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V. ANH and A. INOUE

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DYNAMIC MODELS OF ASSET PRICES WITH LONG MEMORY

V. ANH AND A. INOUE

ABSTRACT. This paper introduces a class of AR(∞)-type models for mean-square continuous processes with stationary increments. The models allow for short- or long-memory dynamics in the processes. Their solutions are shown to have a semimartingale representation. The models are used to describe the dynamics of asset prices, which reduce to the traditional Black-Scholes model as a special case. It is shown that there exists an equivalent martingale measure under which the behaviour of the discounted price process is equal to that in the Black-Scholes environment. As a result, the European option price is given by the Black-Scholes formula. The variance of the log price ratio is also obtained.

1. INTRODUCTION

We consider a risky asset with price $S(t)$ at time t. We suppose that $S(t)$ is of the form

$$S(t) = S(0) \exp Z(t) \qquad (t \geq 0), \tag{1.1}$$

where $S(0)$ is a positive constant and $(Z(t) : t \in \mathbf{R})$ is a zero-mean, mean-square continuous process with stationary increments such that $Z(0) = 0$. Let $\sigma \in (0, \infty)$, $m \in \mathbf{R}$, and $(W(t) : t \in \mathbf{R})$ be a one-dimensional standard Brownian motion such that $W(0) = 0$. If $Z(t)$ is of the form

$$Z(t) = mt + \sigma W(t), \tag{1.2}$$

then this is the Black–Scholes stock price model. In this case, the dynamics of $(Z(t))$ is described by the equation

$$\frac{dZ}{dt}(t) - m = \sigma \frac{dW}{dt}(t). \tag{1.3}$$

In order to allow for long memory (Beran [2], Anh and Heyde [1]) in the dynamics of $Z(t)$, attempts have been made to replace Brownian motion $W(t)$ by fractional Brownian motion $W_H(t)$ in (1.2) with Hurst index $1/2 < H < 1$ (Lin [13], Cutland *et al.* [4], Comte and Renault [5, 6], Willinger *et al.* [18]). However this approach is not entirely satisfactory since fractional Brownian motion is not a semimartingale (Liptser and Shiryayev [14], Lin [13], Rogers [16]), and as a result, the market is not arbitrage free (Cutland *et al.* [4], Rogers [16]).

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In this paper, we consider a stock price model in which the process $(Z(t))$ is determined by the equation

$$\frac{dZ}{dt}(t) - m = \int_{-\infty}^{t} k_{\mu}(t-s)\left\{\frac{dZ}{dt}(s) - m\right\} ds + \sigma\frac{dW}{dt}(t), \tag{1.4}$$

where dZ/dt and dW/dt are the derivatives of $Z(t)$ and $W(t)$ respectively in the random distribution sense (to be defined in § 4). Here, in general, for a Borel measure μ on $(0, \infty)$ such that $\int_0^\infty (s+1)^{-1}\mu(ds) < \infty$, we write

$$k_{\mu}(t) := I_{(0,\infty)}(t)\int_0^\infty e^{-ts}\mu(ds) \qquad (t \in \mathbf{R}). \tag{1.5}$$

The integral on the right-hand side of (1.4) has the effect of incorporating memory into the dynamics of the process, and the constant m corresponds to the *trend*. The simplest case $\mu = 0$ or $k_{\mu}(\cdot) = 0$ gives the Black-Scholes model (1.3). The assumption that $S(0)$ is a constant implies that we model the risky asset under the setting that we know its price at $t = 0$.

We use the following two kinds of assumptions on μ:

$$\text{(S)} \begin{cases} \mu \text{ is a (possibly zero) finite Borel measure on } (0,\infty) \\ \text{such that } \int_0^\infty s^{-1}\mu(ds) < 1; \end{cases}$$

$$\text{(L)} \begin{cases} \mu \text{ is a finite Borel measure on } (0,\infty) \text{ satisfying} \\ \int_0^\infty s^{-1}\mu(ds) = 1,\ \int_0^\infty s^{-2}\mu(ds) = \infty, \text{ and (L1).} \end{cases}$$

with condition (L1) in (L) being given later in §5. Examples of μ satisfying (S) or (L) are given in Examples 6.9, 6.5 and 5.4.

For μ satisfying (L) or (S), we show that the solution $(Z(t))$ (in a proper sense) to the equation (1.4) is of the form

$$Z(t) = mt + \sigma\int_0^t U_{\nu}(s)ds + \sigma W(t), \tag{1.6}$$

where $(U_{\nu}(t) : t \in \mathbf{R})$ is a stationary process of the form

$$U_{\nu}(t) = \int_{-\infty}^{t} k_{\nu}(t-s)dW(s) \tag{1.7}$$

with some finite Borel measure ν on $(0, \infty)$ such that

$$\int_0^\infty\int_0^\infty \frac{1}{s_1+s_2}\nu(ds_1)\nu(ds_2) = \int_0^\infty k_{\nu}(t)^2 dt < \infty$$

(Theorems 5.1 and 6.1). We write $\gamma_{\nu}(\cdot)$ for the autocovariance function of $(U_{\nu}(t))$:

$$\gamma_{\nu}(t) := E[U_{\nu}(t)U_{\nu}(0)] \qquad (t \in \mathbf{R}).$$

Then, by simple calculation, we have

$$\int_0^\infty \gamma_{\nu}(t)dt = \frac{1}{2}\left\{\int_0^\infty \frac{1}{s}\nu(ds)\right\}^2. \tag{1.8}$$

Hence if $\int_0^\infty s^{-1}\nu(ds) < \infty$, then $(U_{\nu}(t))$ is a *short-memory* process in the sense that $\int_0^\infty \gamma_{\nu}(t)dt < \infty$, while if $\int_0^\infty s^{-1}\nu(ds) = \infty$, then $(U_{\nu}(t))$ is a *long-memory* process in the sense that $\int_0^\infty \gamma_{\nu}(t)dt = \infty$ (see [10] and the references cited there). We show that $(U_{\nu}(t))$ is a short-memory process under (S) (Theorem 6.1), while it is a long-memory process under (L) (Theorem 5.1). We determine the asymptotics for $k_{\nu}(t)$ and $\gamma_{\nu}(t)$, as $t \to \infty$, in some typical cases (Theorems 5.3, 6.4, and 6.8).

The representation (1.6) with (1.7) implies that (Z_t), hence (S_t), is a semimartingale. In §7, using Girsanov's theorem, we show that there exists an equivalent martingale measure P^* under which the behavior of the discounted price process $(e^{-rt}S(t) : 0 \leq t < \infty)$ with $r \geq 0$ is equal to that in the Black-Scholes environment with volatility σ. In particular, the European option price is given by the Black-Scholes formula, and the constant σ serves as the *implied* volatility.

If $(S(t))$ follows the Black-Scholes model (1.2), then the variance of $\log(S(t)/S(s))$ with $t > s \geq 0$ is given by $(t-s)\sigma^2$, and so σ is also the historical volatility. Of course this is not so unless the model is Black-Scholes. For the stock price process $(S(t))$ in our model, we investigate the variance of $\log(S(t)/S(s))$, in particular, its asymptotic behavior as $t - s \to \infty$, in §7.

2. Correspondence between two measures (1)

In this and next sections, we consider correspondences between two measures μ and ν on $(0, \infty)$ through the relation

$$\left\{1 + \int_0^\infty \frac{1}{s - iz}\nu(ds)\right\}\left\{1 - \int_0^\infty \frac{1}{s - iz}\mu(ds)\right\} = 1 \quad (\Im z > 0). \tag{2.1}$$

This kind of results is needed in studying the correspondence between the forms (1.4) and (1.6).

Lemma 2.1. *Let $n \in \mathbf{N}$. Let μ be a Borel measure on $(0, \infty)$ of the form*

$$\mu = \sum_{k=1}^{n} a_k \delta_{r_k}, \tag{2.2}$$

with

$$a_k \in (0, \infty) \quad (k = 1, 2, \ldots, n), \tag{2.3}$$

$$0 < r_1 < r_2 < \cdots < r_n < \infty, \tag{2.4}$$

$$\int_0^\infty \frac{1}{s}\mu(ds) < 1. \tag{2.5}$$

Then there exists a Borel measure ν on $(0, \infty)$ of the form

$$\nu = \sum_{k=1}^{n} b_k \delta_{p_k} \tag{2.6}$$

$$b_k \in (0, \infty) \quad (k = 1, 2, \ldots, n), \tag{2.7}$$

$$0 < p_1 < r_1 < p_2 < r_2 < \cdots < p_n < r_n, \tag{2.8}$$

satisfying (2.1).

