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TAKESHI IZAWA AND TATSUO SUWA

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MULTIPLICITY OF FUNCTIONS ON SINGULAR VARIETIES

TAKESHI IZAWA AND TATSUO SUWA

The multiplicity, also called the Milnor number, of a holomorphic function f on a complex manifold M at an isolated critical point is an important invariant and there are many ways to interpret it (see for example [O]). One way is to look at it as the "residue" of the differential df on the top Chern class of the cotangent bundle $T^\vee M$ of M. Then it is expressed as a Grothendieck residue (see for example [S2-3]). In the global situation where there is a holomorphic map of a compact complex manifold M onto a complex curve, with only isolated critical points, this point of view leads to the "Multiplicity formula" (or the "Milnor number formula"), which is stated in Corollary 2.4 below. In the algebraic category, this formula is a special case of the one in [I], see also [F, Example 14.1.5]. In the analytic category, this is proved in [HL, VI 3] using the "Milnor current". Here we reprove it along the above line.

There have been a number of works on the critical points of holomorphic functions on singular varieties and generalization of multiplicities so that they have some properties similar to the ones the usual multiplicity has (see for example [Dim], [Lê], [BR], [Z]).

In this paper, we introduce a multiplicity for a function on a local complete intersection variety, compute the number at an isolated singular point and prove an analogous multiplicity formula, from the above viewpoint. In fact, to define the multiplicity, the function needs not be holomorphic, it only needs to be the restriction of a C^∞ function on the ambient manifold.

More precisely, let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$ such that there exist a holomorphic vector bundle N of rank k and a holomorphic section s of N, generically transverse to the zero section, with $V = s^{-1}(0)$ (see Section 4 below). Note that V is then a local complete intersection defined by the local components of s. We denote the virtual bundle $(TW - N)|_V$ by τ_V and call it the virtual tangent bundle of V. Let g be a C^∞ function on W and let f be its restriction to V. We define the singular set $S(f)$ of f to be the union of the singular set $\mathrm{Sing}(V)$ of V and the critical set $C(f')$ of the restriction f' of f to the regular part $V' = V \setminus \mathrm{Sing}(V)$ of V. Let S be a compact connected component of $S(f)$. Using the differential df', we may define the localization of the

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n-th Chern class of the virtual cotangent bundle $\tau_V^\vee$ of V, which in turn defines the residue $\mathrm{Res}_{c^n}(df, \tau_V^\vee; S)$. We denote this residue by $\widetilde{m}(f,S)$ and call it the virtual multiplicity of f at S. On the other hand, we have the (generalized) Milnor number $\mu(V,S)$ of V at S ([BLSS], see also [P], [PP] in the case of hypersurfaces). We define the multiplicity $m(f,S)$ of f at S as the difference of $\widetilde{m}(f,S)$ and $\mu(V,S)$ (Definition 4.4).

Suppose that S consists of a point p_0 and that V is defined by the equations $h_1 = \cdots = h_k = 0$ in a neighborhood of p_0. Then it is shown that the virtual multiplicity $\widetilde{m}(f,p_0)$ coincides with the residue of the n-th Chern class of the restriction $TW|_V$ with respect to the $(k+1)$-frame $\mathbf{s} = (dg|_V, dh_1|_V, \dots, dh_k|_V)$ (Proposition 4.6) so that it can be computed using formulas in [S3]. Note that in this case, the Milnor number $\mu(V,p_0)$ coincides with the usual one, introduced in [M] when $k=1$, and in [H] for arbitrary k. We may also interpret $\widetilde{m}(f,p_0)$ in the spirit of the "GSV"-index of a vector field on a singular variety, namely, as the sum of the multiplicities of the function on a (local) smoothing (Proposition 4.9). We see that both the virtual multiplicity $\widetilde{m}(f,p_0)$ and the multiplicity $m(f,p_0)$ are integers from either one of the above propositions. Note moreover that, if g is holomorphic, $\widetilde{m}(f,p_0)$ is non-negative.

With this definition of multiplicities, in the global situation where V is compact and where there is a holomorphic map of V onto a complex curve, we have the multiplicity formula, which takes exactly the same form as in the non-singular case (Corollary 5.6).

Let us finally remark that, for an isolated singular point p_0 of a function f, we could have defined $\widetilde{m}(f,p_0)$ simply by the identity in Proposition 4.9. However, the way we adopted here has the following advantages. First, we can express the multiplicity only in terms of some data on V, as Grothendieck residues. Second, to prove a formula as in Corollary 5.6, we do not need a "global smoothing" of V, which is not available in general.

We would like to thank J.-P. Brasselet and J. Seade for helpful conversations.

1. Preliminaries

For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration theory on the Čech-de Rham cohomology is developed in [Le1-2]. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [Bo] and [GH]. See also [S1] for the material in this section. We freely use the notation and facts there.

(A) Integration on the Čech-de Rham cohomology

Let M be a (connected) oriented C^∞ manifold of dimension m. For an open set U in M, we denote by $A^q(U)$ the space of complex valued C^∞ q-forms on U. For an open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of M, we denote by $A^*(\mathcal{U})$ the Čech-de Rham complex, with differential D, associated to the covering $\mathcal{U}$ and by $H^q(A^*(\mathcal{U}))$ its cohomology. We have a canonical isomorphism

$$H^q(M,\mathbb{C}) \xrightarrow{\sim} H^q(A^*(\mathcal{U})), \tag{1.1}$$

where $H^q(M,\mathbb{C})$ denotes the de Rham cohomology of M.

If M is compact, taking a "system of honey-comb cells" $\{R_\alpha\}_\alpha$ adapted to $\mathcal{U}$, we may define the integration

$$\int_M : H^m(A^*(\mathcal{U})) \to \mathbb{C},$$

which is compatible, via (1.1), with the usual integration on the de Rham cohomology.

Now let S be a closed set in M. Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. In this case, a cochain σ in $A^q(\mathcal{U})$ may be written as $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ with $\sigma_i \in A^q(U_i)$, $i = 0, 1$, and $\sigma_{01} \in A^{q-1}(U_{01})$, $U_{01} = U_0 \cap U_1$. The differential $D : A^q(\mathcal{U}) \to A^{q+1}(\mathcal{U})$ is given by

$$D\sigma = (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}).$$

If we set

$$A^q(\mathcal{U}, U_0) = \{\, \sigma \in A^q(\mathcal{U}) \mid \sigma_0 = 0 \,\},$$

$A^*(\mathcal{U}, U_0)$ forms a subcomplex of $A^*(\mathcal{U})$ and we have a canonical isomorphism

$$H^q(A^*(\mathcal{U}, U_0)) \simeq H^q(M, M \setminus S; \mathbb{C}).$$

Suppose S is compact (M may not be). Let R_1 be a compact manifold of dimension m with C^∞ boundary ∂R_1 in U_1, containing S in its interior $\operatorname{Int} R_1$, and set $R_0 = M \setminus \operatorname{Int} R_1$. Then $\{R_0, R_1\}$ is a system of honey-comb cells adapted to $\mathcal{U}$. In this situation, we have the integration on $A^m(\mathcal{U}, U_0)$ defined by

$$\int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01},$$

where $R_{01} = R_0 \cap R_1 = -\partial R_1$ (∂R_1 with opposite orientation). This induces the integration

$$\int_M : H^m(M, M \setminus S; \mathbb{C}) \to \mathbb{C}.$$

If M is compact, the following diagram is commutative :

$$\begin{array}{ccc} H^m(M, M \setminus S; \mathbb{C}) & \xrightarrow{j^*} & H^m(M, \mathbb{C}) \\ \downarrow{\scriptstyle \int_M} & & \downarrow{\scriptstyle \int_M} \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C}. \end{array} \tag{1.2}$$

(B) Chern classes in the Čech-de Rham cohomology

Let M be a C^∞ manifold of dimension m and E a C^∞ complex vector bundle of (complex) rank r over M. For a connection ∇ for E and for $i = 1, \dots, r$, we denote by $c^i(\nabla)$ the i-th Chern form defined by ∇. It is obtained from the i-th symmetric function of the curvature matrix of ∇ and is a closed $2i$-form on M. Its class in $H^{2i}(M, \mathbb{C})$ is the i-th Chern class $c^i(E)$ of E.

