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SPIRAL SOLUTIONS FOR A WEAKLY ANISOTROPIC CURVATURE FLOW EQUATION

Yoshikazu Giga, Naoyuki Ishimura, and Yoshihito Kohsaka

Abstract. The presence of steps associated with screw dislocations plays a key role for the growth of crystal surfaces. In geometric model the motion of curves describing location of steps is governed by curvature flow equations with a driving force term. We show the existence of spiral-shaped solutions for such an equation when anisotropic effect is small. Such a spiral-shaped solution is shown to be stable and unique up to translation of the time.

1 Introduction

More than 50 years ago Frank [4] claimed that the presence of dislocations in a crystal plays a key role in the growth of crystal surfaces; see also [1]. Monomolecular steps on a crystal surface associated with a dislocation move by supersaturation of molecules outside crystal. In geometric model the location of the steps on a crystal surface is represented as a curve $\Gamma(t)$ depending on time t. In [2] it was proposed that the evolution of $\Gamma(t)$ is governed by a curvature flow equation with a driving force term :

$$V = V_0(1 + d_0\kappa). \tag{1.1}$$

Here V and κ denotes the normal velocity and the curvature of $\Gamma(t)$ respectively in the direction of the unit normal vector field $\boldsymbol{n}$ of $\Gamma(t)$ and V_0 and d_0 is a positive constant. If 0 is the only dislocation, we consider (1.1) in $\mathbf{R}^2 \setminus \{0\}$ such that one of the end point of $\Gamma(t)$ is zero. In [2] it is suggested that there is an essentially unique rotating solution for (1.1). Such a solution is called a spiral (-shaped) solution. In modern analysis this problem can be solved by a shooting method as suggested in [18, Appendix AVI, p.190-203]; see also [17].

In this paper we study the existence of spiral (-shaped) solution when the growth equation (1.1) takes the anisotropy into account. Such an extension is very natural in the theory of crystal growth. For technical reasons we postulate that the dislocation is not a point but a closed disk B and that crystal surface D is a large disk having common centers with dislocation disk. Moreover, we postulate that $\Gamma(t)$ is orthogonal to the boundary of the crystal surface D and B. Under these assumptions evolution of $\Gamma(t)$ in an annulus $\Omega = \{x \in \mathbf{R}^2; \rho < |x| < R\} (= D \setminus B)$ is governed by

$$\begin{cases} V = M(\boldsymbol{n})(D_0\kappa_\gamma + V_0) & \text{on } \Gamma(t), \\ \Gamma(t) \perp \partial\Omega, \end{cases} \tag{1.2}$$

Here κ_γ denotes the anisotropic curvature of $\Gamma(t)$ in the direction of $\boldsymbol{n}$. It is defined by

$$\kappa_\gamma := -\mathrm{div}_s \nabla\gamma(\boldsymbol{n}), \tag{1.3}$$

with the interfacial energy density $\gamma : \mathbf{R}^2 \to \mathbf{R}^+ = \{\sigma \in \mathbf{R}; \sigma \geq 0\}$ which is positively homogeneous of degree one, i.e., $\gamma(\lambda p) = \lambda\gamma(p)$ for all $p \in \mathbf{R}^2$ and $\lambda \in \mathbf{R}^+$; ∇ denotes the gradient and div_s denotes the surface divergence. For a vector field f on a curve in $\mathbf{R}^2$ the surface divergence is defined by

$$\mathrm{div}_s f := \langle \partial_s f, \boldsymbol{\tau} \rangle,$$

where ∂_s is the derivative with respect to arclength and $\boldsymbol{\tau}$ is the unit tangent vector to the curve; $\langle , \rangle$ denotes the inner product of $\mathbf{R}^2$. The function $M(\boldsymbol{n}) > 0$ is called the mobility and it depends on $\boldsymbol{n}$; D_0 is a positive constant. If $M \equiv 1$ and $\gamma(p) = |p|$, then the curvature flow equation in (1.2) is nothing but (1.1) with $D_0 = d_0V_0$. For more applications of these equations the reader is referred to a nice monograph of M. E. Gurtin [6] and a review article by J. Taylor, J. Cahn and A. Handwerker [21].

Our goal in this paper is to seek a spiral-shaped solution. Contrary to isotropic case there might be no rotating solution. For anisotropic case it is natural to say that $\Gamma(t)$ is spiral-shaped solution if $\Gamma(t)$ is a periodic-in-time solution of (1.2). In this paper we consider more special spiral solution of the form

$$\Gamma(t) = \{(r\cos\theta(r,t), r\sin\theta(r,t)) \mid \rho \leq r \leq R\} \tag{1.4}$$

where $r = |x|$ and θ represents the argument or the angle of $x \in \mathbf{R}^2$.

Definition 1.1. *We call $\Gamma(t)$ a spiral solution of (1.2) of $\theta(r,t)$ in (1.4) if $\theta(r,t)$ is monotone with respect to r, and periodic in time t, that is, there exists $T > 0$ such that $\theta(r, t+T) = \theta(r,t) + 2\pi$ for all $t > 0$.*

We remark that other kind of spirals, such as those are not included in above category, do exist in reality; certain crystals usually involve facets, where the angle θ is not a monotone function of r. It is not expected that a spiral solution of form (1.4) always exists unless anisotropic effect is small. We here confine ourselves to investigating the existence of spirals within the above somewhat restricted family when anisotropic effect is small. Our main results read as follows.

Theorem 1.2. *(Existence of a spiral solution). Assume that M and γ are smooth on a unit circle. Assume that the equation (1.2) is close to isotropy in the sense defined in Section 3 below. Then there exists a spiral solution $\hat{\Gamma}(t)$ of (1.2), which is unique up to translation of time.*

The assumption that (1.2) is close to isotropic is necessary to obtain the solution of form (1.4).

If there is no driving force term, i.e., $V_0 = 0$ in (1.2), then we can proceed a little further and are able to determine what $\hat{\Gamma}(t)$ precisely is. The shape of $\hat{\Gamma}(t)$ is unrelated to $M(\boldsymbol{n})$, which is stated in the next corollary.

Corollary 1.3. *If $V_0 = 0$, then a spiral solution $\hat{\Gamma}(t)$ is a straight line independent of time t.*

By definition $\hat{\Gamma}(t)$ is of the form

$$\hat{\Gamma}(t) = \hat{\Gamma}_* = \{(r\cos\hat{\theta}_*, r\sin\hat{\theta}_*) \mid \rho \le r \le R\}$$

with some constant $\hat{\theta}_*$.

