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Two Dimensional Commutative Banach Algebras and von Neumann Inequality

by

Takahiko Nakazi* and Takanori Yamamoto*

Dedicated to Professor Tsuyoshi Ando on the occasion of his 70th birthday

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Abstract. We show the following: Let $\mathcal{B}$ be a two dimensional commutative Banach algebra with identity. If $\mathcal{B}$ satisfies

$$T \in \mathcal{B}, \quad \|T\| \leq 1 \quad \Rightarrow \quad \|f(T)\| \leq 1$$

whenever f is a polynomial satisfying $|f(z)| \leq 1$ ($|z| \leq 1$) then $\mathcal{B}$ is isometric to a subalgebra of the algebra $B(H)$ of all bounded linear operators on some Hilbert space H, and $\mathcal{B}$ satisfies

$$T_k \in \mathcal{B}, \quad \|T_k\| \leq 1 \quad (k = 1, ..., n) \quad \Rightarrow \quad \|f(T_1, ..., T_n)\| \leq 1$$

whenever f is a polynomial in n variables satisfying $|f(z_1, ..., z_n)| \leq 1 \quad (|z_k| \leq 1, \ k = 1, ..., n)$, for all n.

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1. Introduction

J.von Neumann's cerebrated inequality on Hilbert space operators asserts that

$$T \in B(H), \quad \|T\| \leq 1 \quad \Rightarrow \quad \|f(T)\| \leq 1$$

whenever $f = f(z)$ is a polynomial satisfying $|f(z)| \leq 1 \quad (|z| \leq 1)$, where H denote a Hilbert space and $B(H)$ denote the algebra of all bounded linear operators on H. Throughout this paper, it is supposed that an identity I in a commutative Banach algebra $\mathcal{B}$ satisfies $\|I\| = 1$.

Definition 1. *Let $\mathcal{B}$ be a commutative Banach algebra with identity. If $\mathcal{B}$ satisfies*

$$T_k \in \mathcal{B}, \quad \|T_k\| \leq 1 \quad (k = 1, ..., n) \quad \Rightarrow \quad \|f(T_1, ..., T_n)\| \leq 1$$

whenever f is a polynomial in n variables satisfying $|f(z_1, ..., z_n)| \leq 1 \quad (|z_k| \leq 1, \ k = 1, ..., n)$, then we say that $\mathcal{B}$ satisfies n-von Neumann inequality. n-von Neumann inequality is denoted by n-(vN) for short.

Hence if $\mathcal{B}$ satisfies n-(vN) then $\mathcal{B}$ satisfies k-(vN) (k=1,...,n). By the theorem of I.G.Craw (see [2, p.271]) and the theorem of B.Cole (see [2, p.272]), if $\mathcal{B}$ is a commutative Banach algebra with identity and $\mathcal{B}$ satisfies n-(vN) for all n then $\mathcal{B}$ is isometric to a subalgebra of $B(H)$. J.von Neumann [9] proved that if $\mathcal{B}$ is a subalgebra of $B(H)$ then $\mathcal{B}$ satisfies 1-(vN). T.Ando [1] proved that if $\mathcal{B}$ is a commutative subalgebra of $B(H)$ then $\mathcal{B}$ satisfies 2-(vN). N.Th.Varopoulos found examples of a finite dimensional commutative subalgebra of $B(H)$ which does not satisfy 3-(vN) (see G.Pisier [10, p.23], I.Suciu [11, p.252]). Suppose $\mathcal{B}$ is a Banach algebra with identity. If $\dim \mathcal{B} = 1$ then it is clear that $\mathcal{B}$ satisfies n-(vN) for all n. By the first author, T.Nakazi [8, Corollary 2], if $\mathcal{B}$ is a two dimensional commutative subalgebra of $B(H)$ with identity then $\mathcal{B}$ satisfies n-(vN) for all n.

Question. *Suppose $\mathcal{B}$ is a commutative Banach algebra with identity and $\mathcal{B}$ satisfies n-(vN) for some n. Does there exist a Hilbert space H such that $\mathcal{B}$ is isometric to a subalgebra of $B(H)$?*

In Section 2, we shall study 1-(vN). In Theorem 1, we shall prove that if $\mathcal{B}$ is a two dimensional commutative subalgebra with identity and if $\mathcal{B}$ satisfies 1-(vN) then $\mathcal{B}$ is isometric to a subalgebra of $B(H)$. By Theorem 1, if $\dim \mathcal{B} = 2$ then the answer to this question is yes with $n = 1$. If $\dim \mathcal{B} \geq 3$ then we do not know the answer. C.Foias [6] (see G.Pisier [10, p.26]) proved that if a Banach algebra $B(X)$ of all bounded linear operators on a Banach space X satisfies 1-(vN) then X is isometric to a Hilbert space. Hence if $\mathcal{B} = B(X)$ then the answer is yes with $n = 1$. In Section 3, we shall study n-(vN). In Theorem 2, we shall prove that if $\mathcal{B}$ is a two dimensional commutative subalgebra of $B(H)$ with identity then $\mathcal{B}$ satisfies n-(vN) for all n. Theorem 2 was proved by T.Nakazi [8, Corollary 2] in a different way. By Theorems 1 and 2, if $\mathcal{B}$ satisfies 1-(vN) then $\mathcal{B}$ satisfies n-(vN) for all n. In Section 4, we shall study the isometric Möbius transformation. In Theorem 3, we shall prove that if $\mathcal{B}$ has an identity and $\mathcal{B}$ is isometric to a subalgebra of $B(H)$, then the Möbius transformation operates isometrically on the unit sphere of $\mathcal{B}$, and conversely.

2. 1-von Neumann inequality

In this section, we shall give Lemmas 1 $\sim$ 8 to prove Theorem 1. Let α, β be complex numbers, and let x be a nonnegative number. Then we define a function $m_{\alpha,\beta}(x)$ which is the generalization of $\max(|\alpha|, |\beta|)$.

Definition 2. *Let*

$$m_{\alpha,\beta}(x) = \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}.$$

If $0 \leq x_1 \leq x_2$ then $\max(|\alpha|, |\beta|) = m_{\alpha,\beta}(0) \leq m_{\alpha,\beta}(x_1) \leq m_{\alpha,\beta}(x_2) < \infty$. If $\alpha = \beta$ then

$$m_{\alpha,\alpha}(x) = \frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + |\alpha|^2}.$$

I.A.Feldman, N.Ja.Krupnik and A.S.Markus [5] (see I.C.Gohberg-N.Ja.Krupnik [7, Vol.II, p.215]) established the following Lemma 1. T.Nakazi [8, Theorem 1 (2)] established the following Lemma 2 which follows from Lemma 1.