Proof. For $w = iz$, we have

$$\begin{aligned}
\left\{1 - \int_0^\infty \frac{1}{s - w}\mu(ds)\right\}^{-1} - 1 &= \left\{\int_0^\infty \frac{1}{s - w}\mu(ds)\right\}\left\{1 - \int_0^\infty \frac{1}{s - w}\mu(ds)\right\}^{-1} \\
&= \left\{\sum_{k=1}^{n} \frac{a_k}{r_k - w}\right\}\left\{1 - \sum_{k=1}^{n} \frac{a_k}{r_k - w}\right\}^{-1} \\
&= f(w)^{-1} \sum_{k=1}^{n} a_k \prod_{m \neq k} (r_m - w),
\end{aligned}$$

where $f(w)$ is a polynomial in w, of degree n, given by

$$f(w) := \prod_{k=1}^{n}(r_k - w) - \sum_{k=1}^{n} a_k \prod_{m\neq k}(r_m - w).$$

Now we have

$$f(0) = \prod_{k=1}^{n} r_k - \sum_{k=1}^{n} a_k \prod_{m\neq k} r_m = \left(\prod_{k=1}^{n} r_k\right)\left\{1 - \int_0^\infty \frac{1}{s}\mu(ds)\right\} > 0,$$

and

$$\operatorname{sgn} f(r_k) = (-1)^k \qquad (k = 1, 2, \ldots, n).$$

Therefore there exist positive numbers p_k $(k = 1, 2, \ldots, n)$ satisfying (2.8) and

$$f(w) = \prod_{k=1}^{n}(p_k - w).$$

¿From $f(p_l) = 0$, it follows that

$$\sum_{k=1}^{n} a_k \prod_{m\neq k}(r_m - p_l) = \prod_{k=1}^{n}(r_k - p_l) \qquad (l = 1, 2, \ldots, n).$$

So, in the partial fraction decomposition

$$f(w)^{-1}\sum_{k=1}^{n} a_k \prod_{m\neq k}(r_k - w) = \sum_{l=1}^{n}\frac{b_l}{p_l - w},$$

the coefficients b_l are given by

$$b_l = \frac{\sum_{k=1}^{n} a_k \prod_{m\neq k}(r_m - p_l)}{\prod_{k\neq l}(p_k - p_l)} = \frac{\prod_{k=1}^{n}(r_k - p_l)}{\prod_{k\neq l}(p_k - p_l)} > 0 \qquad (l = 1, 2, \ldots, n).$$

With these p_l and b_l, the measure ν defined by (2.6) gives the desired measure. □

Conversely, we have the following lemma.

Lemma 2.2. *Let $n \in \mathbf{N}$. Let ν be a Borel measure on $(0, \infty)$ of the form* (2.6) *with* (2.7) *and*

$$0 < p_1 < p_2 < \cdots < p_n < \infty.$$

Then there exists a Borel measure μ on $(0, \infty)$ satisfying (2.1)–(2.3), (2.5), *and* (2.8).

Proof. For $w = iz$, we have

$$\begin{aligned} 1 + \left\{1 + \int_0^\infty \frac{1}{s-w}\nu(ds)\right\}^{-1} &= \left\{\int_0^\infty \frac{1}{s-w}\nu(ds)\right\}\left\{1 + \int_0^\infty \frac{1}{s-w}\nu(ds)\right\}^{-1} \\ &= \left\{\sum_{k=1}^{n}\frac{b_k}{p_k - w}\right\}\left\{1 + \sum_{k=1}^{n}\frac{b_k}{p_k - w}\right\}^{-1} \\ &= g(w)^{-1}\sum_{k=1}^{n} b_k \prod_{m\neq k}(p_m - w), \end{aligned}$$

where $g(w)$ is a polynomial in w, of degree n, given by

$$g(w) := \prod_{k=1}^{n}(p_k - w) + \sum_{k=1}^{n} b_k \prod_{m\neq k}(p_m - w).$$

Since

$$g(w) = (-1)^n w^n + \cdots ,$$
$$\operatorname{sgn} g(p_k) = (-1)^{k-1} \qquad (k = 1, 2, \ldots, n),$$

there exist positive numbers r_k $(k = 1, 2, \ldots, n)$ satisfying (2.8) and

$$g(w) = \prod_{k=1}^{n} (r_k - w).$$

¿From $g(r_l) = 0$, it follows that

$$\sum_{k=1}^{n} a_k \prod_{m \neq k} (p_m - r_l) = - \prod_{k=1}^{n} (p_k - r_l) \qquad (l = 1, 2, \ldots, n).$$

Therefore, in the partial fraction decomposition

$$g(w)^{-1} \sum_{k=1}^{n} b_k \prod_{m \neq k} (p_m - w) = \sum_{l=1}^{n} \frac{a_l}{r_l - w},$$

the coefficients a_l are given by

$$a_l = \frac{\sum_{k=1}^{n} b_k \prod_{m \neq k} (p_m - r_l)}{\prod_{k \neq l} (r_k - r_l)} = - \frac{\prod_{k=1}^{n} (p_k - r_l)}{\prod_{k \neq l} (r_k - r_l)} > 0 \qquad (l = 1, 2, \ldots, n).$$

With these r_l and a_l, we define the measure μ by (2.2). Then

$$\begin{aligned} 1 - \int_0^\infty s^{-1} \mu(ds) &= \left\{ 1 + \lim_{y \downarrow 0} \int_0^\infty \frac{1}{s + y} \nu(ds) \right\}^{-1} \\ &= \left\{ 1 + \int_0^\infty \frac{1}{s} \nu(ds) \right\}^{-1} > 0. \end{aligned} \tag{2.9}$$

Thus μ satisfies (2.5). □

We call a Borel measure μ on $(0, \infty)$ *simple* if it is of the form (2.2), for some $n \in \mathbf{N}$, with (2.3) and (2.4). We define

$$\mathcal{M}_{\rm s} = \left\{ \mu : \begin{array}{l} \mu \text{ is a (possibly zero) simple measure on } (0, \infty) \\ \text{such that } \int_0^\infty s^{-1} \mu(ds) < 1 \end{array} \right\},$$

$$\mathcal{N}_{\rm s} = \{ \nu : \nu \text{ is a (possibly zero) simple measure on } (0, \infty) \}.$$

Definition 2.3. *We define the one-to-one and onto map*

$$\theta_{\rm s} : \mathcal{M}_{\rm s} \ni \mu \mapsto \nu = \theta_{\rm s}(\mu) \in \mathcal{N}_{\rm s}$$

by (2.1).

Example 2.4. *Let* $\mu = a\delta_r$ *with* $0 < a < r$. *Then* $\int_0^\infty s^{-1} \mu(ds) < 1$, *and so* $\mu \in \mathcal{M}_{\rm s}$. *Since*

$$\left\{ 1 - \int_0^\infty \frac{1}{s - w} \mu(ds) \right\}^{-1} - 1 = \frac{a}{r - a - w},$$

we have $\theta_{\rm s}(\mu) = a\delta_{r-a}$.

3. Correspondence between two measures (2)

In the proofs of this and next sections, we regard Borel measures η on $(0,\infty)$ as Borel measures on $[0,\infty]$ by $\eta\{0\}=\eta\{\infty\}=0$ if necessary.

We define

$$\mathcal{M}_0=\left\{\mu:\begin{array}{l}\mu \text{ is a (possibly zero) Borel measure on } (0,\infty)\\ \text{such that } \int_0^\infty s^{-1}\mu(ds)<1\end{array}\right\},$$

$$\mathcal{N}_0=\left\{\nu:\begin{array}{l}\nu \text{ is a (possibly zero) Borel measure on } (0,\infty)\\ \text{such that } \int_0^\infty s^{-1}\nu(ds)<\infty\end{array}\right\}.$$

First we consider the correspondence between μ in $\mathcal{M}_0$ and ν in $\mathcal{N}_0$ through the relation (2.1).

Theorem 3.1. *For $\mu\in\mathcal{M}_0$, there exists a unique $\nu\in\mathcal{N}_0$ satisfying* (2.1). *Conversely, for $\nu\in\mathcal{N}_0$, there exists a unique $\mu\in\mathcal{M}_0$ satisfying* (2.1).

Proof. (I) Let $\mu\in\mathcal{M}_0$. We define the finite Borel measure $\tilde{\mu}$ on $[0,\infty]$ by

$$\tilde{\mu}(ds)=s^{-1}I_{(0,\infty)}(s)\mu(ds).$$

Take a sequence of simple measures μ_n $(n=1,2,\dots)$ such that $s^{-1}\mu_n(ds)$ converges weakly to $\tilde{\mu}$ on $[0,\infty]$. Since

$$\tilde{\mu}[0,\infty]=\int_0^\infty s^{-1}\mu(ds)<1,$$

we may assume that $\int_0^\infty s^{-1}\mu_n(ds)<1$ for $n=1,2,\dots$. We put $\nu_n:=\theta_{\mathrm{s}}(\mu_n)$ and $\tilde{\nu}_n(ds):=s^{-1}\nu_n(ds)$. Then we have, for $n=1,2,\dots$,

$$\left\{1+\int_0^\infty \frac{s}{s-iz}\tilde{\nu}_n(ds)\right\}\left\{1-\int_0^\infty \frac{s}{s-iz}\tilde{\mu}_n(ds)\right\}=1 \quad (\Im z>0). \tag{3.1}$$

Letting $y\downarrow 0$ in (3.1) with $z=iy$, we see that

$$\sup_n \tilde{\nu}_n[0,\infty]=\sup_n \frac{\tilde{\mu}_n[0,\infty]}{1-\tilde{\mu}_n[0,\infty]}<\infty.$$

Therefore, by the Helly selection principle, we can find a subsequence n' such that $\tilde{\nu}_{n'}$ converges weakly to $\tilde{\nu}$, say, on $[0,\infty]$. It follows that

$$\left\{1+\tilde{\nu}\{\infty\}+\int_0^\infty \frac{1}{s-iz}\nu(ds)\right\}\left\{1-\int_0^\infty \frac{1}{s-iz}\mu(ds)\right\}=1 \quad (\Im z>0),$$

where ν is the measure on $(0,\infty)$ defined by

$$\nu(ds):=I_{(0,\infty)}(s)s\tilde{\nu}(ds).$$

Letting $y\uparrow\infty$ in this with $z=iy$, we see that $1+\tilde{\nu}\{\infty\}=1$ or $\tilde{\nu}\{\infty\}=0$. This proves the first half of the theorem.