For $p+1$ connections $\nabla_0, \dots, \nabla_p$ for E, we have the "difference form" $c^i(\nabla_0, \dots, \nabla_p)$, which is a $(2i-p)$-form alternating in the $p+1$ entries and satisfies

$$\sum_{\nu=0}^{p}(-1)^\nu c^i(\nabla_0, \dots, \widehat{\nabla_\nu}, \dots, \nabla_p) + (-1)^p d\, c^i(\nabla_0, \dots, \nabla_p) = 0, \tag{1.3}$$

cf. [Bo]. Here we use a different sign convention, see [S1, Ch.II, (7.10)].

Let $\mathcal{U} = \{U_\alpha\}_\alpha$ be an open covering of M. For each α, we choose a connection ∇_α for E on U_α, and for the collection $\nabla_* = (\nabla_\alpha)_\alpha$, we define the element $c^i(\nabla_*)$ in $A^{2i}(\mathcal{U})$ by

$$c^i(\nabla_*)_{\alpha_0 \dots \alpha_p} = c^i(\nabla_{\alpha_0}, \dots, \nabla_{\alpha_p}).$$

Then we have $D\, c^i(\nabla_*) = 0$ by (1.3). Moreover, it is shown that the class of $c^i(\nabla_*)$ in $H^{2i}(A^*(\mathcal{U}))$ does not depend on the choice of the collection of connections ∇_*. It corresponds to the class $c^i(E)$ in $H^{2i}(M, \mathbb{C})$ under the isomorphism (1.1).

(C) Localization of the top Chern class

Let E be a C^∞ complex vector bundle of rank n over a complex manifold M of dimension n. Also, let S be a closed set in M and suppose we have a C^∞ section s of E on M which is non-vanishing on $M \setminus S$. Sometimes S is called the singular set of s. In this situation, we have the "localization" $c^n(E, s)$ in $H^{2n}(M, M \setminus S; \mathbb{C})$ of the Chern class $c^n(E)$ in $H^{2n}(M, \mathbb{C})$ at S with respect to s, which is described as follows [S2-3].

Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Then the Chern class $c^n(E)$ is represented by the cocycle $c^n(\nabla_*)$ in $A^{2n}(\mathcal{U})$ given by $c^n(\nabla_*) = (c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1))$, where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an s-trivial connection (i.e., a connection with $\nabla_0(s) = 0$), we have $c^n(\nabla_0) = 0$ and the the cocycle $c^n(\nabla_*)$ is in $A^{2n}(\mathcal{U}, U_0)$. The localization $c^n(E, s)$ is represented by this cocycle

$$c^n(\nabla_*) = (0, c^n(\nabla_1), c^n(\nabla_0, \nabla_1)).$$

In the above situation, suppose that S is compact and let $(S_\lambda)_\lambda$ be the connected components of S. Then, for each λ, $c^n(E, s)$ defines the residue of s at S_λ with respect to c^n, denote by $\mathrm{Res}_{c^n}(s, E; S_\lambda)$. Namely, for each λ, we choose a neighborhood U_λ of S_λ in U_1, so that the U_λ's are mutually disjoint, and let R_λ be a compact $2n$-dimensional manifold with C^∞ boundary in U_λ containing S_λ in its interior. We set $R_{0\lambda} = -\partial R_\lambda$. Then the residue is given by

$$\mathrm{Res}_{c^n}(s, E; S_\lambda) = \int_{R_\lambda} c^n(\nabla_1) + \int_{R_{0\lambda}} c^n(\nabla_0, \nabla_1). \tag{1.4}$$

Note that the above residues are in fact integers (cf. [S3, Remark 2.4.3]). From the commutativity of (1.2), we have the "residue formula" :

Proposition 1.5. *If M is compact,*

$$\sum_\lambda \mathrm{Res}_{c^n}(s, E; S_\lambda) = \int_M c^n(E).$$

(D) Residue at an isolated singularity

In general, let U be a neighborhood of 0 in $\mathbb{C}^n$ and $f_1, \ldots, f_n$ holomorphic functions on U such that $\{\, p \in U \mid f_i(p) = 0,\ 1 \le i \le n \,\} = \{0\}$. For a holomorphic n-from ω on U, the Grothendieck residue is defined by

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ denotes the n-cycle in U given by

$$\Gamma = \{\, p \in U \mid |f_i(p)| = \varepsilon_i,\ \ 1 \le i \le n \,\}$$

for small positive numbers ε_i. It is oriented so that $d\arg(f_1) \wedge \cdots \wedge d\arg(f_n) \ge 0$ (see for example [GH, Ch.5]).

In the situation of (C) above, if S_λ consists of a single point p_0 and if E and s are holomorphic, the residue $\mathrm{Res}_{c^n}(s, E; p_0)$ is given as follows [S2-3]. Let $\mathbf{e} = (e_1, \ldots, e_n)$ be a holomorphic frame of E on a sufficiently small open neighborhood U of p_0 and write $s = \sum_{i=1}^n f_i\, e_i$, with f_i holomorphic functions on U. Then we have

$$\mathrm{Res}_{c^n}(s, E; p_0) = \mathrm{Res}_{p_0} \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \ldots, f_n \end{bmatrix}.$$

Note that it is also equal to the dimension of the vector space $\mathcal{O}/(f_1, \ldots, f_n)$, where $\mathcal{O}$ denotes the ring of germs of holomorphic functions at p_0 and $(f_1, \ldots, f_n)$ the ideal generated by $f_1, \ldots, f_n$.

In particular, let $E = T^\vee M$, the (holomorphic) cotangent bundle of M, and let f be a holomorphic function near p_0 with only critical point at p_0. Then the differential df is a section of $T^\vee M$ with an isolated singularity at p_0. If $(z_1, \ldots, z_n)$ denotes a coordinate system around p_0, we have $f_i = \frac{\partial f}{\partial z_i}$ in the above so that

$$\mathrm{Res}_{c^n}(df, T^\vee M; p_0) = m(f, p_0), \tag{1.6}$$

the multiplicity (or the Milnor number) of f at p_0.

2. Multiplicity formula

Let M be an almost complex manifold of dimension $2n$ so that its cotangent bundle $T^\vee M$ may be naturally thought of as a complex vector bundle of rank n. For a C^∞ function f on M, its differential df is a section of $T^\vee M$, which is non-vanishing away from its critical set $C(f)$. For a compact connected component S of $C(f)$ having a neighborhood disjoint from the other components, we define the multiplicity $m(f,S)$ of f at S by

$$m(f,S) = \mathrm{Res}_{c^n}(df, T^\vee M; S).$$

Note that, if M is a complex manifold, f is holomorphic and if S consists of a point p_0, it coinsides with the usual multiplicity of f at p_0 (see (1.6)).