Lemma 1. *Let $P \in B(H)$ satisfy $P^2 = P$ and let $Q = I - P$ where I is an identity operator. Let $\alpha, \beta \in \mathbf{C}$. If $x = |\alpha - \beta|\sqrt{\|P\|^2 - 1}$ then $\|\alpha P + \beta Q\| = m_{\alpha,\beta}(x)$.*

Lemma 2. *Let $N \in B(H)$ satisfy $N^2 = 0$. Let $\alpha, \beta \in \mathbf{C}$. If $x = |\beta| \cdot \|N\|$ then $\|\alpha I + \beta N\| = m_{\alpha,\alpha}(x)$.*

Lemma 3. *If $a > 1$ then the transformation:*

$$w = \rho(z) = \frac{1 - az}{|z|^2 - az}$$

maps $\{z \neq 0, a\}$ onto the complex plane. For every complex number w, there exist at least two complex numbers z.

Proof. Let $w = u + iv$ be a complex number. If $v = 0$ then by the elementary calculation there exist at least two real numbers z satisfying $w = \rho(z)$. It is sufficient to prove when $v \neq 0$. We shall show that there exist at least two complex numbers $z = x + iy$ satisfying the equality. This is equivalent to show that there exist at least two pairs (x, y) satisfying

$$u + iv = \frac{1 - ax - iay}{x^2 + y^2 - ax - iay}.$$

This is equivalent to show that there exist at least two pairs (x, y) satisfying

$$1 - ax = u(x^2 + y^2 - ax) + avy \quad \text{and} \quad -ay = v(x^2 + y^2 - ax) - auy.$$

If $u=0$ then this is equivalent to show that there exist at least two pairs (x,y) satisfying

$$1-ax=avy \quad \text{and} \quad -ay=v(x^2+y^2-ax).$$

This is equivalent to show that there exist at least two real numbers x satisfying

$$a^2(v^2+1)x^2-a(a^2(v^2+1)+2)x+a^2+1=0.$$

Since $a>1$,

$$\begin{aligned} D &= a^2\left((a^2(v^2+1)+2)^2-4(v^2+1)(a^2+1)\right) \\ &= a^2\left(a^4(v^2+1)^2+4-4(v^2+1)\right) \\ &\geq a^2\left((v^2-1)^2+(a^4-1)(v^2+1)\right)>0. \end{aligned}$$

It is sufficient to show that if $uv\neq 0$ then there exist at least two pairs (x,y) satisfying

$$1-ax=u(x^2+y^2-ax)+avy \quad \text{and} \quad v(1-ax)=a(u^2+v^2-u)y.$$

If $u^2+v^2-u=0$ then $x=\frac{1}{a}$ and y is arbitrary. It is sufficient to show that if $uv\neq 0$ and $u^2+v^2-u\neq 0$ then there exist at least two pairs (x,y) satisfying the equalities. This is equivalent to show that there exist at least two real numbers x satisfying

$$\begin{aligned} & a^2(u^2+v^2-u)^2(1-ax) \\ =\ & a^2(u^2+v^2-u)^2u(x^2-ax)+uv^2(1-ax)^2+a^2(u^2+v^2-u)v^2(1-ax). \end{aligned}$$

Let $b=a^2(u^2+v^2-u)^2+v^2$, and let $c=a^2(u^2+v^2-u)(1-u)+v^2$. Then it is sufficient to show that there exist at least two real numbers x satisfying

$$((a^2-1)v^2+b)x^2-a(b+c)x+c=0.$$

Then

$$\begin{aligned} D &= a^2(b+c)^2-4((a^2-1)v^2+b)c \\ &= (a^2-1)((b+c)^2-4v^2c)+(b-c)^2. \end{aligned}$$

Since $b>v^2$, $(b+c)^2-4v^2c>(b-c)^2$. Hence $D>a^2(b-c)^2\geq 0$. Lemma 3 is proved. □

Lemma 4. *If $a>1$ then the transformation:*

$$w=\rho(z)=\frac{1-az}{|z|^2-az}$$

maps $\{0<|z|\leq 1\}$ onto the complex plane.

Proof. If $|z|=1$ then $w=1$. Let $w\neq 1$. By Lemma 3, for every complex number w, there exist at least two complex numbers z satisfying the equality. Let s and t be distinct complex numbers satisfying $w = \rho(s) = \rho(t)$. Then $a(|s|^2 t - s|t|^2 + s - t) = |s|^2 - |t|^2$. Let $p = 1 - |s|^2$, and let $q = 1 - |t|^2$. Then $a(sq - tp) = q - p$. Suppose $sq - tp = 0$. Then $p = q = 0$. Hence $|s| = |t| = 1$. Hence $w = 1$. This contradiction implies that $sq - tp \neq 0$. Since $a > 1$, $|sq - tp| < |q - p|$. Hence $pq(p + q - 2(1 - \mathrm{Re} s\bar{t})) > 0$. Hence $(1 - |s|^2)(1 - |t|^2)|s - t|^2 < 0$. Since $s \neq t$, $(1 - |s|)(1 - |t|) < 0$. Hence $|s| < 1$ or $|t| < 1$. Lemma 4 is proved. $\square$

Lemma 5. *If $a > 1$ and*

$$w = \frac{1 - az}{|z|^2 - az}, \quad (0 < |z| < 1)$$

then

$$m_{w,1}\left(|w - 1|\sqrt{a^2 - 1}\right) = \frac{1}{|z|}.$$

Proof. Since $az(1 - w) = 1 - w|z|^2$, it follows that

$$a^2|z|^2|1 - w|^2 = 1 - 2\mathrm{Re} w|z|^2 + |w|^2|z|^4.$$

Hence

$$|w|^2|z|^4 - \left(2\mathrm{Re} w + a^2|1 - w|^2\right)|z|^2 + 1 = 0.$$

Hence

$$|w|^2|z|^4 - \left(|w - 1|^2(a^2 - 1) + |w|^2 + 1\right)|z|^2 + 1 = 0.$$

Hence

$$|w|^2 - \left(|w - 1|^2(a^2 - 1) + |w|^2 + 1\right)\frac{1}{|z|^2} + \frac{1}{|z|^4} = 0.$$

