)^{1-p} \left(\frac{1}{m_w(F)} \int_F W_2^{-\frac{1}{p-1}} dm_w\right)^{1-p},$$
$$\frac{1}{m_z(E)} \int_E dm_z \left(\frac{1}{m_w(F)} \int_F W^{-\frac{1}{p-1}} dm_w\right)^{1-p} = \frac{1}{m_z(E)} \int_E W_1 dm_z \left(\frac{1}{m_w(F)} \int_F W_2^{-\frac{1}{p-1}} dm_w\right)^{1-p}$$

and

$$\frac{1}{m(E\times F)}\int_{E\times F} Wdm = \frac{1}{m_z(E)}\int_E W_1 dm_z \frac{1}{m_w(F)}\int_F W_2 dm_w.$$

Suppose $1 < p < \infty$ and W is a nonnegative function in L^1. We say that W satisfies (A_p)-condition for w if there exists a positive finite constant γ such that

$$\frac{1}{m(E\times I)}\int_{E\times I} Wdm \le \gamma \frac{1}{m_z(E)}\int_E dm_z \left(\frac{1}{m_w(I)}\int_I W^{-\frac{1}{p-1}} dm_w\right)^{1-p}$$

where E is a Borel set in T_z and I is an interval in T_w. Similarly (A_p)-condition for z can be defined. If W satisfies (A_p)-condition for w and z, then we say that W satisfies (A_p)-condition.

Using a theorem of Hunt, Muckenhoupt and Wheeden [6] on T, we give the generalization to T^2. This is known essentially in [3]. This will be used in Section 4.

Lemma 3. *Suppose $1 < p < \infty$ and W is a nonnegative function in L^1.*

$$\int |f|^p W dm \le \gamma_p \int |f+\bar{g}|^p W dm \quad (f \in \mathbf{H}_w^\infty,\ g \in w\mathbf{H}_w^\infty)$$

with γ_p independent of f and g if and only if W satisfies (A_p)-condition for w.

Proof. It is easy to see that W satisfies (A_p)-condition for w if and only if for $a.e. m_z$ W satisfies (A_p)-condition of one variable. In a theorem of Hunt, Muckenhoupt and Wheeden (cf. [5, Theorem 6.1 in Chapter VI]), the constant Γ_p of (A_p)-condition and the constant γ_p of a weighted norm inequality are equivalent, that is, $0 < \varepsilon \le \Gamma_p/\gamma_p \le 1/\varepsilon$. This implies the lemma with Fubini's theorem.

Theorem 3. *Suppose $1 < p < \infty$ and W is a nonnegative function in L^1.*

$$\int |f|^p W dm \le \gamma_p \int |f+\bar{g}|^p W dm \quad (f \in H,\ g \in K)$$

with γ_p independent of f and g if and only if W satisfies (A_p)-condition.

Proof. Suppose that

$$\int |f|^p W dm \le \gamma_p \int |f+\bar{g}|^p W dm \quad (f \in H,\ g \in K).$$

Then for any nonnegative integer n,

$$\int |\bar{z}^n f|^p W dm \le \gamma_p \int |\bar{z}^n f + \bar{z}^n \bar{g}|^p W dm.$$

Since $K^\infty \supset w\mathbf{H}_w^\infty$, if $g \in w\mathbf{H}_w^\infty$ then $z^n g \in w\mathbf{H}_w^\infty$ and $\bar{z}^n f \in \mathbf{H}_w^\infty$ for any $n \ge 0$. Since $\bigcup_{n=1}^{\infty} \bar{z}^n H$ is dense in $\mathbf{H}_w^\infty$, for $F \in \mathbf{H}_w^\infty$ and $G \in w\mathbf{H}_w^\infty$

$$\int |F|^p W dm \le \gamma_p \int |F+\bar{G}|^p W dm.$$

Now Lemma 3 implies that W satisfies (A_p)-condition for w. The same argument implies that W satisfies (A_p)-condition for z. Conversely, suppose that W satisfies (A_p)-condition. By Lemma 3, for the weight W we have weighted norm inequalities for $\mathbf{H}_w^\infty + \bar{w}\bar{\mathbf{H}}_w^\infty \to \mathbf{H}_w^\infty$ and $\mathbf{H}_z^\infty + \bar{z}\bar{\mathbf{H}}_z^\infty \to \mathbf{H}_z^\infty$. This implies the weighted norm inequality for $H + \bar{K} \to H$. In fact, this is a simple result of the following decomposition. If $f \in H$ and $g \in K$, then $\bar{g} = g_1 + g_2$ where $g_1 \in \mathbf{H}_w^\infty \cap \bar{K}$ and $g_2 \in \bar{w}\bar{\mathbf{H}}_w^\infty \cap \bar{K}$, and so

$$f + \bar{g} = (f + g_1) + g_2$$

where $f + g_1 \in \mathbf{H}_w^\infty \cap L$. Then $f \in H^\infty$ and $g_1 \in \bar{z}\bar{H}_z^\infty$.