We briefly describe our strategy of the proof. First, we derive the equation for $\theta(r,t)$ appeared in the formula (1.4), and establish its gradient estimates under the condition that the equation (1.2) is close to isotropy in some sense. The precise assumption is presented properly in Section 3. The gradient estimate is a consequence of the use of the weak maximum principle, and plays a key role to obtain a global-in-time solution to (1.2). Next, we show a time-monotonicity of the infimum of $\theta(r,t)$ on $\bar{I} := \{r \in \mathbf{R} \mid \rho \le r \le R\}$ and an order-preserving property of θ. The strong maximum principle is involved in the proof. Based on the gradient estimate as well as these properties on θ we apply the theory developed in [17] to obtain a time-periodic solution $\hat{\theta}(r,t)$, which is unique up to translation of time. Finally, the result for $V_0 = 0$ is established by virtue of the Lyapunov functional, which is the length of the interface $\Gamma(t)$.

In [17] they studied the Neumann problem for the Allen–Cahn type equation

$$u_t = \Delta u + g(u - \theta), \quad x \in \Omega$$

when g is 2π-periodic function. Here θ denotes the angle of x. The function g is the derivative of a multi-well potential and $\int_0^{2\pi} g(v)dv > 0$. They proved the unique existence of a spiral traveling wave solution u in the sense that u is of the form

$$u(x,t) = \varphi(r, \theta - \omega t) + \omega t$$

with some $\omega > 0$ (independent of x and t) and a function φ with $r = |x|$. To construct such a solution they use strong maximum principle in a smart way. We shall use their argument in the proof of Theorem 1.2 as mentioned in the previous paragraph. In [17] they also prove the existence of rotating solution for (1.2) when it is isotropic (i.e. $\gamma(p) = |p|$, $M \equiv 1$) by a shooting argument for an ordinary differential equation. We remark that such an ODE argument does not work when the equation is anisotropic.

We take this opportunity to mention several related works on spirals. There are numerical calculations based on the Allen–Cahn equation by A. Karma and M. Plapp

[12], R. Kobayashi [13]; the latter also treats the case when there are several dislocation. In this case two steps may collide. To treat such a phenomena two level set methods are proposed numerically by P. Smereka [20] and analytically by T. Ohtsuka [19]; their methods are different each other. Other aspects of spiral shaped solutions for various interface equations, we refer for instance to [7, 8, 10, 11, 22] and references therein.

This paper is organized as follows. In Section 2, we derive the equation for $\theta(r,t)$. In Section 3, we clarify the meaning of "weakly anisotropic" and establish the gradient estimate of $\theta(r,t)$. In Section 4, we prove the existence result of a spiral solution to (1.2). Section 5 is devoted to the analysis in the case of $V_0 = 0$.

2 Derivation of the equation

In this section, we derive the equations for $\theta(r,t)$ in the expression (1.4) of $\Gamma(t)$.

Since γ is homogeneous of degree 1, we first get

$$\langle \nabla\gamma(p), p\rangle = \gamma(p). \tag{2.1}$$

for $p \in \mathbf{R}^2$. Let $p^\perp$ be rotation of p by $-\pi/2$. We observe that

$$\nabla\gamma(p) = \langle \nabla\gamma(p), \frac{p}{|p|}\rangle \frac{p}{|p|} + \langle \nabla\gamma(p), \frac{p^\perp}{|p^\perp|}\rangle \frac{p^\perp}{|p^\perp|}.$$

Since $|p| = |p^\perp|$, we obtain

$$|p|^2\nabla\gamma(p) = \langle \nabla\gamma(p), p\rangle p + \langle \nabla\gamma(p), p^\perp\rangle p^\perp. \tag{2.2}$$

Combining (2.1) and (2.2), we are led to

$$|p|^2\nabla\gamma(p) = \gamma(p)p + \langle \nabla\gamma(p), p^\perp\rangle p^\perp. \tag{2.3}$$

Here, the unit normal vector $\boldsymbol{n}$ and the unit tangent vector $\boldsymbol{\tau}$ of $\Gamma(t)$ is represented as

$$\begin{aligned} \boldsymbol{n} = \boldsymbol{n}(r,\theta,\theta_r) &= \frac{1}{(1+r^2\theta_r^2)^{1/2}}\begin{pmatrix} -\sin\theta - r\theta_r\cos\theta \\ \cos\theta - r\theta_r\sin\theta \end{pmatrix}, \\ \boldsymbol{\tau} = \boldsymbol{\tau}(r,\theta,\theta_r) &= (\boldsymbol{n}(r,\theta,\theta_r))^\perp = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\begin{pmatrix} \cos\theta - r\theta_r\sin\theta \\ \sin\theta + r\theta_r\cos\theta \end{pmatrix}. \end{aligned}$$

This implies

$$\frac{\partial \boldsymbol{n}}{\partial\theta} = -\boldsymbol{\tau}, \quad \frac{\partial \boldsymbol{\tau}}{\partial\theta} = \boldsymbol{n}.$$

Setting $\hat\gamma(\theta) := \gamma(\boldsymbol{n}(\cdot,\theta,\cdot))$, we find

$$\hat\gamma'(\theta) = \langle \nabla\gamma(\boldsymbol{n}), \frac{\partial \boldsymbol{n}}{\partial\theta}\rangle = -\langle \nabla\gamma(\boldsymbol{n}), \boldsymbol{\tau}\rangle \tag{2.4}$$

where $'$ denotes the derivative with respect to θ. Thus, by virtue of (2.3) and (2.4), we are led to

$$\nabla\gamma(\boldsymbol{n}) = \gamma(\boldsymbol{n})\boldsymbol{n} - \hat\gamma'(\theta)\boldsymbol{\tau} = \hat\gamma(\theta)\boldsymbol{n} - \hat\gamma'(\theta)\boldsymbol{\tau}. \tag{2.5}$$

We also note

$$\partial_s \boldsymbol{n} = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\frac{\partial \boldsymbol{n}}{\partial r} = -\frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\boldsymbol{\tau},$$
$$\partial_s \boldsymbol{\tau} = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\frac{\partial \boldsymbol{\tau}}{\partial r} = \frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\boldsymbol{n}.$$