Now we consider the global situation. Let $f : M \to C$ be a holomorphic map of a complex manifold M of dimension n onto a complex curve C (complex manifold of dimension one). The differential

$$df : TM \to f^*TC$$

of f determines a section of the bundle $T^\vee M \otimes f^*TC$, which is also denoted by df. The set of zeros of df is the critical set $C(f)$ of f. Suppose $C(f)$ is compact and denote by $(S_\lambda)_\lambda$ its connected components. Then we have the residue $\mathrm{Res}_{c^n}(df, T^\vee M \otimes f^*TC; S_\lambda)$ for each λ. By Proposition 1.5, we have :

Proposition 2.1. *If M is compact,*

$$\sum_\lambda \mathrm{Res}_{c^n}(df, T^\vee M \otimes f^*TC; S_\lambda) = \int_M c^n(T^\vee M \otimes f^*TC).$$

We look at the both sides of the above more closely. In the sequel, we set $D(f) = f(C(f))$, the set of critical values. Then, if M is compact, f defines a C^∞ fiber bundle structure on $M \setminus C(f) \to C \setminus D(f)$.

Lemma 2.2. *If M is compact, and if $D(f)$ consists of isolated points,*

$$\int_M c^n(T^\vee M \otimes f^*TC) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)),$$

where F denotes a general fiber of f.

Proof. Using properties of Chern classes and noting that C is complex one dimensional, we have

$$c^n(T^\vee M \otimes f^*TC) = (-1)^n \left(c^n(M) - c^{n-1}(M) \cdot f^*c^1(C)\right).$$

Thus it suffices to show that

$$\int_M c^{n-1}(M) \cdot f^* c^1(C) = \chi(F)\,\chi(C),$$

which follows essentially from the projection formula. See Lemma 5.2 below, where this is proved precisely in a more general situation. □

Suppose that $f(S_\lambda)$ is a point. Taking a coordinate on C around $f(S_\lambda)$, we think of f as a holomorphic function near S_λ. Then we may write

$$\mathrm{Res}_{c^n}(df, T^\vee M \otimes f^*TC; S_\lambda) = \mathrm{Res}_{c^n}(df, T^\vee M; S_\lambda) = m(f, S_\lambda),$$

the multiplicity of f at S_λ. Thus we proved :

Theorem 2.3. *Let $f : M \to C$ be a holomorphic map of a compact complex manifold M of dimension n onto a complex curve C. If the critical values $D(f)$ of f consists of only isolated points, then*

$$\sum_\lambda m(f, S_\lambda) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)),$$

where the sum is taken over the connected components S_λ of $C(f)$.

In particular, we have ([I], see also [F, Example 14.1.5] and [HL, VI 3]) :

Corollary 2.4. *In the above situation, if the critical set $C(f)$ of f consists of only isolated points,*

$$\sum_{p \in C(f)} m(f, p) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)).$$

Analogous formulas are studied in [Dio] when the target space is higher dimensional.

3. Characteristic numbers on singular varieties

We refer to [S1, Ch.IV, 2, Ch.VI, 4] and [S3] for details of the material in this section. Let V be an analytic subvariety of pure dimension n in a complex manifold W of dimension $n + k$. We denote by $\mathrm{Sing}(V)$ the singular set of V and set $V' = V \setminus \mathrm{Sing}(V)$.

(A) Integration

First, we suppose that V is compact. Let $\tilde{U}$ be a neighborhood of V in W and let $\mathcal{U} = \{\tilde{U}_\alpha\}_\alpha$ be an open covering of $\tilde{U}$. Taking a system $\{\tilde{R}_\alpha\}_\alpha$ of honeycomb cells adapted to $\mathcal{U}$ such that V is transverse to each $\tilde{R}_{\alpha_0\cdots\alpha_p} = \tilde{R}_{\alpha_0} \cap \cdots \cap \tilde{R}_{\alpha_p}$, we may define the integration

$$\int_V : H^{2n}(A^*(\mathcal{U})) \to \mathbb{C}.$$

We have $H^{2n}(A^*(\mathcal{U})) \simeq H^{2n}(\tilde{U}, \mathbb{C})$ and the above integration is compatible with the integration on $H^{2n}(\tilde{U}, \mathbb{C})$ induced from the integration of $2n$-forms on $\tilde{U}$ over the $2n$-cycle V.

Second, suppose V may not be compact. Let S be a compact set in V such that there is an open set U in V with $S \subset U$ and $U \setminus S \subset V'$. Letting $\tilde{U}_1$ be a neighborhood of S in W with $\tilde{U}_1 \cap V \subset U$ and $\tilde{U}_0$ a tubular neighborhood of $U_0 = U \setminus S$ in W, we consider the covering $\mathcal{U} = \{\tilde{U}_0, \tilde{U}_1\}$ of $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$. We define the subcomplex $A^*(\mathcal{U}, \tilde{U}_0)$ of $A^*(\mathcal{U})$ as in Section 1 (A).

Let $\tilde{R}_1$ be a compact real $2(n+k)$ dimensional manifold with C^∞ boundary in $\tilde{U}_1$ such that S is in its interior and that $\partial\tilde{R}_1$ is transverse to U. We set $R_1 = \tilde{R}_1 \cap U$, $R_{01} = -\partial R_1 = -\partial\tilde{R}_1 \cap U$. Then we have the integration on $A^{2n}(\mathcal{U}, \tilde{U}_0)$ given by

$$\int_U \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}$$

for $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{2n}(\mathcal{U}, \tilde{U}_0)$. This induces the integration on the cohomology

$$\int_U : H^{2n}(A^*(\mathcal{U}, \tilde{U}_0)) \to \mathbb{C}.$$

Suppose V is compact again and let S be a compact set in V which contains $\mathrm{Sing}(V)$. In this situation we set $U = V$ and we have the following commutative diagram :

$$\begin{array}{lccc}
(3.1) \qquad & H^{2n}(A^*(\mathcal{U}, \tilde{U}_0)) & \longrightarrow & H^{2n}(A^*(\mathcal{U})) \\
& \Big\downarrow \int_V & & \Big\downarrow \int_V \\
& \mathbb{C} & \xrightarrow{=} & \mathbb{C}.
\end{array}$$

Remarks 3.2. 1. In the above, the assumption that $U_0 = U \setminus S$ is in the regular part $V' = V \setminus \mathrm{Sing}(V)$ is not necessary. However, with this condition, to define a cochain $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^q(\mathcal{U})$ we only need to define σ_0 on U_0, since there is a C^∞ retraction $\tilde{U}_0 \to U_0$.

2. If S admits a regular neighborhood in W, we may choose $\tilde{U}_1$ so that it is a regular neighborhood of S and that U is a deformation retract of $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$. In this case, we have canonical isomorphisms

$$H^q(A^*(\mathcal{U})) \simeq H^q(U, \mathbb{C}) \qquad \text{and} \qquad H^q(A^*(\mathcal{U}, \tilde{U}_0)) \simeq H^q(U, U \setminus S; \mathbb{C}).$$

(B) Localization of Chern classes

Let E be a vector bundle of rank r. In general, an ℓ-frame of E is an ℓ-tuple of sections $\mathbf{s} = (s_1, \dots, s_\ell)$ of E linearly independent everywhere on the domain of definition. An r-frame is simply called a frame. A connection ∇ for E is $\mathbf{s}$-trivial if $\nabla(s_i) = 0$ for all i.

Let S, $\tilde{U}_1$, $\tilde{U}_0$, $\mathcal{U}$ and $\tilde{U}$ be as in (A). For a complex vector bundle E over $\tilde{U}$ of rank r ($\geq n$), the Chern class $c^n(E)$ is represented by the cocycle $c^n(\nabla_*)$ in $A^{2n}(\mathcal{U})$ given by

$$c^n(\nabla_*) = (c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on $\tilde{U}_0$ and $\tilde{U}_1$, respectively. Note that it is sufficient if ∇_0 is defined only on $U_0 = U \setminus S$ (see Remark 3.2.1).