(II) Let $\nu\in\mathcal{N}_0$. We define the finite Borel measure $\tilde{\nu}$ on $[0,\infty]$ by

$$\tilde{\nu}(ds)=s^{-1}I_{(0,\infty)}(s)\nu(ds).$$

Take a sequence of simple measures ν_n $(n=1,2,\dots)$ such that $s^{-1}\nu_n(ds)$ converges weakly to $\tilde{\nu}$ on $[0,\infty]$. We put $\mu_n:=\theta_{\mathrm{s}}^{-1}(\nu_n)$ and $\tilde{\mu}_n(ds):=s^{-1}\mu_n(ds)$. Then we have (3.1) for $n=1,2,\dots$. Letting $y\downarrow 0$ in (3.1) with $z=iy$, we see that

$$\sup_n \tilde{\mu}_n[0,\infty]=\sup_n \frac{\tilde{\nu}_n[0,\infty]}{1+\tilde{\nu}_n[0,\infty]}<\infty.$$

Therefore, again by the Helly selection principle, we can find a subsequence n' such that $\tilde{\mu}_{n'}$ converges weakly to $\tilde{\mu}$, say, on $[0, \infty]$. It follows that

$$\left\{1 + \int_0^\infty \frac{1}{s - iz}\nu(ds)\right\}\left\{1 - \tilde{\mu}\{\infty\} - \int_0^\infty \frac{1}{s - iz}\mu(ds)\right\} = 1 \quad (\Im z > 0),$$

where μ is the measure on $(0, \infty)$ defined by

$$\mu(ds) := I_{(0,\infty)}(s)s\tilde{\mu}(ds).$$

Letting $y \uparrow \infty$ in this with $z = iy$, we see that $1 - \tilde{\mu}\{\infty\} = 1$ or $\tilde{\mu}\{\infty\} = 0$. Finally, by the same argument as (2.9), it follows that $\int_0^\infty s^{-1}\mu(ds) < 1$. This proves the second half of the theorem. □

Definition 3.2. *We define the one-to-one and onto map*

$$\theta_0 : \mathcal{M}_0 \ni \mu \mapsto \nu = \theta_0(\mu) \in \mathcal{N}_0$$

by (2.1).

We define

$$\mathcal{M}_1 = \left\{\mu : \begin{array}{c}\mu \text{ is a Borel measure on } (0,\infty) \text{ such that} \\ \int_0^\infty s^{-1}\mu(ds) = 1,\ \int_0^\infty s^{-2}\mu(ds) = \infty\end{array}\right\},$$

$$\mathcal{N}_1 = \left\{\nu : \begin{array}{c}\nu \text{ is a Borel measure on } (0,\infty) \text{ such that} \\ \int_0^\infty (s+1)^{-1}\nu(ds) < \infty,\ \int_0^\infty s^{-1}\nu(ds) = \infty\end{array}\right\}.$$

Next we consider the correspondence between μ in $\mathcal{M}_1$ and ν in $\mathcal{N}_1$ through the relation (2.1).

Theorem 3.3. *For $\mu \in \mathcal{M}_1$, there exists a unique $\nu \in \mathcal{N}_1$ satisfying* (2.1). *Conversely, for $\nu \in \mathcal{N}_1$, there exists a unique $\mu \in \mathcal{M}_1$ satisfying* (2.1).

Proof. (I) Let $\mu \in \mathcal{M}_1$. Set $m := \inf\{s : s \in \operatorname{supp}(\mu)\}$. If $m = 0$, then

$$\int_0^\infty s^{-2}\mu(ds) \le m^{-1}\int_{[m,\infty)} s^{-1}\mu(ds) < \infty,$$

contradicting the condition $\int_0^\infty s^{-2}\mu(ds) = \infty$. Thus $m = 0$. Therefore there exists an $N \in \mathbf{N}$, such that, for $\mu_n(ds) := I_{(1/n,\infty)}(s)\mu(ds)$,

$$\int_0^\infty s^{-1}\mu_n(ds) < \int_0^\infty s^{-1}\mu(ds) = 1 \quad (n \ge N),$$

whence $\mu_n \in \mathcal{M}_0$ for $n \ge N$. We define $\nu_n := \theta_0(\mu_n) \in \mathcal{N}_0$. Then, as $n \to \infty$,

$$\int_0^\infty \frac{1}{s+1}\nu_n(ds) = \frac{\int_0^\infty (1+s)^{-1}\mu_n(ds)}{1 - \int_0^\infty (1+s)^{-1}\mu_n(ds)} \to \frac{\int_0^\infty (1+s)^{-1}\mu(ds)}{1 - \int_0^\infty (1+s)^{-1}\mu(ds)} \in (0,\infty),$$

so that

$$\sup_n \int_0^\infty \frac{1}{1+s}\nu_n(ds) < \infty.$$

Therefore, for $\tilde{\nu}_n(ds) := (s+1)^{-1}I_{(0,\infty)}(s)\nu_n(ds)$, there exists a subsequence n' such that $\tilde{\nu}_{n'}$ converges weakly to a finite Borel measure $\tilde{\nu}$, say, on $[0, \infty]$. It follows that, for $\Im z > 0$,

$$\left\{1 - \frac{\tilde{\nu}\{0\}}{iz} + \tilde{\nu}\{\infty\} + \int_0^\infty \frac{1}{s - iz}\nu(ds)\right\}\left\{1 - \int_0^\infty \frac{1}{s - iz}\mu(ds)\right\} = 1, \tag{3.2}$$

where ν is the measure on $(0, \infty)$ defined by $\nu(ds) := (1 + s)I_{(0,\infty)}(s)\tilde{\nu}(ds)$.

Letting $y \uparrow \infty$ in (3.2) with $z = iy$, we have $\tilde{\nu}\{\infty\} = 0$. From $\int_0^\infty s^{-1}\mu(ds) = 1$, it follows that

$$1 - \int_0^\infty \frac{1}{s+y}\mu(ds) = y\int_0^\infty \frac{1}{s(s+y)}\mu(ds),$$

hence

$$\left\{y + \tilde{\nu}\{0\} + \int_0^\infty \frac{1}{(s/y)+1}\nu(ds)\right\}\int_0^\infty \frac{1}{s(s+y)}\mu(ds) = 1 \quad (y > 0),$$

and so

$$\tilde{\nu}\{0\} = \lim_{y\downarrow 0}\left\{\int_0^\infty \frac{1}{s(s+y)}\mu(ds)\right\}^{-1} = 0.$$

Finally,

$$\int_0^\infty \frac{1}{s}\nu(ds) = \lim_{y\downarrow 0}\left\{\int_0^\infty \frac{1}{s+y}\mu(ds)\right\}\left\{1 - \int_0^\infty \frac{1}{s+y}\mu(ds)\right\}^{-1} = \infty.$$

Thus ν is the desired element of $\mathcal{N}_1$.

(II) Conversely, for $\nu \in \mathcal{N}_1$, define $\nu_n(ds) := I_{(1/n,\infty)}(s)\nu(ds)$ $(n = 1, 2, \ldots)$. Then $\nu_n \in \mathcal{N}_0$. We put $\mu_n := \theta_0^{-1}(\nu_n) \in \mathcal{M}_0$ for $n = 1, 2, \ldots$. Then, as $n \to \infty$,

$$\int_0^\infty \frac{1}{s+1}\mu_n(ds) = \frac{\int_0^\infty (1+s)^{-1}\nu_n(ds)}{1 + \int_0^\infty (1+s)^{-1}\nu_n(ds)} \to \frac{\int_0^\infty (1+s)^{-1}\nu(ds)}{1 + \int_0^\infty (1+s)^{-1}\nu(ds)} \in (0, \infty),$$

hence

$$\sup_n \int_0^\infty \frac{1}{1+s}\mu_n(ds) < \infty.$$

Therefore, for $\tilde{\mu}_n(ds) := (s+1)^{-1}I_{(0,\infty)}(s)\mu_n(ds)$, there exists a subsequence n' such that $\tilde{\mu}_{n'}$ converges weakly to a finite Borel measure $\tilde{\mu}$, say, on $[0, \infty]$. It follows that, for $\Im z > 0$,

$$\left\{1 + \int_0^\infty \frac{1}{s - iz}\nu(ds)\right\}\left\{1 + \frac{\tilde{\mu}\{0\}}{iz} - \tilde{\mu}\{\infty\} - \int_0^\infty \frac{1}{s - iz}\mu(ds)\right\} = 1, \tag{3.3}$$

where μ is the measure on $(0, \infty)$ defined by $\mu(ds) := (1+s)I_{(0,\infty)}(s)\tilde{\mu}(ds)$.