Then, by (1.3) and (2.5), we derive

$$\begin{aligned}\kappa_\gamma &= -\mathrm{div}_s(\hat{\gamma}(\theta)\boldsymbol{n} - \hat{\gamma}'(\theta)\boldsymbol{\tau}) \\ &= -\langle \partial_s(\hat{\gamma}(\theta)\boldsymbol{n} - \hat{\gamma}'(\theta)\boldsymbol{\tau}), \boldsymbol{\tau}\rangle \\ &= -\left\{-\hat{\gamma}(\theta)\frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}} - \partial_s\hat{\gamma}'(\theta)\right\}|\tau|^2.\end{aligned}$$

On the other hand, differentiating the both sides of (2.4) with respect to the arclength parameter s, we deduce that

$$\partial_s\hat{\gamma}'(\theta) = \frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\{\langle(\nabla^2\gamma(\boldsymbol{n}))\boldsymbol{\tau},\boldsymbol{\tau}\rangle - \langle\nabla\gamma(\boldsymbol{n}),\boldsymbol{n}\rangle\}. \tag{2.6}$$

Moreover, differentiating the both sides of (2.4) with respect to θ, we obtain

$$\hat{\gamma}''(\theta) = \langle(\nabla^2\gamma(\boldsymbol{n}))\boldsymbol{\tau},\boldsymbol{\tau}\rangle - \langle\nabla\gamma(\boldsymbol{n}),\boldsymbol{n}\rangle. \tag{2.7}$$

Thanks to (2.6) and (2.7), we are led to

$$\partial_s\hat{\gamma}'(\theta) = \frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\hat{\gamma}''(\theta),$$

from where we conclude that

$$\kappa_\gamma = (\hat{\gamma}(\theta)+\hat{\gamma}''(\theta))\frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}}. \tag{2.8}$$

Consequently, since the normal velocity of $\Gamma(t)$ is

$$V = \langle\frac{\partial\Gamma}{\partial t},\boldsymbol{n}\rangle = \frac{r\theta_t}{(1+r^2\theta_r^2)^{1/2}},$$

the interface equation (1.2) become

$$\begin{cases}\theta_t = M(\boldsymbol{n})\left(\dfrac{a(\boldsymbol{n})(r\theta_{rr}+r^2\theta_r^3+2\theta_r)}{r(1+r^2\theta_r^2)} + \dfrac{V_0(1+r^2\theta_r^2)^{1/2}}{r}\right), \\ \theta_r(\rho,t) = \theta_r(R,t) = 0.\end{cases} \tag{2.9}$$

where $a(\boldsymbol{n}) := D_0(\gamma(\theta)+\gamma''(\theta)) = D_0\langle[\nabla^2\gamma(\boldsymbol{n})]\boldsymbol{\tau},\boldsymbol{\tau}\rangle$.

Remark 2.1. *(Existence and uniqueness of the solution of (2.9)).* We describe the existence and uniqueness of the solution of (2.9). By using the optimal regularity theory of analytic semigroups as in [16], we get a unique and smooth local-in-time solution of (2.9) with existence time T, which depends on $1/\|\theta_0\|_{C^{1+\alpha}(I)}$. This implies that if we obtain $C^{1+\alpha}$-a priori estimate of the solution $\theta(\cdot,t)$ for $t>0$, there exists a unique global-in-time solution of (2.9). We find the gradient bound under some assumptions in Section 3 and derive the estimate of θ in Section 4. According to the theory of parabolic equations (see [3, 14, 15]), the Hölder norm of the gradient is estimated by a constant depending on the maximum norm of the gradient and some given constants in the assumptions for $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$. This means that the gradient and θ's estimates assure the existence of the global-in-time solution of (2.9). □

We conclude this section with the following proposition which will be invoked in Section 5.

Proposition 2.2. *Let* $L[\Gamma(t)]$ *be the length of* $\Gamma(t)$*, that is,*

$$L[\Gamma(t)] := \int_\rho^R (1+r^2\theta_r^2)^{1/2}dr.$$

If $\Gamma(t)$ *is a solution of (1.2), the following formula is valid :*

$$\frac{d}{dt}L[\Gamma(t)] = -\int_\rho^R M(\boldsymbol{n})(1+r^2\theta_r^2)^{1/2}(a(\boldsymbol{n})\kappa^2+\kappa V_0)dr.$$

Proof. By means of the direct calculation and the integration by parts with the boundary condition $\theta_r(\rho,t)=\theta_r(R,t)=0$, we are led to

$$\frac{d}{dt}L[\Gamma(t)] = \int_\rho^R \frac{r^2\theta_r}{(1+r^2\theta_r^2)^{1/2}}\cdot\theta_{rt}dr = -\int_\rho^R \frac{\partial}{\partial r}\left\{\frac{r^2\theta_r}{(1+r^2\theta_r^2)^{1/2}}\right\}\cdot\theta_t dr.$$

On the other hand, we see that

$$\begin{aligned}\frac{\partial}{\partial r}\left\{\frac{r^2\theta_r}{(1+r^2\theta_r^2)^{1/2}}\right\} &= r\cdot\frac{\{2\theta_r(1+r^2\theta_r^2)+r\theta_{rr}(1+r^2\theta_r^2)-r\theta_r(r\theta_r^2+r^2\theta_r\theta_{rr})\}}{(1+r^2\theta_r^2)^{3/2}}\\ &= r\cdot\frac{r\theta_{rr}+r^2\theta_r^3+2\theta_r}{(1+r^2\theta_r^2)^{3/2}} =: r\kappa.\end{aligned}$$

Note that κ is the usual curvature. Applying $D_0\kappa_\gamma = a(\boldsymbol{n})\kappa$, which is derived by (2.8) and the definition of $a(\boldsymbol{n})$, and using the equation (2.9), it follows that

$$\begin{aligned}\frac{d}{dt}L[\Gamma(t)] &= -\int_\rho^R r\kappa\cdot\frac{(1+r^2\theta_r^2)^{1/2}}{r}M(\boldsymbol{n})(a(\boldsymbol{n})\kappa+V_0)dr\\ &= -\int_\rho^R M(\boldsymbol{n})(1+r^2\theta_r^2)^{1/2}(a(\boldsymbol{n})\kappa^2+\kappa V_0)dr\end{aligned}$$

The desired formula is thus established. □

3 Gradient estimate

The goal of this section is to obtain the gradient estimate. For this purpose, we first provide the precise assumption that the equation of (2.9) is close to isotropic.

For λ, μ, $\varepsilon > 0$ and $\Lambda < \infty$, we set

$(A^{\Lambda}_{\lambda,\mu})$ $\lambda \le M(\boldsymbol{n}) \le \Lambda$ and $\mu \le a(\boldsymbol{n}) \le \Lambda$ for all $\boldsymbol{n} \in S^1$.

$$(A^{\varepsilon}) \quad \left|\frac{\langle \nabla M(\boldsymbol{n}), \boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}\right| + \left|\frac{\langle \nabla a(\boldsymbol{n}), \boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right| \le \varepsilon.$$

If (A^{ε}) holds for small $\varepsilon > 0$, this implies that M and a are close to isotropic. The next proposition is the main ingredient of our current exposition.

Proposition 3.1. *(Gradient estimate). Assume that $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ satisfy $(A^{\Lambda}_{\lambda,\mu})$. Let ε_* be*

$$\varepsilon_* = \frac{\mu\rho}{2R^2V_0(d+\sqrt{d^2-1})} \quad \text{where } d := 1 + \frac{2R^2}{\rho^2}\left(2 + \frac{RV_0}{\mu}\right) (> 1).$$

Assume that $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ satisfy (A^{ε}) for some $\varepsilon \in (0, \varepsilon_*]$. Then if $\theta(r,t)$ is a solution of (2.9) with the initial data $\theta(\cdot,0) = \theta_0$ satisfying $-L \le \theta_{0r} \le 0$ for $L \in [L_1, L_2]$, the gradient estimate $-L \le \theta_r(\cdot,t) \le 0$ holds for $t > 0$. Here, constants L_1, L_2 $(0 < L_1 \le L_2)$ are solutions of quadratic equation

$$\frac{2\varepsilon}{\rho}\left(2 + \frac{RV_0}{\mu}\right)L^2 - \left(\frac{1}{R^2} - \frac{2\varepsilon V_0}{\mu\rho}\right)L + \frac{V_0}{\mu\rho^2} = 0. \tag{3.1}$$

Remark 3.2. If $0 < \varepsilon \le \varepsilon_*$, the quadratic equation (3.1) has two positive real-valued solutions. Indeed, (3.1) has two positive real-valued solutions if and only if

$$\frac{1}{R^2} - \frac{2\varepsilon V_0}{\mu\rho} > 0, \quad \text{and} \quad \left(\frac{1}{R^2} - \frac{2\varepsilon V_0}{\mu\rho}\right)^2 - 4\cdot\frac{2\varepsilon}{\rho}\left(2 + \frac{RV_0}{\mu}\right)\cdot\frac{V_0}{\mu\rho^2} \ge 0.$$

If ε satisfies $0 < \varepsilon \le \varepsilon_*$, these conditions holds. We also stress that ε in (A^{ε}) is determined a posteriori from ε_* in Proposition 3.1.

Proof. Differentiating both sides of (2.9) and setting $v := \theta_r$, we have

$$\alpha(r,t)v_{rr} + \beta(r,t)v_r + \gamma(r,t)v - v_t = \frac{M(\boldsymbol{n})V_0}{r^2(1+r^2v^2)^{1/2}} \ge 0,$$

where

$$\begin{aligned}
\alpha(r,t) &= \frac{M(\boldsymbol{n})a(\boldsymbol{n})}{1+r^2v^2}, \\
\beta(r,t) &= M(\boldsymbol{n})a(\boldsymbol{n})\frac{2 + r^4v^4 - 2r^3vv_r - r^2v^2}{r(1+r^2v^2)^2}
\end{aligned}$$

$$
\begin{aligned}
& -M(\boldsymbol{n})a(\boldsymbol{n})\left(\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}+\frac{\langle\nabla a(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right)\frac{r^2v_r+2rv(2+r^2v^2)}{r(1+r^2v^2)^2} \\
& +\frac{M(\boldsymbol{n})V_0}{(1+r^2v^2)^{1/2}}\left(v-\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}\right) \\
\gamma(r,t) \;=\; & -\frac{M(\boldsymbol{n})a(\boldsymbol{n})}{r^2}\left(1+\frac{1+3r^2v^2}{(1+r^2v^2)^2}\right) \\
& -M(\boldsymbol{n})a(\boldsymbol{n})\left(\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}+\frac{\langle\nabla a(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right)\frac{(2+r^2v^2)^2v}{(1+r^2v^2)^2r} \\
& -\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle\frac{V_0(1+r^2v^2)^{1/2}(2+r^2v^2)}{r(1+r^2v^2)}.
\end{aligned}
$$

Since $\alpha(r,t)>0$ and $|\alpha(r,t)|$, $|\beta(r,t)|$, $|\gamma(r,t)|<\infty$, we can appeal to the weak maximum principle; we deduce that $v(\cdot,t)\le 0$ for $t>0$ if the initial data of v satisfies $v(\cdot,0)\le 0$. That is, we are led to $\theta_r(\cdot,t)\le 0$ for $t>0$ if the initial data θ_0 satisfies $\theta_{0r}\le 0$.

Next, we prove that if the initial data θ_0 satisfies $\theta_{0r}\ge -L$ for $L\in[L_1,L_2]$, the minimum of $\theta_r(\cdot,t)$ is estimated by $-L$ for $t>0$. To prove this, we set $w:=-v-L$. Then w satisfies

$$
\begin{aligned}
& \hat\alpha(r,t)w_{rr}+\hat\beta(r,t)w_r+\hat\gamma(r,t)w-w_t \\
= \;& \frac{M(\boldsymbol{n})a(\boldsymbol{n})}{r^2}\left(1+\frac{1+3r^2(w+L)^2}{(1+r^2(w+L)^2)^2}\right)\cdot L \\
& -\frac{M(\boldsymbol{n})a(\boldsymbol{n})}{r}\left(\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}+\frac{\langle\nabla a(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right)\frac{(2+r^2(w+L)^2)^2}{(1+r^2(w+L)^2)^2}\cdot L^2 \\
& +\frac{M(\boldsymbol{n})V_0(1+r^2(w+L)^2)^{1/2}}{r}\cdot\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}\cdot\frac{2+r^2(w+L)^2}{1+r^2(w+L)^2}\cdot L \\
& -\frac{M(\boldsymbol{n})V_0}{r^2(1+r^2(w+L)^2)^{1/2}} \\
\ge \;& M(\boldsymbol{n})a(\boldsymbol{n})\left\{\frac{1}{R^2}L-\frac{4\varepsilon}{\rho}L^2-\frac{2\varepsilon V_0(1+R(w+L))}{\mu\rho}L-\frac{V_0}{\mu\rho^2}\right\}
\end{aligned}
$$

where $\hat\alpha$, $\hat\beta$, $\hat\gamma$ are the terms evaluated at r, w, *etc.* Transposing the term of w in the right hand side to the left hand side, we see that

$$
\begin{aligned}
& \hat\alpha(r,t)w_{rr}+\hat\beta(r,t)w_r+\left(\hat\gamma(r,t)+\frac{2\varepsilon RV_0L}{\mu\rho}\right)w-w_t \\
& \ge -M(\boldsymbol{n})a(\boldsymbol{n})\left\{\frac{2\varepsilon}{\rho}\left(2+\frac{RV_0}{\mu}\right)L^2-\left(\frac{1}{R^2}-\frac{2\varepsilon V_0}{\mu\rho}\right)L+\frac{V_0}{\mu\rho^2}\right\} \qquad (3.2) \\
& =: -M(\boldsymbol{n})a(\boldsymbol{n})Q(L).
\end{aligned}
$$

According to the condition of L, we find that $Q(L)\le 0$. Since $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ are positive, it follows that

$$
\hat\alpha(r,t)w_{rr}+\hat\beta(r,t)w_r+\left(\hat\gamma(r,t)+\frac{2\varepsilon RV_0L}{\mu\rho}\right)w-w_t\ge -M(\boldsymbol{n})a(\boldsymbol{n})Q(L)\ge 0.
$$

Using the weak maximum principle for this equation, we obtain that $w(\cdot,t) \le 0$ for $t>0$ if the initial data of w satisfies $w(\cdot,0) \le 0$. The proof is now complete. $\square$

4 Existence of spiral solutions

In this section, our goal is to obtain a periodic solution of (2.9). For this purpose, we shall apply the idea of [17]. To do so, we first derive the useful properties on θ.

Lemma 4.1. *(i) (Monotonicity for time). Let $m(t) := \inf_{\rho \le r \le R} \theta(r,t)$ and assume that $\theta_r(\cdot,t) \le 0$ for each $t>0$. Then there exists a constant $\nu > 0$ such that*

$$\frac{d}{dt} m(t) \ge \nu \quad \textit{for } t > 0.$$

(ii) (Order-preserving). Let $\theta^{(1)}(\cdot,t)$, $\theta^{(2)}(\cdot,t)$ be solutions of (2.9) with the initial data $\theta_0^{(1)}$, $\theta_0^{(2)}$ respectively. If $\theta_0^{(1)} \le \theta_0^{(2)}$ with $\theta_0^{(1)} \not\equiv \theta_0^{(2)}$, then the order is preserved for $t>0$. In fact,

$$\theta^{(1)}(\cdot,t) < \theta^{(2)}(\cdot,t) \quad \textit{for } t > 0.$$

Proof. We first prove (i). By virtue of the assumption that $\theta_r(\cdot,t) \le 0$ for each $t>0$, we are led to

$$\inf_{\rho \le r \le R} \theta(r,t) = \theta(R,t)(=: m(t)).$$

Letting $r \uparrow R$ in (2.9), we have

$$\theta_t(R,t) = M(\boldsymbol{n}) \left(a(\boldsymbol{n}) \theta_{rr}(R,t) + \frac{V_0}{R} \right). \tag{4.1}$$

Now, we claim that

$$\theta_{rr}(R,t) \ge 0 \quad \text{for } t>0. \tag{4.2}$$

In fact, suppose that $\theta_{rr}(R,t) < 0$. Then there exists a constant $\delta > 0$ such that $\theta_{rr}(r,t) < 0$ for $R-\delta < r \le R$. Since $\theta_r(R,t) = 0$, we see that $\theta_r(r,t) > 0$ for $R - \delta < r \le R$. This contradicts $\theta_r(r,t) \le 0$ for $\rho \le r \le R$, which verifies (4.2). In view of (4.1) and (4.2), we thus obtain

$$\theta_t(R,t) \ge M(\boldsymbol{n}) \frac{V_0}{R} \ge \frac{\lambda V_0}{R} > 0.$$

Since θ is a C^1-function with respect to t for $t>0$, the desired inequality is established.

(ii) is proved by using the strong maximum principle. We may safely omit the details. $\square$

Moreover we obtain the estimate of θ as follows.

Lemma 4.2. *Assume that $\theta_r(\cdot,t) \le 0$ for each $t>0$. Then we have*

$$\theta_0(R) + \nu_1 t \le \theta(\cdot,t) \le \theta_0(\rho) + \nu_2 t \quad \textit{for } t>0.$$

where $\nu_1 = \lambda V_0/R$ and $\nu_2 = \Lambda V_0/\rho$.

Proof. By means of the assumption that $\theta_r(\cdot,t) \le 0$ for each $t>0$, we see that

$$\theta(R,t) \le \theta(r,t) \le \theta(\rho,t) \quad \text{for } \rho \le r \le R.$$

It follows from Lemma 4.1(i) that $\theta_0(R)+\nu_1 t \le \theta(R,t)$. On the other hand, applying the similar argument to the proof of Lemma 4.1(i), $\theta_{rr}(\rho,t) \le 0$ is verified. Letting $r \downarrow \rho$ in (2.9), we have

$$\theta_t(\rho,t) = M(\boldsymbol{n})\left(a(\boldsymbol{n})\theta_{rr}(\rho,t) + \frac{V_0}{\rho}\right) \le M(\boldsymbol{n})\frac{V_0}{\rho} \le \frac{\Lambda V_0}{\rho}.$$

This implies that $\theta(\rho,t) \le \theta_0(\rho)+\nu_2 t$ and completes the proof. □

We set

$$\mathcal{D} = \{\psi \in C^{1+\alpha}(\bar{I}) \mid -L \le \psi_r \le 0,\ \psi_r(\rho)=\psi_r(R)=0\}$$

where $I=\{r \in \mathbf{R} \mid \rho < r < R\}$ and define the map Φ_t on $\mathcal{D}$ as

$$\Phi_t(\theta_0) = \theta(\cdot,t) \quad \text{for each } t>0 \tag{4.3}$$

where $\theta(\cdot,t)$ is the solution of (2.9) with the initial data $\theta(\cdot,0)=\theta_0$. It follows from Proposition 3.1 and Lemma 4.2 that if the initial data is in $\mathcal{D}$, there exists a unique global-in-time solution of (2.9) (see Remark 2.1) and this solution stays also in $\mathcal{D}$. The definition of the mapping Φ_t and the uniqueness of the solution of (2.9) imply that a family of the mappings Φ_t from $\mathcal{D}$ to itself satisfies the semigroup property :

$$\Phi_0(\theta) = \theta \quad \text{for all } \theta \in \mathcal{D}, \quad \Phi_t \circ \Phi_s = \Phi_{t+s} \quad \text{for any } t,s \in [0,\infty). \tag{4.4}$$

The invariance of (2.9) for 2π-periodicity and the uniqueness of the solution of (2.9) justify

$$\Phi_t(\theta \pm 2n\pi) = \Phi_t(\theta) \pm 2n\pi \quad \text{for any } t>0,\ \theta \in \mathcal{D} \text{ and } n \in \mathbf{N}. \tag{4.5}$$

In addition, the standard parabolic estimate implies that Φ_t is a compact map on $\mathcal{D}$ for each $t>0$. Recalling Lemma 4.1(ii), Φ_t is also order-preserving for each $t>0$, which means that $\theta_1 < \theta_2$ implies $\Phi_t(\theta_1) < \Phi_t(\theta_2)$ for each $t>0$. To obtain a periodic solution of (2.9), we need the following proposition.

Proposition 4.3. *Let $\{\Phi_t\}_{t\in[0,\infty)}$ be a family of mappings Φ_t defined by (4.3). Then there exists a unique $T_0>0$ such that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ for a function $\varphi \in \mathcal{D}$.*

In order to prove this proposition, we apply the idea of [17].

Proof. Let θ be a solution of (2.9) with the initial data $\theta(\cdot,0)=\theta_0 \in \mathcal{D}$. According to Proposition 3.1, we have

$$\max\{\theta(r,t) \mid r \in \bar{I}\} - \min\{\theta(r,t) \mid r \in \bar{I}\} \le 2LR \quad \text{for } t>0.$$

Set $\theta_k(r) := \theta(r,k) - 2\pi n_k$ and choose $n_k \in \mathbf{Z}$ satisfying

$$\theta_k(r) \in [0, 2LR+2\pi].$$

Note that $\theta_k \in \mathcal{D}$. Let $s \in (0,1)$ be fixed. Since $\{\Phi_s(\theta_k)\}_{k=1}^{\infty}$ is relatively compact in $C^{1+\alpha}(\bar{I})$ for each $s \in [\varepsilon, 1)$ where $\varepsilon > 0$ is arbitrary, there exists a subsequence $\{\Phi_s(\theta_{k_j})\}_{j=1}^{\infty} \subset \{\Phi_s(\theta_k)\}_{k=1}^{\infty}$ and a function $\varphi \in C^{1+\alpha}(\bar{I})$ such that

$$||\Phi_s(\theta_{k_j}) - \varphi||_{C^{1+\alpha}(\bar{I})} \to 0 \quad \text{as } j \to \infty.$$

Note that also $\varphi \in \mathcal{D}$ because a constant $L > 0$ is independent of time t.

We first show that $\varphi + 2\pi \leq \Phi_T(\varphi)$ for a $T > 0$. Lemma 4.1(i) implies that $\theta_0 + 2\pi < \theta(\cdot, T)$ for a $T > 0$. Using Lemma 4.1(ii) and (4.4), $\Phi_{s+k_j}(\theta_0) + 2\pi < \Phi_{s+k_j+T}(\theta_0)$. Adding $-2\pi n_{k_j}$ to the both side of this inequality and recalling $\Phi_{s+k_j}(\theta_0) - 2\pi n_{k_j} = \Phi_s(\theta_{k_j})$, we have

$$\Phi_s(\theta_{k_j}) + 2\pi < \Phi_T(\Phi_s(\theta_{k_j})).$$

Letting $j \uparrow \infty$, we see $\varphi + 2\pi \leq \Phi_T(\varphi)$.

Now we define

$$T_0 := \inf\{t \geq 0 \mid \varphi + 2\pi \leq \Phi_t(\varphi)\}.$$

It is the completely same argument as in [17, Section 3] to prove that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ and $T_0 > 0$ is unique. However, we present its proof for the reader's convenience. By definition $0 < T_0 \leq T$ and $\varphi + 2\pi \leq \Phi_{T_0}(\varphi)$. Suppose that $\varphi + 2\pi \not\equiv \Phi_{T_0}(\varphi)$. Then by Lemma 4.1 (ii)

$$\Phi_\delta(\varphi + 2\pi) = \Phi_\delta(\varphi) + 2\pi < \Phi_{T_0+\delta}(\varphi) \quad \text{in } \bar{I}$$

for any $\delta > 0$. Thus for sufficiently large k_0 we see that

$$\Phi_{\delta+s+t}(\theta_{k_0}) + 2\pi < \Phi_{T_0+\delta+s-\varepsilon+t}(\theta_{k_0}) \quad \text{in } \bar{I},\ t > 0$$

for sufficiently small $\varepsilon > 0$. We now add $2\pi n_{k_0} - 2\pi n_k$ to both sides and put $t = t_k - t_{k_0} - \delta$ to get $\Phi_s(\theta_k) + 2\pi < \Phi_{T_0-\varepsilon}(\Phi_s(\theta_k))$. Sending $k \uparrow \infty$ yields $\varphi + 2\pi \leq \Phi_{T_0-\varepsilon}(\varphi)$ which contradicts the definition of T_0. We have thus proved that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$.

It remains to prove the uniqueness of T_0. Assume that $\varphi_i + 2\pi = \Phi_{T_i}(\varphi_i)$ $(i = 1, 2)$ for some $\varphi_i \in \mathcal{D}$. Suppose that $T_1 \neq T_2$. We may assume that $T_1 < T_2$ and that $\varphi_1 < \varphi_2$ by translating its value modulo 2π. By definition of T_1 and T_2 it follows that $\Phi_t(\varphi_1) > \Phi_t(\varphi_2)$ for sufficiently large $t > 0$. This would contradict the order preserving property. □

Proposition 4.3 implies the following theorem.

Theorem 4.4. *(Existence of a periodic solution). There exists a periodic solution $\hat{\theta}$ of (2.9) satisfying $\hat{\theta}(\cdot, t) \in \mathcal{D}$ for $t \in \mathbf{R}$, which is unique up to translation of time.*

Proof. Choose $\varphi \in \mathcal{D}$, which satisfies $\varphi + 2\pi = \Phi_{T_0}(\varphi)$, as the initial data. Then, we can obtain a solution $\hat{\theta}$ of (2.9) with $\hat{\theta}(\cdot, 0) = \varphi$ and $\hat{\theta}(\cdot, t) \in \mathcal{D}$ for $t \geq 0$. This solution fulfills

$$\hat{\theta}(\cdot, t) + 2\pi = \Phi_t(\varphi) + 2\pi = \Phi_t(\varphi + 2\pi) = \Phi_t(\Phi_{T_0}(\varphi)) = \Phi_{t+T_0}(\varphi) = \hat{\theta}(\cdot, t + T_0).$$

That is, $\hat{\theta}$ is a periodic solution of (2.9) for $t \geq 0$. Using the uniqueness of the solution of (2.9) and the periodicity of the map Φ_t, we can extend this periodic solution $\hat{\theta}$ to $t < 0$. Note that Φ_t is also order-preserving for $t < 0$.

Finally we prove that this periodic solution is unique up to translation to the t-direction for $t \in \mathbf{R}$. The argument is essentially similar to that of [17, Section 3]. However, since the argument in [17] is based on an abstract theory, we reproduce their idea directly without appealing the abstract theory. Assume that $\hat{\theta}_i(\cdot, t) \in \mathcal{D}$ $(i = 1, 2)$ are the periodic solutions of (2.9) with the period $T_0 > 0$. We can take $\hat{\theta}_1(r, -kT_0) \leq \hat{\theta}_2(r, 0)$ for sufficiently large $k \in \mathbf{N}$. Rewrite $\hat{\theta}_1(r, t - kT_0)$ as $\hat{\theta}_1(r, t)$. Then $\hat{\theta}_1(r, 0) \leq \hat{\theta}_2(r, 0)$. The order-preserving property implies $\hat{\theta}_1(r, t) \leq \hat{\theta}_2(r, t)$ for $r \in \bar{I}$ and $t \in \mathbf{R}$. We may consider the attainable time t in the time-interval $[0, T_0]$ instead of $\mathbf{R}$ by the periodicity. Set

$$s_0 := \sup\{s \geq 0 \mid \hat{\theta}_1(r, t + s) \leq \hat{\theta}_2(r, t),\ r \in \bar{I},\ t \in [0, T_0]\}.$$

Clearly $\hat{\theta}_1(r, t + s_0) \leq \hat{\theta}_2(r, t)$ and it follows from the compactness of $\bar{I} \times [0, T_0]$ that there exists $(r_0, t_0) \in I \times [0, T_0]$ satisfying $\hat{\theta}_1(r_0, t_0 + s_0) = \hat{\theta}_2(r_0, t_0)$. Applying the strong maximum principle, we have $\hat{\theta}_1(r, t + s_0) \equiv \hat{\theta}_2(r, t)$ for all $r \in \bar{I}$ and $t \leq t_0$. Using the weak maximum principle as the initial time t_0, we see $\hat{\theta}_1(r, t + s_0) \equiv \hat{\theta}_2(r, t)$ for all $r \in \bar{I}$ and $t \geq t_0$. Thus, we obtain the desired result. □

Consequently, by virtue of Theorem 4.4, we can obtain the spiral solution of (1.2) of the form (1.4), which completes the proof of Theorem 1.2.

Remark 4.5. *(Stability of the spiral solution).* We can derive the stability of the spiral solution given by Theorem 1.2. That is, for any $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ such that if $d(\Gamma(0), \hat{\Gamma}(0)) < \delta_0$, then $d(\Gamma(t), \hat{\Gamma}(t)) < \varepsilon_0$ for all $t > 0$. Indeed, by means of applying the similar argument as in [17, Section 3], we deduce that for any $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ such that $\|\theta(\cdot, t) - \hat{\theta}(\cdot, t)\|_{C(\bar{I})} < \varepsilon_0$ for all $t > 0$ whenever $\|\theta(\cdot, 0) - \hat{\theta}(\cdot, 0)\|_{C(\bar{I})} < \delta_0$. Since

$$d(\Gamma(t), \hat{\Gamma}(t)) \leq C\|\theta(\cdot, t) - \hat{\theta}(\cdot, t)\|_{C(\bar{I})} \quad \text{for all } t \geq 0$$

where C is a positive constant independent of t, it follows that the spiral solution given by Theorem 1.2 is stable. □

5 Problem with no driving force term

In this section, we discuss the special case : $V_0 = 0$ in (2.9). That is,

$$\begin{cases} \theta_t = \dfrac{M(\boldsymbol{n})a(\boldsymbol{n})(r\theta_{rr} + r^2\theta_r^3 + 2\theta_r)}{r(1 + r^2\theta_r^2)} \\ \theta_r(\rho, t) = \theta_r(R, t) = 0. \end{cases} \tag{5.1}$$

We first derive the gradient estimate of (5.1)

Proposition 5.1. *(Gradient estimate for the case $V_0 = 0$). Assume that $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ satisfy $(A^{\Lambda}_{\lambda,\mu})$. For each $\varepsilon > 0$, let*

$$0 < L \leq \frac{\rho}{4\varepsilon R^2}.$$

Then if $\theta(r, t)$ is a solution of (5.1) with the initial data $\theta(\cdot, 0) = \theta_0$ satisfying $|\theta_{0r}| \leq L$, the gradient estimate $|\theta_r(\cdot, t)| \leq L$ holds for $t > 0$.

Proof. Set $w := -\theta_r - L$ and recall the proof of Proposition 3.1. Putting $V_0 = 0$ in (3.3), we are led to

$$\begin{aligned}
&\hat{\alpha}(r,t)w_{rr} + \hat{\beta}(r,t)w_r + \left(\hat{\gamma}(r,t) + \frac{2\varepsilon RV_0L}{\mu\rho}\right)w - w_t \\
\geq\ & -M(\boldsymbol{n})a(\boldsymbol{n})\left(\frac{4\varepsilon}{\rho}L^2 - \frac{1}{R^2}L\right) = -M(\boldsymbol{n})a(\boldsymbol{n})\cdot\frac{4\varepsilon}{\rho}L\left(L - \frac{\rho}{4\varepsilon R^2}\right) \geq 0.
\end{aligned}$$

It follows from the weak maximum principle that $w(\cdot,t) \leq 0$ for $t > 0$ if $w(\cdot,0) \leq 0$. That is, we see that $\theta_r(\cdot,t) \geq -L$ for $t > 0$ if $\theta_{0r} \geq -L$. Applying the same argument to $\bar{w} := \theta_r - L$, we derive the desired result. □

Using the weak maximum principle, the following lemma is verified.

Lemma 5.2. *(Boundness for the case $V_0 = 0$). Assume that $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ satisfy $(A^{\Lambda}_{\lambda,\mu})$ and that $\theta(r,t)$ is a solution of (5.1) with the initial data $\theta(\cdot,0) = \theta_0$. Then we have*

$$\max_{r\in[\rho,R]}|\theta(r,t)| \leq \max_{r\in[\rho,R]}|\theta_0(r)|.$$

Note that Proposition 5.1 and Lemma 5.2 assure the solvability of (5.1) and the standard parabolic estimate for the time-interval which does not include $t = 0$.

Setting $V_0 = 0$ in Proposition 2.2, we obtain the following Lyapunov function for (1.2).

Proposition 5.3. *Let $L[\Gamma(t)]$ be the length of $\Gamma(t)$. If $\Gamma(t)$ is a solution of (1.2) with $V_0 = 0$, the following formula is valid :*

$$\frac{d}{dt}L[\Gamma(t)] = -\int_\rho^R M(\boldsymbol{n})a(\boldsymbol{n})(1 + r^2\theta_r^2)^{1/2}\kappa^2 dr \leq 0.$$

Now we are ready to prove Corollary 1.3. It first follows from Proposition 5.3 and the assumption $(A^{\Lambda}_{\lambda,\mu})$ that

$$\int_0^\infty \int_\rho^R \kappa^2 dr dt \leq C_1 < \infty$$

where a constant C_1 (> 0) depends only on λ, μ and $L[\Gamma_0]$. Recalling the equation (1.2), we are led to

$$\int_0^\infty \int_\rho^R \theta_t^2 dr dt \leq C_2 < \infty \tag{5.2}$$

where a constant C_2 (> 0) is independent of time t. We consider the sequence of function $\{\theta_k\}$ defined on $E = [\rho, R] \times (0,1)$ with

$$\theta_k(r,t) = \theta(r, t+k), \quad k \in \mathbf{N}.$$

Applying the similar argument as in [9, Section 4] and [5, Section 2] with (5.2), we see that a subsequence $\{\theta_{k_j}\} \subset \{\theta_k\}$ converges uniformly to a function $\hat{\theta}_*$, which is independent of time t and a solution of

$$r\hat{\theta}_{*rr} + r^2\hat{\theta}_{*r}^3 + 2\hat{\theta}_{*r} = 0, \quad \hat{\theta}_{*r}(\rho) = \hat{\theta}_{*r}(R) = 0. \tag{5.3}$$

Solve (5.3) as the ordinary differential equation of $\hat{\theta}_{*r}$. We find that $\hat{\theta}_{*r} \equiv 0$ is a unique solution of (5.3). This means that $\hat{\theta}_*$ is also independent of r. That is, $\hat{\theta}_*$ is a constant. It remains to argue the uniform convergence of θ to a constant $\hat{\theta}_*$ along the full sequence of time. For each $\delta > 0$ there is sufficiently large j that satisfies

$$|\theta_{k_j}(r,t) - \hat{\theta}_*| < \delta \quad \text{for all } (r,t) \in \overline{E}.$$

In particular, we have

$$\hat{\theta}_* - \delta < \theta_{k_j}(r,0) = \theta(r,k_j) < \hat{\theta}_* + \delta \quad \text{for all } r \in [\rho, R].$$

Since $\hat{\theta}_* \pm \delta$ is a solution of (5.1), the comparison implies

$$\hat{\theta}_* - \delta < \theta(r,t) < \hat{\theta}_* + \delta \quad \text{for all } t \geq k_j,\ r \in [\rho, R].$$

This means that $\theta(r,t)$ converges uniformly to $\hat{\theta}_*$ as $t \to \infty$ in $[\rho, R]$ and completes the proof of Corollary 1.3.

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References

[1] W. K. Burton, N. Cabrera and F. C. Frank, Role of dislocations in crystal growth, Nature, 163 (1949), 398-399.

[2] W. K. Burton, N. Cabrera and F. C. Frank, The growth of crystals and equilibrium structure of their surfaces, Philos. Trans. Roy. Soc. London A, 243 (1951), 299-358.

[3] G. Dong, Nonlinear Partial Differential Equations of Second Order, Translations of mathematical monographs, vol. 95, American Mathematical Society, 1991.

[4] F. C. Frank, The influence of dislocations on crystal growth, Disc. Faraday Soc., 5 (1949), 48-54.

[5] Y. Giga, M. Ohnuma, and M.-H. Sato, On strong maximum principle and large time behaviour of generalized mean curvature flow with the Neumann boundary condition, J. Differential Equations, 154 (1999), 107-131.

[6] M. E. Gurtin, Thermomechanics of Evolving Phase Boundaries in the Plane, Clarendon, Oxford, 1993.

[7] N. Hungerbühler and K. Smoczyk, Soliton solutions for the mean curvature flow, Differential and Integral Equations 13 (2000), 1321-1345.

[8] R. Ikota, N. Ishimura, and T. Yamaguchi, On the structure of steady solutions for the kinematic model of spiral waves in excitable media, Japan J. Indust. Appl. Math., 15 (1998), 317-330.

[9] T. Ilmanen, P. Sternberg, and W. P. Ziemer, Equilibrium solutions to generalized motion by mean curvature, J. Geom. Anal., 8 (1998), 845-858.

[10] H. Imai, N. Ishimura, and T. K. Ushijima, A crystalline motion of spiral-shaped curves with symmetry, J. Math. Anal. Appl., 240 (1999), 115-127.

[11] N. Ishimura, Shape of spirals, Tohoku Math. J., 50 (1998), 197-202.

[12] A. Karma and M. Plapp, Spiral surface growth without desorption, Physical Review Letters, 81 (1998), 4444-4447.

[13] R. Kobayashi, work in preparation.

[14] O. A. Ladyzhenskaia, V. A. Solonnikov, and N. N. Uraltseva, Linear and Quasi-Linear Equations of Parabolic Type, AMS, 1968.

[15] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific.

[16] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, 1995.

[17] T. Ogiwara and K-I. Nakamura, Spiral traveling wave solutions of some parabolic equations on annuli, Josai Mathematical Monographs, 2 (2000), 15-34.

[18] M. Ohara and R. C. Reid, Modelling Crystal Growth Rates from Solution, Prentice-Hall, Englewood Cliffs, N.J. (1973).

[19] T. Ohtsuka, A level set method for spiral growth, in preparation

[20] P. Smereka, Spiral crystal growth, Physica D: Nonlinear Phenomena, 138 (2000), 282-301.

[21] J. E. Taylor, J. W. Cahn, and C. A. Handwerker, Geometric models of crystal growth, Acta Metall. Mater., 40 (1992), 1443-1474.

[22] J. Urbas, Convex curves moving homothetically by negative powers of their curvature, Asian J. Math., 3 (1999), 635-656.

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