Since $0 < |z| < 1$, this implies that

$$\begin{aligned} \frac{1}{|z|^2} &= \frac{1}{2}\left(|w - 1|^2(a^2 - 1) + |w|^2 + 1 + \sqrt{(|w - 1|^2(a^2 - 1) + |w|^2 + 1)^2 - 4|w|^2}\right) \\ &= \left(m_{w,1}\left(|w - 1|\sqrt{a^2 - 1}\right)\right)^2. \end{aligned}$$

Lemma 5 is proved. $\square$

Lemma 6. *Let $\mathcal{B}$ be a two dimensional commutative Banach algebra which is spanned by P and Q such that $P^2 = P \neq 0, I$ and $Q = I - P$ where I is an identity. Let $\alpha, \beta \in \mathbf{C}$. Then*

(1) $$m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right) \leq 1 \quad \textit{if and only if} \quad \left|\frac{\alpha - \beta}{1 - \bar{\beta}\alpha}\right| \leq \frac{1}{\|P\|}.$$

(2) $$m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right) = 1 \quad \textit{if and only if} \quad \left|\frac{\alpha - \beta}{1 - \bar{\beta}\alpha}\right| = \frac{1}{\|P\|}.$$

(3) *If $\mathcal{B}$ satisfies 1-(vN) then* $\|\alpha P+\beta Q\| = m_{\alpha,\beta}\left(|\alpha-\beta|\sqrt{\|P\|^2-1}\right)$.

Proof. (1): Since $m_{\alpha,\beta}(\sqrt{1-|\alpha|^2}\sqrt{1-|\beta|^2}) = 1$, it follows that

$$m_{\alpha,\beta}\left(|\alpha-\beta|\sqrt{\|P\|^2-1}\right) \leq 1$$

is equivalent to

$$|\alpha-\beta|\sqrt{\|P\|^2-1} \leq \sqrt{1-|\alpha|^2}\sqrt{1-|\beta|^2}.$$

This is equivalent to

$$\left|\frac{\alpha-\beta}{1-\bar{\beta}\alpha}\right| \leq \frac{1}{\|P\|}.$$

(2): Because the above equivalence holds not only for inequalities but also for equalities.
(3): Let $c = \|\alpha P+\beta Q\|$. Let $a = \frac{\alpha}{c}$ and $b = \frac{\beta}{c}$. Then $\|aP+bQ\| = 1$. Since $P^2 = P$ and $Q = I-P$, it follows that

$$\|aP\| = \|(aP+bQ)P\| \leq \|P\|,$$
$$\|bQ\| = \|(aP+bQ)Q\| \leq \|Q\|,$$

Since $P, Q \neq 0$, this implies that $|a|, |b| \leq 1$. It is sufficient to prove when $|a|, |b| < 1$. Since

$$\phi_b(z) = \frac{z-b}{1-\bar{b}z}, \quad (|z|<1)$$

is analytic and $\phi_b(b) = 0$, it follows that

$$\phi_b(aP+bQ) = \phi_b(a)P + \phi_b(b)Q = \frac{a-b}{1-\bar{b}a}P.$$

Since $\|aP+bQ\| = 1$ and $\mathcal{B}$ satisfies 1-(vN), it follows that $\|\phi_b(aP+bQ)\| \leq 1$, and hence

$$\left|\frac{a-b}{1-\bar{b}a}\right| \leq \frac{1}{\|P\|}.$$

By (1), $m_{a,b}\left(|a-b|\sqrt{\|P\|^2-1}\right) \leq 1$. Hence $m_{ca,cb}\left(|ca-cb|\sqrt{\|P\|^2-1}\right) \leq c$. Hence

$$m_{\alpha,\beta}\left(|\alpha-\beta|\sqrt{\|P\|^2-1}\right) \leq \|\alpha P+\beta Q\|.$$

We shall prove the reverse inequality. If $|\lambda| \leq 1$ then

$$\phi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}, \quad (|z|<1)$$

is analytic. Let $t = \|P\|$. Then $\|\phi_\lambda(t^{-1}P)\| \leq 1$. Since

$$\phi_\lambda(t^{-1}P) = \phi_\lambda(t^{-1}P+0Q) = \phi_\lambda(t^{-1})P + \phi_\lambda(0)Q = -\lambda\left(\frac{1-t\lambda}{|\lambda|^2-t\lambda}P+Q\right),$$

it follows that

$$\left\| \frac{1-t\lambda}{|\lambda|^2 - t\lambda} P + Q \right\| \le \frac{1}{|\lambda|}, \quad (0 < |\lambda| \le 1).$$

By Lemmas 4 and 5, for all $\alpha \in \mathbf{C}$, $\|\alpha P + Q\| \le m_{\alpha,1}\left(|\alpha - 1|\sqrt{\|P\|^2 - 1}\right)$.
Hence, for all $\alpha, \beta \in \mathbf{C}$, $\|\alpha P + \beta Q\| \le m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right)$. Lemma 6 is proved. □

Lemma 7. *Let $a > 0$. Then the transformation:*

$$w = \sigma(z) = \frac{1 - |z|^2}{az}$$

maps $\{0 < |z| \le 1\}$ onto the complex plane, and

$$\frac{1}{|z|} = \left|\frac{w}{2}\right| a + \sqrt{\left|\frac{w}{2}\right|^2 a^2 + 1}.$$

Proof. It is sufficient to prove when $a = 1$. Let $z = x + iy$ and $w = u + iv$. Then $ux - vy = 1 - x^2 - y^2$ and $uy + vx = 0$. If $u = v = 0$ then $x^2 + y^2 = 1$. If $u = 0$ and $v \ne 0$ then $x = 0$ and $y^2 - vy - 1 = 0$. Hence $y = (v \pm \sqrt{v^2 + 4})/2$. If $u \ne 0$ then $y = -vx/u$ and

$$x = -\frac{u}{2} \pm \sqrt{\left(\frac{u}{2}\right)^2 + \frac{u^2}{u^2 + v^2}}.$$

Then

$$|z|^2 = x^2 + y^2 = 1 - \frac{u^2 + v^2}{2u^2}\left(-u^2 \pm u\sqrt{u^2 + \frac{4u^2}{u^2 + v^2}}\right).$$

Hence σ maps $\{0 < |z| \le 1\}$ onto the complex plane. Since $0 < |z| \le 1$, it follows that

$$\frac{1}{|z|^2} - |w| \cdot \frac{1}{|z|} - 1 = 0.$$

Lemma 7 is proved. □

Lemma 8. *Let $\mathcal{B}$ be a two dimensional Banach algebra which is spanned by an identity I and N such that $N \ne 0$ and $N^2 = 0$. Let $\alpha, \beta \in \mathbf{C}$. Then*

(1) $$m_{\alpha,\alpha}(|\beta| \cdot \|N\|) \le 1 \quad \textit{if and only if} \quad \frac{|\beta|}{1 - |\alpha|^2} \le \frac{1}{\|N\|}.$$

(2) $$m_{\alpha,\alpha}(|\beta| \cdot \|N\|) = 1 \quad \textit{if and only if} \quad \frac{|\beta|}{1 - |\alpha|^2} = \frac{1}{\|N\|}.$$

(3) *If $\mathcal{B}$ satisfies 1-(vN) then* $\|\alpha I + \beta N\| = m_{\alpha,\alpha}(|\beta| \cdot \|N\|)$.

Proof. (1): Since $m_{\alpha,\alpha}(1 - |\alpha|^2) = 1$, it follows that $m_{\alpha,\alpha}(|\beta| \cdot \|N\|) \leq 1$ if and only if $|\beta| \cdot \|N\| \leq 1 - |\alpha|^2$ if and only if

$$\frac{|\beta|}{1 - |\alpha|^2} \leq \frac{1}{\|N\|}.$$

(2): Because the above equivalence holds not only for inequalities but also for equalities.
(3): Let $c = \|\alpha I + \beta N\|$. Let $a = \frac{\alpha}{c}$ and $b = \frac{\beta}{c}$. Then $\|aI + bN\| = 1$. Since $N^2 = 0$, it follows that

$$\|aN\| = \|(aI + bN)N\| \leq \|N\|.$$

Since $N \neq 0$, this implies that $|a| \leq 1$. It is sufficient to prove when $|a| < 1$. Since

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad (|z| < 1)$$

is analytic and $\phi_a(a) = 0$, it follows that

$$\phi_a(aI + bN) = \phi_a(a)I + b\phi_a'(a)N = b\phi_a'(a)N.$$

Since $\|aI + bN\| = 1$ and $\mathcal{B}$ satisfies 1-(vN), it follows that $\|\phi_a(aI + bN)\| \leq 1$, and hence

$$\frac{|b|}{1 - |a|^2} = |b\phi_a'(a)| \leq \frac{1}{\|N\|}.$$

By (1), $m_{a,a}(|b| \cdot \|N\|) \leq 1$. Hence $m_{ca,ca}(|cb| \cdot \|N\|) \leq c$. Hence

$$m_{\alpha,\alpha}(|\beta| \cdot \|N\|) \leq \|\alpha I + \beta N\|.$$

We shall prove the reverse inequality. If $|\lambda| \leq 1$ then

$$\phi_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda}z}, \quad (|z| < 1)$$

is analytic. Let $t = \|N\|$. Then $\|\phi_\lambda(t^{-1}N)\| \leq 1$. Since

$$\phi_\lambda(t^{-1}N) = \phi_\lambda(0I + t^{-1}N) = \phi_\lambda(0)I + t^{-1}\phi_\lambda'(0)N = -zI + \frac{1 - |z|^2}{t}N,$$

it follows that

$$\left\| I - \frac{1 - |z|^2}{tz}N \right\| \leq \frac{1}{|z|}, \quad (0 < |z| \leq 1).$$

By Lemma 7, for all $\beta \in \mathbf{C}$, $\|I + \beta N\| = m_{1,1}(|\beta| \cdot \|N\|)$.
Hence, for all $\alpha, \beta \in \mathbf{C}$, $\|\alpha I + \beta N\| = m_{\alpha,\alpha}(|\beta| \cdot \|N\|)$. Lemma 8 is proved. □

Theorem 1. *Let $\mathcal{B}$ be a two dimensional commutative Banach algebra with identity. If $\mathcal{B}$ satisfies 1-(vN) then $\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some Hilbert space H.*

Proof. By T.Nakazi [8, Proposition 1], it follows that

$$\mathcal{B} = \{\alpha P + \beta Q \ ; \ \alpha, \beta \in \mathbf{C}\} \quad \text{with} \quad P^2 = P \neq 0, I \quad \text{and} \quad Q = I - P,$$

or

$$\mathcal{B} = \{\alpha I + \beta N \ ; \ \alpha, \beta \in \mathbf{C}\} \quad \text{with} \quad N \neq 0 \quad \text{and} \quad N^2 = 0,$$

where I denotes an identity. Suppose $\mathcal{B} = \{\alpha P + \beta Q \ ; \ \alpha, \beta \in \mathbf{C}\}$. By Lemma 6(3),

$$\|\alpha P + \beta Q\| = m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right).$$

Let H be a Hilbert space satisfying $\dim H \geq 2$. Then there exists an identity operator I_0 and $P_0 \in B(H)$ satisfying $P_0^2 = P_0 \neq 0, I_0$ and $\|P_0\| = \|P\|$, and $N_0 \in B(H)$ satisfying $N_0 \neq 0$, $N_0^2 = 0$ and $\|N_0\| = \|N\|$. Let $Q_0 = I_0 - P_0$. Let $\mathcal{B}_1$ be a two dimensional commutative subalgebra of $B(H)$ which is spanned by P_0 and Q_0. Define Φ by

$$\Phi(\alpha P + \beta Q) = \alpha P_0 + \beta Q_0 \quad (\alpha, \beta \in \mathbf{C}).$$

By Lemma 1,

$$\begin{aligned} \|\Phi(\alpha P + \beta Q)\| &= \|\alpha P_0 + \beta Q_0\| = m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P_0\|^2 - 1}\right) \\ &= m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right) = \|\alpha P + \beta Q\|. \end{aligned}$$

Hence Φ is an isometry from $\mathcal{B}$ onto $\mathcal{B}_1$. Suppose $\mathcal{B} = \{\alpha I + \beta N \ ; \ \alpha, \beta \in \mathbf{C}\}$. By Lemma 8(3),

$$\|\alpha I + \beta N\| = m_{\alpha,\alpha}(|\beta| \cdot \|N\|).$$

Let $\mathcal{B}_2$ be a two dimensional commutative subalgebra of $B(H)$ which is spanned by I_0 and N_0. Define Ψ by

$$\Psi(\alpha I + \beta N) = \alpha I_0 + \beta N_0 \quad (\alpha, \beta \in \mathbf{C}).$$

By Lemma 2,

$$\|\Psi(\alpha I + \beta N)\| = \|\alpha I_0 + \beta N_0\| = m_{\alpha,\alpha}(|\beta| \cdot \|N_0\|) = m_{\alpha,\alpha}(|\beta| \cdot \|N\|) = \|\alpha I + \beta N\|.$$

Hence Ψ is an isometry from $\mathcal{B}$ onto $\mathcal{B}_2$. Theorem 1 is proved. $\square$

3. n-von Neumann inequality

In this section, we shall give Lemmas 9, 10, 11 to prove Theorem 2. Let $P \in B(H)$ satisfy $P^2 = P \neq 0, I$. Let $Q = I - P$. Let $\mathcal{B}$ be a two dimensional commutative subalgebra of $B(H)$ which is spanned by P and Q. By Theorem 2, $\mathcal{B}$ satisfies n-(vN) for all n. Let $T_k = \alpha_k P + \beta_k Q$. Then

$$T_k \in \mathcal{B}, \quad \|T_k\| \leq 1 \quad (k = 1, ..., n) \quad \Rightarrow \quad \|f(T_1, ..., T_n)\| \leq 1.$$

Since $f(T_1, ..., T_n) = f(\alpha_1, ..., \alpha_n)P + f(\beta_1, ..., \beta_n)Q$, it follows that

$$\|\alpha_k P + \beta_k Q\| \leq 1 \quad (k = 1, ..., n) \quad \Rightarrow \quad \|f(\alpha_1, ..., \alpha_n)P + f(\beta_1, ..., \beta_n)Q\| \leq 1$$

By Lemmas 1 and 6(1),

$$\left|\frac{\alpha_k - \beta_k}{1 - \bar{\beta}_k \alpha_k}\right| \leq \frac{1}{\|P\|} \quad (k = 1, ..., n) \quad \Rightarrow \quad \left|\frac{f(\alpha_1, ..., \alpha_n) - f(\beta_1, ..., \beta_n)}{1 - \overline{f(\beta_1, ..., \beta_n)} f(\alpha_1, ..., \alpha_n)}\right| \leq \frac{1}{\|P\|},$$

for all P satisfying $P^2 = P$. This implies Lemma 9(1). Similarly, let $N \in B(H)$ satisfy $N \neq 0$ and $N^2 = 0$. Let $\mathcal{B}$ be a two dimensional commutative subalgebra of $B(H)$ which is spanned by I and N. By the similar argument, Lemma 9(2) follows from Theorem 2 and Lemmas 2 and 8(1). In the following proof of Lemma 9, Prof. S. Takahashi gave the proof of (1). Lemma 9 may be well known. But for the self-containedness, we shall give its proof which does not use Theorem 2. If we use the theorem of I.G.Craw (see [2, p.271]) and Theorem 2 then Lemmas 10 and 11 follow. But we shall give another proofs which do not use Theorem 2. We prove Lemma 10 by Lemma 9(1), and we prove Lemma 11 by Lemma 9(2). We prove Theorem 2 by Lemmas 10,11 and T.Nakazi [8, Proposition 1].

Lemma 9. *If $|\alpha_k|, |\beta_k| \leq 1,\ k = 1, ..., n$ and f is a polynomial in n variables satisfying*

$$|f(z_1, ..., z_n)| \leq 1 \quad (|z_k| \leq 1,\ k = 1, ..., n)$$

then

(1)
$$\left|\frac{f(\alpha_1, ..., \alpha_n) - f(\beta_1, ..., \beta_n)}{1 - \overline{f(\beta_1, ..., \beta_n)} f(\alpha_1, ..., \alpha_n)}\right| \leq \max_{k=1,..,n} \left|\frac{\alpha_k - \beta_k}{1 - \bar{\beta}_k \alpha_k}\right|.$$

(2)
$$\frac{|(\beta_1 f_{z_1} + ... + \beta_n f_{z_n})(\alpha_1, ..., \alpha_n)|}{1 - |f(\alpha_1, ..., \alpha_n)|^2} \leq \max_{k=1,\ ,n} \frac{|\beta_k|}{1 - |\alpha_k|^2}.$$

Proof. Let $p(z) = f(\alpha_1 z, ..., \alpha_n z)$. Then p is a polynomial satisfying $|p(z)| \leq 1 \quad (|z| < R)$ where

$$R = \frac{1}{\max(|\alpha_1|, ..., |\alpha_n|)} \geq 1.$$

(1): By the Schwarz lemma,

$$\left|\frac{p(z)-p(a)}{1-\overline{p(a)}p(z)}\right| \leq \left|\frac{R(z-a)}{R^2-\bar{a}z}\right| \quad (|z|<R,\ |a|<R).$$

Hence

$$p(0)=0 \quad \Rightarrow \quad |p(z)| \leq \frac{|z|}{R} \quad (|z|<R).$$

Hence

$$f(0,...,0)=0 \quad \Rightarrow \quad |f(\alpha_1 z,...,\alpha_n z)| \leq \max(|\alpha_1 z|,...,|\alpha_n z|) \quad (|z|<R).$$

Hence, if $f(0,...,0)=0$ and $|z_1|,...,|z_n|<1$ then

$$|f(z_1,...,z_n)| \leq \max(|z_1|,...,|z_n|).$$

Let

$$w_k = \frac{z_k-\beta_k}{1-\overline{\beta_k}z_k} \quad (k=1,...,n),$$

and let

$$\psi(w_1,...,w_n) = \frac{f(z_1,...,z_n)-f(\beta_1,...,\beta_n)}{1-\overline{f(\beta_1,...,\beta_n)}f(z_1,...,z_n)}.$$

Since $\psi(0,...,0)=0$ and $|w_1|,...,|w_n|<1$, it follows that

$$|\psi(w_1,...,w_n)| \leq \max(|w_1|,...,|w_n|).$$

(2): By the Schwarz lemma,

$$\frac{|p'(z)|}{1-|p(z)|^2} \leq \frac{R}{R^2-|z|^2} \quad (|z|<R).$$

Since $p'(z)=(\alpha_1 f_{z_1}+...+\alpha_n f_{z_n})(\alpha_1 z,...,\alpha_n z)$, it follows that

$$\frac{|(\alpha_1 f_{z_1}+...+\alpha_n f_{z_n})(0,...,0)|}{1-|f(0,...,0)|^2} = \frac{|p'(0)|}{1-|p(0)|^2} \leq \frac{1}{R} = \max(|\alpha_1|,...,|\alpha_n|),$$

for all $\alpha_1,...,\alpha_n$. Hence

$$f(0,...,0)=0 \quad \Rightarrow \quad |f_{z_1}(0,...,0)|+...+|f_{z_n}(0,...,0)| \leq 1.$$

Let

$$g(z_1,...,z_n) = f\left(\frac{z_1+\alpha_1}{1+\overline{\alpha_1}z_1},...,\frac{z_n+\alpha_n}{1+\overline{\alpha_n}z_n}\right).$$

Then

$$g_{z_k}(z_1,...,z_n) = f_{z_k}\left(\frac{z_1+\alpha_1}{1+\overline{\alpha_1}z_1},...,\frac{z_n+\alpha_n}{1+\overline{\alpha_n}z_n}\right)\frac{1-|\alpha_k|^2}{(1+\overline{\alpha_k}z_k)^2}.$$

Hence

$$g_{z_k}(0,...,0) = f_{z_k}(\alpha_1,...,\alpha_n)(1-|\alpha_k|^2).$$

Since

$$g(0,...,0)=0 \quad \Rightarrow \quad |g_{z_1}(0,...,0)|+...+|g_{z_n}(0,...,0)| \leq 1,$$

it follows that

$$f(\alpha_1, ..., \alpha_n) = 0 \quad \Rightarrow \quad (1 - |\alpha_1|^2)|f_{z_1}(\alpha_1, ..., \alpha_n)| + ... + (1 - |\alpha_n|^2)|f_{z_n}(\alpha_1, ..., \alpha_n)| \leq 1.$$

Hence, for all complex numbers $t_1, ..., t_n$,

$$|(t_1(1 - |\alpha_1|^2)f_{z_1} + ... + t_n(1 - |\alpha_n|^2)f_{z_n})(\alpha_1, ..., \alpha_n)| \leq \max_{1\leq k\leq n} |t_k|.$$

Let

$$h(z_1, ..., z_n) = \frac{f(z_1, ..., z_n) - f(\alpha_1, ..., \alpha_n)}{1 - \overline{f(\alpha_1, ..., \alpha_n)}f(z_1, ..., z_n)}.$$

Then

$$h_{z_k}(\alpha_1, ..., \alpha_n) = \frac{f_{z_k}(\alpha_1, ..., \alpha_n)}{1 - |f(\alpha_1, ..., \alpha_n)|^2}.$$

Since $h(\alpha_1, ..., \alpha_n) = 0$, it follows that for all complex numbers $t_1, ..., t_n$,

$$\frac{|(t_1(1 - |\alpha_1|^2)f_{z_1} + ... + t_n(1 - |\alpha_n|^2)f_{z_n})(\alpha_1, ..., \alpha_n)|}{1 - |f(\alpha_1, ..., \alpha_n)|^2} \leq \max_{1\leq k\leq n} |t_k|.$$

This implies (2). Lemma 9 is proved. □

Lemma 10. *Let $P \in B(H)$ satisfy $P \neq 0, I$ and $P^2 = P$. Let $Q = I - P$. Let $T_k = \alpha_k P + \beta_k Q$. Then for all n,*

$$\|T_k\| \leq 1 \quad (k = 1, ..., n) \quad \Rightarrow \quad \|f(T_1, ..., T_n)\| \leq 1$$

whenever f is a polynomial in n variables satisfying $|f(z_1, ..., z_n)| \leq 1 \quad (|z_k| \leq 1,\ k = 1, ..., n)$.

Proof. By Lemma 1, if $\|T_k\| \leq 1 \quad (k = 1, ..., n)$ then

$$m_{\alpha_k,\beta_k}\left(|\alpha_k - \beta_k|\sqrt{\|P\|^2 - 1}\right) \leq 1 \quad (k = 1, ..., n).$$

By Lemma 6(1),

$$\left|\frac{\alpha_k - \beta_k}{1 - \overline{\beta_k}\alpha_k}\right| \leq \frac{1}{\|P\|} \quad (k = 1, ..., n).$$

By Lemma 9(1),

$$\left|\frac{f(\alpha_1, ..., \alpha_n) - f(\beta_1, ..., \beta_n)}{1 - \overline{f(\beta_1, ..., \beta_n)}f(\alpha_1, ..., \alpha_n)}\right| \leq \frac{1}{\|P\|}.$$

By Lemmas 1 and 6(1),

$$\|f(\alpha_1, ..., \alpha_n)P + f(\beta_1, ..., \beta_n)Q\| \leq 1.$$

Since

$$f(T_1, ..., T_n) = f(\alpha_1, ..., \alpha_n)P + f(\beta_1, ..., \beta_n)Q,$$

it follows that $\|f(T_1, ..., T_n)\| \leq 1$. Lemma 10 is proved. □

Lemma 11. *Let $N \in B(H)$ satisfy $N \neq 0$ and $N^2 = 0$. Let $T_k = \alpha_k I + \beta_k N$. Then for all n,*

$$\|T_k\| \leq 1 \quad (k = 1, ..., n) \quad \Rightarrow \quad \|f(T_1, ..., T_n)\| \leq 1$$

whenever f is a polynomial in n variables satisfying $|f(z_1, ..., z_n)| \leq 1 \quad (|z_k| \leq 1,\ k = 1, ..., n)$.

Proof. By Lemma 2, if $\|T_k\| \leq 1 \quad (k = 1, ..., n)$ then

$$m_{\alpha_k, \alpha_k}(|\beta_k| \cdot \|N\|) \leq 1 \quad (k = 1, ..., n).$$

By Lemma 8(1),

$$\frac{|\beta_k|}{1 - |\alpha_k|^2} \leq \frac{1}{\|N\|} \quad (k = 1, ..., n).$$

By Lemma 9(2),

$$\frac{|(\beta_1 f_{z_1} + ... + \beta_n f_{z_n})(\alpha_1, ..., \alpha_n)|}{1 - |f(\alpha_1, ..., \alpha_n)|^2} \leq \frac{1}{\|N\|}.$$

By Lemmas 2 and 8(1),

$$\|f(\alpha_1, ..., \alpha_n)I + (\beta_1 f_{z_1} + ... + \beta_n f_{z_n})(\alpha_1, ..., \alpha_n)N\| \leq 1.$$

Since

$$f(T_1, ..., T_n) = f(\alpha_1, ..., \alpha_n)I + (\beta_1 f_{z_1} + ... + \beta_n f_{z_n})(\alpha_1, ..., \alpha_n)N,$$

it follows that $\|f(T_1, ..., T_n)\| \leq 1$. Lemma 11 is proved. □

The following Theorem 2 was given by S.W.Drury [4] (see B.Cole-K.Lewis-J.Wermer [3]) when $\mathcal{B}$ is semi-simple, and by T.Nakazi [8, Corollary 2] when $\mathcal{B}$ is not semi-simple in a different way. We shall give an another proof.

Theorem 2. *If $\mathcal{B}$ is a two dimensional commutative subalgebra of $B(H)$ with identity then $\mathcal{B}$ satisfies n-(vN) for all n.*

Proof. By T.Nakazi [8, Proposition 1], it follows that $\mathcal{B} = \{\alpha P + \beta Q\ ;\ \alpha, \beta \in \mathbf{C}\}$ with $P^2 = P \neq 0, I$ and $Q = I - P$, or $\mathcal{B} = \{\alpha I + \beta N\ ;\ \alpha, \beta \in \mathbf{C}\}$ with $N \neq 0$ and $N^2 = 0$. By Lemma 10, if $\mathcal{B} = \{\alpha P + \beta Q\ ;\ \alpha, \beta \in \mathbf{C}\}$ then Theorem 2 is proved. By Lemmas 11, if $\mathcal{B} = \{\alpha I + \beta N\ ;\ \alpha, \beta \in \mathbf{C}\}$ then Theorem 2 is proved. □

By Theorems 1 and 2, we have the following corollary immediately.

Corollary 1. *Let $\mathcal{B}$ be a two dimensional commutative Banach algebra with identity. If $\mathcal{B}$ satisfies 1-(vN) then $\mathcal{B}$ satisfies n-(vN) for all n.*

Remark. If $n = 1$ then Theorem 2 gives an another proof of the von Neumann inequality for a two dimensional subalgebra $\mathcal{B}$ of $B(H)$ with identity: indeed, by T.Nakazi [8, Proposition 1], it follows that $\mathcal{B} = \{\alpha P + \beta Q \ ; \ \alpha, \beta \in \mathbf{C}\}$ with $P^2 = P \neq 0, I$ and $Q = I - P$, or $\mathcal{B} = \{\alpha I + \beta N \ ; \ \alpha, \beta \in \mathbf{C}\}$ with $N \neq 0$ and $N^2 = 0$. We shall prove that

$$T \in \mathcal{B}, \quad \|T\| \leq 1 \quad \Rightarrow \quad \|f(T)\| \leq 1$$

whenever f is a polynomial satisfying $|f(z)| \leq 1 \quad (|z| \leq 1)$.
It is sufficient to prove that for $\alpha, \beta \in \mathbf{C}$,

$$\|\alpha P + \beta Q\| \leq 1 \quad \Rightarrow \quad \|f(\alpha P + \beta Q)\| \leq 1$$

or

$$\|\alpha I + \beta N\| \leq 1 \quad \Rightarrow \quad \|f(\alpha I + \beta N)\| \leq 1.$$

By Lemmas 1 and 6(1), if $|\alpha|, |\beta| \leq 1$ then

$$\|\alpha P + \beta Q\| \leq 1 \quad \Leftrightarrow \quad \left|\frac{\alpha - \beta}{1 - \overline{\beta}\alpha}\right| \leq \frac{1}{\|P\|}.$$

Since $f(\alpha P + \beta Q) = f(\alpha)P + f(\beta)Q$, it follows that

$$\|f(\alpha P + \beta Q)\| \leq 1 \quad \Leftrightarrow \quad \left|\frac{f(\alpha) - f(\beta)}{1 - \overline{f(\beta)}f(\alpha)}\right| \leq \frac{1}{\|P\|}.$$

By the Schwarz lemma, if $\|\alpha P + \beta Q\| \leq 1$ then

$$\left|\frac{f(\alpha) - f(\beta)}{1 - \overline{f(\beta)}f(\alpha)}\right| \leq \left|\frac{\alpha - \beta}{1 - \overline{\beta}\alpha}\right| \leq \frac{1}{\|P\|},$$

and hence $\|f(\alpha P + \beta Q)\| \leq 1$.
If $\|\alpha I + \beta N\| \leq 1$ then $|\beta| \cdot \|N\| \leq 1 - |\alpha|^2$. By the Schwarz lemma,

$$\frac{|f'(\alpha)|}{1 - |f(\alpha)|^2} \leq \frac{1}{1 - |\alpha|^2} \leq \frac{1}{|\beta| \cdot \|N\|}.$$

Hence

$$\begin{aligned}
\|f(\alpha I + \beta N)\| &= \|f(\alpha)I + f'(\alpha)\beta N\| \\
&\leq \left|\frac{f'(\alpha)\beta}{2}\right| \|N\| + \sqrt{\left|\frac{f'(\alpha)\beta}{2}\right|^2 \|N\|^2 + |f(\alpha)|^2} \\
&\leq \frac{1 - |f(\alpha)|^2}{2} + \sqrt{\left(\frac{1 - |f(\alpha)|^2}{2}\right)^2 + |f(\alpha)|^2} \\
&= 1.
\end{aligned}$$

4. Isometric Möbius transformation

Suppose $\mathcal{B}$ is a two dimensional commutative Banach algebra with identity. By von Neumann inequality and Theorem 1, it follows that $\mathcal{B}$ satisfies 1-(vN) if and only if $\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some Hilbert space H. In this section, we shall show the another equivalent assertions using Möbius transformations. Let

$$\phi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}, \quad \text{for}(|\lambda| < 1).$$

In Theorem 3, (3) implies that the Möbius transformation ϕ_λ operates isometrically on the unit sphere of $\mathcal{B}$.

Theorem 3. *Let $\mathcal{B}$ be a two dimensional commutative Banach algebra with identity. Then the following assertions are equivalent:*

(1) *$\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some Hilbert space H.*

(2) $T \in \mathcal{B}, \quad \|T\| \le 1 \quad \Rightarrow \quad \|\phi_\lambda(T)\| \le 1 \quad (|\lambda| < 1).$

(3) $T \in \mathcal{B}, \quad \|T\| = 1 \quad \Rightarrow \quad \|\phi_\lambda(T)\| = 1 \quad (|\lambda| < 1).$

Proof. (1) $\Rightarrow$ (2): Even if we do not use the von Neumann inequality, it is easy to prove

$$T \in B(H), \quad \|T\| \le 1 \quad \Rightarrow \quad \|\phi_\lambda(T)\| \le 1$$

whenever $|\lambda| \le 1$: indeed (see Pisier [10, p.21]), if $T \in B(H), \|T\| \le 1$ and $x \in H$

$$\|(T-\lambda I)x\|^2 - \|(I-\bar{\lambda}T)x\|^2 = (\|Tx\|^2 - \|x\|^2)(1-|\lambda|^2) \le 0$$

hence $\|(T-\lambda)(1-\bar{\lambda}T)^{-1}\| \le 1$ which means that $\|\phi_\lambda(T)\| \le 1$.

(2) $\Rightarrow$ (1): By T.Nakazi [8, Proposition 1], it follows that $\mathcal{B} = \{\alpha P + \beta Q \ ; \ \alpha, \beta \in \mathbf{C}\}$ with $P^2 = P \ne 0, I$ and $Q = I - P$, or $\mathcal{B} = \{\alpha I + \beta N \ ; \ \alpha, \beta \in \mathbf{C}\}$ with $N \ne 0$ and $N^2 = 0$. Suppose $\mathcal{B} = \{\alpha P + \beta Q \ ; \ \alpha, \beta \in \mathbf{C}\}$. By the proof of Lemma 6(3), if

$$T \in \mathcal{B}, \quad \|T\| \le 1 \quad \Rightarrow \quad \|\phi_\lambda(T)\| \le 1 \quad (|\lambda| < 1),$$

then $\|\alpha P + \beta Q\| = m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right)$. By the proof of Theorem 1, $\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some Hilbert space H. Suppose $\mathcal{B} = \{\alpha I + \beta N \ ; \ \alpha, \beta \in \mathbf{C}\}$. By the proof of Lemma 8(3), if

$$T \in \mathcal{B}, \quad \|T\| \le 1 \quad \Rightarrow \quad \|\phi_\lambda(T)\| \le 1 \quad (|\lambda| < 1),$$

then $\|\alpha I + \beta B\| = m_{\alpha,\alpha}(|\beta| \cdot \|N\|)$. By the proof of Theorem 1, $\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some Hilbert space H.

(3) $\Rightarrow$ (1): Suppose $\mathcal{B} = \{\alpha P + \beta Q \ ; \ \alpha, \beta \in \mathbf{C}\}$. By (3), if $\|\alpha P + \beta Q\| = 1$ then $\|\phi_\beta(\alpha P + \beta Q)\| = 1$. By the similar proof to Lemma 6(3), $\|\alpha P + \beta Q\| = m_{\alpha,\beta}\left(|\alpha - \beta|\sqrt{\|P\|^2 - 1}\right)$. By Lemma 1 and the similar proof to Theorem 1, $\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some Hilbert space H. Suppose $\mathcal{B} = \{\alpha I + \beta N \ ; \ \alpha, \beta \in \mathbf{C}\}$. By (3), if $\|\alpha P + \beta Q\| = 1$ then $\|\phi_\alpha(\alpha P + \beta Q)\| = 1$. By the similar proof to Lemma 8(3), $\|\alpha I + \beta N\| = m_{\alpha,\alpha}(|\beta| \cdot \|N\|)$. By Lemma 2 and the similar proof to Theorem 1, $\mathcal{B}$ is isometric to a subalgebra of $B(H)$ for some

Hilbert space H.
(1) $\Rightarrow$ (3): By T.Nakazi [8, Proposition 1], it follows that $\mathcal{B} = \{\alpha P + \beta Q\ ;\ \alpha, \beta \in \mathbf{C}\}$ with $P^2 = P \neq 0, I$ and $Q = I - P$, or $\mathcal{B} = \{\alpha I + \beta N\ ;\ \alpha, \beta \in \mathbf{C}\}$ with $N \neq 0$ and $N^2 = 0$. Suppose $\|\alpha P + \beta Q\| = 1$. By Lemma 6(2), this implies that

$$\left|\frac{\alpha - \beta}{1 - \bar{\beta}\alpha}\right| = \frac{1}{\|P\|}.$$

For $|\lambda| < 1$, let

$$\phi_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda} z}.$$

By the calculation,

$$\left|\frac{\phi_\lambda(\alpha) - \phi_\lambda(\beta)}{1 - \overline{\phi_\lambda(\beta)}\phi_\lambda(\alpha)}\right| = \left|\frac{\alpha - \beta}{1 - \bar{\beta}\alpha}\right|.$$

Hence

$$\left|\frac{\phi_\lambda(\alpha) - \phi_\lambda(\beta)}{1 - \overline{\phi_\lambda(\beta)}\phi_\lambda(\alpha)}\right| = \frac{1}{\|P\|}.$$

By Lemma 6(2), $\|\phi_\lambda(\alpha P + \beta Q)\| = \|\phi_\lambda(\alpha)P + \phi_\lambda(\beta)Q\| = 1$.
Next suppose $\|\alpha I + \beta N\| = 1$. By Lemma 8(2), this implies that

$$\frac{|\beta|}{1 - |\alpha|^2} = \frac{1}{\|N\|}.$$

By the calculation,

$$\frac{|\beta\phi'_\lambda(\alpha)|}{1 - |\phi_\lambda(\alpha)|^2} = \frac{|\beta|}{1 - |\alpha|^2}.$$

Hence

$$\frac{|\beta\phi'_\lambda(\alpha)|}{1 - |\phi_\lambda(\alpha)|^2} = \frac{1}{\|N\|}.$$

By Lemma 8(2), $\|\phi_\lambda(\alpha I + \beta N)\| = \|\phi_\lambda(\alpha)I + \beta\phi'_\lambda(\alpha)N\| = 1$. Theorem 3 is proved. $\square$

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