Letting $y \uparrow \infty$ in (3.3) with $z = iy$, we have $\tilde{\mu}\{\infty\} = 0$. Moreover, letting $y \downarrow 0$ in

$$\tilde{\mu}\{0\} + \int_0^\infty \frac{1}{(s/y)+1}\mu(ds) = \frac{\int_0^\infty \{(s/y)+1\}^{-1}\nu(ds)}{1 + \int_0^\infty \{(s/y)+1\}^{-1}\nu(ds)},$$

we get $\tilde{\mu}\{0\} = 0$. Since $\int_0^\infty s^{-1}\nu(ds) = \infty$, we have

$$1 - \int_0^\infty \frac{1}{s}\mu(ds) = \lim_{y\downarrow 0}\left\{1 + \int_0^\infty \frac{1}{s+y}\nu(ds)\right\}^{-1} = 0.$$

In particular, this implies

$$1 - \int_0^\infty \frac{1}{s+y}\mu(ds) = \int_0^\infty \frac{y}{s(s+y)}\mu(ds),$$

and so

$$\int_0^\infty \frac{1}{s^2}\mu(ds) = \lim_{y\downarrow 0}\left\{y + \int_0^\infty \frac{1}{(s/y)+1}\nu(ds)\right\}^{-1} = \infty.$$

Thus μ is the desired element of $\mathcal{M}_1$. □

Definition 3.4. *We define the one-to-one and onto map*

$$\theta_1 : \mathcal{M}_1 \ni \mu \mapsto \nu = \theta_1(\mu) \in \mathcal{N}_1$$

by (2.1).

Lemma 3.5. *For $\mu \in \mathcal{M}_0$ (resp. $\mu \in \mathcal{M}_1$), we set $\nu := \theta_0(\mu) \in \mathcal{N}_0$ (resp. $\nu := \theta_1(\mu) \in \mathcal{N}_1$). Then $\mu(0,\infty) = \nu(0,\infty)$. In particular, $\mu(0,\infty) < \infty$ if and only if $\nu(0,\infty) < \infty$.*

Proof. It follows from (2.1) that

$$\nu(0,\infty) = \lim_{y\uparrow\infty} \int_0^\infty \frac{y}{y+s}\nu(ds) = \lim_{y\uparrow\infty} \frac{\int_0^\infty y/(y+s)\mu(ds)}{1-\int_0^\infty 1/(y+s)\mu(ds)} = \mu(0,\infty).$$

Thus the lemma follows. □

4. Stationary random distributions

We recall some notation in the theory of stationary random distributions (cf. [11] and [10]). We denote by H the Hilbert space of $\mathbf{C}$-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$, with expectation zero and finite variance:

$$H := \{a \in L^2(\Omega, \mathcal{F}, P) : E[a] = 0\},$$
$$(a,b)_H := E\left[a\bar{b}\right], \quad \|a\|_H := (a,a)_H^{1/2} \qquad (a, b \in H).$$

By $\mathcal{D}(\mathbf{R})$, we denote the space of all $\phi \in C^\infty(\mathbf{R})$ with compact support, endowed with the usual topology. A *random distribution* (with expectation zero) is a linear continuous map from $\mathcal{D}(\mathbf{R})$ to H. We write $\mathcal{D}'(H)$ for the class of random distributions on $(\Omega, \mathcal{F}, P)$. For $X \in \mathcal{D}'(H)$, we define $DX \in \mathcal{D}'(H)$ by $DX(\phi) := -X(d\phi/dx)$. We call DX the derivative of X. For $X \in \mathcal{D}'(H)$ and $t \in \mathbf{R}$, we write $M(X)$ (resp. $M_t(X)$) for the closed linear hull of $\{X(\phi) : \phi \in \mathcal{D}(\mathbf{R})\}$ (resp. $\{X(\phi) : \phi \in \mathcal{D}(\mathbf{R}),\ \mathrm{supp}\ \phi \subset (-\infty, t]\}$) in H. For $X, Y \in \mathcal{D}'(H)$, we define $P_Y X \in \mathcal{D}'(H)$ by $P_Y X(\phi) := p_Y(X(\phi))$, where p_Y is the orthogonal projection operator from H onto $M(Y)$. It holds that $P_Y DX = DP_Y X$ for $X, Y \in \mathcal{D}'(H)$. Two random distributions X and Y are said to be *stationarily correlated* if

$$(X(\tau_h\phi), Y(\tau_h\psi))_H = (X(\phi), Y(\psi))_H \qquad (\phi, \psi \in \mathcal{D}(\mathbf{R}),\ h \in \mathbf{R}),$$

where τ_h is the shift operator defined by $\tau_h\phi(t) := \phi(t+h)$.

A random distribution X is called *stationary* if

$$(X(\tau_h\phi), X(\tau_h\psi))_H = (X(\phi), X(\psi))_H \qquad (\phi, \psi \in \mathcal{D}(\mathbf{R}),\ h \in \mathbf{R}).$$

We write $\mathcal{S}$ for the class of stationary random distributions on $(\Omega, \mathcal{F}, P)$. For $X, Y \in \mathcal{S}$, the random distribution $P_Y X$ is also stationary if X and Y are stationarily correlated (see [10, Theorem 2.1]). For $X \in \mathcal{S}$, we write μ_X for the spectral measure of X:

$$(X(\phi), X(\psi))_H = \int_{-\infty}^\infty \hat{\phi}(\xi)\overline{\hat{\psi}}(\xi)\mu_X(d\xi) \qquad (\phi, \psi \in \mathcal{D}(\mathbf{R})),$$

where $\hat{\phi}$ is the Fourier transform of ϕ: $\hat{\phi}(\xi) := \int_{-\infty}^\infty e^{-it\xi}\phi(\xi)d\xi$. For $k \in \mathbf{N} \cup \{0\}$, we define

$$\mathcal{S}_k := \left\{X \in \mathcal{S} : \int_{-\infty}^\infty (1+\xi^2)^{-k}\mu_X(d\xi) < \infty\right\}.$$

Then $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$ and $\mathcal{S} = \cup_{k=0}^\infty \mathcal{S}_k$ (see [11, Theorem 3.2]). The class $\mathcal{S}_0$ can be naturally identified with that of mean-square continuous weakly stationary processes with zero expectation ([11, Theorem 4.2]). Any stationary random distribution X has the spectral representation of the form

$$X(\phi) = \int_{-\infty}^\infty \hat{\phi}(\xi)Z_X(d\xi) \qquad (\phi \in \mathcal{D}(\mathbf{R})),$$

where Z_X is the associated random measure. It holds that

$$M(X) = \left\{ \int_{-\infty}^{\infty} g(\xi) Z_X(d\xi) : g \in L^2(\mu_X) \right\}.$$

We say that X in $\mathcal{S}$ is *purely nondeterministic* if $\cap_{t\in\mathbf{R}} M_t(X) = \{0\}$.

For $k \in L^1(\mathbf{R}, dt)$ and $X \in \mathcal{S}$, we wish to define the convolution $k * S \in \mathcal{S}$. Formally we have

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} k(u) X(t-u) du \right) \phi(t) dt = \int_{-\infty}^{\infty} k(u) \left(\int_{-\infty}^{\infty} X(t) \phi(t+u) dt \right) du$$
$$= \int_{-\infty}^{\infty} k(u) X(\tau_u \phi) du.$$

With this in mind, we define $k * X \in \mathcal{S}$ by

$$(k * X)(\phi) := \int_{-\infty}^{\infty} k(u) X(\tau_u \phi) du \qquad (\phi \in \mathcal{D}(\mathbf{R})),$$

where the integral on the right-hand side is in the sense of H-valued Bochner integral.

We set

$$\mathcal{M} = \left\{ \rho : \rho \text{ is a Borel measure on } (0,\infty) \text{ such that } \textstyle\int_0^\infty (1+s)^{-1} \rho(ds) < \infty \right\}.$$

For $\rho \in \mathcal{M}$, we define

$$F_\rho(z) := \int_0^\infty \frac{1}{s - iz} \rho(ds) \qquad (\Im z \geq 0).$$

Recall $k_\rho(t)$ from (1.5). We have

$$\rho(0,\infty) = k_\rho(0+), \tag{4.1}$$

$$\int_0^\infty s^{-1} \rho(ds) = \int_0^\infty k_\rho(t) dt, \tag{4.2}$$

$$\int_0^\infty s^{-2} \rho(ds) = \int_0^\infty dt \int_t^\infty k_\rho(u) du. \tag{4.3}$$

As in [10, Proposition 2.3], we have the following proposition.

