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# DYNAMIC MODELS OF ASSET PRICES WITH LONG MEMORY

V. ANH AND A. INOUE

**ABSTRACT.** This paper introduces a class of  $AR(\infty)$ -type models for mean-square continuous processes with stationary increments. The models allow for short- or long-memory dynamics in the processes. Their solutions are shown to have a semimartingale representation. The models are used to describe the dynamics of asset prices, which reduce to the traditional Black-Scholes model as a special case. It is shown that there exists an equivalent martingale measure under which the behaviour of the discounted price process is equal to that in the Black-Scholes environment. As a result, the European option price is given by the Black-Scholes formula. The variance of the log price ratio is also obtained.

## 1. INTRODUCTION

We consider a risky asset with price  $S(t)$  at time  $t$ . We suppose that  $S(t)$  is of the form

$$(1.1) \quad S(t) = S(0) \exp Z(t) \quad (t \geq 0),$$

where  $S(0)$  is a positive constant and  $(Z(t) : t \in \mathbf{R})$  is a zero-mean, mean-square continuous process with stationary increments such that  $Z(0) = 0$ . Let  $\sigma \in (0, \infty)$ ,  $m \in \mathbf{R}$ , and  $(W(t) : t \in \mathbf{R})$  be a one-dimensional standard Brownian motion such that  $W(0) = 0$ . If  $Z(t)$  is of the form

$$(1.2) \quad Z(t) = mt + \sigma W(t),$$

then this is the Black–Scholes stock price model. In this case, the dynamics of  $(Z(t))$  is described by the equation

$$(1.3) \quad \frac{dZ}{dt}(t) - m = \sigma \frac{dW}{dt}(t).$$

In order to allow for long memory (Beran [2], Anh and Heyde [1]) in the dynamics of  $Z(t)$ , attempts have been made to replace Brownian motion  $W(t)$  by fractional Brownian motion  $W_H(t)$  in (1.2) with Hurst index  $1/2 < H < 1$  (Lin [13], Cutland *et al.* [4], Comte and Renault [5, 6], Willinger *et al.* [18]). However this approach is not entirely satisfactory since fractional Brownian motion is not a semimartingale (Liptser and Shiriyayev [14], Lin [13], Rogers [16]), and as a result, the market is not arbitrage free (Cutland *et al.* [4], Rogers [16]).

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In this paper, we consider a stock price model in which the process  $(Z(t))$  is determined by the equation

$$(1.4) \quad \frac{dZ}{dt}(t) - m = \int_{-\infty}^t k_{\mu}(t-s) \left\{ \frac{dZ}{dt}(s) - m \right\} ds + \sigma \frac{dW}{dt}(t),$$

where  $dZ/dt$  and  $dW/dt$  are the derivatives of  $Z(t)$  and  $W(t)$  respectively in the random distribution sense (to be defined in § 4). Here, in general, for a Borel measure  $\mu$  on  $(0, \infty)$  such that  $\int_0^{\infty} (s+1)^{-1} \mu(ds) < \infty$ , we write

$$(1.5) \quad k_{\mu}(t) := I_{(0, \infty)}(t) \int_0^{\infty} e^{-ts} \mu(ds) \quad (t \in \mathbf{R}).$$

The integral on the right-hand side of (1.4) has the effect of incorporating memory into the dynamics of the process, and the constant  $m$  corresponds to the *trend*. The simplest case  $\mu = 0$  or  $k_{\mu}(\cdot) = 0$  gives the Black-Scholes model (1.3). The assumption that  $S(0)$  is a constant implies that we model the risky asset under the setting that we know its price at  $t = 0$ .

We use the following two kinds of assumptions on  $\mu$ :

$$(S) \left\{ \begin{array}{l} \mu \text{ is a (possibly zero) finite Borel measure on } (0, \infty) \\ \text{such that } \int_0^{\infty} s^{-1} \mu(ds) < 1; \end{array} \right.$$

$$(L) \left\{ \begin{array}{l} \mu \text{ is a finite Borel measure on } (0, \infty) \text{ satisfying} \\ \int_0^{\infty} s^{-1} \mu(ds) = 1, \int_0^{\infty} s^{-2} \mu(ds) = \infty, \text{ and (L1).} \end{array} \right.$$

with condition (L1) in (L) being given later in §5. Examples of  $\mu$  satisfying (S) or (L) are given in Examples 6.9, 6.5 and 5.4.

For  $\mu$  satisfying (L) or (S), we show that the solution  $(Z(t))$  (in a proper sense) to the equation (1.4) is of the form

$$(1.6) \quad Z(t) = mt + \sigma \int_0^t U_{\nu}(s) ds + \sigma W(t),$$

where  $(U_{\nu}(t) : t \in \mathbf{R})$  is a stationary process of the form

$$(1.7) \quad U_{\nu}(t) = \int_{-\infty}^t k_{\nu}(t-s) dW(s)$$

with some finite Borel measure  $\nu$  on  $(0, \infty)$  such that

$$\int_0^{\infty} \int_0^{\infty} \frac{1}{s_1 + s_2} \nu(ds_1) \nu(ds_2) = \int_0^{\infty} k_{\nu}(t)^2 dt < \infty$$

(Theorems 5.1 and 6.1). We write  $\gamma_{\nu}(\cdot)$  for the autocovariance function of  $(U_{\nu}(t))$ :

$$\gamma_{\nu}(t) := E[U_{\nu}(t)U_{\nu}(0)] \quad (t \in \mathbf{R}).$$

Then, by simple calculation, we have

$$(1.8) \quad \int_0^{\infty} \gamma_{\nu}(t) dt = \frac{1}{2} \left\{ \int_0^{\infty} \frac{1}{s} \nu(ds) \right\}^2.$$

Hence if  $\int_0^{\infty} s^{-1} \nu(ds) < \infty$ , then  $(U_{\nu}(t))$  is a *short-memory* process in the sense that  $\int_0^{\infty} \gamma_{\nu}(t) dt < \infty$ , while if  $\int_0^{\infty} s^{-1} \nu(ds) = \infty$ , then  $(U_{\nu}(t))$  is a *long-memory* process in the sense that  $\int_0^{\infty} \gamma_{\nu}(t) dt = \infty$  (see [10] and the references cited there). We show that  $(U_{\nu}(t))$  is a short-memory process under (S) (Theorem 6.1), while it is a long-memory process under (L) (Theorem 5.1). We determine the asymptotics for  $k_{\nu}(t)$  and  $\gamma_{\nu}(t)$ , as  $t \rightarrow \infty$ , in some typical cases (Theorems 5.3, 6.4, and 6.8).

The representation (1.6) with (1.7) implies that  $(Z_t)$ , hence  $(S_t)$ , is a semimartingale. In §7, using Girsanov's theorem, we show that there exists an equivalent martingale measure  $P^*$  under which the behavior of the discounted price process  $(e^{-rt}S(t) : 0 \leq t < \infty)$  with  $r \geq 0$  is equal to that in the Black-Scholes environment with volatility  $\sigma$ . In particular, the European option price is given by the Black-Scholes formula, and the constant  $\sigma$  serves as the *implied* volatility.

If  $(S(t))$  follows the Black-Scholes model (1.2), then the variance of  $\log(S(t)/S(s))$  with  $t > s \geq 0$  is given by  $(t-s)\sigma^2$ , and so  $\sigma$  is also the historical volatility. Of course this is not so unless the model is Black-Scholes. For the stock price process  $(S(t))$  in our model, we investigate the variance of  $\log(S(t)/S(s))$ , in particular, its asymptotic behavior as  $t-s \rightarrow \infty$ , in §7.

## 2. CORRESPONDENCE BETWEEN TWO MEASURES (1)

In this and next sections, we consider correspondences between two measures  $\mu$  and  $\nu$  on  $(0, \infty)$  through the relation

$$(2.1) \quad \left\{ 1 + \int_0^\infty \frac{1}{s-iz} \nu(ds) \right\} \left\{ 1 - \int_0^\infty \frac{1}{s-iz} \mu(ds) \right\} = 1 \quad (\Im z > 0).$$

This kind of results is needed in studying the correspondence between the forms (1.4) and (1.6).

**Lemma 2.1.** *Let  $n \in \mathbf{N}$ . Let  $\mu$  be a Borel measure on  $(0, \infty)$  of the form*

$$(2.2) \quad \mu = \sum_{k=1}^n a_k \delta_{r_k},$$

with

$$(2.3) \quad a_k \in (0, \infty) \quad (k = 1, 2, \dots, n),$$

$$(2.4) \quad 0 < r_1 < r_2 < \dots < r_n < \infty,$$

$$(2.5) \quad \int_0^\infty \frac{1}{s} \mu(ds) < 1.$$

Then there exists a Borel measure  $\nu$  on  $(0, \infty)$  of the form

$$(2.6) \quad \nu = \sum_{k=1}^n b_k \delta_{p_k}$$

$$(2.7) \quad b_k \in (0, \infty) \quad (k = 1, 2, \dots, n),$$

$$(2.8) \quad 0 < p_1 < r_1 < p_2 < r_2 < \dots < p_n < r_n,$$

satisfying (2.1).

*Proof.* For  $w = iz$ , we have

$$\begin{aligned} \left\{ 1 - \int_0^\infty \frac{1}{s-w} \mu(ds) \right\}^{-1} - 1 &= \left\{ \int_0^\infty \frac{1}{s-w} \mu(ds) \right\} \left\{ 1 - \int_0^\infty \frac{1}{s-w} \mu(ds) \right\}^{-1} \\ &= \left\{ \sum_{k=1}^n \frac{a_k}{r_k - w} \right\} \left\{ 1 - \sum_{k=1}^n \frac{a_k}{r_k - w} \right\}^{-1} \\ &= f(w)^{-1} \sum_{k=1}^n a_k \prod_{m \neq k} (r_m - w), \end{aligned}$$

where  $f(w)$  is a polynomial in  $w$ , of degree  $n$ , given by

$$f(w) := \prod_{k=1}^n (r_k - w) - \sum_{k=1}^n a_k \prod_{m \neq k} (r_m - w).$$

Now we have

$$f(0) = \prod_{k=1}^n r_k - \sum_{k=1}^n a_k \prod_{m \neq k} r_m = \left( \prod_{k=1}^n r_k \right) \left\{ 1 - \int_0^\infty \frac{1}{s} \mu(ds) \right\} > 0,$$

and

$$\operatorname{sgn} f(r_k) = (-1)^k \quad (k = 1, 2, \dots, n).$$

Therefore there exist positive numbers  $p_k$  ( $k = 1, 2, \dots, n$ ) satisfying (2.8) and

$$f(w) = \prod_{k=1}^n (p_k - w).$$

From  $f(p_l) = 0$ , it follows that

$$\sum_{k=1}^n a_k \prod_{m \neq k} (r_m - p_l) = \prod_{k=1}^n (r_k - p_l) \quad (l = 1, 2, \dots, n).$$

So, in the partial fraction decomposition

$$f(w)^{-1} \sum_{k=1}^n a_k \prod_{m \neq k} (r_m - w) = \sum_{l=1}^n \frac{b_l}{p_l - w},$$

the coefficients  $b_l$  are given by

$$b_l = \frac{\sum_{k=1}^n a_k \prod_{m \neq k} (r_m - p_l)}{\prod_{k \neq l} (p_k - p_l)} = \frac{\prod_{k=1}^n (r_k - p_l)}{\prod_{k \neq l} (p_k - p_l)} > 0 \quad (l = 1, 2, \dots, n).$$

With these  $p_l$  and  $b_l$ , the measure  $\nu$  defined by (2.6) gives the desired measure.  $\square$

Conversely, we have the following lemma.

**Lemma 2.2.** *Let  $n \in \mathbf{N}$ . Let  $\nu$  be a Borel measure on  $(0, \infty)$  of the form (2.6) with (2.7) and*

$$0 < p_1 < p_2 < \dots < p_n < \infty.$$

*Then there exists a Borel measure  $\mu$  on  $(0, \infty)$  satisfying (2.1)–(2.3), (2.5), and (2.8).*

*Proof.* For  $w = iz$ , we have

$$\begin{aligned} 1 + \left\{ 1 + \int_0^\infty \frac{1}{s-w} \nu(ds) \right\}^{-1} &= \left\{ \int_0^\infty \frac{1}{s-w} \nu(ds) \right\} \left\{ 1 + \int_0^\infty \frac{1}{s-w} \nu(ds) \right\}^{-1} \\ &= \left\{ \sum_{k=1}^n \frac{b_k}{p_k - w} \right\} \left\{ 1 + \sum_{k=1}^n \frac{b_k}{p_k - w} \right\}^{-1} \\ &= g(w)^{-1} \sum_{k=1}^n b_k \prod_{m \neq k} (p_m - w), \end{aligned}$$

where  $g(w)$  is a polynomial in  $w$ , of degree  $n$ , given by

$$g(w) := \prod_{k=1}^n (p_k - w) + \sum_{k=1}^n b_k \prod_{m \neq k} (p_m - w).$$

Since

$$\begin{aligned} g(w) &= (-1)^n w^n + \dots, \\ \text{sgn } g(p_k) &= (-1)^{k-1} \quad (k = 1, 2, \dots, n), \end{aligned}$$

there exist positive numbers  $r_k$  ( $k = 1, 2, \dots, n$ ) satisfying (2.8) and

$$g(w) = \prod_{k=1}^n (r_k - w).$$

From  $g(r_l) = 0$ , it follows that

$$\sum_{k=1}^n a_k \prod_{m \neq k} (p_m - r_l) = - \prod_{k=1}^n (p_k - r_l) \quad (l = 1, 2, \dots, n).$$

Therefore, in the partial fraction decomposition

$$g(w)^{-1} \sum_{k=1}^n b_k \prod_{m \neq k} (p_m - w) = \sum_{l=1}^n \frac{a_l}{r_l - w},$$

the coefficients  $a_l$  are given by

$$a_l = \frac{\sum_{k=1}^n b_k \prod_{m \neq k} (p_m - r_l)}{\prod_{k \neq l} (r_k - r_l)} = - \frac{\prod_{k=1}^n (p_k - r_l)}{\prod_{k \neq l} (r_k - r_l)} > 0 \quad (l = 1, 2, \dots, n).$$

With these  $r_l$  and  $a_l$ , we define the measure  $\mu$  by (2.2). Then

$$\begin{aligned} (2.9) \quad 1 - \int_0^\infty s^{-1} \mu(ds) &= \left\{ 1 + \lim_{y \downarrow 0} \int_0^\infty \frac{1}{s+y} \nu(ds) \right\}^{-1} \\ &= \left\{ 1 + \int_0^\infty \frac{1}{s} \nu(ds) \right\}^{-1} > 0. \end{aligned}$$

Thus  $\mu$  satisfies (2.5). □

We call a Borel measure  $\mu$  on  $(0, \infty)$  *simple* if it is of the form (2.2), for some  $n \in \mathbb{N}$ , with (2.3) and (2.4). We define

$$\mathcal{M}_s = \left\{ \mu : \begin{array}{l} \mu \text{ is a (possibly zero) simple measure on } (0, \infty) \\ \text{such that } \int_0^\infty s^{-1} \mu(ds) < 1 \end{array} \right\},$$

$$\mathcal{N}_s = \{ \nu : \nu \text{ is a (possibly zero) simple measure on } (0, \infty) \}.$$

**Definition 2.3.** We define the one-to-one and onto map

$$\theta_s : \mathcal{M}_s \ni \mu \mapsto \nu = \theta_s(\mu) \in \mathcal{N}_s$$

by (2.1).

**Example 2.4.** Let  $\mu = a\delta_r$  with  $0 < a < r$ . Then  $\int_0^\infty s^{-1} \mu(ds) < 1$ , and so  $\mu \in \mathcal{M}_s$ . Since

$$\left\{ 1 - \int_0^\infty \frac{1}{s-w} \mu(ds) \right\}^{-1} - 1 = \frac{a}{r-a-w},$$

we have  $\theta_s(\mu) = a\delta_{r-a}$ .



### 3. CORRESPONDENCE BETWEEN TWO MEASURES (2)

In the proofs of this and next sections, we regard Borel measures  $\eta$  on  $(0, \infty)$  as Borel measures on  $[0, \infty]$  by  $\eta\{0\} = \eta\{\infty\} = 0$  if necessary.

We define

$$\mathcal{M}_0 = \left\{ \mu : \begin{array}{l} \mu \text{ is a (possibly zero) Borel measure on } (0, \infty) \\ \text{such that } \int_0^\infty s^{-1}\mu(ds) < 1 \end{array} \right\},$$

$$\mathcal{N}_0 = \left\{ \nu : \begin{array}{l} \nu \text{ is a (possibly zero) Borel measure on } (0, \infty) \\ \text{such that } \int_0^\infty s^{-1}\nu(ds) < \infty \end{array} \right\}.$$

First we consider the correspondence between  $\mu$  in  $\mathcal{M}_0$  and  $\nu$  in  $\mathcal{N}_0$  through the relation (2.1).

**Theorem 3.1.** *For  $\mu \in \mathcal{M}_0$ , there exists a unique  $\nu \in \mathcal{N}_0$  satisfying (2.1). Conversely, for  $\nu \in \mathcal{N}_0$ , there exists a unique  $\mu \in \mathcal{M}_0$  satisfying (2.1).*

*Proof.* (I) Let  $\mu \in \mathcal{M}_0$ . We define the finite Borel measure  $\tilde{\mu}$  on  $[0, \infty]$  by

$$\tilde{\mu}(ds) = s^{-1}I_{(0, \infty)}(s)\mu(ds).$$

Take a sequence of simple measures  $\mu_n$  ( $n = 1, 2, \dots$ ) such that  $s^{-1}\mu_n(ds)$  converges weakly to  $\tilde{\mu}$  on  $[0, \infty]$ . Since

$$\tilde{\mu}[0, \infty] = \int_0^\infty s^{-1}\mu(ds) < 1,$$

we may assume that  $\int_0^\infty s^{-1}\mu_n(ds) < 1$  for  $n = 1, 2, \dots$ . We put  $\nu_n := \theta_s(\mu_n)$  and  $\tilde{\nu}_n(ds) := s^{-1}\nu_n(ds)$ . Then we have, for  $n = 1, 2, \dots$ ,

$$(3.1) \quad \left\{ 1 + \int_0^\infty \frac{s}{s-iz} \tilde{\nu}_n(ds) \right\} \left\{ 1 - \int_0^\infty \frac{s}{s-iz} \tilde{\mu}_n(ds) \right\} = 1 \quad (\Im z > 0).$$

Letting  $y \downarrow 0$  in (3.1) with  $z = iy$ , we see that

$$\sup_n \tilde{\nu}_n[0, \infty] = \sup_n \frac{\tilde{\mu}_n[0, \infty]}{1 - \tilde{\mu}_n[0, \infty]} < \infty.$$

Therefore, by the Helly selection principle, we can find a subsequence  $n'$  such that  $\tilde{\nu}_{n'}$  converges weakly to  $\tilde{\nu}$ , say, on  $[0, \infty]$ . It follows that

$$\left\{ 1 + \tilde{\nu}\{\infty\} + \int_0^\infty \frac{1}{s-iz} \nu(ds) \right\} \left\{ 1 - \int_0^\infty \frac{1}{s-iz} \mu(ds) \right\} = 1 \quad (\Im z > 0),$$

where  $\nu$  is the measure on  $(0, \infty)$  defined by

$$\nu(ds) := I_{(0, \infty)}(s)s\tilde{\nu}(ds).$$

Letting  $y \uparrow \infty$  in this with  $z = iy$ , we see that  $1 + \tilde{\nu}\{\infty\} = 1$  or  $\tilde{\nu}\{\infty\} = 0$ . This proves the first half of the theorem.

(II) Let  $\nu \in \mathcal{N}_0$ . We define the finite Borel measure  $\tilde{\nu}$  on  $[0, \infty]$  by

$$\tilde{\nu}(ds) = s^{-1}I_{(0, \infty)}(s)\nu(ds).$$

Take a sequence of simple measures  $\nu_n$  ( $n = 1, 2, \dots$ ) such that  $s^{-1}\nu_n(ds)$  converges weakly to  $\tilde{\nu}$  on  $[0, \infty]$ . We put  $\mu_n := \theta_s^{-1}(\nu_n)$  and  $\tilde{\mu}_n(ds) := s^{-1}\mu_n(ds)$ . Then we have (3.1) for  $n = 1, 2, \dots$ . Letting  $y \downarrow 0$  in (3.1) with  $z = iy$ , we see that

$$\sup_n \tilde{\mu}_n[0, \infty] = \sup_n \frac{\tilde{\nu}_n[0, \infty]}{1 + \tilde{\nu}_n[0, \infty]} < \infty.$$

Therefore, again by the Helly selection principle, we can find a subsequence  $n'$  such that  $\tilde{\mu}_{n'}$  converges weakly to  $\tilde{\mu}$ , say, on  $[0, \infty]$ . It follows that

$$\left\{ 1 + \int_0^\infty \frac{1}{s - iz} \nu(ds) \right\} \left\{ 1 - \tilde{\mu}\{\infty\} - \int_0^\infty \frac{1}{s - iz} \mu(ds) \right\} = 1 \quad (\Im z > 0),$$

where  $\mu$  is the measure on  $(0, \infty)$  defined by

$$\mu(ds) := I_{(0, \infty)}(s) s \tilde{\mu}(ds).$$

Letting  $y \uparrow \infty$  in this with  $z = iy$ , we see that  $1 - \tilde{\mu}\{\infty\} = 1$  or  $\tilde{\mu}\{\infty\} = 0$ . Finally, by the same argument as (2.9), it follows that  $\int_0^\infty s^{-1} \mu(ds) < 1$ . This proves the second half of the theorem.  $\square$

**Definition 3.2.** We define the one-to-one and onto map

$$\theta_0 : \mathcal{M}_0 \ni \mu \mapsto \nu = \theta_0(\mu) \in \mathcal{N}_0$$

by (2.1).

We define

$$\mathcal{M}_1 = \left\{ \mu : \begin{array}{l} \mu \text{ is a Borel measure on } (0, \infty) \text{ such that} \\ \int_0^\infty s^{-1} \mu(ds) = 1, \int_0^\infty s^{-2} \mu(ds) = \infty \end{array} \right\},$$

$$\mathcal{N}_1 = \left\{ \nu : \begin{array}{l} \nu \text{ is a Borel measure on } (0, \infty) \text{ such that} \\ \int_0^\infty (s+1)^{-1} \nu(ds) < \infty, \int_0^\infty s^{-1} \nu(ds) = \infty \end{array} \right\}.$$

Next we consider the correspondence between  $\mu$  in  $\mathcal{M}_1$  and  $\nu$  in  $\mathcal{N}_1$  through the relation (2.1).

**Theorem 3.3.** For  $\mu \in \mathcal{M}_1$ , there exists a unique  $\nu \in \mathcal{N}_1$  satisfying (2.1). Conversely, for  $\nu \in \mathcal{N}_1$ , there exists a unique  $\mu \in \mathcal{M}_1$  satisfying (2.1).

*Proof.* (I) Let  $\mu \in \mathcal{M}_1$ . Set  $m := \inf\{s : s \in \text{supp}(\mu)\}$ . If  $m = 0$ , then

$$\int_0^\infty s^{-2} \mu(ds) \leq m^{-1} \int_{[m, \infty)} s^{-1} \mu(ds) < \infty,$$

contradicting the condition  $\int_0^\infty s^{-2} \mu(ds) = \infty$ . Thus  $m = 0$ . Therefore there exists an  $N \in \mathbb{N}$ , such that, for  $\mu_n(ds) := I_{(1/n, \infty)}(s) \mu(ds)$ ,

$$\int_0^\infty s^{-1} \mu_n(ds) < \int_0^\infty s^{-1} \mu(ds) = 1 \quad (n \geq N),$$

whence  $\mu_n \in \mathcal{M}_0$  for  $n \geq N$ . We define  $\nu_n := \theta_0(\mu_n) \in \mathcal{N}_0$ . Then, as  $n \rightarrow \infty$ ,

$$\int_0^\infty \frac{1}{s+1} \nu_n(ds) = \frac{\int_0^\infty (1+s)^{-1} \mu_n(ds)}{1 - \int_0^\infty (1+s)^{-1} \mu_n(ds)} \rightarrow \frac{\int_0^\infty (1+s)^{-1} \mu(ds)}{1 - \int_0^\infty (1+s)^{-1} \mu(ds)} \in (0, \infty),$$

so that

$$\sup_n \int_0^\infty \frac{1}{1+s} \nu_n(ds) < \infty.$$

Therefore, for  $\tilde{\nu}_n(ds) := (s+1)^{-1} I_{(0, \infty)}(s) \nu_n(ds)$ , there exists a subsequence  $n'$  such that  $\tilde{\nu}_{n'}$  converges weakly to a finite Borel measure  $\tilde{\nu}$ , say, on  $[0, \infty]$ . It follows that, for  $\Im z > 0$ ,

$$(3.2) \quad \left\{ 1 - \frac{\tilde{\nu}\{0\}}{iz} + \tilde{\nu}\{\infty\} + \int_0^\infty \frac{1}{s - iz} \nu(ds) \right\} \left\{ 1 - \int_0^\infty \frac{1}{s - iz} \mu(ds) \right\} = 1,$$

where  $\nu$  is the measure on  $(0, \infty)$  defined by  $\nu(ds) := (1+s) I_{(0, \infty)}(s) \tilde{\nu}(ds)$ .

Letting  $y \uparrow \infty$  in (3.2) with  $z = iy$ , we have  $\tilde{\nu}\{\infty\} = 0$ . From  $\int_0^\infty s^{-1}\mu(ds) = 1$ , it follows that

$$1 - \int_0^\infty \frac{1}{s+y}\mu(ds) = y \int_0^\infty \frac{1}{s(s+y)}\mu(ds),$$

hence

$$\left\{ y + \tilde{\nu}\{0\} + \int_0^\infty \frac{1}{(s/y)+1}\nu(ds) \right\} \int_0^\infty \frac{1}{s(s+y)}\mu(ds) = 1 \quad (y > 0),$$

and so

$$\tilde{\nu}\{0\} = \lim_{y \downarrow 0} \left\{ \int_0^\infty \frac{1}{s(s+y)}\mu(ds) \right\}^{-1} = 0.$$

Finally,

$$\int_0^\infty \frac{1}{s}\nu(ds) = \lim_{y \downarrow 0} \left\{ \int_0^\infty \frac{1}{s+y}\mu(ds) \right\} \left\{ 1 - \int_0^\infty \frac{1}{s+y}\mu(ds) \right\}^{-1} = \infty.$$

Thus  $\nu$  is the desired element of  $\mathcal{N}_1$ .

(II) Conversely, for  $\nu \in \mathcal{N}_1$ , define  $\nu_n(ds) := I_{(1/n, \infty)}(s)\nu(ds)$  ( $n = 1, 2, \dots$ ). Then  $\nu_n \in \mathcal{N}_0$ . We put  $\mu_n := \theta_0^{-1}(\nu_n) \in \mathcal{M}_0$  for  $n = 1, 2, \dots$ . Then, as  $n \rightarrow \infty$ ,

$$\int_0^\infty \frac{1}{s+1}\mu_n(ds) = \frac{\int_0^\infty (1+s)^{-1}\nu_n(ds)}{1 + \int_0^\infty (1+s)^{-1}\nu_n(ds)} \rightarrow \frac{\int_0^\infty (1+s)^{-1}\nu(ds)}{1 + \int_0^\infty (1+s)^{-1}\nu(ds)} \in (0, \infty),$$

hence

$$\sup_n \int_0^\infty \frac{1}{1+s}\mu_n(ds) < \infty.$$

Therefore, for  $\tilde{\mu}_n(ds) := (s+1)^{-1}I_{(0, \infty)}(s)\mu_n(ds)$ , there exists a subsequence  $n'$  such that  $\tilde{\mu}_{n'}$  converges weakly to a finite Borel measure  $\tilde{\mu}$ , say, on  $[0, \infty]$ . It follows that, for  $\Im z > 0$ ,

$$(3.3) \quad \left\{ 1 + \int_0^\infty \frac{1}{s-iz}\nu(ds) \right\} \left\{ 1 + \frac{\tilde{\mu}\{0\}}{iz} - \tilde{\mu}\{\infty\} - \int_0^\infty \frac{1}{s-iz}\mu(ds) \right\} = 1,$$

where  $\mu$  is the measure on  $(0, \infty)$  defined by  $\mu(ds) := (1+s)I_{(0, \infty)}(s)\tilde{\mu}(ds)$ .

Letting  $y \uparrow \infty$  in (3.3) with  $z = iy$ , we have  $\tilde{\mu}\{\infty\} = 0$ . Moreover, letting  $y \downarrow 0$  in

$$\tilde{\mu}\{0\} + \int_0^\infty \frac{1}{(s/y)+1}\mu(ds) = \frac{\int_0^\infty \{(s/y)+1\}^{-1}\nu(ds)}{1 + \int_0^\infty \{(s/y)+1\}^{-1}\nu(ds)},$$

we get  $\tilde{\mu}\{0\} = 0$ . Since  $\int_0^\infty s^{-1}\nu(ds) = \infty$ , we have

$$1 - \int_0^\infty \frac{1}{s}\mu(ds) = \lim_{y \downarrow 0} \left\{ 1 + \int_0^\infty \frac{1}{s+y}\nu(ds) \right\}^{-1} = 0.$$

In particular, this implies

$$1 - \int_0^\infty \frac{1}{s+y}\mu(ds) = \int_0^\infty \frac{y}{s(s+y)}\mu(ds),$$

and so

$$\int_0^\infty \frac{1}{s^2}\mu(ds) = \lim_{y \downarrow 0} \left\{ y + \int_0^\infty \frac{1}{(s/y)+1}\nu(ds) \right\}^{-1} = \infty.$$

Thus  $\mu$  is the desired element of  $\mathcal{M}_1$ . □

**Definition 3.4.** We define the one-to-one and onto map

$$\theta_1 : \mathcal{M}_1 \ni \mu \mapsto \nu = \theta_1(\mu) \in \mathcal{N}_1$$

by (2.1).

**Lemma 3.5.** For  $\mu \in \mathcal{M}_0$  (resp.  $\mu \in \mathcal{M}_1$ ), we set  $\nu := \theta_0(\mu) \in \mathcal{N}_0$  (resp.  $\nu := \theta_1(\mu) \in \mathcal{N}_1$ ). Then  $\mu(0, \infty) = \nu(0, \infty)$ . In particular,  $\mu(0, \infty) < \infty$  if and only if  $\nu(0, \infty) < \infty$ .

*Proof.* It follows from (2.1) that

$$\nu(0, \infty) = \lim_{y \uparrow \infty} \int_0^\infty \frac{y}{y+s} \nu(ds) = \lim_{y \uparrow \infty} \frac{\int_0^\infty y/(y+s) \mu(ds)}{1 - \int_0^\infty 1/(y+s) \mu(ds)} = \mu(0, \infty).$$

Thus the lemma follows.  $\square$

#### 4. STATIONARY RANDOM DISTRIBUTIONS

We recall some notation in the theory of stationary random distributions (cf. [11] and [10]). We denote by  $H$  the Hilbert space of  $\mathbf{C}$ -valued random variables, defined on a probability space  $(\Omega, \mathcal{F}, P)$ , with expectation zero and finite variance:

$$H := \{a \in L^2(\Omega, \mathcal{F}, P) : E[a] = 0\},$$

$$(a, b)_H := E[ab], \quad \|a\|_H := (a, a)_H^{1/2} \quad (a, b \in H).$$

By  $\mathcal{D}(\mathbf{R})$ , we denote the space of all  $\phi \in C^\infty(\mathbf{R})$  with compact support, endowed with the usual topology. A *random distribution* (with expectation zero) is a linear continuous map from  $\mathcal{D}(\mathbf{R})$  to  $H$ . We write  $\mathcal{D}'(H)$  for the class of random distributions on  $(\Omega, \mathcal{F}, P)$ . For  $X \in \mathcal{D}'(H)$ , we define  $DX \in \mathcal{D}'(H)$  by  $DX(\phi) := -X(d\phi/dx)$ . We call  $DX$  the derivative of  $X$ . For  $X \in \mathcal{D}'(H)$  and  $t \in \mathbf{R}$ , we write  $M(X)$  (resp.  $M_t(X)$ ) for the closed linear hull of  $\{X(\phi) : \phi \in \mathcal{D}(\mathbf{R})\}$  (resp.  $\{X(\phi) : \phi \in \mathcal{D}(\mathbf{R}), \text{supp } \phi \subset (-\infty, t]\}$ ) in  $H$ . For  $X, Y \in \mathcal{D}'(H)$ , we define  $P_Y X \in \mathcal{D}'(H)$  by  $P_Y X(\phi) := p_Y(X(\phi))$ , where  $p_Y$  is the orthogonal projection operator from  $H$  onto  $M(Y)$ . It holds that  $P_Y DX = DP_Y X$  for  $X, Y \in \mathcal{D}'(H)$ . Two random distributions  $X$  and  $Y$  are said to be *stationarily correlated* if

$$(X(\tau_h \phi), Y(\tau_h \psi))_H = (X(\phi), Y(\psi))_H \quad (\phi, \psi \in \mathcal{D}(\mathbf{R}), h \in \mathbf{R}),$$

where  $\tau_h$  is the shift operator defined by  $\tau_h \phi(t) := \phi(t+h)$ .

A random distribution  $X$  is called *stationary* if

$$(X(\tau_h \phi), X(\tau_h \psi))_H = (X(\phi), X(\psi))_H \quad (\phi, \psi \in \mathcal{D}(\mathbf{R}), h \in \mathbf{R}).$$

We write  $\mathcal{S}$  for the class of stationary random distributions on  $(\Omega, \mathcal{F}, P)$ . For  $X, Y \in \mathcal{S}$ , the random distribution  $P_Y X$  is also stationary if  $X$  and  $Y$  are stationarily correlated (see [10, Theorem 2.1]). For  $X \in \mathcal{S}$ , we write  $\mu_X$  for the spectral measure of  $X$ :

$$(X(\phi), X(\psi))_H = \int_{-\infty}^{\infty} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \mu_X(d\xi) \quad (\phi, \psi \in \mathcal{D}(\mathbf{R})),$$

where  $\hat{\phi}$  is the Fourier transform of  $\phi$ :  $\hat{\phi}(\xi) := \int_{-\infty}^{\infty} e^{-it\xi} \phi(t) dt$ . For  $k \in \mathbf{N} \cup \{0\}$ , we define

$$\mathcal{S}_k := \left\{ X \in \mathcal{S} : \int_{-\infty}^{\infty} (1 + \xi^2)^{-k} \mu_X(d\xi) < \infty \right\}.$$

Then  $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots$  and  $\mathcal{S} = \cup_{k=0}^{\infty} \mathcal{S}_k$  (see [11, Theorem 3.2]). The class  $\mathcal{S}_0$  can be naturally identified with that of mean-square continuous weakly stationary processes with zero expectation ([11, Theorem 4.2]). Any stationary random distribution  $X$  has the spectral representation of the form

$$X(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) Z_X(d\xi) \quad (\phi \in \mathcal{D}(\mathbf{R})),$$

where  $Z_X$  is the associated random measure. It holds that

$$M(X) = \left\{ \int_{-\infty}^{\infty} g(\xi) Z_X(d\xi) : g \in L^2(\mu_X) \right\}.$$

We say that  $X$  in  $\mathcal{S}$  is *purely nondeterministic* if  $\cap_{t \in \mathbf{R}} M_t(X) = \{0\}$ .

For  $k \in L^1(\mathbf{R}, dt)$  and  $X \in \mathcal{S}$ , we wish to define the convolution  $k * X \in \mathcal{S}$ . Formally we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} k(u) X(t-u) du \right) \phi(t) dt &= \int_{-\infty}^{\infty} k(u) \left( \int_{-\infty}^{\infty} X(t) \phi(t+u) dt \right) du \\ &= \int_{-\infty}^{\infty} k(u) X(\tau_u \phi) du. \end{aligned}$$

With this in mind, we define  $k * X \in \mathcal{S}$  by

$$(k * X)(\phi) := \int_{-\infty}^{\infty} k(u) X(\tau_u \phi) du \quad (\phi \in \mathcal{D}(\mathbf{R})),$$

where the integral on the right-hand side is in the sense of  $H$ -valued Bochner integral.

We set

$$\mathcal{M} = \left\{ \rho : \rho \text{ is a Borel measure on } (0, \infty) \text{ such that } \int_0^{\infty} (1+s)^{-1} \rho(ds) < \infty \right\}.$$

For  $\rho \in \mathcal{M}$ , we define

$$F_{\rho}(z) := \int_0^{\infty} \frac{1}{s - iz} \rho(ds) \quad (\Im z \geq 0).$$

Recall  $k_{\rho}(t)$  from (1.5). We have

$$(4.1) \quad \rho(0, \infty) = k_{\rho}(0+),$$

$$(4.2) \quad \int_0^{\infty} s^{-1} \rho(ds) = \int_0^{\infty} k_{\rho}(t) dt,$$

$$(4.3) \quad \int_0^{\infty} s^{-2} \rho(ds) = \int_0^{\infty} dt \int_t^{\infty} k_{\rho}(u) du.$$

As in [10, Proposition 2.3], we have the following proposition.