Suppose that we have a C^∞ ℓ-frame $\mathbf{s} = (s_1, \dots, s_\ell)$ on U_0, $\ell = r - n + 1$, and let ∇_0 be $\mathbf{s}$-trivial. Then we have the vanishing $c^n(\nabla_0) = 0$ and the above cocycle $c^n(\nabla_*)$ is in $A^{2n}(\mathcal{U}, \tilde{U}_0)$ and it defines a class $c^n(E, \mathbf{s})$ in $H^{2n}(A^*(\mathcal{U}, \tilde{U}_0))$, which is independent of the choice of the $\mathbf{s}$-trivial connection ∇_0 or the connection ∇_1.

If we let $(S_\lambda)_\lambda$ be the connected components of S, integrating $c^n(E, \mathbf{s})$ over U, we have the residue $\mathrm{Res}_{c^n}(\mathbf{s}, E|_U; S_\lambda)$ of s at S_λ with respect to c^n, for each λ. For each λ, we choose a neighborhood $\tilde{U}_\lambda$ of S_λ in $\tilde{U}_1$, so that the $\tilde{U}_\lambda$'s are mutually disjoint. Let $\tilde{R}_\lambda$ be a real $2(n+k)$-dimensional manifold with C^∞ boundary in $\tilde{U}_\lambda$ containing S_λ in its interior such that the boundary $\partial \tilde{R}_\lambda$ is transverse to V. We set $R_{0\lambda} = -\partial \tilde{R}_\lambda \cap V$. Then the residue is given by

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_U; S_\lambda) = \int_{R_\lambda} c^n(\nabla_1) + \int_{R_{0\lambda}} c^n(\nabla_0, \nabla_1).$$

Note that the above residues are in fact integers. From the commutativity of (3.1), we have the "residue formula" :

Proposition 3.4. *If V is compact,*

$$\sum_\lambda \mathrm{Res}_{c^n}(\mathbf{s}, E|_V; S_\lambda) = \int_V c^n(E).$$

(C) Residue at an isolated singularity

In general, let $\tilde{U}$ be a neighborhood of 0 in $\mathbb{C}^{n+k}$ and V a subvariety of dimension n in $\tilde{U}$ which contains 0 as at most an isolated singular point. Also, let $f_1, \dots, f_n$ be holomorphic functions on $\tilde{U}$ such that

$$\{\, p \in \tilde{U} \mid f_i(p) = 0,\ 1 \le i \le n \,\} \cap V = \{0\}.$$

For a holomorphic n-from ω on $\tilde{U}$, the Grothendieck residue relative to V is defined by

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ denotes the n-cycle expressed as the one in Section 1 (D) with $U = \tilde{U} \cap V$ ([LS], [Su1, Ch.IV, 8]).

If V is a complete intersection defined by $h_1 = \cdots = h_k = 0$ in $\tilde{U}$, by an iterated use of the projection formula, we see that

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \operatorname{Res}_0 \begin{bmatrix} \omega \wedge dh_1 \wedge \cdots \wedge dh_k \\ f_1, \dots, f_n, h_1, \dots, h_k \end{bmatrix}.$$

Now we go back to the previous situation. Let p_0 be a point in V and assume that V has at most an isolated singularity at p_0. Let E be a holomorphic vector bundle of rank r, $1 \le n \le r$, over a neighborhood $\tilde{U}$ of p_0 in W. Let $\ell = r - n + 1$ and suppose we have ℓ holomorphic sections $s_1, \dots, s_\ell$ of E on $\tilde{U}$ such that

$$\{\, p \in \tilde{U} \mid s_1 \wedge \cdots \wedge s_\ell(p) = 0 \,\} \cap V = \{p_0\}.$$

Thus $\mathbf{s} = (s_1, \dots, s_\ell)$ is an ℓ-frame of E on $U_0 = U \setminus \{p_0\}$, $U = \tilde{U} \cap V$. In this situation, the residue $\operatorname{Res}_{c^n}(\mathbf{s}, E|_V; p_0)$ is given as follows [S3, Theorem 5.7]. Let $\mathbf{e} = (e_1, \dots, e_r)$ be a holomorphic frame of E on $\tilde{U}$ and write $s_i = \sum_{j=1}^r f_{ij}\, e_j$, $i = 1, \dots, \ell$, with f_{ij} holomorphic functions on $\tilde{U}$. Let F be the $\ell \times r$ matrix whose (i,j)-entry is f_{ij}. We set

$$\mathcal{I} = \{\, (i_1, \dots, i_\ell) \mid 1 \le i_1 < \cdots < i_\ell \le r \,\}.$$

For an element $I = (i_1, \dots, i_\ell)$ in $\mathcal{I}$, let F_I denote the $\ell \times \ell$ matrix consisting of the columns of F corresponding to I and set $f_I = \det F_I$. We have [S3, Lemma 5.6] :

Lemma 3.5. *We may choose a holomorphic frame $\mathbf{e} = (e_1, \dots, e_r)$ of E so that there exist n elements $I^{(1)}, \dots, I^{(n)}$ in $\mathcal{I}$ with the property*

$$\{\, p \in \tilde{U} \mid f_{I^{(1)}}(p) = \cdots = f_{I^{(n)}}(p) = 0 \,\} \cap V = \{p_0\}.$$

Note that we may assume that $I^{(1)}, \ldots, I^{(n)}$ are the first n elements in $\mathcal{I}$ with the lexicographic order. Let $\mathbf{e} = (e_1, \ldots, e_r)$ be a frame of E on $\tilde{U}$ as in Lemma 3.5. Let us write $I^{(\alpha)} = (i_1^{(\alpha)}, \ldots, i_\ell^{(\alpha)})$, $\alpha = 1, \ldots, n$, and let $F^{(\alpha)}$ be the $r \times r$ matrix obtained by replacing the $i_j^{(\alpha)}$-th row of the $r \times r$ identity matrix by the j-th row of F, $j = 1, \ldots, \ell$. Note that $\det F^{(\alpha)} = f_{I^{(\alpha)}}$. Let $\check{F}^{(\alpha)}$ denote the adjoint matrix of $F^{(\alpha)}$ and set

$$\Theta^{(\alpha)} = \check{F}^{(\alpha)} \cdot dF^{(\alpha)},$$

which is an $r \times r$-matrix whose entries are holomorphic 1-forms. For an n-tuple of integers $A = (a_1, \ldots, a_n)$ with $1 \leq a_1 < \cdots < a_n \leq r$, we denote by $\Theta_A^{(\alpha)}$ the $n \times n$ matrix whose (i, j)-entry is the (a_i, a_j)-entry of $\Theta^{(\alpha)}$. Also, for a permutation ρ of degree n, we denote by $\Theta_A(\rho)$ the $n \times n$-matrix whose i-th column is that of $\Theta_A^{(\rho(i))}$. For the collection $\Theta = \{\Theta^{(\alpha)}\}_\alpha$, we set

$$\sigma_n(\Theta) = \frac{1}{n!} \sum_A \sum_\rho \operatorname{sgn} \rho \cdot \det \Theta_A(\rho),$$

where the sums are taken over the n-tuples A as above and the permutations ρ of degree n, and the determinant is defined as in [S3, 4 (C)]. Note that $\sigma_n(\Theta)$ is a holomorphic n-form on $\tilde{U}$. Then we have [S3, Theorem 5.7] :

Theorem 3.6. *In the above situation, the residue is given by*

$$\operatorname{Res}_{c^n}(\mathbf{s}, E; p_0) = \operatorname{Res}_{p_0} \begin{bmatrix} \sigma_n(\Theta) \\ f_{I^{(1)}}, \ldots, f_{I^{(n)}} \end{bmatrix}_V.$$

We refer to [S3, 6] for more explicit expressions of the above in some special cases.

(D) Virtual bundles

Let E and E' be vector bundles. The total Chern class $c^*(\xi)$ of the virtual bundle $\xi = E - E'$ is defined by $c^*(\xi) = c^*(E)/c^*(E')$. The i-th Chern class $c^i(\xi)$ is the component of $c^*(\xi)$ of degree $2i$. It is also define by taking connections ∇ and ∇' of E and E', respectively. From the pair $\nabla^\bullet = (\nabla', \nabla)$ we may define a closed $2i$-form $c^i(\nabla^\bullet)$ and its class in the de Rham cohomology is $c^i(\xi)$ [S1, Ch.II, 8 (C)].

For $p + 1$ pairs of connections $\nabla_0^\bullet, \ldots, \nabla_p^\bullet$ for E, the "difference form" $c^i(\nabla_0^\bullet, \ldots, \nabla_p^\bullet)$ is defined similarly as in the case of a single vector bundle. It is a $(2i - p)$-form satisfying an identity similar to (1.3).

Suppose we have an exact sequence of vector bundles

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0.$$

If we have $p+1$ triples of connections $(\nabla'_i, \nabla_i, \nabla''_i)$, $i = 0, \dots, p$, compatible with the above sequence [BB], we have

$$c^i(\nabla^\bullet_0, \dots, \nabla^\bullet_p) = c^i(\nabla''_0, \dots, \nabla''_p), \tag{3.7}$$

where $\nabla^\bullet_i = (\nabla'_i, \nabla_i)$.

Chern numbers of virtual bundles on a singular variety can be computed similarly as in the case of single vector bundles.

4. Multiplicity of functions on local complete intersections

In Sections 4 and 5, the varieties we consider are "local complete intersections defined by a section". Namely, let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$. We say that V is a local complete intersection defined by a section, if there exist a holomorphic vector bundle N of rank k over W and a holomorphic section s of N such that s is generically transverse to the zero section and that V is the zero set of s. Note that, in this case, the ideal sheaf of germs of holomorphic functions vanishing on V is generated by the local components of s and V is in fact a local complete intersection (cf. [T]).

Let V be as above and let ι denote the embedding $V \hookrightarrow W$. We denote by $\mathrm{Sing}(V)$ the singular set of V and by V' the regular part of V ; $V' = V \setminus \mathrm{Sing}(V)$. Then the restriction $N|_{V'}$ coincides with the normal bundle $N_{V'}$ of V' in W. We set $N_V = N|_V$. On V', there is a commutative diagram of vector bundles whose first row is exact :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TV' & \xrightarrow{d\iota} & TW|_{V'} & \longrightarrow & N_{V'} & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \text{incl.}} & & \big\downarrow{\scriptstyle \text{incl.}} & & \\ & & & & TW|_V & \xrightarrow{\rho} & N_V & & . \end{array} \tag{4.1}$$

We set $\tau_V = TW|_V - N_V$ and call it the virtual tangent bundle of V. Taking the duals of the above, we have :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^\vee_{V'} & \longrightarrow & T^\vee W|_{V'} & \xrightarrow{\iota^*} & T^\vee V' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \text{incl.}} & & \big\downarrow{\scriptstyle \text{incl.}} & & & & \\ & & N^\vee_V & \xrightarrow{\rho^\vee} & T^\vee W|_V & & & & . \end{array} \tag{4.2}$$

We set $\tau^\vee_V = T^\vee W|_V - N^\vee_V$ call it the virtual cotangent bundle of V.

Let g be a C^∞ function on W and let f and f' denote the restrictions of g to V and V', respectively. We set $S(f) = \mathrm{Sing}(V) \cup C(f')$ and call it the singular set of f. We let $V_0 = V \setminus S(f)$. Let S be a compact connected component of $S(f)$ which admits an open neighborhood U in V with $U \setminus S \subset V_0$. Letting $\tilde{U}_1$

be a neighborhood of S in W with $\tilde{U}_1 \cap V \subset U$ and $\tilde{U}_0$ a tubular neighborhood of $U_0 = U \setminus S$ in W, we consider the covering $\mathcal{U} = \{\tilde{U}_0, \tilde{U}_1\}$ of $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$.

We denote by f_0 the restriction of g to V_0. Thus df_0 is a non-vanishing section of the bundle $T^\vee U_0$, which is of rank n. We now try to compute $c^n(\tau^\vee)$, $\tau^\vee = T^\vee W - N^\vee$, and will see that there is a canonical localization $c^n(\tau^\vee, df)$ in $H^{2n}(A^*(\mathcal{U}, \tilde{U}_0))$ of $c^n(\tau^\vee)$. Let ∇ be a df_0-trivial connection for $T^\vee U_0$ on U_0. Let ∇_0 and ∇_0' be connections for $T^\vee W$ and $N^\vee$ on U_0, respectively, such that the triple $(\nabla_0', \nabla_0, \nabla)$ is compatible with the first row in (4.2). Let ∇_1 and ∇_1' be arbitrary connections for $T^\vee W$ and $N^\vee$ on $\tilde{U}_1$, respectively. Then the class $c^n(\tau^\vee)$ is represented in $H^{2n}(A^*(\mathcal{U}))$ by the cocycle

$$c^n(\nabla_*^\bullet) = (c^n(\nabla_0^\bullet), c^n(\nabla_1^\bullet), c^n(\nabla_0^\bullet, \nabla_1^\bullet)),$$

where we set $\nabla_i^\bullet = (\nabla_i', \nabla_i)$, $i = 0, 1$. Since $(\nabla_0', \nabla_0, \nabla)$ is compatible and ∇ is df_0-trivial, we have

$$c^n(\nabla_0^\bullet) = c^n(\nabla) = 0.$$

Hence the cocycle is in $A^{2n}(\mathcal{U}, U_0)$ and it defines a class in $H^{2n}(A^*(\mathcal{U}, \tilde{U}_0))$ which is denoted by $c^n(\tau^\vee, df)$ and called the localization of $c^n(\tau^\vee)$ by df at S. It is independent of the choces of various connections.

Let $\tilde{R}_1$ be a compact real $2(n+k)$ dimensional manifold with C^∞ boundary in $\tilde{U}_1$ such that S is in its interior and that $\partial \tilde{R}_1$ is transverse to U. We set $R_1 = \tilde{R}_1 \cap U$, $R_{01} = -\partial R_1 = -\partial \tilde{R}_1 \cap U$. Then, integrating $c^n(\tau^\vee, df)$ over U, we have the residue $\mathrm{Res}_{c^n}(df, \tau_V^\vee; S)$, which is given by

$$\mathrm{Res}_{c^n}(df, \tau_V^\vee; S) = \int_{R_1} c^n(\nabla_1^\bullet) + \int_{R_{01}} c^n(\nabla_0^\bullet, \nabla_1^\bullet). \tag{4.3}$$

We define the virtual multiplicity $\tilde{m}(f, S)$ of f at S to be this residue :

$$\tilde{m}(f, S) = \mathrm{Res}_{c^n}(df, \tau_V^\vee; S).$$

On the other hand, for such a component S, we have the (generalized) Milnor number $\mu(V, S)$ of V at S [BLSS]. Note that if S consists of a point p, it is the usual Milnor number $\mu(V, p)$ of the isolated complete intersection singularity (V, p) ([M], [H], see also [Lo]). In the hypersurface case, it is studied in detail in [P], [PP].

Definition 4.4. The multiplicity $m(f, S)$ of f at S is defined by

$$m(f, S) = \tilde{m}(f, S) - \mu(V, S).$$

Remark 4.5. If S is in the regular part V', we may choose the above connections ∇_1' and ∇_1 and a connection for $T^\vee V$ so that they are compatible with the first row of (4.2) on $\tilde{U}_1 \cap V$. Then we have, by (3.7),

$$\mathrm{Res}_{c^n}(df, \tau_V^\vee; S) = \mathrm{Res}_{c^n}(df, T^\vee V'; S).$$

On the other hand, in this case we have $\mu(V, S) = 0$ so that $m(f, S)$ coincides with the one in Section 2.

If S consists of a single point p_0, the residue $\mathrm{Res}_{c^n}(df, \tau_V^\vee; p_0)$ is given as follows. We may assume that $\tilde{U}$ is small enough so that the bundle N admits a frame $\nu = (\nu_1, \dots, \nu_k)$ on $\tilde{U}$. We write $s = \sum_{i=1}^k h_i \nu_i$ with h_i holomorphic functions on $\tilde{U}$. Thus V is defined by $h_1 = \cdots = h_k = 0$ on $\tilde{U}$. Then the $(k+1)$-tuple of sections

$$\mathbf{s} = (dg|_U, dh_1|_U, \cdots, dh_k|_U)$$

of $T^\vee W|_U$ form a $(k+1)$-frame on U_0. Here it should be emphasized that these are not restrictions of differential forms but restrictions of sections. Since the rank of $T^\vee W|_V$ is $n+k$, we have the residue $\mathrm{Res}_{c^n}(\mathbf{s}, T^\vee W|_V; p_0)$ (see Section 3).

Proposition 4.6.

$$\tilde{m}(f, p_0) = \mathrm{Res}_{c^n}(\mathbf{s}, T^\vee W|_V; p_0).$$

Proof. Recall that the left hand side is given by (4.3) and the right hand side by a similar formula, except $\nabla_0^\bullet$ is replaced by an $\mathbf{s}$-trivial connection for $T^\vee W|_U$ on U_0 and $\nabla_1^\bullet$ by a connection for $T^\vee W$ on $\tilde{U}_1$.

First we take connections on U_0. Note that, on U, the homomorphism ρ in (4.1) sends a tangent vector v of W to $\sum_{i=1}^k v(h_i)\nu_i|_U$. Hence, if we let $\nu^\vee = (\nu_1^\vee, \dots, \nu_k^\vee)$ be the frame of $N^\vee$ on $\tilde{U}$ which is dual to ν, the homomorphism $\rho^\vee$ in (4.2) sends $\nu_i^\vee|_U$ to $dh_i|_U$, $i = 1, \dots, k$. On the other hand, $dg|_{U_0}$ is sent to df_0 by ι^*. Thus there exist, on U_0, a $\nu^\vee$-trivial connection ∇_0' for $N^\vee|_U$, an $\mathbf{s}$-trivial connection ∇_0 for $T^\vee W|_U$ and a df-trivial connection ∇ for $T^\vee U_0$ such that the triple $(\nabla_0', \nabla_0, \nabla)$ is compatible with the first row of (4.2) on U_0. If we set $\nabla_0^\bullet = (\nabla_0', \nabla_0)$, we have $c^n(\nabla_0^\bullet) = c^n(\nabla) = 0$, since ∇ is df_0-trivial. We also have $c^n(\nabla_0) = 0$, since ∇_0 is $\mathbf{s}$-trivial

Next we take connections on $\tilde{U}_1$. Let ∇_1 be an arbitrary connection for $T^\vee W$ and let ∇_1' be the $\nu^\vee$-trivial connection for $N^\vee$ on $\tilde{U}_1$ and set $\nabla_1^\bullet = (\nabla_1', \nabla_1)$.

From $c^*(\tau_V^\vee) = c^*(T^\vee W)/c^*(N^\vee)$, we may write

$$c^n(\tau_V^\vee) = c^n(T^\vee W) + \sum_{i=1}^n c^{n-i}(T^\vee W) \cdot \varphi^i(N^\vee),$$

where φ^i denotes a Chern polynomial homogeneous of degree i. Then, since ∇_1' is $\nu^\vee$-trivial, we have $\varphi^i(\nabla_1') = 0$ and

$$c^n(\nabla_1^\bullet) = c^n(\nabla_1) + \sum_{i=1}^n c^{n-i}(\nabla_1) \cdot \varphi^i(\nabla_1') = c^n(\nabla_1). \tag{4.7}$$

To compute $c^n(\nabla_0^\bullet, \nabla_1^\bullet)$, recall that it is given by integrating $c^n(\tilde{\nabla}^\bullet)$ over the 1-simplex $[0,1]$, where $\tilde{\nabla}^\bullet = (\tilde{\nabla}', \tilde{\nabla})$ with $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$ and $\tilde{\nabla}' = (1-t)\nabla_0' +$

$t\nabla'_1$. Since $\nabla'_0 = \nabla'_1$, which is $\nu^\vee$-trivial, we have $\tilde{\nabla}' = \nabla'_1$ and $\varphi^i(\tilde{\nabla}') = \varphi^i(\nabla'_1) = 0$ for all i. Hence we have

$$c^n(\tilde{\nabla}^\bullet) = c^n(\tilde{\nabla}) + \sum_{i=1}^{n} c^{n-i}(\tilde{\nabla}) \cdot \varphi^i(\nabla'_1) = c^n(\tilde{\nabla}).$$

Therefore we have $c^n(\nabla_0^\bullet, \nabla_1^\bullet) = c^n(\nabla_0, \nabla_1)$, which together with (4.7) imply the equality of the residues. □

Corollary 4.8. *The multiplicity of f at p_0 is given by*

$$m(f, p_0) = \mathrm{Res}_{c^n}(\mathbf{s}, T^\vee W|_V; p_0) - \mu(V, p_0).$$

From the above we see that $\widetilde{m}(f, p_0)$ and thus $m(f, p_0)$ are integers. If g is holomorphic, we may compute $\widetilde{m}(f, p_0) = \mathrm{Res}_{c^n}(\mathbf{s}, T^\vee W|_V; p_0)$ using the formula in Theorem 3.6. We give an explicit expression in some cases.

(1) The case $n = 1$ with k arbitrary.

Thus V is a curve of arbitrary codimension in W. We may assume that $\tilde{U}$ is small enough so that it admits a coordinate system $(z_1, \dots, z_{k+1})$. Let φ denote the Jacobian :

$$\varphi = \det \frac{\partial(g, h_1, \dots, h_k)}{\partial(z_1, \dots, z_{k+1})}.$$

Then we have [S3, Section 6, (2)] :

$$\widetilde{m}(f, p_0) = \mathrm{Res}_{p_0} \begin{bmatrix} d\varphi \\ \varphi \end{bmatrix}_V = \mathrm{Res}_{p_0} \begin{bmatrix} d\varphi \wedge dh_1 \wedge \cdots \wedge dh_k \\ \varphi, h_1, \dots, h_k \end{bmatrix}.$$

Thus, in this case, $\widetilde{m}(f, p_0)$ can be interpreted as the intersection number at p_0 of V and the divisor defined by φ in $\tilde{U}$.

(2) The case $k = 1$ with n arbitrary.

Thus V is a hypersurface in W of arbitrary dimension. We may assume that $\tilde{U}$ admits a coordinate system $(z_1, \dots, z_{n+1})$ such that the frame $(dz_1, \dots, dz_{n+1})$ satisfies the condition of Lemma 3.5. Set $h = h_1$ and

$$\varphi_{ij} = \begin{vmatrix} \frac{\partial g}{\partial z_i} & \frac{\partial g}{\partial z_j} \\ \frac{\partial h}{\partial z_i} & \frac{\partial h}{\partial z_j} \end{vmatrix}, \qquad \theta_{ij} = \begin{vmatrix} \frac{\partial g}{\partial z_i} & d\frac{\partial g}{\partial z_j} \\ \frac{\partial h}{\partial z_i} & d\frac{\partial h}{\partial z_j} \end{vmatrix}.$$

We may assume that

$$\{\, p \in \tilde{U} \mid \varphi_{12}(p) = \cdots = \varphi_{1,n+1}(p) = 0 \,\} \cap V = \{p_0\}.$$

Then we have [S3, Section 6, (3)] :

$$\widetilde{m}(f, p_0) = \mathrm{Res}_{p_0} \begin{bmatrix} \sigma_n(\Theta) \\ \varphi_{12}, \dots, \varphi_{1,n+1} \end{bmatrix}_V = \mathrm{Res}_{p_0} \begin{bmatrix} \sigma_n(\Theta) \wedge dh \\ \varphi_{12}, \dots, \varphi_{1,n+1}, h \end{bmatrix},$$

where

$$\sigma_n(\Theta) = \frac{1}{n}\Big(\sum_{i=2}^{n+1} \theta_{12}\wedge\cdots\wedge\theta_{1,i-1}\wedge d\varphi_{1i}\wedge\theta_{1,i+1}\wedge\cdots\wedge\theta_{1,n+1}$$
$$+\sum_{2\le i<j\le n+1}(-1)^{i+j}\theta_{11}\wedge d\varphi_{ij}\wedge\theta_{12}\wedge\cdots\wedge\widehat{\theta_{1i}}\wedge\cdots\wedge\widehat{\theta_{1j}}\wedge\cdots\wedge\theta_{1,n+1}\Big).$$

We finish this section by giving an interpretation of the number $\widetilde{m}(f,p_0)$ similar to the one for the "GSV-index" of a vector field ([GSV], [SS1]) so that it may be called the "virtual multiplicity" of f at p_0 (cf. [LSS], [SS2], where the virtual index of a vector field is defined and is shown to coincide with the GSV-index). The integrality of $\widetilde{m}(f,p_0)$ and $m(f,p_0)$ can be also seen from this.

Let $\tilde{U}$ and $\nu=(\nu_1,\dots,\nu_k)$ be as before and write $s=\sum_{i=1}^k h_i\nu_i$. We consider the holomorphic map $h=(h_1,\dots,h_k):(\tilde{U},p_0)\to(\mathbb{C}^k,0)$. For a point t in $\mathbb{C}^k$ near 0, we set $V_t=h^{-1}(t)$ and dente by f_t the restriction of g to V_t. We set $S(f_t)=\mathrm{Sing}(V_t)\cup C(f_t)$. Note that $S(f_t)$ consists of a finite number of points. For a general point t (a point not in $h(C(h))$), V_t is non-singular (the Milnor fiber of h, [M], [H], [Lo]).

Proposition 4.9. *In the above situation, we have*

$$\widetilde{m}(f,p_0)=\sum_{p\in S(f_t)}\widetilde{m}(f_t,p).$$

In particular, for general t,

$$\widetilde{m}(f,p_0)=\sum_{p\in C(f_t)}m(f_t,p)\quad\text{and thus}\quad m(f,p_0)=\sum_{p\in C(f_t)}m(f_t,p)-\mu(V,p_0).$$

Proof. We set $\tilde{U}'=\tilde{U}\setminus C(h)$ and denote by h' the restriction of h to $\tilde{U}'$. Then there is a commutative diagram of vector bundles whose first row is exact and which extends (4.1) (restricted to $V'\cap\tilde{U}$) :

$$\text{(4.10)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & N^\vee|_{\tilde{U}'} & \longrightarrow & T^\vee W|_{\tilde{U}'} & \xrightarrow{\pi} & T^\vee h' & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle\text{incl.}} & & \big\downarrow{\scriptstyle\text{incl.}} & & & & \\ & & N^\vee|_{\tilde{U}} & \xrightarrow{\tilde{\rho}^\vee} & T^\vee W|_{\tilde{U}} & & & & , \end{array}$$

where Th' denotes the bundle of vectors tangent to the fibers of h'. We compute the localization $c^n(\tau_{V_t}^\vee, df_t)$ using (the restrictions to V_t of) the connections deifned as follows. On $\tilde{U}_0=\tilde{U}'\setminus C(g)$, $\pi(dg)$ is non-vanishing and its restrction to V_t is

df_t. Let $\tilde{\nabla}$ be a $\pi(dg)$-trivial connection for $T^\vee h'$ on $\tilde{U}_0$ and choose connections $\tilde{\nabla}_0$ and $\tilde{\nabla}'_0$, respectively, for $T^\vee W$ and $N^\vee$ on $\tilde{U}_0$ so that the triple $(\tilde{\nabla}, \tilde{\nabla}_0, \tilde{\nabla}'_0)$ is compatible with the first row of (4.10) on $\tilde{U}_0$. We let ∇_1 and ∇'_1 be arbitrary connections for $T^\vee W$ and $N^\vee$, respectively, on $\tilde{U}$.

Then we see that the class $c^n(\tau_{V_t}^\vee, df_t)$ gives the sum $\sum_{p\in S(f_t)} \widetilde{m}(f_t, p)$, which depends continuously on t. For a regular value t of h, we have $c^n(\tau_{V_t}^\vee, df_t) = c^n(T^\vee V_t, df_t)$ (see Remark 4.5) and the sum becomes $\sum_{p\in C(f_t)} m(f_t, p)$, which is an integer (see Section 2). Thus the sum remains constant, since the regular values are dense. $\square$

5. Singular multiplicity formula

Let V be a local complete intersection of dimension n in W defined by a section of a holomorphic vector bundle N over W. Also, let $g : W \longrightarrow C$ be a holomorphic map from W onto a complex curve C and let f denote the restrictions of g to V. We denote by f' the restiction of f to V' and set $S(f) = \mathrm{Sing}(V) \cup C(f')$. Note that $C(g) \cap V \subset S(f)$. We assume that $S(f)$ is compact. We set $V_0 = V \setminus S(f)$, which is a subset of V'. We denote by f_0 the restriction of g to V_0. Thus df_0 is a non-vanishing section of the bundle $T^\vee V_0 \otimes f_0^* TC$, which is of rank n. We now try to compute $c^n(\varepsilon)$, $\varepsilon = \tau_V^\vee \otimes f^*TC$ and will see that there is a canonical localization $c^n(\varepsilon, df)$ in $H^{2n}(V, V \setminus S; \mathbb{C})$ of $c^n(\varepsilon)$. This is done exactly in the same way, except all the bundles are tensored by g^*TC or f^*TC.

Let $(S_\lambda)_\lambda$ be the connected components of S and let $(R_\lambda)_\lambda$ be as in Section 3 (B). Then we have, for each λ, the residue $\mathrm{Res}_{c^n}(df, \tau_V^\vee \otimes f^*TC; S_\lambda)$, which is given by a formula similar to (4.3). Note that, if S_λ is in the regular part V', it coincides with the one in Section 2. From the definition, we have :

Proposition 5.1. *If V is compact,*

$$\sum_\lambda \mathrm{Res}_{c^n}(df, \tau_V^\vee \otimes f^*TC; S_\lambda) = \int_V c^n(\tau_V^\vee \otimes f^*TC).$$

We now compute the global number in the right hand side of the equality in Proposition 5.1.

Lemma 5.2. *If V is compact, and if $f(S(f)) = f(\mathrm{Sing}(V) \cup C(f'))$ consists of isolated points,*

$$\int_V c^n(\tau_V^\vee \otimes f^*TC) = (-1)^n \left(\chi(V) - \chi(F)\,\chi(C)\right) + \sum_\lambda \mu(V, S_\lambda),$$

where F is a general fiber of f.

Proof. First, using the properties of the Chern classes and noting that C is complex one dimensional, we have

$$c^n(\tau_V^\vee \otimes f^*TC) = (-1)^n \left(c^n(\tau_V) - c^{n-1}(\tau_V)\cdot f^*c^1(C)\right).$$

By [BLSS] (see also [SS2]),

$$\int_V c^n(\tau_V) = \chi(V) + (-1)^n \sum_\lambda \mu(V, S_\lambda).$$

Thus it suffices to prove

$$\int_V c^{n-1}(\tau_V) \cdot f^* c^1(C) = \chi(F)\,\chi(C). \tag{5.3}$$

Let $f(S) = \{t_1 \cdots t_k\}$ and, for each t_i, let C_i be a small open disk around t_i. We set $C_0 = C \setminus f(S)$ and consider the covering $\mathcal{C} = \{C_0, \{C_i\}_{i=1}^k\}$ of C. In this proof, we set $V_0 = V \setminus f^{-1}(C_0)$ so that $f_0 : V_0 \longrightarrow C_0$ is a holomorphic submersion and is a C^∞ fiber bundle with typical fiber F. Let $W_i = g^{-1}(C_i)$ and let W_0 be a tubular neighborhood of V_0 in W small enough so that g does not have critical points in W_0. We let g_0 be the restriction of g to W_0. Let $\mathcal{W}$ denote the covering $\{W_0, \{W_i\}_{i=1}^k\}$ of W. In the following, we work on the Čech-de Rham cohomologies associated with the coverings $\mathcal{C}$ and $\mathcal{W}$. We have an exact sequence of vector bundles on V_0 :

$$0 \longrightarrow Tf_0 \longrightarrow TV_0 \longrightarrow f_0^* TC \longrightarrow 0. \tag{5.4}$$

where Tf_0 denotes the bundles of vectors tangent to the fibers of f_0.

To compute the left hand side of (5.3), we take various connections. We choose a connection ∇_0^C for TC on C_0 and connections ∇_0^V, ∇_0^f, ∇_0 and ∇_0', respectively, for TV_0, Tf_0, $TW|_{V_0}$ and N_{V_0} on V_0 so that $(\nabla_0^f, \nabla_0^V, f^*\nabla_0^C)$ is compatible with (5.4) and $(\nabla_0^V, \nabla_0, \nabla_0')$ is compatible with (5.1). For each $i = 1, \dots, k$, let ∇_i^C be a connection for TC on C_i which is trivial with respect to some local frame of TC. Also, let ∇_i and ∇_i' be connections for TW and N, respectively, on W_i. We set $\nabla_0^\bullet = (\nabla_0', \nabla_0)$ and $\nabla_i^\bullet = (\nabla_i', \nabla_i)$. Then $c^1(C)$ is represented by the cocycle

$$c^1(\nabla_*^C) = \left(c^1(\nabla_0^C), (c^1(\nabla_i^C))_i, (c^1(\nabla_0^C, \nabla_i^C))_i\right)$$

in $A^2(\mathcal{C})$ and $c^{n-1}(\tau_V)$ is represented by the cocycle

$$c^{n-1}(\nabla_*^\bullet) = \left(c^{n-1}(\nabla_0^\bullet), (c^{n-1}(\nabla_i^\bullet))_i, (c^{n-1}(\nabla_0^\bullet, \nabla_i^\bullet))_i\right)$$

in $A^{2(n-1)}(\mathcal{W})$ (cf. Remark 3.2.1). Note that $c^1(\nabla_i^C) = 0$. The class $c^{n-1}(\tau_V) \cdot f^*c^1(C)$ is then represented by the cup product

$$c^{n-1}(\nabla_*^\bullet) \cdot f^*c^1(\nabla_*^C) = \left(c^{n-1}(\nabla_0^\bullet) \cdot f^*c^1(\nabla_0^C), 0, (c^{n-1}(\nabla_0^\bullet) \cdot f^*c^1(\nabla_0^C, \nabla_i^C))_i\right).$$

From the compatibilities of the connections and the fact that C is complex one dimensional, we have

$$c^{n-1}(\nabla_0^\bullet) = c^{n-1}(\nabla_0^V) = c^{n-1}(\nabla_0^f) + c^{n-2}(\nabla_0^f) \cdot f^*c^1(\nabla_0^C).$$

Hence we have

$$c^{n-1}(\nabla^{\bullet}_{*})\cdot f^{*}c^{1}(\nabla^{C}_{*})=\left(c^{n-1}(\nabla^{f}_{0})\cdot f^{*}c^{1}(\nabla^{C}_{0}),0,(c^{n-1}(\nabla^{f}_{0})\cdot f^{*}c^{1}(\nabla^{C}_{0},\nabla^{C}_{i}))_{i}\right).$$

For each $i=1,\dots,k$, let D_i be a closed disk in C_i centered at t_i and let $D_0=C\setminus\cup_{i=1}^{k}\operatorname{Int}D_i$. Then $\{D_0,\{D_i\}_{i=1}^{k}\}$ is a system of honey-comb cells adapted to the covering $\mathcal{C}$. If we set $R_i=f^{-1}(D_i)$ and $R_{0i}=f^{-1}(D_{0i})$, $D_{0i}=-\partial D_i$, by the projection formula, the left hand side of (5.3) becomes

$$\begin{aligned}\int_{R_0}&c^{n-1}(\nabla^{f}_{0})\cdot f^{*}c^{1}(\nabla^{C}_{0})+\sum_{i=1}^{k}\int_{R_{0i}}c^{n-1}(\nabla^{f}_{0})\cdot f^{*}c^{1}(\nabla^{C}_{0},\nabla^{C}_{i})\\&=\int_{D_0}f_{*}c^{n-1}(\nabla^{f}_{0})\cdot c^{1}(\nabla^{C}_{0})+\sum_{i=1}^{k}\int_{D_{0i}}f_{*}c^{n-1}(\nabla^{f}_{0})\cdot c^{1}(\nabla^{C}_{0},\nabla^{C}_{i}).\end{aligned}$$

In the above, $f_{*}c^{n-1}(\nabla^{f}_{0})$ denotes the integration along the fibers of f and it is equal to the constant $\chi(F)$. On the other hand,

$$\int_{D_0}c^{1}(\nabla^{C}_{0})+\sum_{i=1}^{k}\int_{D_{0i}}c^{1}(\nabla^{C}_{0},\nabla^{C}_{i})=\int_{C}c^{1}(C)=\chi(C).$$

Thus we proved (5.3). □

Suppose that $f(S_\lambda)$ is a point. Taking a coordinate on C around $f(S_\lambda)$, we think of g and f as functions near S_λ. Then we may write

$$\operatorname{Res}_{c^n}(df,\tau_V^{\vee}\otimes f^{*}TC;S_\lambda)=\widetilde{m}(f,S_\lambda)=m(f,S_\lambda)-\mu(V,S_\lambda).$$

Summarizing the above, we have :

Theorem 5.5. *Let V be a local complete intersection of dimension n defined by a section of a holomorphic vector bundle over a complex manifold W. Let $g:W\longrightarrow C$ be a holomorphic map and let f be the restriction of g to V. If V is compact and if $f(S(f))$, $S(f)=\operatorname{Sing}(V)\cup C(f')$, consists only of isolated points,*

$$\sum_{\lambda}m(f,S_\lambda)=(-1)^{n}\left(\chi(V)-\chi(F)\,\chi(C)\right),$$

where F denotes a general fiber of f.

Corollary 5.6. *In the above situation, if $S(f)$ consists only of isolated points,*

$$\sum_{p\in S(f)}m(f,p)=(-1)^{n}\left(\chi(V)-\chi(F)\,\chi(C)\right).$$

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