Proposition 4.1. *Let $k \in L^1(\mathbf{R}, dt)$ and $X \in \mathcal{S}$. Then*

$$(k * X)(\phi) = \int_{-\infty}^{\infty} \hat{k}(-\xi) \hat{\phi}(\xi) Z_X(d\xi) \qquad (\phi \in \mathcal{D}(\mathbf{R})).$$

In particular, for $\mu \in \mathcal{M}$ such that $\int_0^\infty s^{-1} \mu(ds) < \infty$, we have

$$(k_\mu * X)(\phi) = \int_{-\infty}^{\infty} F_\mu(\xi) \hat{\phi}(\xi) Z_X(d\xi) \qquad (\phi \in \mathcal{D}(\mathbf{R})).$$

Notice that $k_\mu * X$ can be written formally as

$$k_\mu * X(t) = \int_{-\infty}^{t} k_\mu(t-s) X(s) ds.$$

Proposition 4.2. *Let $X \in \mathcal{S}$, and let $\mu \in \mathcal{M}$ such that $\int_0^\infty s^{-1} \mu(ds) < \infty$. Then X is stationarily correlated with $X - k_\mu * X$.*

Proof. By Proposition 4.1, we have

$$(X - k_\mu * X)(\phi) = \int_{-\infty}^{\infty} \{1 - F_\mu(\xi)\} \hat{\phi}(\xi) Z_X(d\xi).$$

The lemma follows from this. ∎

Proposition 4.3. *Let $X \in \mathcal{S}$.*

(1) *For $\mu \in \mathcal{M}_0$, X satisfies $X = k_\mu * X$ if and only if $X = 0$.*

(2) *For $\mu \in \mathcal{M}_1$, X satisfies $X = k_\mu * X$ if and only if $X = a$ for some $a \in H$.*

Proof. By Proposition 4.1, we have

$$\|(X - k_\mu * X)(\phi)\|_H^2 = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 |1 - F_\mu(\xi)|^2 \mu_X(d\xi).$$

If $\mu \in \mathcal{M}_0$, then

$$\Re\{1 - F_\mu(\xi)\} = 1 - \int_0^\infty \frac{s}{s^2 + \xi^2} \mu(ds) \geq 1 - \int_0^\infty s^{-1} \mu(ds) > 0,$$

while if $\mu \in \mathcal{M}_1$, then

$$|1 - F_\mu(\xi)|^2 > 0 \quad (\xi \neq 0), \quad = 0 \quad (\xi = 0).$$

The lemma follows easily from these. □

5. Long-memory model

Let $\sigma > 0$ and let $W = (W(t) : t \in \mathbf{R})$ be a one-dimensional standard Brownian motion such that $W(0) = 0$, defined on $(\Omega, \mathcal{F}, P)$. Since W is a process with stationary increments, the derivative DW is a stationary random distribution (see [11]). We are concerned with the following equation

$$X = k_\mu * X + \sigma DW. \tag{5.1}$$

It should be noted that the equation (5.1) can be written formally as

$$X(t) = \int_{-\infty}^{t} k_\mu(t-s) X(s) ds + \sigma \frac{dW}{dt}(t). \tag{5.2}$$

For $\nu \in \mathcal{M}$ such that

$$\int_0^\infty k_\nu(t)^2 dt < \infty, \tag{5.3}$$

we define a real, centered, weakly stationary process $(U_\nu(t) : t \in \mathbf{R})$ by

$$U_\nu(t) := \int_{-\infty}^{t} k_\nu(t-s) dW(s) \qquad (t \in \mathbf{R}). \tag{5.4}$$

Then $(U_\nu(t))$ is purely nondeterministic, and (5.4) corresponds to the so-called *canonical representation* of $(U_\nu(t))$; thus, $M_t(U_\nu) = M_t(DW)$ for $t \in \mathbf{R}$. On the other hand, the spectral representation of U_ν, as a stationary random distribution, is given by

$$U_\nu(\phi) = \int_{-\infty}^{\infty} F_\nu(\xi) \hat{\phi}(\xi) Z_{DW}(d\xi) \qquad (\phi \in \mathcal{D}(\mathbf{R})). \tag{5.5}$$

We refer to [10] for these results.

If ν is a finite measure in $\mathcal{N}_1$ such that

$$\int_1^\infty k_\nu(t)^2 dt < \infty, \tag{5.6}$$

then ν satisfies (5.3) since

$$\int_0^1 k_\nu(t)^2 dt \leq k_\nu(0+) \int_0^1 k_\nu(t) dt < \infty.$$

In this case, since $\int_0^\infty k_\nu(t)dt = \infty$, the stationary process $(U_\nu(t))$ defined by (5.4) is long-memory. Now recall the condition (L) in §1; we define the condition (L1) for $\mu \in \mathcal{M}_1$ there by

$$\text{(L1)} \qquad \nu = \theta_1(\mu) \text{ satisfies (5.6).}$$

Theorem 5.1. *Let $\sigma > 0$. Let μ be a measure satisfying (L), and let $\nu := \theta_1(\mu)$. Then a stationary random distribution X satisfies* (5.1) *if and only if $X = X_0 + a$, where a is an arbitrary element of $M(DW)^\perp$ and X_0 is the stationary random distribution defined by*

$$X_0 = \sigma U_\nu + \sigma DW. \tag{5.7}$$

In particular, X_0 is the only purely nondeterministic stationary random distribution that satisfies (5.1).

Proof. Let X be a stationary random distribution satisfying (5.1). Then, by Proposition 4.2, X and $DW = X - k_\mu * X$ are stationarily correlated. We define $X_1 := X - P_{DW}X$. Then, by [10, Theorem 2.1], X_1 is a stationary random distribution satisfying $X_1 = k_\mu * X_1$. So, by Proposition 4.3 (2), we see that $X - P_{DW}X = a$ for some $a \in M(DW)^\perp$.

We set $X_0 := P_{DW}X$. Then, again by [10, Theorem 2.1], there exists $g \in L^2(\mathbf{R}, (1+x^2)^{-k}d\xi)$, for some $k \in \mathbf{N} \cup \{0\}$, such that

$$X_0(\phi) = \int_{-\infty}^{\infty} g(\xi)\hat{\phi}(\xi) Z_{DW}(d\xi).$$

By Proposition 4.1, we have

$$k_\mu * X_0(\phi) = \int_{-\infty}^{\infty} F_\mu(\xi) g(\xi)\hat{\phi}(\xi) Z_{DW}(d\xi).$$

Since $\mu_{DW}(d\xi) = (2\pi)^{-1}d\xi$, it follows that, for $\phi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned} 0 &= \|X_0(\phi) - k_\mu * X_0(\phi) - \sigma DW(\phi)\|_H^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 |\{(1 - F_\mu(\xi))g(\xi) - \sigma\}|^2 d\xi. \end{aligned}$$

This implies

$$g(\xi) = \frac{\sigma}{1 - F_\mu(\xi)} = \sigma F_\nu(\xi) + \sigma,$$

hence

$$X_0(\phi) = \int_{-\infty}^{\infty} \{\sigma F_\nu(\xi) + \sigma\}\hat{\phi}(\xi) Z_{DW}(d\xi) = \sigma U_\nu(\phi) + \sigma DW(\phi).$$

Thus X_0 is given by (5.7). Conversely, we can easily show that $X = X_0 + a$ with (5.7) and $a \in M(DW)^\perp$ is a stationary random distribution that satisfies (5.1).

For $z = x + iy$ with $y > 0$, it holds that

$$\Re\{\sigma F_\nu(z) + \sigma\} = \sigma + \int_0^\infty \frac{s + y}{(s+y)^2 + x^2}\nu(ds) > 0.$$

By Theorem A in the appendix, this implies

$$M_t(X_0) = M_t(DW) \qquad (t \in \mathbf{R}),$$

(see, e.g., [9]), hence $M_t(X_0 + a) = M_t(DW) + \mathbf{C}a$ for $a \in M(DW)^\perp$. Therefore

$$\bigcap_t M_t(X_0 + a) = \left\{\bigcap_t M_t(X_0)\right\} \oplus \mathbf{C}a = \left\{\bigcap_t M_t(DW)\right\} \oplus \mathbf{C}a = \mathbf{C}a.$$

Thus $X_0 + a$ with $a \in M(DW)^{\perp}$ is purely nondeterministic if and only if $a = 0$. □

We give a sufficient condition for (L1).