Our (A_p)-condition on T^2 seems to be strange if we compare with that on T. A natural (A_p)-condition on T^2 may be the following : There exists a positive constant γ such that

$$\begin{aligned} &\frac{1}{m(I \times I)} \int_{I \times I} W dm \\ &\leq \gamma \left(\frac{1}{m(I \times I)} \int_{I \times I} W^{\frac{1}{p-1}} dm \right)^{1-p}. \end{aligned}$$

However this is too weak for one weighted norm inequality.

Corollary 5. *Suppose $1 < p < \infty$ and $W = W_w W_z$ where W_w is a nonnegative function in $L^1(T_w)$ and W_z is a nonnegative function in $L^1(T_w)$.*

$$\int |f|^p W dm \leq \gamma_p \int |f + \bar{g}|^p w dm \quad (f \in H, \ g \in K)$$

if and only if W_w and W_z satisefy one variable (A_p)-condition.

§4. Toeplitz operator

For ϕ in L^∞, $\mathbf{T}_\phi^w$ and $\mathbf{T}_\phi^z$ are Toeplitz operators on $\mathbf{H}_w^p$ and $\mathbf{H}_z^p$, respectively. That is , $\mathbf{T}_\phi^w f = \mathbf{P}^w(\phi f)$ $(f \in \mathbf{H}_w^p)$ and $\mathbf{T}_\phi^z f = \mathbf{P}^z(\phi f)(f \in \mathbf{H}_z^p)$. $\sigma(T_\phi)$, $\sigma(\mathbf{T}_\phi^w)$ and $\sigma(\mathbf{T}_\phi^z)$ denote the spectrums, respectively. For a nonzero function h in H^p, we call it an outer function if

$$\int \log |h| dm = \log |\int h dm|$$

(cf. [14, p73]). For a nonzero function h in $\mathbf{H}_w^p$, we call it a w-outer function if

$$\int_{T^2} \log |h| dm = \int_T \log |\int_T h dm_w| dm_z > -\infty.$$

Similary we can define an z-outer function. When a function in H^p is outer for w and z, it is called weakly outer (see [11]). If h is an outer function in H^p, then h is weakly outer.

In order to prove Lemma 4, we use a general theory of an extended weak $*$ Dirichlet algebra [7]. However we can also prove this using a general theory of a weak $*$ Dirichlet algebra. Since

$$\int_T \log \left| \int_T h dm_w \right| dm_z = \int_{T^2} \log |\mathcal{E}(h)| dm$$

where $\mathcal{E}$ is a conditional expectation from $\mathbf{H}_w^\infty$ onto $\mathcal{L}_z^\infty = \mathbf{H}_w^\infty \cap \bar{\mathbf{H}}_w^\infty$, if h is a w-outer function, then $h\mathbf{H}_w^\infty$ is dense in $\mathbf{H}_w^p$ [7].

Lemma 4. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. If $\mathbf{T}_\phi^\infty$ is left invertible on $\mathbf{H}_w^p$, $\phi = k\phi_0$ where k is invertible in $\mathbf{H}_w^\infty$, ϕ_0 is a unimodular function, and $\mathbf{T}_{\phi_0}^w$ is left invertible on $\mathbf{H}_w^p$.*

Proof. If $\mathbf{T}_\phi^w$ is left invertible on $\mathbf{H}_w^p$, then there exist a positive constant ε such that

$$\int |\phi f + \bar{g}|^p dm \geq \varepsilon \int |f|^p dm$$

for $f \in \mathbf{H}_w^p$ and $g \in w\mathbf{H}_w^p$. As $g = 0$, $\int |\phi|^p |f|^p dm \geq \varepsilon \int |f|^p dm$ for $f \in \mathbf{H}_w^p$. If v is a nonnegative function in L^1 with $\log v \in L^1$, then by Theorem 4′ in [7] $v^{1/p} = |f|$ for some function f in $\mathbf{H}_w^1 \cap L^p$. By Theorem 5 in [7], $\mathbf{H}_w^1 \cap L^p = \mathbf{H}_w^p$ and so $f \in \mathbf{H}_w^p$. Hence

$$\int |\phi|^p v dm \geq \varepsilon \int v dm$$

for all $v \in L^1$ with $v \geq 0$. This implies that ϕ is invertible in L^∞. Again by Theorem $4'\phi = k\phi_0$ for some $k \in \mathbf{H}_w^\infty$ and for some unimodular function ϕ_0. It is easy to see that k is invertible in $\mathbf{H}_w^\infty$. Since $\mathbf{T}_\phi^w = \mathbf{T}_{\phi_0}^w \mathbf{T}_k^w$ and $\mathbf{T}_k^w$ is invertible, $\mathbf{T}_{\phi_0}^w$ is left invertible.

Lemma 5. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. $\mathbf{T}_\phi^w$ is invertible on $\mathbf{H}_w^p$ if and only if there exist an invertible function k in $\mathbf{H}_w^\infty$ and a w-outer function h in $\mathbf{H}_w^p$ such that $\phi = k\bar{h}/h$ and $|h|^p$ satisfies (A_p)-condition for w.*

Proof. By Lemma 4, we may assume that ϕ is a unimodular function. If $\mathbf{T}_\phi^w$ is invertible on $\mathbf{H}_w^p$, then there exist $f \in \mathbf{H}_w^p$ and $g \in w\mathbf{H}_w^p$ such that $\phi f = 1 + \bar{g}$. Since $(\mathbf{T}_\phi^w)^*$ is invertible on $\mathbf{H}_w^q$ where $1/p + 1/q = 1$, there exist $f' \in \mathbf{H}_w^q$ and $g' \in w\mathbf{H}_w^q$ such that $\bar{\phi} f' = 1 + \bar{g}'$. Therefore $ff' = \overline{(1+g)}\ \overline{(1+g')}$ belongs to $\mathbf{H}_w^{p/2} \cap \bar{\mathbf{H}}_w^{q/2}$. When $p \geq 2$, $\mathbf{H}_w^{p/2} \cap \bar{\mathbf{H}}_w^{q/2} = \mathbf{H}_w^{p/2} \cap \bar{\mathbf{H}}_w^{p/2} = \mathcal{L}_z^{p/2}$. Hence ff' belongs to $\mathcal{L}_z^{p/2}$ and so $ff' = 1$ *a.e.* on T^2 because $g, g' \in w\mathbf{H}_w^q$. When $1 < p < 2$, $\mathbf{H}_w^{p/2} \cap \bar{\mathbf{H}}_w^{q/2} = \mathcal{L}_z^{q/2}$. This also implies $ff' = 1$ *a.e.* on T^2. Hence $f \in \mathbf{H}_w^p$ and $f^{-1} \in \mathbf{H}_w^q$, and $1 + g \in \mathbf{H}_w^p$ and $(1+g)^{-1} \in \mathbf{H}_w^q$. Since $\phi = (1+\bar{g})/f$ and $|\phi| = 1$ *a.e.* on T^2, $|f| = |1+g|$ *a.e.* on T^2 and so $1 + g = \alpha f$ for some unimodular $\alpha \in \mathcal{L}_z^\infty$. Therefore $\phi = \bar{h}/h$ where $h = \beta f$ for some unimodular $\beta \in \mathcal{L}_z^\infty$. Then h is a w-outer function in $\mathbf{H}_w^p$.

Since $\mathbf{T}_\phi^w$ is invertible on $\mathbf{H}_w^p$, there exist positive constants γ and ε such that

$$\gamma \|\phi f + \bar{g}\|_p \geq \|\mathbf{T}_\phi^w f\|_p \geq \varepsilon \|f\|_p$$

where $f \in \mathbf{H}^\infty_w$ and $g \in w\mathbf{H}^\infty_w$. Then

$$\gamma^p \int |h^{-1}f + \bar{h}^{-1}\bar{g}|^p |h|^p dm \geq \varepsilon^p \int |h^{-1}f|^p |h|^p dm$$

and hence we can show that

$$\gamma^p \int |F + \bar{G}|^p |h|^p dm \geq \varepsilon^p \int |F|^p |h|^p dm$$

where $F \in \mathbf{H}^\infty_w$ and $G \in w\mathbf{H}^\infty_w$, because h is w-outer. By Lemma 3, $|h|^p$ satisfies (A_p)-condition for w.

Conversely if $\phi = \bar{h}/h$ and $|h|^p$ satisfies (A_p)-condition for w, then by Lemma 3

$$\gamma_p \int |f + \bar{g}|^p |h|^p dm \geq \int |f|^p |h|^p dm \quad (f \in \mathbf{H}^\infty_w,\ g \in w\mathbf{H}^\infty_w)$$

and so

$$\gamma_p \int |\phi h f + \bar{h}\bar{g}|^p dm \geq \int |hf|^p dm \quad (f \in \mathbf{H}^\infty_w,\ g \in w\mathbf{H}^\infty_w).$$

Since h is a w-outer function in $\mathbf{H}^p_w$, $h\mathbf{H}^\infty_w$ is dense in $\mathbf{H}^p_w$ by [7] and so

$$\gamma_p \int |\phi F + \bar{G}|^p dm \geq \int |F|^p dm \quad (F \in \mathbf{H}^\infty_w,\ G \in w\mathbf{H}^\infty_w).$$

This implies that $\mathbf{T}^w_\phi$ is left invertible because $L^p/\bar{w}\bar{\mathbf{H}}^p_w \cong \mathbf{H}^p_w$. Since $[\mathbf{T}^w_\phi \mathbf{H}^p_w]_p \supseteq [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$, we will prove that $[\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p = \mathbf{H}^p_w$. Then the invertibility of $\mathbf{T}^w_\phi$ follows. For any n, we can write $h = \sum_{j=0}^{n} h_j w^j + w^{n+1}k_{n+1}$ where $h_j \in \mathcal{L}^p_z (0 \leq j \leq n)$ and $k_{n+1} \in \mathbf{H}^p_w$. Since h is a w-outer function, $|h_0| > 0$ *a.e.* on T^2 and $h_0 \in [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$. $[\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$ is invariant under multiplication by $u \in \mathcal{L}^\infty_z$ and so $\mathcal{L}^p_z \subset [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$. Since $\mathbf{P}^w(w\bar{h}) = w\bar{h}_0 + \bar{h}_1$, $w\bar{h}_0 \in [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$ and so $w\mathcal{L}^p_z \subset [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$. Since $\mathbf{P}^w(w^2\bar{h}) = w^2\bar{h}_0 + w\bar{h}_1 + \bar{h}_2$, similarly we can show that $w^2\mathcal{L}^p_z \subset [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$. By repeating this method, we can prove that $\mathbf{H}^p_w \subseteq [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$.

Theorem 4. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. If T_ϕ is invertible on H^p, then*

$$\phi = k_w \frac{\bar{h}_w}{h_w} = k_z \frac{\bar{h}_z}{h_z}$$

where k_t is invertible in $\mathbf{H}^\infty_t$ for $t = w, z$ and h_t is a t-outer function in $\mathbf{H}^p_t$ for $t = w, z$ such that $|h_t|^p$ satisfies (A_p)-condition for $t = w, z$.

Proof. If T_ϕ is invertible on H^p, then there exist positive constants γ and ε such that

$$\gamma \|\phi f + \bar{g}\|_p \geq \|T_\phi f\|_p \geq \varepsilon \|f\|_p$$

where $f \in H$ and $g \in K$. As in the proof of Theorem 3, for any nonegative integer n

$$\gamma^p \int |\phi \bar{z}^n f + \bar{z}^n g|^p dm \geq \varepsilon^p \int |\bar{z}^n f|^p dm$$

where $f \in H$ and $g \in (w\mathbf{H}^\infty_w) \cap L$, and so we can show that

$$\gamma\|\phi f + \bar{g}\|_p \geq \varepsilon\|f\|_p$$

where $f \in \mathbf{H}^\infty_w$ and $g \in w\mathbf{H}^\infty_w$. This implies that T^w_ϕ is left invertible on $\mathbf{H}^p_w$. Since $\mathbf{P}^w(\phi\mathbf{H}^\infty_w) \supset P(\phi\mathbf{H}^\infty_w) \supset P(\phi H)$ and $T_\phi H^p = H^p$, $\mathbf{T}^w_\phi\mathbf{H}^p_w$ is dense in $\mathbf{H}^p_w$ and so $\mathbf{T}^w_\phi$ is invertible on $\mathbf{H}^p_w$. Now Lemma 5 implies that $\phi = k_w\bar{h}_w/h_w$ where k_w is invertible in $\mathbf{H}^\infty_w$ and h_w is a w-outer function in $\mathbf{H}^p_w$ such that $|h_w|^p$ satisfies (A_p)-condition for w. The same method implies the statement about z.

Corollary 6. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. Then*

$$\sigma(T_\phi) \supseteq \sigma(\mathbf{T}^w_\phi) \cup \sigma(\mathbf{T}^z_\phi).$$

Theorem 5. *Suppose $1 < p < \infty$.*

(1) Suppose $\phi = \bar{h}/h$ for some nonzero function h in H^p. If T_ϕ is left invertible on H^p, then $|h|^p$ satisfies (A_p)-condition.

(2) Suppose k is an invertible function in H^∞, h is an outer function in H^∞ and $|h|^p$ satisfies (A_p)-condition. If $\phi = k\bar{h}/h$, then T_ϕ is invertible on H^p.

Proof. (1) If T_ϕ is invertible on H^p, then there exists a positive constant γ such that

$$\gamma\|\phi f + \bar{g}\|_p \geq \|f\|_p \quad (f \in H,\ g \in K).$$

As in the proof of Theorem 3, for any nonnegative integer n,

$$\gamma^p \int |\phi\bar{z}^n f + \bar{z}^n\bar{g}|^p dm \geq \int |\bar{z}^n f|^p dm$$

where $f \in H$ an $g \in (w\mathbf{H}^\infty_w) \cap L$, and so we can show that

$$\gamma\|\phi f + \bar{g}\|_p \geq \|f\|_p \quad (f \in \mathbf{H}^\infty_w,\ g \in w\mathbf{H}^\infty_w).$$

and

$$\gamma^p \int |h^{-1}f + \bar{h}^{-1}\bar{g}|^p|h|^p dm \geq \int |h^{-1}f|^p|h|^p dm$$

where $f \in \mathbf{H}^\infty_w$ and $g \in w\mathbf{H}^\infty_w$. For any $F \in \mathbf{H}^\infty_w$ and any $G \in w\mathbf{H}^\infty_w$,

$$\inf_{f \in \mathbf{H}^\infty_w} \int |h^{-1}f - F|^p|h|^p dm = \inf_{f \in \mathbf{H}^\infty_w} \int |f - hF|^p dm = 0$$

and

$$\inf_{g \in w\mathbf{H}^\infty_w} \int |\bar{h}^{-1}\bar{g} - \bar{G}|^p|h|^p dm = \inf_{g \in w\mathbf{H}^\infty_w} \int |\bar{g} - \bar{h}\bar{G}|^p dm = 0.$$

Hence

$$\gamma^p \int |F + \bar{G}|^p|h|^p dm \geq \int |F|^p|h|^p dm \quad (F \in \mathbf{H}^\infty_w,\ G \in w\mathbf{H}^\infty_w).$$

By the same argument, we can give the above inequality for $\mathbf{H}_z^\infty + \bar{z}\bar{\mathbf{H}}_z^\infty$ instead of $\mathbf{H}_w^\infty + \bar{w}\bar{\mathbf{H}}_w^\infty$. By Lemma 3, $|h|^p$ satisfies (A_p)-condition.

(2) Since $T_\phi = T_{\frac{\bar{h}}{h}} T_k$ and T_k is invertible on H^p, we may assume that $\phi = \bar{h}/h$. If $|h|^p$ satisfies (A_p)-condition, by Theorem 3

$$\int |f|^p |h|^p dm \le \gamma_p \int |f + \bar{g}|^p |h|^p dm \quad (f \in H,\ g \in K)$$

and so

$$\int |hf|^p dm \le \gamma_p \int |\phi hf + \bar{h}\bar{g}|^p dm \quad (f \in H,\ g \in K).$$

Since h is outer, h^{-1} belongs to N_*. Since $|h|^p$ satisfies (A_p)-condition, h^{-1} belongs to $N_* \cap L_p = H_p$. This implies that hH^p is dense in H^p because $h \in H^\infty$. Thus

$$\int |F|^p dm \le \gamma_p \int |\phi F + \bar{G}|^p dm \quad (F \in H,\ G \in K).$$

This implies that T_ϕ is left invertible because $L^p/[\bar{K}]_p \cong H^p$. If we can prove that $[T_\phi(H)]_p = [P(\bar{h}H)]_p = H^p$, then the invertibity of T_ϕ follows.

Let $h = \sum_{j=0}^{\infty} h_j$ be a homogeneous expansion of h where h_j is a homogeneous polynomial of degree j. Since h is outer, h_0 is a nonzero constant and $1 \in [P(\bar{h}H)]_p$. $P(z\bar{h}) = z\bar{h}_0 + P(z\bar{h}_1) = z\bar{h}_0 + c$ for some constant c because

$$z\bar{h} = z\bar{h}_0 + z\bar{h}_1 + z\sum_{j=2}^{\infty} \bar{h}_j.$$

Hence $z \in [P(\bar{h}H)]_p$ because $1 \in [P(\bar{h}H)]_p$. Similarly $w \in [P(\bar{h}H)]_p$. $P(z^2\bar{h}) = z^2\bar{h}_0 + P(z^2\bar{h}_1 + z^2\bar{h}_2) = z^2\bar{h}_0 + cz + d$ for some constant c and d because

$$z^2\bar{h} = z^2\bar{h}_0 + z^2(\bar{h}_1 + \bar{h}_2) + z^2\sum_{j=3}^{\infty} \bar{h}_j$$

Hence $z^2 \in [P(\bar{h}H)]_p$ because $1,\ z \in [P(\bar{h}H)]_p$. Similarly w^2 and zw belong to $[P(\bar{h}H)]_p$. By repeating this method, we can prove that $H \subset [P(\bar{h}H)]_p$.

Corollary 7. *Suppose $\phi = \phi_w\phi_z$ is a function in L^∞ where $\phi_w \in L^\infty(T_w, m_w)$ and $\phi_z \in L^\infty(T_z, m_z)$. T_ϕ is invertible on H^p if and only if T_{ϕ_w} is invertible on $H^p(T_w, m_w)$ and T_{ϕ_z} is invertible on $H^p(T_z, m_z)$.*

Proof. If T_ϕ is invertible on H^p, then both ϕ_z and ϕ_w are invertible on H^p, and there exists a positive constant ε such that

$$\int |\phi f + \bar{g}|^p dm \ge \varepsilon \int |f|^p dm$$

for $f \in H^p$ and $g \in w\mathrm{H}_w^\infty$. This implies that there exists a positive constant ε' such that

$$\int |\phi_w f + \bar{g}|^p dm \ge \varepsilon' \int |f|^p dm$$

for $f \in H^p$ and $g \in w\mathbf{H}_w^\infty$ because $\phi_z^{-1} w\mathbf{H}_w^\infty \subseteq w\mathbf{H}_w^\infty$. Hence T_{ϕ_w} is left invertible on $H^p(T_w, m_w)$. It is easy to see that $T_{\phi_w} H^p(T_w, m_w)$ is dense in $H^p(T_w, m_w)$. Thus T_{ϕ_w} is invertible on $H^p(T_w, m_w)$. Similarly T_{ϕ_z} is also invertible on $H^p(T_z, m_z)$.

Conversely if both T_{ϕ_w} and T_ϕ are invertible on $H^p(T_w, m_w)$ and $H^p(T_z, m_z)$. respectively, then by a theorem of R.Rochberg [13] $\phi = \phi_w \phi_z$ satisfies the condition in (2) of Theorem 5. Hence T_ϕ is invertible on H^p.

Remark

(1) Suppose $\phi = (2 - \bar{z}w)/(2 - z\bar{w})$. By Theorem 4, $\mathbf{T}_\phi^w$ and $\mathbf{T}_\phi^z$ are invertible on $\mathbf{H}_w^p$ and $\mathbf{H}_z^p$, respectively.

(2) If ϕ is a unimodular function and $\|\phi + \mathbf{H}_w^\infty\| + \|\phi + \mathbf{H}_z^\infty\| < 1$, then T_ϕ is left invertible on H^2. For by Theorem 1 $\|H_\phi\| < 1$ and so $\|1 - T_\phi^* T_\phi\| < 1$ because $T_\phi^* T_\phi + H_\phi^* H_\phi = I$. Suppose $\phi_a = (a - \bar{z}w)/(a - z\bar{w})$ and $|a| \geq 2$. Then $\|\phi_a + \mathbf{H}_w^\infty\| + \|\phi_a + \mathbf{H}_z^\infty\| < 1$ for some a and then $\|\bar{\phi}_a + \mathbf{H}_w^\infty\| + \|\bar{\phi}_a + \mathbf{H}_z^\infty\| < 1$. This implies that T_{ϕ_a} is invertible on H^2.

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