**Proposition 4.1.** *Let  $k \in L^1(\mathbf{R}, dt)$  and  $X \in \mathcal{S}$ . Then*

$$(k * X)(\phi) = \int_{-\infty}^{\infty} \hat{k}(-\xi) \hat{\phi}(\xi) Z_X(d\xi) \quad (\phi \in \mathcal{D}(\mathbf{R})).$$

*In particular, for  $\mu \in \mathcal{M}$  such that  $\int_0^{\infty} s^{-1} \mu(ds) < \infty$ , we have*

$$(k_{\mu} * X)(\phi) = \int_{-\infty}^{\infty} F_{\mu}(\xi) \hat{\phi}(\xi) Z_X(d\xi) \quad (\phi \in \mathcal{D}(\mathbf{R})).$$

Notice that  $k_{\mu} * X$  can be written formally as

$$k_{\mu} * X(t) = \int_{-\infty}^t k_{\mu}(t-s) X(s) ds.$$

**Proposition 4.2.** *Let  $X \in \mathcal{S}$ , and let  $\mu \in \mathcal{M}$  such that  $\int_0^{\infty} s^{-1} \mu(ds) < \infty$ . Then  $X$  is stationarily correlated with  $X - k_{\mu} * X$ .*

*Proof.* By Proposition 4.1, we have

$$(X - k_{\mu} * X)(\phi) = \int_{-\infty}^{\infty} \{1 - F_{\mu}(\xi)\} \hat{\phi}(\xi) Z_X(d\xi).$$

The lemma follows from this. □

**Proposition 4.3.** *Let  $X \in \mathcal{S}$ .*

- (1) *For  $\mu \in \mathcal{M}_0$ ,  $X$  satisfies  $X = k_\mu * X$  if and only if  $X = 0$ .*
- (2) *For  $\mu \in \mathcal{M}_1$ ,  $X$  satisfies  $X = k_\mu * X$  if and only if  $X = a$  for some  $a \in H$ .*

*Proof.* By Proposition 4.1, we have

$$\|(X - k_\mu * X)(\phi)\|_H^2 = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 |1 - F_\mu(\xi)|^2 \mu_X(d\xi).$$

If  $\mu \in \mathcal{M}_0$ , then

$$\Re\{1 - F_\mu(\xi)\} = 1 - \int_0^\infty \frac{s}{s^2 + \xi^2} \mu(ds) \geq 1 - \int_0^\infty s^{-1} \mu(ds) > 0,$$

while if  $\mu \in \mathcal{M}_1$ , then

$$|1 - F_\mu(\xi)|^2 > 0 \quad (\xi \neq 0), \quad = 0 \quad (\xi = 0).$$

The lemma follows easily from these.  $\square$

## 5. LONG-MEMORY MODEL

Let  $\sigma > 0$  and let  $W = (W(t) : t \in \mathbf{R})$  be a one-dimensional standard Brownian motion such that  $W(0) = 0$ , defined on  $(\Omega, \mathcal{F}, P)$ . Since  $W$  is a process with stationary increments, the derivative  $DW$  is a stationary random distribution (see [11]). We are concerned with the following equation

$$(5.1) \quad X = k_\mu * X + \sigma DW.$$

It should be noted that the equation (5.1) can be written formally as

$$(5.2) \quad X(t) = \int_{-\infty}^t k_\mu(t-s)X(s)ds + \sigma \frac{dW}{dt}(t).$$

For  $\nu \in \mathcal{M}$  such that

$$(5.3) \quad \int_0^\infty k_\nu(t)^2 dt < \infty,$$

we define a real, centered, weakly stationary process  $(U_\nu(t) : t \in \mathbf{R})$  by

$$(5.4) \quad U_\nu(t) := \int_{-\infty}^t k_\nu(t-s)dW(s) \quad (t \in \mathbf{R}).$$

Then  $(U_\nu(t))$  is purely nondeterministic, and (5.4) corresponds to the so-called *canonical representation* of  $(U_\nu(t))$ ; thus,  $M_t(U_\nu) = M_t(DW)$  for  $t \in \mathbf{R}$ . On the other hand, the spectral representation of  $U_\nu$ , as a stationary random distribution, is given by

$$(5.5) \quad U_\nu(\phi) = \int_{-\infty}^{\infty} F_\nu(\xi) \hat{\phi}(\xi) Z_{DW}(d\xi) \quad (\phi \in \mathcal{D}(\mathbf{R})).$$

We refer to [10] for these results.

If  $\nu$  is a finite measure in  $\mathcal{N}_1$  such that

$$(5.6) \quad \int_1^\infty k_\nu(t)^2 dt < \infty,$$

then  $\nu$  satisfies (5.3) since

$$\int_0^1 k_\nu(t)^2 dt \leq k_\nu(0+) \int_0^1 k_\nu(t) dt < \infty.$$

In this case, since  $\int_0^\infty k_\nu(t)dt = \infty$ , the stationary process  $(U_\nu(t))$  defined by (5.4) is long-memory. Now recall the condition (L) in §1; we define the condition (L1) for  $\mu \in \mathcal{M}_1$  there by

$$(L1) \quad \nu = \theta_1(\mu) \text{ satisfies (5.6).}$$

**Theorem 5.1.** *Let  $\sigma > 0$ . Let  $\mu$  be a measure satisfying (L), and let  $\nu := \theta_1(\mu)$ . Then a stationary random distribution  $X$  satisfies (5.1) if and only if  $X = X_0 + a$ , where  $a$  is an arbitrary element of  $M(DW)^\perp$  and  $X_0$  is the stationary random distribution defined by*

$$(5.7) \quad X_0 = \sigma U_\nu + \sigma DW.$$

*In particular,  $X_0$  is the only purely nondeterministic stationary random distribution that satisfies (5.1).*

*Proof.* Let  $X$  be a stationary random distribution satisfying (5.1). Then, by Proposition 4.2,  $X$  and  $DW = X - k_\mu * X$  are stationarily correlated. We define  $X_1 := X - P_{DW}X$ . Then, by [10, Theorem 2.1],  $X_1$  is a stationary random distribution satisfying  $X_1 = k_\mu * X_1$ . So, by Proposition 4.3 (2), we see that  $X - P_{DW}X = a$  for some  $a \in M(DW)^\perp$ .

We set  $X_0 := P_{DW}X$ . Then, again by [10, Theorem 2.1], there exists  $g \in L^2(\mathbf{R}, (1+x^2)^{-k}d\xi)$ , for some  $k \in \mathbf{N} \cup \{0\}$ , such that

$$X_0(\phi) = \int_{-\infty}^{\infty} g(\xi)\hat{\phi}(\xi)Z_{DW}(d\xi).$$

By Proposition 4.1, we have

$$k_\mu * X_0(\phi) = \int_{-\infty}^{\infty} F_\mu(\xi)g(\xi)\hat{\phi}(\xi)Z_{DW}(d\xi).$$

Since  $\mu_{DW}(d\xi) = (2\pi)^{-1}d\xi$ , it follows that, for  $\phi \in \mathcal{D}(\mathbf{R})$ ,

$$\begin{aligned} 0 &= \|X_0(\phi) - k_\mu * X_0(\phi) - \sigma DW(\phi)\|_H^2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 |(1 - F_\mu(\xi))g(\xi) - \sigma|^2 d\xi. \end{aligned}$$

This implies

$$g(\xi) = \frac{\sigma}{1 - F_\mu(\xi)} = \sigma F_\nu(\xi) + \sigma,$$

hence

$$X_0(\phi) = \int_{-\infty}^{\infty} \{\sigma F_\nu(\xi) + \sigma\}\hat{\phi}(\xi)Z_{DW}(d\xi) = \sigma U_\nu(\phi) + \sigma DW(\phi).$$

Thus  $X_0$  is given by (5.7). Conversely, we can easily show that  $X = X_0 + a$  with (5.7) and  $a \in M(DW)^\perp$  is a stationary random distribution that satisfies (5.1).

For  $z = x + iy$  with  $y > 0$ , it holds that

$$\Re\{\sigma F_\nu(z) + \sigma\} = \sigma + \int_0^\infty \frac{s+y}{(s+y)^2 + x^2} \nu(ds) > 0.$$

By Theorem A in the appendix, this implies

$$M_t(X_0) = M_t(DW) \quad (t \in \mathbf{R}),$$

(see, e.g., [9]), hence  $M_t(X_0 + a) = M_t(DW) + \mathbf{C}a$  for  $a \in M(DW)^\perp$ . Therefore

$$\bigcap_t M_t(X_0 + a) = \left\{ \bigcap_t M_t(X_0) \right\} \oplus \mathbf{C}a = \left\{ \bigcap_t M_t(DW) \right\} \oplus \mathbf{C}a = \mathbf{C}a.$$

Thus  $X_0 + a$  with  $a \in M(DW)^\perp$  is purely nondeterministic if and only if  $a = 0$ .  $\square$

We give a sufficient condition for (L1).

**Lemma 5.2.** *Let  $0 < p < 1$  and let  $\ell(\cdot)$  be a slowly varying function at infinity. Let  $\mu \in \mathcal{M}_1$  and define  $\nu \in \mathcal{N}_1$  by  $\nu := \theta_1(\mu)$ . Then*

$$(5.8) \quad k_\mu(t) \sim t^{-(p+1)} \ell(t) p \quad (t \rightarrow \infty)$$

if and only if

$$(5.9) \quad k_\nu(t) \sim \frac{t^{-(1-p)}}{\ell(t)} \cdot \frac{\sin(p\pi)}{\pi} \quad (t \rightarrow \infty).$$

*Proof.* We prove only (5.8)  $\Rightarrow$  (5.9); the converse implication (5.9)  $\Rightarrow$  (5.8) can be proved in the same way. Since  $\int_0^\infty k_\mu(t) dt = 1$ , we have, by integration by parts,

$$1 - \int_0^\infty e^{-ty} k_\mu(t) dt = y \int_0^\infty e^{-ty} \left( \int_t^\infty k_\mu(s) ds \right) dt \quad (y > 0).$$

Hence

$$\int_0^\infty e^{-ty} k_\nu(t) dt = \frac{\int_0^\infty e^{-ty} k_\mu(t) dt}{1 - \int_0^\infty e^{-ty} k_\mu(t) dt} = \frac{\int_0^\infty e^{-ty} k_\mu(t) dt}{y \int_0^\infty e^{-ty} \left( \int_t^\infty k_\mu(s) ds \right) dt}.$$

Now (5.8) implies

$$\int_t^\infty k_\mu(s) ds \sim t^{-p} \ell(t) \quad (t \rightarrow \infty),$$

so that

$$y \int_0^\infty e^{-ty} \left( \int_t^\infty k_\mu(s) ds \right) dt \sim y^p \ell(1/y) \Gamma(1-p) \quad (y \downarrow 0)$$

(cf. [3, Theorem 1.7.6]). On the other hand,

$$\lim_{y \downarrow 0} \int_0^\infty e^{-ty} k_\mu(t) dt = 1.$$

Thus

$$y \int_0^\infty e^{-ty} k_\nu(t) dt \sim \frac{y^{1-p}}{\ell(1/y) \Gamma(1-p)} \quad (y \downarrow 0).$$

By Karamata's Tauberian theorem ([3, Theorem 1.7.6]), this implies (5.9).  $\square$

**Theorem 5.3.** *Let  $0 < p < 1/2$  and let  $\ell(\cdot)$  be a slowly varying function at infinity. Let  $\mu$  be a finite measure in  $\mathcal{M}_1$  satisfying (5.8). Then  $\mu$  satisfies (L1), whence (L). If we put  $\nu := \theta_1(\mu)$ , then  $k_\nu(\cdot)$  satisfies (5.9), and the autocovariance function  $\gamma_\nu(\cdot)$  of the stationary process  $(U_\nu(t))$  in (5.7) satisfies*

$$(5.10) \quad \gamma_\nu(t) \sim \frac{t^{-(1-2p)}}{\ell(t)^2} \left( \frac{\sin(p\pi)}{\pi} \right)^2 B(1-2p, p) \quad (t \rightarrow \infty).$$

*Proof.* By Lemma 5.2,  $k_\nu(\cdot)$  satisfies (5.9). Thus (5.6) holds. Moreover, by [10, Proposition 4.3], we have

$$\gamma_\nu(t) = \int_0^\infty k_\nu(t+s) k_\nu(s) ds \sim \frac{t^{-(1-2p)}}{\ell(t)^2} \left( \frac{\sin(p\pi)}{\pi} \right)^2 B(1-2p, p) \quad (t \rightarrow \infty).$$

Thus the theorem follows.  $\square$



**Example 5.4.** For  $0 < p < 1/2$ , set  $\mu(ds) := \Gamma(p)^{-1} s^p e^{-s} ds$ . Then we have

$$k_\mu(t) = \frac{p}{(t+1)^{p+1}} \quad (t > 0).$$

Since  $k_\mu(0+) < \infty$ ,  $\int_0^\infty k_\mu(t) dt = 1$ ,  $\int_0^\infty dt \int_t^\infty k_\mu(s) ds = \infty$ , and

$$k_\mu(t) \sim pt^{-(p+1)} \quad (t \rightarrow \infty),$$

we find that  $\mu$  satisfies (L); take  $\ell(\cdot) = 1$  in Theorem 5.1.

## 6. SHORT-MEMORY MODEL

If  $\nu$  is a finite measure in  $\mathcal{N}_0$ , then

$$\int_0^\infty k_\nu(t)^2 dt \leq k_\nu(0+) \int_0^\infty k_\nu(t) dt < \infty.$$

Thus  $\nu$  satisfies (5.3). In this case, since  $\int_0^\infty k_\nu(t) dt < \infty$ , the stationary process  $(U_\nu(t))$  is short-memory.

**Theorem 6.1.** Let  $\sigma > 0$  and let  $\mu$  be a measure satisfying (S). Set  $\nu := \theta_0(\mu)$ . Then the stationary random distribution  $X$  defined by

$$(6.1) \quad X = \sigma U_\nu + \sigma DW.$$

is the unique stationary random distribution that satisfies (5.1).

*Proof.* It follows from Lemma 3.5 that  $\nu(0, \infty) < \infty$ . Hence (5.3) holds. Let  $X$  be a stationary random distribution satisfying (5.1). As in the proof of Theorem 5.1 but using Proposition 4.3 (1) instead of (2), we see that  $X_1 = 0$ , i.e.,  $X = P_{DW}X$ . Then, in a similar manner, we find that  $X$  is given by (6.1).  $\square$

We investigate the asymptotics for  $k_\nu(t)$  and  $\gamma_\nu(t)$  as  $t \rightarrow \infty$  when  $k_\mu(t)$  is regularly varying.

**Lemma 6.2.** Let  $\mu \in \mathcal{M}_0$  and  $\nu := \theta_0(\mu)$ . Let  $0 < p < \infty$  and  $\ell(\cdot)$  be a slowly varying function at  $\infty$ . Then

$$(6.2) \quad k_\mu(t) \sim t^{-(p+1)} \ell(t) p \quad (t \rightarrow \infty)$$

if and only if

$$(6.3) \quad k_\nu(t) \sim t^{-(p+1)} \ell(t) \frac{p}{\{1 - \int_0^\infty k_\mu(u) du\}^2} \quad (t \rightarrow \infty).$$

*Proof.* We prove only the implication (6.2)  $\Rightarrow$  (6.3). The converse implication (6.3)  $\Rightarrow$  (6.2) can be proved in a similar fashion. Thus we assume (6.2). We set  $n := [p]$ , where  $[\cdot]$  denotes the integer part. We define

$$f(y) := 1 - \int_0^\infty e^{-yt} k_\mu(t) dt \quad (y > 0).$$

By differentiating both sides of

$$\int_0^\infty e^{-yt} k_\nu(t) dt = \frac{1}{f(y)} - 1 \quad (y > 0)$$

$n + 1$  times with respect to  $y$ , we obtain

$$\int_0^\infty e^{-yt} t^{n+1} k_\nu(t) dt = \frac{\int_0^\infty e^{-yt} t^{n+1} k_\mu(t) dt}{f(y)^2} + \frac{F_{n+1}(y)}{f(y)^{n+2}} \quad (y > 0),$$

where  $F_{n+1}(y)$  is a polynomial in  $\{f^{(k)}(y) : k = 0, 1, \dots, n\}$  (see [8, Lemma 3.2]). Since  $n+1 - (p+1) = n-p > -1$  and

$$t^{n+1}k_\mu(t) \sim t^{n-p}\ell(t)p \quad (t \rightarrow \infty),$$

it follows that

$$\int_0^\infty e^{-yt}t^{n+1}k_\mu(t)dt \sim y^{-n+p-1}\ell(1/y)p\Gamma(n-p+1) \quad (y \rightarrow 0+)$$

(see [3, Theorem 1.7.6]). On the other hand, for any  $\epsilon > 0$  and  $0 \leq k \leq n$ , we have

$$y^\epsilon f^{(k)}(y) \rightarrow 0 \quad (y \rightarrow 0+)$$

(cf. [8, Lemma 3.5]), and so

$$\frac{F_{n+1}(y)}{y^{-n+p-1}\ell(1/y)} \rightarrow 0 \quad (y \rightarrow 0+).$$

Thus

$$\int_0^\infty e^{-yt}t^{n+1}k_\nu(t)dt \sim y^{-n+p-1}\ell(1/y)\frac{p\Gamma(n-p+1)}{\{1 - \int_0^\infty k_\mu(u)du\}^2} \quad (y \rightarrow 0+).$$

Since the function  $\log\{t^{n+1}k_\nu(t)\}$  is slowly increasing ([3, §1.7.6]), Karamata's Tauberian theorem (cf. [3, Theorem 1.7.6]) yields

$$t^{n+1}k_\nu(t) \sim t^{n-p}\ell(t)\frac{p}{\{1 - \int_0^\infty k_\mu(u)du\}^2} \quad (t \rightarrow \infty),$$

or (6.3), as desired.  $\square$

**Remark 6.3.** *The condition  $\mu \in \mathcal{M}_0$  implies*

$$1 - \int_0^\infty k_\mu(u)du = 1 - \int_0^\infty s^{-1}\mu(ds) > 0.$$

**Theorem 6.4.** *Let  $0 < p < \infty$  and  $\ell(\cdot)$  be a slowly varying function at  $\infty$ . Let  $\mu$  be a measure satisfying (S), and set  $\nu := \theta_0(\mu)$ . Let  $(U_\nu(t) : t \in \mathbf{R})$  be the stationary process in (6.1) with autocovariance function  $\gamma_\nu(\cdot)$ . Then (6.2) implies (6.3) and*

$$(6.4) \quad \gamma_\nu(t) \sim t^{-(p+1)}\ell(t)\frac{p \int_0^\infty k_\mu(u)du}{\{1 - \int_0^\infty k_\mu(u)du\}^3} \quad (t \rightarrow \infty).$$

*Proof.* That (6.2) implies (6.3) is a direct consequence of Lemma 6.2. Now we have

$$\gamma_\nu(t) = \int_0^\infty k_\nu(t+s)k_\nu(s)ds \sim k_\nu(t) \left( \int_0^\infty k_\nu(s)ds \right) \quad (t \rightarrow \infty)$$

(see [8, Lemma 3.8]). Since

$$\int_0^\infty k_\nu(s)ds = \frac{\int_0^\infty k_\mu(s)ds}{1 - \int_0^\infty k_\mu(s)ds},$$

(6.4) follows.  $\square$

**Example 6.5.** *For  $0 < p < \infty$  and  $0 < c < 1$ , we put*

$$\mu(ds) = \frac{c}{\Gamma(p)}s^p e^{-s} ds.$$

*Then  $k_\mu(t) = pc(1+t)^{-p-1}$  for  $t > 0$ . We see that  $\mu$  satisfies (S) and (6.2) with  $\ell(\cdot) = c$ .*

We now investigate the asymptotics for  $k_\nu(t)$  and  $\gamma_\nu(t)$  when  $k_\mu(t)$  decays exponentially as  $t \rightarrow \infty$ . For a finite Borel measure  $\rho$  on  $(0, \infty)$ , we define  $s_0(\rho) \in [0, \infty)$  by

$$s_0(\rho) := \inf\{s : s \in \text{supp } \rho\}.$$

Then it holds that

$$\lim_{t \rightarrow \infty} \frac{\log k_\rho(t)}{t} = -s_0(\rho)$$

(see [15, Proposition 3.2]). This implies that  $k_\rho(t)$  decays exponentially if and only if  $s_0(\rho) > 0$ .

**Lemma 6.6.** *Let  $\mu \in \mathcal{M}_0$ , and let  $\nu := \theta_0(\mu) \in \mathcal{N}_0$ . Then the following are equivalent:*

$$(6.5) \quad s_0(\mu) > 0,$$

$$(6.6) \quad s_0(\nu) > 0.$$

*Proof.* We prove only the implication (6.5)  $\Rightarrow$  (6.6). The converse (6.6)  $\Rightarrow$  (6.5) can be proved in a similar fashion. Assume (6.5). Then, in part (I) of the proof of Theorem 3.1, we can choose the sequence of simple measures  $\mu_n$  so that  $\text{supp } \mu_n \subset [s_0(\mu), \infty)$  for  $n = 1, 2, \dots$ . Moreover, by taking  $\mu_n + (1/n)\delta_{s_0(\mu)}$ , we may (and shall) assume that  $s_0(\mu_n) = s_0(\mu)$  for  $n = 1, 2, \dots$ . Then, by (2.8), we find that the simple measures  $\tilde{\nu}_n$  are of the form

$$\tilde{\nu}_n = b(n)\delta_{p(n)} + \tilde{\eta}_n \quad (n = 1, 2, \dots),$$

where  $b(n) \in (0, \infty)$ ,  $p(n) \in (0, s_0(\mu))$ , and  $\tilde{\eta}_n$  are simple measures such that  $\text{supp } \tilde{\eta}_n \subset (s_0(\mu), \infty)$ . Since

$$b(n) \leq \tilde{\nu}_n(0, \infty) \leq \sup_m \tilde{\nu}_m(0, \infty) < \infty,$$

$$\tilde{\eta}_n(s_0(\mu), \infty) \leq \tilde{\nu}_n(0, \infty) \leq \sup_m \tilde{\nu}_m(0, \infty) < \infty,$$

we can choose the subsequence  $n'$  there so that, for some  $b \in [0, \infty)$ ,  $p \in [0, s_0(\mu)]$  and finite Borel measure  $\tilde{\eta}$  on  $[s_0(\mu), \infty]$ , we have  $b(n') \rightarrow b$ ,  $p(n') \rightarrow p$ , and  $\tilde{\eta}_{n'} \rightarrow \tilde{\eta}$  weakly on  $[s_0(\mu), \infty]$ . By the arguments in the part (I) of the proof of Theorem 3.1, we find that  $\nu := \theta_0(\mu)$  is of the form

$$\nu = \begin{cases} b\delta_p + \eta & \text{if } p > 0, \\ \eta & \text{if } p = 0, \end{cases}$$

where  $\eta$  is the measure on  $(0, \infty)$  defined by

$$\eta(ds) := I_{[s_0(\mu), \infty)}(s) s \tilde{\eta}(ds).$$

Thus (6.6) follows. □

**Remark 6.7.** *For  $\mu \in \mathcal{M}_0$  and  $\nu := \theta_0(\mu) \in \mathcal{N}_0$ , we can prove  $s_0(\nu) \leq s_0(\mu)$ .*

**Theorem 6.8.** *Let  $\mu$  be a measure satisfying (S), and set  $\nu := \theta_0(\mu)$ . Let  $(U_\nu(t) : t \in \mathbf{R})$  be the stationary process in (6.1) with autocovariance function  $\gamma_\nu(\cdot)$ . If  $k_\mu(t)$  decays exponentially as  $t \rightarrow \infty$ , then both  $k_\nu(t)$  and  $\gamma_\nu(t)$  decay exponentially as  $t \rightarrow \infty$ .*

*Proof.* The exponential decay of  $k_\nu(t)$  follows from Lemma 6.6. Now

$$\gamma_\nu(t) = \int_0^\infty e^{-ts} \sigma(ds) \quad (t \in \mathbf{R})$$

with

$$\sigma(ds) := \left\{ \int_0^\infty \frac{1}{s+u} \nu(du) \right\} \nu(ds).$$

Clearly  $s_0(\sigma) = s_0(\nu)$ , and so  $\gamma_\nu(t)$  also decays exponentially.  $\square$

**Example 6.9.** Let  $\mu$  be as in Example 2.4. Clearly  $\mu$  satisfies (S). In this case,  $k_\mu(t) = ae^{-rt}$  for  $t > 0$  and  $k_\nu(t) = ae^{-(r-a)t}$ . The autocovariance function  $\gamma_\nu(t)$  of  $U_\nu$  in (6.1) is given by

$$\gamma_\nu(t) = a^2 \int_0^\infty e^{-(r-a)(|t|+s)} e^{-(r-a)s} ds = \frac{a^2 e^{-(r-a)|t|}}{2(r-a)} \quad (t \in \mathbf{R}).$$

## 7. RISKY ASSET MODEL

Let  $\sigma \in (0, \infty)$ ,  $m \in \mathbf{R}$ , and  $(W(t) : t \in \mathbf{R})$  be a one-dimensional standard Brownian motion such that  $W(0) = 0$ , defined on  $(\Omega, \mathcal{F}, P)$ . We consider a risky asset with price  $S(t)$  at time  $t$ . We suppose that  $S(t)$  is of the form (1.1) with

$$Z(t) = mt + Y(t) \quad (t \in \mathbf{R}),$$

where  $(Y(t) : t \in \mathbf{R})$  is a zero-mean, mean-square continuous process with stationary increments such that  $Y(0) = 0$ . We also suppose that the derivative  $X := DY$  is the (purely nondeterministic) solution to (5.1) for  $\mu$  satisfying (S) (resp. (L)). Then, by Theorem 5.1 (resp. Theorem 6.1) and [11, Theorem 6.1],  $Z(t)$  is of the form (1.6) with  $\nu = \theta_0(\mu)$  (resp.  $\nu = \theta_1(\mu)$ ), whence

$$(7.1) \quad S(t) = S(0) \exp \left\{ mt + \sigma \int_0^t U_\nu(s) ds + \sigma W(t) \right\} \quad (t \geq 0).$$

As before, we write  $\gamma_\nu(\cdot)$  for the autocovariance function of the stationary process  $(U_\nu(t) : t \in \mathbf{R})$ .

Let  $\mathcal{N}$  be the class of all  $P$ -negligible sets from  $\mathcal{F}$ . We use the following  $P$ -augmented filtration  $(\mathcal{F}_t)_{t \geq 0}$ :

$$\mathcal{F}_t := \bigcap_{\epsilon > 0} \sigma(\mathcal{G}_{t+\epsilon} \cup \mathcal{N}) \quad (t \geq 0),$$

where

$$\mathcal{G}_t := \sigma\{W(s) : -\infty < s \leq t\} \quad (t \geq 0).$$

Then, with respect to  $(\mathcal{F}_t)_{t \geq 0}$ , the Brownian motion  $(W(t) : t \geq 0)$  is a  $(\mathcal{F}_t)$ -Brownian motion, and the process  $(U_\nu(t) : t \geq 0)$  is  $(\mathcal{F}_t)$ -adapted, as is desired here.

Suppose that we are in a market in which the riskless asset price  $S_0(t)$  follows  $S_0(t) = \exp(rt)$  for  $t \geq 0$ , where  $r$  is a nonnegative constant. We write  $\tilde{S}(t)$  for the discounted price of the risky asset:  $\tilde{S}(t) := e^{-rt}S(t)$ . We put

$$a := \frac{m - r + \frac{1}{2}\sigma^2}{\sigma},$$

$$W^*(t) := W(t) + \int_0^t \{a + U_\nu(s)\} ds \quad (t \geq 0).$$

Then we have

$$(7.2) \quad \tilde{S}(t) = S(0) \exp \left( \sigma W^*(t) - \frac{1}{2} \sigma^2 t \right).$$

**Lemma 7.1.** Let  $0 < t < \infty$  and  $0 < \delta < \gamma_\nu(0)^{-1}$ . Then

$$(7.3) \quad E \left[ \exp \left\{ \frac{1}{2} \int_t^{t+\delta} (a + U_\nu(s))^2 ds \right\} \right] < \infty.$$

*Proof.* By Jensen's inequality, we have

$$\exp \left\{ \frac{1}{2} \int_t^{t+\delta} (a + U_\nu(s))^2 ds \right\} \leq \frac{1}{\delta} \int_t^{t+\delta} \exp \left\{ \frac{\delta}{2} (a + U_\nu(s))^2 \right\} ds.$$

Since  $(U_\nu(s))$  is a stationary Gaussian process, it follows that

$$\begin{aligned} E \left[ \exp \left\{ \frac{1}{2} \int_t^{t+\delta} (a + U_\nu(s))^2 ds \right\} \right] &\leq \frac{1}{\delta} \int_t^{t+\delta} E \left[ \exp \left\{ \frac{\delta}{2} (a + U_\nu(s))^2 \right\} \right] ds \\ &= E \left[ \exp \left\{ \frac{\delta}{2} (a + U_\nu(0))^2 \right\} \right] \\ &= \frac{1}{\sqrt{2\pi\gamma_\nu(0)}} \int_{-\infty}^{\infty} \exp \left\{ \frac{\delta}{2} (a + x)^2 \right\} \exp \left\{ -\frac{x^2}{2\gamma_\nu(0)} \right\} dx. \end{aligned}$$

Thus the lemma follows.  $\square$

**Remark 7.2.** Let  $\{t_n\}_{n=0}^\infty$  be a sequence of real numbers with  $0 = t_0 < \dots < t_n \uparrow \infty$ , such that  $t_n - t_{n-1} < \gamma_\nu(0)^{-1}$  for  $n = 1, 2, \dots$ . Then, by Lemma 7.1, we have

$$E \left[ \exp \left\{ \frac{1}{2} \int_{t_{n-1}}^{t_n} (a + U_\nu(s))^2 ds \right\} \right] < \infty \quad (n = 1, 2, \dots).$$

Therefore, by [12, Corollary 5.14] and Girsanov's theorem (cf. [12, §3.5]), there exists a probability measure  $P^*$ , equivalent to  $P$ , under which  $(W^*(t) : 0 \leq t < \infty)$  is a standard Brownian motion. From (7.2), we see that the behaviour of  $(\tilde{S}(t) : 0 \leq t < \infty)$  under  $P^*$  is equal to that in the Black-Scholes environment with volatility  $\sigma$ . In particular, for any  $T > 0$ , the prices of European calls and puts with maturity  $T$  are given by the Black-Scholes formulas with implied volatility  $\sigma$ .

We now turn to the variance of  $\log(S(t)/S(s))$  for  $t > s \geq 0$ .

**Lemma 7.3.** Let  $t > s \geq 0$ . Then

$$(7.4) \quad \begin{aligned} &\text{Var}\{\log(S(t)/S(s))\} \\ &= \left\{ (t-s) + 2 \int_0^{t-s} du \int_0^u k_\nu(v) dv + 2 \int_0^{t-s} du \int_0^u \gamma_\nu(v) dv \right\} \sigma^2. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} \text{Var}\{\log(S(t)/S(s))\} &= E \left[ \left\{ W(t) - W(s) + \int_s^t U_\nu(u) du \right\}^2 \right] \sigma^2 \\ &= \left\{ (t-s) + 2E \left[ (W(t) - W(s)) \int_s^t U_\nu(u) du \right] + \int_s^t \int_s^t \gamma_\nu(u-v) dudv \right\} \sigma^2. \end{aligned}$$

By simple calculation, we get

$$\int_s^t \int_s^t \gamma_\nu(u-v) dudv = 2 \int_0^{t-s} du \int_0^u \gamma_\nu(v) dv.$$

Now, for  $s \leq u \leq t$ ,

$$E[(W(t) - W(s))U_\nu(u)] = E[(W(u) - W(s))U_\nu(u)] = \int_s^u k_\nu(u-v) dv,$$

whence

$$E \left[ (W(t) - W(s)) \int_s^t U_\nu(u) du \right] = \int_s^t \int_s^u k_\nu(u-v) dudv = \int_0^{t-s} du \int_0^u k_\nu(v) dv.$$

Thus the lemma follows.  $\square$

From (7.3), we find that, in our model,

$$\text{Var}\{\log(S(t)/S(s))\} \geq (t-s)\sigma^2 \quad (t > s \geq 0).$$

In other words, the historical volatility  $\geq$  the implied volatility. The equality holds only when the model is Black-Scholes.

Now we investigate the asymptotic behavior of  $\text{Var}\{\log(S(t)/S(s))\}$  as  $t-s \rightarrow \infty$ . First we consider the short-memory case.

**Proposition 7.4.** *We assume (S). Then*

$$(7.5) \quad \begin{aligned} \text{Var}\{\log(S(t)/S(s))\} &\sim (t-s) \left\{ 1 + \int_0^\infty k_\nu(u) du \right\}^2 \sigma^2 \\ &= (t-s) \left\{ 1 - \int_0^\infty k_\mu(u) du \right\}^{-2} \sigma^2 \quad (t-s \rightarrow \infty). \end{aligned}$$

*Proof.* The assumption (S) implies that  $\int_0^\infty k_\nu(t) dt < \infty$ . Now

$$\int_0^\infty \gamma_\nu(t) dt = \frac{1}{2} \left\{ \int_0^\infty k_\nu(u) du \right\}^2.$$

Thus (7.5) follows from (7.4).  $\square$

Next we consider the long-memory case.

**Proposition 7.5.** *Let  $0 < p < 1/2$  and let  $\ell(\cdot)$  be a slowly varying function at infinity. We assume (L) and (5.8), hence (5.9). Then*

$$(7.6) \quad \text{Var}\{\log(S(t)/S(s))\} \sim \frac{(t-s)^{2p+1}}{\ell(t-s)^2} \left( \frac{\sin(p\pi)}{\pi} \right)^2 \frac{B(1-2p, p)}{p(2p+1)} \sigma^2 \quad (t-s \rightarrow \infty).$$

*Proof.* By Theorem 5.3, (5.9) and (5.10) hold. We then find that, among the three terms on the right-hand side of (7.4), the first and second terms are negligible relative to the third. Thus

$$\begin{aligned} \text{Var}\{\log(S(t)/S(s))\} &\sim 2\sigma^2 \int_0^{t-s} du \int_0^u \gamma_\nu(v) dv \\ &\sim \frac{(t-s)^{2p+1}}{\ell(t-s)^2} \left( \frac{\sin(p\pi)}{\pi} \right)^2 \frac{B(1-2p, p)}{p(2p+1)} \sigma^2 \end{aligned}$$

as  $t-s \rightarrow \infty$ , hence (7.6).  $\square$

It should be noted that, in (7.6), the index of  $(t-s)$  is  $2p+1$  unlike the short-memory case (7.5).

## 8. APPENDIX

If  $\phi$  is a positive measurable function on  $\mathbf{R}$  such that  $(1+t^2)^{-1} \log \phi(t) \in L^1(\mathbf{R}, dt)$ , and if

$$(8.1) \quad f(z) := \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \cdot \frac{\log \phi(t)}{1+t^2} dt \right\} \quad (\Im z > 0),$$

then we call  $f$  an *outer function*. For completeness, we prove the following (seemingly well-known) theorem.

**Theorem A.** *Let  $f(z)$  be analytic and  $\Re\{f(z)\} > 0$  in  $\{z \in \mathbf{C} : \Im z > 0\}$ . Then  $f$  is an outer function.*

*Proof.* As in Duren [7, p. 189], we consider the following mappings from  $\{z \in \mathbf{C} : \Im z > 0\}$  onto  $\{w \in \mathbf{C} : |w| < 1\}$  :

$$w = p(z) = \frac{z-i}{z+i}, \quad z = q(w) = \frac{i(1+w)}{1-w}.$$

Set

$$g(w) := f(q(w)) \quad (|w| < 1).$$

Since  $\Re\{F(w)\} > 0$ ,  $F$  is an outer function on  $\{|w| < 1\}$  (see, e.g., Duren [7, p. 51]). Hence

$$\psi(e^{i\theta}) := \lim_{r \uparrow 1} |F(re^{i\theta})|$$

exists for almost every  $\theta \in (-\pi, \pi)$  and it holds that

$$g(w) := \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} \log |\psi(e^{i\theta})| d\theta \right\} \quad (|w| < 1)$$

(cf. Rudin [17, 17.16 Theorem]). Now with the change of variables  $e^{i\theta} = p(t)$  and  $w = p(z)$ , it follows that

$$\frac{e^{i\theta} + w}{e^{i\theta} - w} = \frac{1+tz}{i(t-z)}$$

and

$$d\theta = \frac{2t}{1+t^2} dt.$$

Thus  $\phi$  defined by  $\phi(t) := \psi(q(t))$  satisfies  $(1+t^2)^{-1} \log \phi(t) \in L^1(\mathbf{R}, dt)$  and (8.1).  $\square$

From the proof above, we find that, for  $f$  and  $\phi$  in (8.1),

$$\lim_{y \downarrow 0} |f(x+iy)| = \phi(x) \quad \text{a.e. on } \mathbf{R}.$$

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