Lemma 5.2. *Let $0 < p < 1$ and let $\ell(\cdot)$ be a slowly varying function at infinity. Let $\mu \in \mathcal{M}_1$ and define $\nu \in \mathcal{N}_1$ by $\nu := \theta_1(\mu)$. Then*

$$k_\mu(t) \sim t^{-(p+1)}\ell(t)p \qquad (t \to \infty) \tag{5.8}$$

if and only if

$$k_\nu(t) \sim \frac{t^{-(1-p)}}{\ell(t)} \cdot \frac{\sin(p\pi)}{\pi} \qquad (t \to \infty). \tag{5.9}$$

Proof. We prove only (5.8) $\Rightarrow$ (5.9); the converse implication (5.9) $\Rightarrow$ (5.8) can be proved in the same way. Since $\int_0^\infty k_\mu(t)dt = 1$, we have, by integration by parts,

$$1 - \int_0^\infty e^{-ty} k_\mu(t)dt = y \int_0^\infty e^{-ty} \left(\int_t^\infty k_\mu(s)ds \right) dt \qquad (y > 0).$$

Hence

$$\int_0^\infty e^{-ty} k_\nu(t)dt = \frac{\int_0^\infty e^{-ty} k_\mu(t)dt}{1 - \int_0^\infty e^{-ty} k_\mu(t)dt} = \frac{\int_0^\infty e^{-ty} k_\mu(t)dt}{y \int_0^\infty e^{-ty} \left(\int_t^\infty k_\mu(s)ds \right) dt}.$$

Now (5.8) implies

$$\int_t^\infty k_\mu(s)ds \sim t^{-p}\ell(t) \qquad (t \to \infty),$$

so that

$$y \int_0^\infty e^{-ty} \left(\int_t^\infty k_\mu(s)ds \right) dt \sim y^p \ell(1/y)\Gamma(1-p) \qquad (y \downarrow 0)$$

(cf. [3, Theorem 1.7.6]). On the other hand,

$$\lim_{y \downarrow 0} \int_0^\infty e^{-ty} k_\mu(t)dt = 1.$$

Thus

$$y \int_0^\infty e^{-ty} k_\nu(t)dt \sim \frac{y^{1-p}}{\ell(1/y)\Gamma(1-p)} \qquad (y \downarrow 0).$$

By Karamata's Tauberian theorem ([3, Theorem 1.7.6]), this implies (5.9). □

Theorem 5.3. *Let $0 < p < 1/2$ and let $\ell(\cdot)$ be a slowly varying function at infinity. Let μ be a finite measure in $\mathcal{M}_1$ satisfying (5.8). Then μ satisfies (L1), whence (L). If we put $\nu := \theta_1(\mu)$, then $k_\nu(\cdot)$ satisfies (5.9), and the autocovariance function $\gamma_\nu(\cdot)$ of the stationary process $(U_\nu(t))$ in (5.7) satisfies*

$$\gamma_\nu(t) \sim \frac{t^{-(1-2p)}}{\ell(t)^2} \left(\frac{\sin(p\pi)}{\pi} \right)^2 B(1-2p, p) \qquad (t \to \infty). \tag{5.10}$$

Proof. By Lemma 5.2, $k_\nu(\cdot)$ satisfies (5.9). Thus (5.6) holds. Moreover, by [10, Proposition 4.3], we have

$$\gamma_\nu(t) = \int_0^\infty k_\nu(t+s)k_\nu(s)ds \sim \frac{t^{-(1-2p)}}{\ell(t)^2} \left(\frac{\sin(p\pi)}{\pi} \right)^2 B(1-2p, p) \qquad (t \to \infty).$$

Thus the theorem follows. □

Example 5.4. *For $0 < p < 1/2$, set $\mu(ds) := \Gamma(p)^{-1} s^p e^{-s} ds$. Then we have*

$$k_\mu(t) = \frac{p}{(t+1)^{p+1}} \qquad (t > 0).$$

Since $k_\mu(0+) < \infty$, $\int_0^\infty k_\mu(t)dt = 1$, $\int_0^\infty dt \int_t^\infty k_\mu(s)ds = \infty$, and

$$k_\mu(t) \sim pt^{-(p+1)} \qquad (t \to \infty),$$

we find that μ satisfies (L); take $\ell(\cdot) = 1$ in Theorem 5.1.

6. Short-memory model

If ν is a finite measure in $\mathcal{N}_0$, then

$$\int_0^\infty k_\nu(t)^2 dt \le k_\nu(0+) \int_0^\infty k_\nu(t)dt < \infty.$$

Thus ν satisfies (5.3). In this case, since $\int_0^\infty k_\nu(t)dt < \infty$, the stationary process $(U_\nu(t))$ is short-memory.

Theorem 6.1. *Let $\sigma > 0$ and let μ be a measure satisfying (S). Set $\nu := \theta_0(\mu)$. Then the stationary random distribution X defined by*

$$X = \sigma U_\nu + \sigma DW. \tag{6.1}$$

is the unique stationary random distribution that satisfies (5.1).

Proof. It follows from Lemma 3.5 that $\nu(0, \infty) < \infty$. Hence (5.3) holds. Let X be a stationary random distribution satisfying (5.1). As in the proof of Theorem 5.1 but using Proposition 4.3 (1) instead of (2), we see that $X_1 = 0$, i.e., $X = P_{DW} X$. Then, in a similar manner, we find that X is given by (6.1). □

We investigate the asymptotics for $k_\nu(t)$ and $\gamma_\nu(t)$ as $t \to \infty$ when $k_\mu(t)$ is regularly varying.

Lemma 6.2. *Let $\mu \in \mathcal{M}_0$ and $\nu := \theta_0(\mu)$. Let $0 < p < \infty$ and $\ell(\cdot)$ be a slowly varying function at ∞. Then*

$$k_\mu(t) \sim t^{-(p+1)} \ell(t) p \qquad (t \to \infty) \tag{6.2}$$

if and only if

$$k_\nu(t) \sim t^{-(p+1)} \ell(t) \frac{p}{\{1 - \int_0^\infty k_\mu(u)du\}^2} \qquad (t \to \infty). \tag{6.3}$$

Proof. We prove only the implication (6.2) ⇒ (6.3). The converse implication (6.3) ⇒ (6.2) can be proved in a similar fashion. Thus we assume (6.2). We set $n := [p]$, where $[\cdot]$ denotes the integer part. We define

$$f(y) := 1 - \int_0^\infty e^{-yt} k_\mu(t)dt \qquad (y > 0).$$

By differentiating both sides of

$$\int_0^\infty e^{-yt} k_\nu(t)dt = \frac{1}{f(y)} - 1 \qquad (y > 0)$$

$n + 1$ times with respect to y, we obtain

$$\int_0^\infty e^{-yt} t^{n+1} k_\nu(t)dt = \frac{\int_0^\infty e^{-yt} t^{n+1} k_\mu(t)dt}{f(y)^2} + \frac{F_{n+1}(y)}{f(y)^{n+2}} \qquad (y > 0),$$

where $F_{n+1}(y)$ is a polynomial in $\{f^{(k)}(y) : k = 0, 1, \ldots, n\}$ (see [8, Lemma 3.2]). Since $n + 1 - (p + 1) = n - p > -1$ and

$$t^{n+1}k_\mu(t) \sim t^{n-p}\ell(t)p \qquad (t \to \infty),$$

it follows that

$$\int_0^\infty e^{-yt}t^{n+1}k_\mu(t)dt \sim y^{-n+p-1}\ell(1/y)p\Gamma(n-p+1) \qquad (y \to 0+)$$

(see [3, Theorem 1.7.6]). On the other hand, for any $\epsilon > 0$ and $0 \le k \le n$, we have

$$y^\epsilon f^{(k)}(y) \to 0 \qquad (y \to 0+)$$

(cf. [8, Lemma 3.5]), and so

$$\frac{F_{n+1}(y)}{y^{-n+p-1}\ell(1/y)} \to 0 \qquad (y \to 0+).$$

Thus

$$\int_0^\infty e^{-yt}t^{n+1}k_\nu(t)dt \sim y^{-n+p-1}\ell(1/y)\frac{p\Gamma(n-p+1)}{\{1 - \int_0^\infty k_\mu(u)du\}^2} \qquad (y \to 0+).$$

Since the function $\log\{t^{n+1}k_\nu(t)\}$ is slowly increasing ([3, §1.7.6]), Karamata's Tauberian theorem (cf. [3, Theorem 1.7.6]) yields

$$t^{n+1}k_\nu(t) \sim t^{n-p}\ell(t)\frac{p}{\{1 - \int_0^\infty k_\mu(u)du\}^2} \qquad (t \to \infty),$$

or (6.3), as desired. □

Remark 6.3. *The condition $\mu \in \mathcal{M}_0$ implies*

$$1 - \int_0^\infty k_\mu(u)du = 1 - \int_0^\infty s^{-1}\mu(ds) > 0.$$

Theorem 6.4. *Let $0 < p < \infty$ and $\ell(\cdot)$ be a slowly varying function at ∞. Let μ be a measure satisfying (S), and set $\nu := \theta_0(\mu)$. Let $(U_\nu(t) : t \in \mathbf{R})$ be the stationary process in (6.1) with autocovariance function $\gamma_\nu(\cdot)$. Then (6.2) implies (6.3) and*

$$\gamma_\nu(t) \sim t^{-(p+1)}\ell(t)\frac{p\int_0^\infty k_\mu(u)du}{\{1 - \int_0^\infty k_\mu(u)du\}^3} \qquad (t \to \infty). \tag{6.4}$$

Proof. That (6.2) implies (6.3) is a direct consequence of Lemma 6.2. Now we have

$$\gamma_\nu(t) = \int_0^\infty k_\nu(t+s)k_\nu(s)ds \sim k_\nu(t)\left(\int_0^\infty k_\nu(s)ds\right) \qquad (t \to \infty)$$

(see [8, Lemma 3.8]). Since

$$\int_0^\infty k_\nu(s)ds = \frac{\int_0^\infty k_\mu(s)ds}{1 - \int_0^\infty k_\mu(s)ds},$$

(6.4) follows. □

Example 6.5. *For $0 < p < \infty$ and $0 < c < 1$, we put*

$$\mu(ds) = \frac{c}{\Gamma(p)}s^p e^{-s}ds.$$

Then $k_\mu(t) = pc(1+t)^{-p-1}$ for $t > 0$. We see that μ satisfies (S) and (6.2) with $\ell(\cdot) = c$.

We now investigate the asymptotics for $k_\nu(t)$ and $\gamma_\nu(t)$ when $k_\mu(t)$ decays exponentially as $t \to \infty$. For a finite Borel measure ρ on $(0,\infty)$, we define $s_0(\rho) \in [0,\infty)$ by

$$s_0(\rho) := \inf\{s : s \in \text{supp } \rho\}.$$

Then it holds that

$$\lim_{t\to\infty} \frac{\log k_\rho(t)}{t} = -s_0(\rho)$$

(see [15, Proposition 3.2]). This implies that $k_\rho(t)$ decays exponentially if and only if $s_0(\rho) > 0$.

Lemma 6.6. *Let $\mu \in \mathcal{M}_0$, and let $\nu := \theta_0(\mu) \in \mathcal{N}_0$. Then the following are equivalent:*

$$s_0(\mu) > 0, \tag{6.5}$$

$$s_0(\nu) > 0. \tag{6.6}$$

Proof. We prove only the implication (6.5) $\Rightarrow$ (6.6). The converse (6.6) $\Rightarrow$ (6.5) can be proved in a similar fashion. Assume (6.5). Then, in part (I) of the proof of Theorem 3.1, we can choose the sequence of simple measures μ_n so that supp $\mu_n \subset [s_0(\mu),\infty)$ for $n = 1,2,\ldots$. Moreover, by taking $\mu_n + (1/n)\delta_{s_0(\mu)}$, we may (and shall) assume that $s_0(\mu_n) = s_0(\mu)$ for $n = 1,2,\ldots$. Then, by (2.8), we find that the simple measures $\tilde{\nu}_n$ are of the form

$$\tilde{\nu}_n = b(n)\delta_{p(n)} + \tilde{\eta}_n \qquad (n = 1,2,\ldots),$$

where $b(n) \in (0,\infty)$, $p(n) \in (0, s_0(\mu))$, and $\tilde{\eta}_n$ are simple measures such that supp $\tilde{\eta}_n \subset (s_0(\mu),\infty)$. Since

$$b(n) \le \tilde{\nu}_n(0,\infty) \le \sup_m \tilde{\nu}_m(0,\infty) < \infty,$$

$$\tilde{\eta}_n(s_0(\mu),\infty) \le \tilde{\nu}_n(0,\infty) \le \sup_m \tilde{\nu}_m(0,\infty) < \infty,$$

we can choose the subsequence n' there so that, for some $b \in [0,\infty)$, $p \in [0, s_0(\mu)]$ and finite Borel measure $\tilde{\eta}$ on $[s_0(\mu),\infty]$, we have $b(n') \to b$, $p(n') \to p$, and $\tilde{\eta}_n \to \tilde{\eta}$ weakly on $[s_0(\mu),\infty]$. By the arguments in the part (I) of the proof of Theorem 3.1, we find that $\nu := \theta_0(\mu)$ is of the form

$$\nu = \begin{cases} b\delta_p + \eta & \text{if } p > 0, \\ \eta & \text{if } p = 0, \end{cases}$$

where η is the measure on $(0,\infty)$ defined by

$$\eta(ds) := I_{[s_0(\mu),\infty)}(s)s\tilde{\eta}(ds).$$

Thus (6.6) follows. $\square$

Remark 6.7. *For $\mu \in \mathcal{M}_0$ and $\nu := \theta_0(\mu) \in \mathcal{N}_0$, we can prove $s_0(\nu) \le s_0(\mu)$.*

Theorem 6.8. *Let μ be a measure satisfying (S), and set $\nu := \theta_0(\mu)$. Let $(U_\nu(t) : t \in \mathbf{R})$ be the stationary process in (6.1) with autocovariance function $\gamma_\nu(\cdot)$. If $k_\mu(t)$ decays exponentially as $t \to \infty$, then both $k_\nu(t)$ and $\gamma_\nu(t)$ decay exponentially as $t \to \infty$.*

Proof. The exponential decay of $k_\nu(t)$ follows from Lemma 6.6. Now

$$\gamma_\nu(t) = \int_0^\infty e^{-ts}\sigma(ds) \qquad (t \in \mathbf{R})$$

with

$$\sigma(ds) := \left\{\int_0^\infty \frac{1}{s+u}\nu(du)\right\}\nu(ds).$$

Clearly $s_0(\sigma) = s_0(\nu)$, and so $\gamma_\nu(t)$ also decays exponentially. □

Example 6.9. *Let μ be as in Example 2.4. Clearly μ satisfies (S). In this case, $k_\mu(t) = ae^{-rt}$ for $t > 0$ and $k_\nu(t) = ae^{-(r-a)t}$. The autocovariance function $\gamma_\nu(t)$ of U_ν in (6.1) is given by*

$$\gamma_\nu(t) = a^2 \int_0^\infty e^{-(r-a)(|t|+s)} e^{-(r-a)s} ds = \frac{a^2 e^{-(r-a)|t|}}{2(r-a)} \qquad (t \in \mathbf{R}).$$

7. Risky asset model

Let $\sigma \in (0, \infty)$, $m \in \mathbf{R}$, and $(W(t) : t \in \mathbf{R})$ be a one-dimensional standard Brownian motion such that $W(0) = 0$, defined on $(\Omega, \mathcal{F}, P)$. We consider a risky asset with price $S(t)$ at time t. We suppose that $S(t)$ is of the form (1.1) with

$$Z(t) = mt + Y(t) \qquad (t \in \mathbf{R}),$$

where $(Y(t) : t \in \mathbf{R})$ is a zero-mean, mean-square continuous process with stationary increments such that $Y(0) = 0$. We also suppose that the derivative $X := DY$ is the (purely nondeterministic) solution to (5.1) for μ satisfying (S) (resp. (L)). Then, by Theorem 5.1 (resp. Theorem 6.1) and [11, Theorem 6.1], $Z(t)$ is of the form (1.6) with $\nu = \theta_0(\mu)$ (resp. $\nu = \theta_1(\mu)$), whence

$$S(t) = S(0) \exp\left\{ mt + \sigma \int_0^t U_\nu(s) ds + \sigma W(t) \right\} \qquad (t \geq 0). \tag{7.1}$$

As before, we write $\gamma_\nu(\cdot)$ for the autocovariance function of the stationary process $(U_\nu(t) : t \in \mathbf{R})$.

Let $\mathcal{N}$ be the class of all P-negligible sets from $\mathcal{F}$. We use the following P-augmented filtration $(\mathcal{F}_t)_{t\geq 0}$:

$$\mathcal{F}_t := \bigcap_{\epsilon > 0} \sigma(\mathcal{G}_{t+\epsilon} \cup \mathcal{N}) \qquad (t \geq 0),$$

where

$$\mathcal{G}_t := \sigma\{W(s) : -\infty < s \leq t\} \qquad (t \geq 0).$$

Then, with respect to $(\mathcal{F}_t)_{t\geq 0}$, the Brownian motion $(W(t) : t \geq 0)$ is a $(\mathcal{F}_t)$-Brownian motion, and the process $(U_\nu(t) : t \geq 0)$ is $(\mathcal{F}_t)$-adapted, as is desired here.

Suppose that we are in a market in which the riskless asset price $S_0(t)$ follows $S_0(t) = \exp(rt)$ for $t \geq 0$, where r is a nonnegative constant. We write $\tilde{S}(t)$ for the discounted price of the risky asset: $\tilde{S}(t) := e^{-rt} S(t)$. We put

$$a := \frac{m - r + \frac{1}{2}\sigma^2}{\sigma},$$

$$W^*(t) := W(t) + \int_0^t \{a + U_\nu(s)\} ds \qquad (t \geq 0).$$

Then we have

$$\tilde{S}(t) = S(0) \exp\left(\sigma W^*(t) - \frac{1}{2}\sigma^2 t\right). \tag{7.2}$$

Lemma 7.1. *Let $0 < t < \infty$ and $0 < \delta < \gamma_\nu(0)^{-1}$. Then*

$$E\left[\exp\left\{\frac{1}{2}\int_t^{t+\delta} (a + U_\nu(s))^2 ds\right\}\right] < \infty. \tag{7.3}$$

Proof. By Jensen's inequality, we have

$$\exp\left\{\frac{1}{2}\int_t^{t+\delta}(a+U_\nu(s))^2 ds\right\} \leq \frac{1}{\delta}\int_t^{t+\delta}\exp\left\{\frac{\delta}{2}(a+U_\nu(s))^2\right\} ds.$$

Since $(U_\nu(s))$ is a stationary Gaussian process, it follows that

$$\begin{aligned} E\left[\exp\left\{\frac{1}{2}\int_t^{t+\delta}(a+U_\nu(s))^2 ds\right\}\right] &\leq \frac{1}{\delta}\int_t^{t+\delta} E\left[\exp\left\{\frac{\delta}{2}(a+U_\nu(s))^2\right\}\right] ds \\ &= E\left[\exp\left\{\frac{\delta}{2}(a+U_\nu(0))^2\right\}\right] \\ &= \frac{1}{\sqrt{2\pi\gamma_\nu(0)}}\int_{-\infty}^{\infty}\exp\left\{\frac{\delta}{2}(a+x)^2\right\}\exp\left\{-\frac{x^2}{2\gamma_\nu(0)}\right\} dx. \end{aligned}$$

Thus the lemma follows. □

Remark 7.2. *Let $\{t_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $0 = t_0 < \cdots < t_n \uparrow \infty$, such that $t_n - t_{n-1} < \gamma_\nu(0)^{-1}$ for $n = 1, 2, \ldots$. Then, by Lemma 7.1, we have*

$$E\left[\exp\left\{\frac{1}{2}\int_{t_{n-1}}^{t_n}(a+U_\nu(s))^2 ds\right\}\right] < \infty \qquad (n = 1, 2, \ldots).$$

Therefore, by [12, Corollary 5.14] *and Girsanov's theorem (cf.* [12, §3.5]*), there exists a probability measure P^*, equivalent to P, under which $(W^*(t) : 0 \leq t < \infty)$ is a standard Brownian motion . From (7.2), we see that the behaviour of $(\tilde{S}(t) : 0 \leq t < \infty)$ under P^* is equal to that in the Black-Scholes environment with volatility σ. In particular, for any $T > 0$, the prices of European calls and puts with maturity T are given by the Black-Scholes formulas with implied volatility σ.*

We now turn to the variance of $\log(S(t)/S(s))$ for $t > s \geq 0$.

Lemma 7.3. *Let $t > s \geq 0$. Then*

$$\begin{aligned} &\mathrm{Var}\{\log(S(t)/S(s))\} \\ &= \left\{(t-s) + 2\int_0^{t-s} du \int_0^u k_\nu(v)dv + 2\int_0^{t-s} du \int_0^u \gamma_\nu(v)dv\right\}\sigma^2. \end{aligned} \tag{7.4}$$

Proof. We have

$$\begin{aligned} \mathrm{Var}\{\log(S(t)/S(s))\} &= E\left[\left\{W(t) - W(s) + \int_s^t U_\nu(u)du\right\}^2\right]\sigma^2 \\ &= \left\{(t-s) + 2E\left[(W(t)-W(s))\int_s^t U_\nu(u)du\right] + \int_s^t\int_s^t \gamma_\nu(u-v)dudv\right\}\sigma^2. \end{aligned}$$

By simple calculation, we get

$$\int_s^t\int_s^t \gamma_\nu(u-v)dudv = 2\int_0^{t-s} du \int_0^u \gamma_\nu(v)dv.$$

Now, for $s \leq u \leq t$,

$$E[(W(t)-W(s))U_\nu(u)] = E[(W(u)-W(s))U_\nu(u)] = \int_s^u k_\nu(u-v)dv,$$

whence

$$E\left[(W(t)-W(s))\int_s^t U_\nu(u)du\right] = \int_s^t\int_s^u k_\nu(u-v)dudv = \int_0^{t-s} du\int_0^u k_\nu(v)dv.$$

Thus the lemma follows. □

From (7.3), we find that, in our model,

$$\mathrm{Var}\{\log(S(t)/S(s))\} \geq (t-s)\sigma^2 \qquad (t > s \geq 0).$$

In other words, the historical volatility $\geq$ the implied volatility. The equality holds only when the model is Black-Scholes.

Now we investigate the asymptotic behavior of $\mathrm{Var}\{\log(S(t)/S(s))\}$ as $t-s\to\infty$. First we consider the short-memory case.

Proposition 7.4. *We assume (S). Then*

$$\begin{aligned}\mathrm{Var}\{\log(S(t)/S(s))\} &\sim (t-s)\left\{1+\int_0^\infty k_\nu(u)du\right\}^2\sigma^2 \\ &= (t-s)\left\{1-\int_0^\infty k_\mu(u)du\right\}^{-2}\sigma^2 \qquad (t-s\to\infty).\end{aligned} \tag{7.5}$$

Proof. The assumption (S) implies that $\int_0^\infty k_\nu(t)dt<\infty$. Now

$$\int_0^\infty \gamma_\nu(t)dt = \frac{1}{2}\left\{\int_0^\infty k_\nu(u)du\right\}^2.$$

Thus (7.5) follows from (7.4). □

Next we consider the long-memory case.

Proposition 7.5. *Let $0<p<1/2$ and let $\ell(\cdot)$ be a slowly varying function at infinity. We assume (L) and* (5.8), *hence* (5.9). *Then*

$$\mathrm{Var}\{\log(S(t)/S(s))\} \sim \frac{(t-s)^{2p+1}}{\ell(t-s)^2}\left(\frac{\sin(p\pi)}{\pi}\right)^2\frac{B(1-2p,p)}{p(2p+1)}\sigma^2 \quad (t-s\to\infty). \tag{7.6}$$

Proof. By Theorem 5.3, (5.9) and (5.10) hold. We then find that, among the three terms on the right-hand side of (7.4), the first and second terms are negligible relative to the third. Thus

$$\begin{aligned}\mathrm{Var}\{\log(S(t)/S(s))\} &\sim 2\sigma^2\int_0^{t-s}du\int_0^u\gamma_\nu(v)dv \\ &\sim \frac{(t-s)^{2p+1}}{\ell(t-s)^2}\left(\frac{\sin(p\pi)}{\pi}\right)^2\frac{B(1-2p,p)}{p(2p+1)}\sigma^2\end{aligned}$$

as $t-s\to\infty$, hence (7.6). □

It should be noted that, in (7.6), the index of $(t-s)$ is $2p+1$ unlike the short-memory case (7.5).

8. Appendix

If ϕ is a positive measurable function on $\mathbf{R}$ such that $(1+t^2)^{-1}\log\phi(t) \in L^1(\mathbf{R}, dt)$, and if

$$f(z) := \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+tz}{t-z}\cdot\frac{\log\phi(t)}{1+t^2}dt\right\} \qquad (\Im z > 0), \tag{8.1}$$

then we call f an *outer function*. For completeness, we prove the following (seemingly well-known) theorem.

Theorem A. *Let $f(z)$ be analytic and $\Re\{f(z)\} > 0$ in $\{z \in \mathbf{C} : \Im z > 0\}$. Then f is an outer function.*

Proof. As in Duren [7, p. 189], we consider the following mappings from $\{z \in \mathbf{C} : \Im z > 0\}$ onto $\{w \in \mathbf{C} : |w| < 1\}$:

$$w = p(z) = \frac{z-i}{z+i}, \qquad z = q(w) = \frac{i(1+w)}{1-w}.$$

Set

$$g(w) := f(q(w)) \qquad (|w| < 1).$$

Since $\Re\{F(w)\} > 0$, F is an outer function on $\{|w| < 1\}$ (see, e.g., Duren [7, p. 51]). Hence

$$\psi(e^{i\theta}) := \lim_{r\uparrow 1}|F(re^{i\theta})|$$

exists for almost every $\theta \in (-\pi, \pi)$ and it holds that

$$g(w) := \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+w}{e^{i\theta}-w}\log|\psi(e^{i\theta})|d\theta\right\} \qquad (|w| < 1)$$

(cf. Rudin [17, 17.16 Theorem]). Now with the change of variables $e^{i\theta} = p(t)$ and $w = p(z)$, it follows that

$$\frac{e^{i\theta}+w}{e^{i\theta}-w} = \frac{1+tz}{i(t-z)}$$

and

$$d\theta = \frac{2t}{1+t^2}dt.$$

Thus ϕ defined by $\phi(t) := \psi(q(t))$ satisfies $(1+t^2)^{-1}\log\phi(t) \in L^1(\mathbf{R}, dt)$ and (8.1). □

From the proof above, we find that, for f and ϕ in (8.1),

$$\lim_{y\downarrow 0}|f(x+iy)| = \phi(x) \qquad \text{a.e. on } \mathbf{R}.$$

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E-mail address: v.anh@fsc.qut.edu.au
E-mail address: inoue@math.sci.hokudai.ac.jp

School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Queensland 4001, Australia

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan