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On a limiting motion and self-intersections for the intermediate surface diffusion flow

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Abstract

We rigorously prove that the solution surface of the intermediate surface diffusion flow converges to that of the averaged mean curvature flow locally in time as the diffusion coefficient tends to infinity. As an application of this convergence result, we show that the intermediate surface diffusion flow can drive embedded hypersurfaces into self-intersections.

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1 Introduction

We study the following nonlocal geometric evolution equation:

$$V(t) = \Delta_{\Gamma(t)} \Big(\frac{1}{D} - \frac{1}{M}\Delta_{\Gamma(t)}\Big)^{-1} H(t) \qquad \text{on } \ \Gamma(t) \ \text{ for } \ t > 0. \tag{1}$$

Here $\Gamma(t)$ is an unknown, with respect to time $t > 0$ evolving closed compact oriented hypersurface in $\mathbb{R}^n$. We write $\Delta_{\Gamma(t)}$ for the Lapalce-Beltrami operator on $\Gamma(t)$ with respect to the Euclidean metric. The mean curvature of $\Gamma(t)$ is denoted by $H(t)$ and V stands for the normal velocity of the family $\{\Gamma(t)\,;\, t > 0\}$. The evolution equation (1) does not depend on the local choice of orientation. However, if $\Gamma(t)$ encloses a domain $\Omega(t)$ (which is the physical relevant case), we always choose the orientation such that $V(t)$ is positive if $\Omega(t)$ grows and such that $H(t)$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$. We mention that in the plane case sometimes the opposite orientation is used. Finally, $D > 0$ is the diffusion coefficient and $M > 0$ is the mobility constant. The equation (1) is called *intermediate surface diffusion flow* and was first proposed by J. W. Cahn and J. E. Taylor [1]. The evolution equation (1) has to be supplemented by an appropriate initial condition

$$\Gamma(0) = \Gamma_0, \tag{2}$$

where Γ_0 is a (possibly) immersed compact closed oriented hypersurface of class $C^{2+\alpha}$ with $\alpha \in (0, 1)$ fixed. We are interested in the behaviour of solutions to (1), (2) for large D, therefore we restrict ourselves to the case $D \geq 1$.

The purpose of this paper is to prove that problem (1), (2) admits a unique local solution $\Gamma^D := \{\Gamma^D(t)\,;\, t \in [0, T]\}$ of class $C^{1,2+\alpha}$ on a common existence interval $[0, T]$ (i.e. $[0, T]$ is independent of $D \geq 1$), and that this solution Γ^D converges for $D \to \infty$ in $C^{1,2+\alpha}$ to the unique solution $\Gamma := \{\Gamma(t)\,;\, t \in [0, T]\}$ of the *averaged mean curvature flow*, defined by

$$V(t) = M\Big(\overline{H}(t) - H(t)\Big) \quad \text{ on } \ \Gamma(t) \ \text{ for } \ t > 0, \quad \Gamma(0) = \Gamma_0. \tag{3}$$

Here $\overline{H}(t)$ is the spatial average of the mean curvature $H(t)$, i.e.

$$\overline{H}(t) := \int_{\Gamma(t)} H(t)\, d\sigma(t) \Big/ \int_{\Gamma(t)} d\sigma(t),$$

where $d\sigma(t)$ denote s the Euklidean surface measure on $\Gamma(t)$.

Additionally, as an application of the above convergence result, we prove that embedded hypersurfaces can be driven under the flow (1) into self-intersections in finite time for all $D > 0$.

The intermediate surface diffusion flow was introduced to describe a geometric growth law for a moving interface where surface diffusion is the only transport mechanism and the reduction of total surface energy is the only driving force for surface motion, cf. [1, 11]. The first mathematical results for (1), (2) was presented by C. M. Elliott and H. Garcke [2]. They prove both global existence and stabilty results for plane curves when Γ_0 is close to a circle. These results were extended by J. Escher and G. Simonett [5] to the multi-dimensional case. Additionally, a general uniqueness result is obtained in [5].

It was already suggested in [1] that the formal limit of (1) becomes (3) as $D \to \infty$. In this paper we carefully investigate the $D-$dependence of solutions to (1), which enables us to justify this conjecture. More precisely, we show that there exist positive constants T_0 and C, both independent of $D \geq 1$, such that for any $D \geq 1$ problem (1) has a unique solution on $[0, T_0]$, satisfying the uniform a priori estimate $\|\Gamma^D\|_{C^{1,2+\alpha}} \leq C$. We mention that the present results generalize those obtained in [3].

The analysis here is restricted to local-in-time solutions. We have the following two reasons to do that. First, even in the plane case general global existence results for (1) and (3) are not available. Secondly, the flow (3) preserves convexity, cf. [7, 8], whereas (1) does not seem to share this property, and hence the global-in-time convergence of solutions to (1) towards solutions to (3) in $C^{1,2+\alpha}$ may be delicate.

Based on the convergence result for $D \to \infty$, we can investigate particular properties of solutions to (1) with large D by analyzing the averaged mean curvature flow which is (sometimes) easier to treat than (1). We illustrate this procedure here by proving that the flow (1) may drive embedded initial surfaces to immersed but not embedded surfaces. This behaviour was known for the flow (3) by the work of M. Gage [7] for curves and by the paper of U. F. Mayer and G. Simonett [10] for surfaces. These results suggest that a loss of embeddedness for (1) can also happen, since (1) and (3) are linked by the limit $D \to \infty$. In the present paper we prove, using the above mentioned convergence result and a scaling argument, that for solutions to (1) this phenomenon actually occurs for all $D > 0$.

2 Parametrization

For simplicity we set $\delta := 1/D$ with $D \in [1, \infty)$ and $M := 1$. Hence we consider the evolution equation

$$V = \Delta_{\Gamma(t)} \left(\delta - \Delta_{\Gamma(t)}\right)^{-1} H(t) \quad \text{on } \Gamma(t) \text{ for } t > 0, \tag{4}$$

with the initial condition (2). Observe that (4) can be rewritten as

$$V = \overline{H}(t) - H(t) - \delta \left(\delta - \Delta_{\Gamma(t)}\right)^{-1} (\overline{H}(t) - H(t)) \tag{5}$$

on $\Gamma(t)$ fo r $t > 0$. This follows from the fact that Δ and $(\delta - \Delta)^{-1}$ commute and since $\Delta\overline{H} = 0$.

Furthermore, note that (5) is also meaningful for $\delta = 0$ and coincides in that case with the averaged mean curvature flow (3). This follows from the fact that $\Delta^{-1}_{\Gamma(t)} f$ is well defined for continuous functions having zero average. In the following we study (5) for all $\delta \in [0, 1]$.

We parametrize (5) in a neighbourhood of a smooth compact closed immersed oriented hypersurface Σ in $\mathbb{R}^n$ being close to Γ_0. To make this precise, let ν denote the unit outer normal field on Σ. Moreover, given $a > 0$, choose a localization system $\{(U_l, \varphi_l)\,;\, l = 1, \ldots, m\}$ for Σ such that $\Sigma = \cup_{l=1}^m U_l$ and

$$\varphi_l \,:\, (-a, a)^{n-1} \to U_l, \quad l \in \{1, \ldots, m\},$$

is a smooth parametrization of U_l. Shrinking $a > 0$ if nessecary, we may assume that

$$X_l \,:\, U_l \times (-a, a) \to \mathbb{R}^n, \qquad X_l(s, r) := s + r\nu(s).$$

is a smooth diffeomorphism onto its image $\mathcal{R}_l := \mathrm{im}(X_l)$, i.e.

$$X_l \in \mathrm{Diff}^\infty(U_l \times (-a, a), \mathcal{R}_l).$$

The inverse of X_l can be decomposed in the following way. Writing $S_l \in C^\infty(\mathcal{R}_l, U_l)$ and $\Lambda_l \in C^\infty(\mathcal{R}_l, (-a, a))$ for the metric projection of $\mathcal{R}_l$ onto U_l and for the signed distance function with respect to U_l, respectively, we have $X_l^{-1} = (S_l, \Lambda_l)$. In particular, observe that $\mathcal{R} := \cup_{l=1}^m \mathcal{R}_l$ consists of those points in $\mathbb{R}^n$ with distance less than a to Σ.

Let now

$$W(\Sigma) := W_a(\Sigma) := \{\rho \in C^{2+\alpha}(\Sigma)\,;\, \|\rho\|_{C(\Sigma)} \le a/2\}.$$

and define

$$\Gamma_\rho := \bigcup_{l=1}^{m} \{X_l(s,\rho(s))\,;\, s \in U_l\}$$

for $\rho \in W(\Sigma)$. Here $\|\rho\|_{C(\Sigma)}$ denotes the maximum norm of ρ on Σ. Then Γ_ρ is a compact closed oriented immersed hypersurface in $\mathbb{R}^n$ of class $C^{2+\alpha}$, which can be seen as a graph in normal direction over Σ. Of course, ρ measures the signed distance of Σ to Γ_ρ. For convenience let us also introduce the mapping

$$\theta_\rho \,:\, \Sigma \to \Gamma_\rho, \qquad s \mapsto X_l(s,\rho(s)) \quad \text{for} \quad s \in U_l.$$

Then θ_ρ is a well-defined global diffeomorphism of class $C^{2+\alpha}$ from Σ onto Γ_ρ. By means of this diffeomorphism we can pull back the Euclidean metric on Γ_ρ to Σ, producing in that way a Riemannian manifold which we denote in the following by $\Sigma(\rho)$. We now consider a family of hypersurfaces in $\mathcal{R}$. More precisely, let $T > 0$ be given, and define $\Sigma_T := [0,T] \times \Sigma$, as well as

$$W(\Sigma_T) := W_a(\Sigma_T) := \{\rho \in C^{1,2+\alpha}(\Sigma_T)\,;\, \|\rho\|_{C(\Sigma_T)} \le a/2\}.$$

Then we transform the evolution equation (5) on the family $\Gamma := \{\Gamma_{\rho(t)}\,;\, t \in [0,T]\}$ into an evolution equation on Σ. For this we first calculate the normal velocity of Γ. We have, cf. [4],

$$V(t,s) = \partial_t \rho(t,s) \Big/ |\nabla_x \Phi_\rho(x,t)|_{x=\theta_{\rho(t)}(s)} \qquad \text{for} \;\; (t,s) \in \Sigma_T,$$

where we used the function

$$\Phi_\rho \,:\, \mathcal{R} \times [0,T] \to \mathbb{R}, \quad (x,t) \mapsto \Lambda_l(x) - \rho(t, S_l(x))$$

to represent $\Gamma_{\rho(t)}$ as the $0-$level set of $\Phi_\rho(\cdot,t)$, i.e. $\Gamma_{\rho(t)} = \Phi^{-1}(\cdot,t)(0)$. To shorten our notation, let

$$L(\rho)(t,s) := |\nabla_x \Phi_\rho(x,t)|_{x=\theta_{\rho(t)}(s)}, \quad (t,s) \in \Sigma_T.$$

Moreover, we write

$$K(\rho) := \theta_\rho^* H \quad \text{and} \quad \overline{K}(\rho) := \int\limits_\Sigma K(\rho) g(\rho)\, d\sigma \Bigg/ \int\limits_\Sigma g(\rho)\, d\sigma$$

for the mean curvature of $\Sigma(\rho)$ and its average over $\Sigma(\rho)$, respectively. Here, θ_ρ^* denotes the pull-back operator induced by the diffeomorphism θ_ρ, i.e. $\theta_\rho^* f = f \circ \theta_\rho$. We further set

$$g(\rho) := \sqrt{\det([D_s\theta_\rho]^\top [D_s\theta_\rho])}$$

with $A^\top$ being the transpose of A, and $d\sigma$ stands for the Euclidean surface volume element on Σ. Finally, let Δ_ρ denote the Laplace-Beltrami operator on $\Sigma(\rho)$. This means in particular that we have

$$\theta_\rho^* \Delta_{\Gamma_\rho} = \Delta_\rho \theta_\rho^*.$$

Consider now the following nonlinear nonlocal partial differential equation

$$\frac{d\rho}{dt} = F^\delta(\rho) \quad \text{in} \quad \Sigma_T, \qquad \rho(0) = \rho_0 \quad \text{on} \quad \Sigma, \tag{6}$$

where ρ_0 is determined by Γ_0, and $F^\delta := F_1 - F_2 - F_3^\delta$ with

$$\begin{aligned} &F_1(\rho) := L(\rho)\overline{K}(\rho), \qquad F_2(\rho) := L(\rho)K(\rho), \\ &F_3^\delta(\rho) := \delta L(\rho)(\delta - \Delta_\rho)^{-1}(\overline{K}(\rho) - K(\rho)). \end{aligned}$$

Obviously, (6) is a transfomed version of (1), (2), defined on Σ_T. For later purposes we express the terms $L(\rho)$, $K(\rho)$, and Δ_ρ in local coordinates. To make this precise, let

$$\hat\rho_l(s) := \rho(\varphi_l(s)), \quad \hat X_l(s,r) := X_l(\varphi_l(s), r), \quad (s,r) \in (-a,a)^n,$$

be the local representations of ρ_l and X_l with respect to U_l. In the following we do not always distinguish between ρ_l, X_l and their local representations $\hat\rho_l$, $\hat X_l$, as well as between local coordinates $s \in (-a,a)^{n-1}$ and the corresponding points $\varphi_l(s)$ on U_l. Moreover, we suppress the index $l \in \{1, \dots, m\}$ if no confusion seems likely. Given $\rho \in W(\Sigma)$, define

$$w_{jk}(\rho)(s) := (\partial_j X|\partial_k X)|_{(s,\rho(s))}, \quad s \in (-a,a)^{n-1},$$

for where $(\cdot|\cdot)$ stands for the Euclidean metric in $\mathbb{R}^n$ and ∂_j denotes the partial derivative with respect to the j-th variable of s. Since ρ belongs to $W(\Sigma)$, the matrix $[w_{jk}(\rho)]$ is invertible and we write $w^{jk}(\rho)$ for the entries of its inverse. Then we have

$$L(\rho) = \sqrt{1 + w^{jk}(\rho)\partial_j\rho\partial_k\rho} \tag{7}$$

cf. (2.3) in [6]. Moreover, using the summation convention over repeated indices we introduce the corresponding Christoffel symbols:

$$\Gamma^i_{jk}(\rho) := \frac{1}{2}w^{il}(\rho)\Big(\partial_k w_{lj}(\rho) - \partial_l w_{jk}(\rho) + \partial_j w_{ki}(\rho)\Big).$$

Then Lemma 2.1 in [6] shows that $K(\rho)$ carries a quasi-linear structure, i.e. letting $U(\Sigma) := \{\rho \in C^{1+\gamma}(\Sigma)\,;\, \|\rho\|_{C(\Sigma)} < a/2\}$, with $\gamma \in (\alpha, 1)$, there exist

$$P \in C^\infty(U(\Sigma), \mathcal{L}(C^{2+\alpha}(\Sigma), C^\alpha(\Sigma))) \quad \text{and} \quad Q \in C^\infty(U(\Sigma), C^\gamma(\Sigma)) \tag{8}$$

such that

$$K(\rho) = P(\rho)\rho + Q(\rho) \quad \text{for} \quad \rho \in U \cap C^{2+\alpha}(\Sigma). \tag{9}$$

Moreover, given $\rho \in U(\Sigma)$, the operator $P(\rho)$ is uniformly strong elliptic, cf. Lemma 3.2 in [4]. In the chosen local coordinates these operators are represented as:

$$\begin{aligned} P(\rho) = \tfrac{1}{(n-1)L(\rho)^3}\Big[&\Big\{ - L(\rho)^2 w^{jk}(\rho) + w^{jl}(\rho)w^{km}(\rho)\partial_l\rho\partial_m\rho\Big\}\partial_j\partial_k \\ &+\Big\{L(\rho)^2 w^{jk}(\rho)\Gamma^i_{jk}(\rho) + w^{jl}(\rho)w^{ki}(\rho)\Gamma^n_{jk}(\rho)\partial_l\rho \\ &+2w^{km}(\rho)\Gamma^i_{nk}(\rho)\partial_m\rho - w^{jl}(\rho)w^{km}(\rho)\Gamma^i_{jk}(\rho)\partial_l\rho\partial_m\rho\Big\}\partial_i\Big] \end{aligned} \tag{10}$$

and

$$Q(\rho) = -\frac{1}{(n-1)L(\rho)}w^{jk}(\rho)\Gamma^n_{jk}(\rho). \tag{11}$$

In order to express Δ_ρ in local coordinates, let η be the Euclidean metric and write $\sigma(\rho) := \theta^*_\rho\eta$ for the Riemannian metric on Σ induced by the diffeomorphism θ_ρ. This means that, using the above introduced notation, we have

$\Sigma(\rho) = (\Sigma, \sigma(\rho))$. Let further $\sigma_{jk}(\rho)$ denote the components of $\sigma(\rho)$ in local coordinates and write $\sigma^{jk}(\rho)$ for the components of the inverse of $[\sigma_{jk}(\rho)]$. Using summation convention over repeated indices, the Christoffel symbols of $\sigma(\rho)$ in the chosen coordinates are given by

$$\gamma^l_{jk} = \frac{\sigma^{lm}}{2}\Big[\frac{\partial \sigma_{km}}{\partial s^j} + \frac{\partial \sigma_{jm}}{\partial s^k} - \frac{\partial \sigma_{jk}}{\partial s^m}\Big],$$

and we have

$$\Delta_\rho = \sigma^{jk}(\rho)\Big[\frac{\partial^2}{\partial s^j \partial s^k} - \gamma^l_{jk}\frac{\partial}{\partial s^l}\Big], \quad \rho \in U \cap C^{2+\alpha}(\Sigma), \tag{12}$$

cf. the proof of Lemma 2.1 in [6].

3 Uniform local existence

In this section we establish the existence of a unique local solution of (6). Additionally, we prove that there exists a common interval of existence for all $\delta \in [0,1]$. More precisely, letting

$$C^{1,2+\alpha}(\Sigma_T) := \{f \in C([0,T] \times \Sigma)\,;\, \partial_t f,\ D^2 f \in C^{0,\alpha}([0,T] \times \Sigma)\}$$

endowed with the norm

$$\|f\|_{C^{1,2+\alpha}(\Sigma_T)} := \|f\|_{C(\Sigma_T)} + \|\nabla f\|_{C(\Sigma_T)} + \|\partial_t f\|_{C^{0,\alpha}(\Sigma_T)} + \|D^2 f\|_{C^{0,\alpha}(\Sigma_T)}$$

we have

Theorem 3.1 (Uniform local existence) *Given $\rho_0 \in W_{a/2}(\Sigma)$, there are positive constants $T_0 = T_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$ and $N_0 = N_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$, which are independent of $\delta \in [0,1]$, such that (6) admits a unique solution ρ^δ in $C^{1,2+\alpha}(\Sigma_{T_0})$ satisfying*

$$\|\rho^\delta\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \le N_0 \quad \text{for} \quad \delta \in [0,1].$$

Remarks 3.2 (a) Local existence results for (4) were previously obtained in [2, 5]. But in these articles the dependence of solutions to (4) with respect to δ is not considered.

(b) Using well-known bootstrapping arguments for quasi-linear uniformly parabolic equations (based e.g. on [12]) it can be shown that the solution constructed in Theorem 3.1 is in fact smooth, i.e.

$$\rho^\delta \in C^\infty((0,T_0)\times\Sigma) \quad for \quad \delta\in[0,1].$$

Since we do not need this result in the present paper we omit its proof. $\square$

Before we proceed to the proof of Theorem 3.1, we provide some preparation. To economize our notation we fix t and suppress it whenever possible. Our plan is to use the quasi-linear structure of the principal part of F^δ in order to apply the Banach contraction principle on a closed subspace of $C^{1,2+\alpha}(\Sigma_T)$. More precisely, fix $\rho_0 \in W_{a/2}(\Sigma)$ and let $A := -L(\rho_0)P(\rho_0)$. Then it follows from (10), the fact that $\Sigma(\rho)$ is a smooth compact closed Riemannian manifold, and well-known Schauder-estimates that A is sectorial in $C^\alpha(\Sigma)$ with

dom$(A) = C^{2+\alpha}(\Sigma)$, cf. Theorem 3.1.12 and Corollary 3.1.16 in [9]. Next we remark that $[\rho\mapsto L(\rho)\overline{K}(\rho)]$ is of "lower order" with respect to A. More precisely, let $\rho\in W(\Sigma)$ be given. Using the fact that $P(\rho)$ is a second order differential operator with coefficients depending smoothly on ρ and on $\nabla\rho$, cf. (10), we find a positive constant $C = C(\|\rho\|_{C^{1+\alpha}(\Sigma)})$ such that

$$\|P(\rho)r\|_{C(\Sigma)} \le C\|r\|_{C^2(\Sigma)}, \quad r\in C^2(\Sigma).$$

Further we have $\|L(\rho)\|_{C^\alpha(\Sigma)} \le C$ and therefore

$$\|L(\rho)\overline{P(\rho)\rho}\|_{C^\alpha(\Sigma)} = \|L(\rho)\|_{C^\alpha(\Sigma)}\frac{|\int_\Sigma P(\rho)\rho g(\rho)\,d\sigma|}{\int_\Sigma g(\rho)\,d\sigma} \le C\|\rho\|_{C^2(\Sigma)}, \tag{13}$$

for each $\rho\in W(\Sigma)$. We already noted that Q is of lower order with respect to $P(\rho)$. Hence

$$\hat{F}_1(\rho) := L(\rho)[\overline{K}(\rho) - Q(\rho)], \quad \rho\in W(\Sigma) \tag{14}$$

consists only of such lower order terms. Moreover, using (10), (11), and (13) it is easily verified that, given ρ, $\sigma\in W(\Sigma)$, there exists a constant C depending continuously on $\|\rho\|_{C^{1+\alpha}(\Sigma)}$, $\|\sigma\|_{C^{1+\alpha}(\Sigma)}$ such that

$$\|\hat{F}_1(\rho) - \hat{F}_1(\sigma)\|_{C^\alpha(\Sigma)} \le C\|\rho-\sigma\|_{C^2(\Sigma)}. \tag{15}$$

We proceed in decomposing the nonlinear operator $F^\delta(\rho)$, cf. (6). For this let

$$\hat{F}_2(\rho) := [L(\rho_0)P(\rho_0) - L(\rho)P(\rho)]\rho, \tag{16}$$

where $\rho \in W(\Sigma)$. Here we obtain from (10) and (11) that there is a constant C, depending continuously on $\|\rho_0\|_{C^{1+\alpha}(\Sigma)}$ and $\|\rho\|_{C^{1+\alpha}(\Sigma)}$, such that

$$\|\hat{F}_2(\rho)\|_{C^\alpha(\Sigma)} \leq C\|\rho - \rho_0\|_{C^{1+\alpha}(\Sigma)}\|\rho\|_{C^{2+\alpha}(\Sigma)}, \quad \rho \in W(\Sigma). \tag{17}$$

Finally, we collect those terms in $F^\delta(\rho)$ containing the parameter δ:

$$\hat{F}_3^\delta(\rho) := \delta L(\rho)(\delta - \Delta_\rho)^{-1}(\overline{K}(\rho) - K(\rho))$$

so that we have

$$F^\delta(\rho) = A\rho + \hat{F}_1(\rho) + \hat{F}_2(\rho) - \hat{F}_3^\delta(\rho), \quad \rho \in W(\Sigma). \tag{18}$$

We now consider time dependent distance functions. Given any $\rho \in W(\Sigma_T)$, define

$$\hat{F}^\delta(\rho) := \hat{F}_1(\rho) + \hat{F}_2(\rho) - \hat{F}_3^\delta(\rho), \quad \text{if } \delta \in (0, 1],$$

and

$$\hat{F}^0(\rho) := \hat{F}_1(\rho) + \hat{F}_2(\rho), \quad \text{if } \delta = 0,$$

and observe that $\hat{F}^\delta(\rho) \in C^{0,\alpha}(\Sigma_T)$ for $\rho \in W(\Sigma_T)$ and $\delta \in [0, 1]$.Therefore, applying general results for linear inhomogeneous evolution equations involving sectorial operators, in particular Theorem 5.1.4(iv) in [9], we obtain

Lemma 3.3 *Given $\rho_0 \in W(\Sigma)$ and $\rho \in W(\Sigma_T)$, there is a unique solution $r^\delta \in C^{1,2+\alpha}(\Sigma_T)$ of*

$$\frac{dr}{dt} = Ar + \hat{F}^\delta(\rho) \quad in \quad \Sigma_T, \qquad r(0) = \rho_0 \quad on \quad \Sigma, \tag{19}$$

satisfying

$$\|r^\delta\|_{C^{1,2+\alpha}(\Sigma_T)} \leq c\left(\|\rho_0\|_{C^{2+\alpha}(\Sigma)} + \|\hat{F}^\delta(\rho)\|_{C^{0,\alpha}(\Sigma_T)}\right), \tag{20}$$

where the constant $c = c(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$ is independent of $\delta \in [0, 1]$.

Observe that, given $\rho \in W(\Sigma)$, we have

$$\ker(\delta - \Delta_\rho) = \begin{cases} 0 & \text{if } \delta > 0, \\ \mathbb{R}\,1 & \text{if } \delta = 0. \end{cases}$$

Thus there is no hope to have a uniform resolvent estimate of $(\delta - \Delta_\rho)^{-1}$ with respect to $\delta \in (0, 1]$. However, our next result shows that such a uniform resolvent estimate holds true on the $L_2(\Sigma_\rho)$−complement of $\mathbb{R}\,1$ in $C^\alpha(\Sigma)$. For this let

$$M_\rho(\Sigma) := \{h \in C^\alpha(\Sigma)\,;\, \overline{h} := \int_\Sigma hg(\rho)\,d\sigma = 0\} \quad \text{for} \quad \rho \in W(\Sigma)$$

and

$$B_R(\Sigma) := W(\Sigma) \cap \{u \in C^{2+\alpha}(\Sigma)\,;\, \|u\|_{C^{2+\alpha}(\Sigma)} \leq R\}.$$

Further we note that, given $\delta > 0$ and $\rho \in W(\Sigma)$, we have

$$(\delta - \Delta_\rho)^{-1}\, M_\rho(\Sigma) \subset M_\rho(\Sigma). \tag{21}$$

Indeed, given $h \in M_\rho(\Sigma)$, set $u := (\delta - \Delta_\rho)^{-1}h$. Since Σ_ρ is a closed Riemannian manifold, we know that $\int_\Sigma [\Delta_\rho u] g(\rho)\,d\sigma = 0$, showing that $\delta \int_\Sigma ug(\rho)\,d\sigma$ vanishes as well. By assumption δ is positive, and hence we find that $u \in M_\rho(\Sigma)$.

Lemma 3.4 *Given $R > 0$, there exists a positive constant $C(R)$ independent of $\delta \in (0, 1]$ such that*

$$\|(\delta - \Delta_\rho)^{-1}h\|_{C^{2+\alpha}(\Sigma)} \leq C(R)\|h\|_{C^\alpha(\Sigma)}$$

for all $\rho \in B_R(\Sigma),\ h \in M_\rho(\Sigma),\ \delta \in (0, 1]$.

Proof. (i) Let $\rho \in B_R(\Sigma),\ h \in M_\rho(\Sigma)$, and $\delta \in (0, 1]$ be given. By standard Schauder theory for elliptic equations, cf. Theorem 3.1.34 in [9], we have $(\delta - \Delta_\rho)^{-1}h \in C^{2+\alpha}(\Sigma)$ and there is a constant $\tilde{c} := \tilde{c}(R, a)$, independent of (ρ, h, δ), such that

$$\|(\delta - \Delta_\rho)^{-1}h\|_{C^{2+\alpha}(\Sigma)} \leq \tilde{c}\left(\|h\|_{C^\alpha(\Sigma)} + \|(\delta - \Delta_\rho)^{-1}h\|_{C(\Sigma)}\right) \tag{22}$$

(ii) We would like to prove that there is a c such that

$$\|(\delta - \Delta_\rho)^{-1}h\|_{C^{2+\alpha}(\Sigma)} \leq c\|h\|_{C^\alpha(\Sigma)} \tag{23}$$

for all $\rho \in B_R(\Sigma),\ h \in M_\rho(\Sigma)$, and $\delta \in (0, 1]$. To prove (23) we argue by contradiction. If (23) does not hold true there is a sequence $(\rho_m, \hat{h}_m, \delta_m)$ such that $\rho_m \in B_R(\Sigma)$, $\hat{h}_m \in M_{\rho_m}$, $\delta_m \in (0, 1]$ and such that

$$\|(\delta_m - \Delta_{\rho_m})^{-1}\hat{h}_m\|_{C^{2+\alpha}(\Sigma)} > m\|\hat{h}_m\|_{C^\alpha(\Sigma)}, \quad m \in \mathbb{N}.$$

Set $h_m := \hat{h}_m \Big/ \|(\delta_m - \Delta_{\rho_m})^{-1}\hat{h}_m\|_{C^{2+\alpha}(\Sigma)}$ and $u_m := (\delta_m - \Delta_{\rho_m})^{-1}h_m$. Then we have

$$\|u_m\|_{C^{2+\alpha}(\Sigma)} = 1 \qquad \text{and} \qquad \|h_m\|_{C^{\alpha}(\Sigma)} < 1/m, \quad m \in \mathbb{N}. \tag{24}$$

By the Theorem of Arzela-Ascoli there is a subsequence of (u_m, ρ_m) still denoted by (u_m, ρ_m) and u, $\rho \in C^2(\Sigma)$ such that $u_m \to u$ and $\rho_m \to \rho$ in $C^2(\Sigma)$. Of course, we have $\|\rho\|_{C(\Sigma)} \le a$. Hence by means of local coordinates we conclude that

$$\|\left(\Delta_{\rho_m} - \Delta_\rho\right) u\|_{C(\Sigma)} \to 0 \quad \text{as} \quad m \to \infty.$$

Moreover, using the bound $\|\rho_m\|_{C^{2+\alpha}(\Sigma)} \le R$, we conclude that

$$\|\Delta_{\rho_m}(u_m - u)\|_{C(\Sigma)} \le C_R\|u_m - u\|_{C^2(\Sigma)} \to 0 \quad \text{as} \quad m \to \infty.$$

Consequently, we see that

$$\Delta_{\rho_m} u_m \to \Delta_\rho u \quad \text{in} \quad C(\Sigma) \quad \text{as} \quad m \to \infty. \tag{25}$$

Moreover, there is a subsequence of (δ_m), again still denoted by (δ_m), and a $\delta \in [0,1]$ with $\lim_m \delta_m = \delta$. Since (h_m) converges in $C^\alpha(\Sigma)$ to 0, cf. (24), we conclude from (25) that

$$(\delta - \Delta_\rho)u = 0. \tag{26}$$

In addition, (22) applied to (ρ_m, h_m, δ_m) yields

$$1 \le \tilde{c}\left(\|h_m\|_{C^\alpha(\Sigma)} + \|u_m\|_{C(\Sigma)}\right).$$

Passing to the limit $m \to \infty$, we see that

$$1 \le \tilde{c}\|u\|_{C(\Sigma)}. \tag{27}$$

(iii) Assume first that $\delta > 0$. Then by (26) we must have $u = 0$, which contradicts (27).

In the case $\delta = 0$, (26) implies that u is a constant. By (21) we know that $\int_\Sigma u_m g(\rho_m)\, d\sigma = 0$. Sending $m \to \infty$, we find

$$\int_\Sigma u g(\rho)\, d\sigma = u \int_\Sigma g(\rho)\, d\sigma = 0,$$

which is for a constant u only possible if $u = 0$. This contradicts again (27), and the proof is complete. $\square$

Recalling that

$$\overline{h} = \int_\Sigma hg(\rho)\, d\sigma \Big/ \int_\Sigma g(\rho)\, d\sigma$$

for $h \in C^\alpha(\Sigma)$, we have the following

Corollary 3.5 *Given $R > 0$, there exists a positive constant $C = C(R)$ such that*

$$\|\delta(\delta - \Delta_\rho)^{-1} h\|_{C^{2+\alpha}(\Sigma)} \le C\|h - \overline{h}\|_{C^\alpha(\Sigma)} + |\overline{h}|$$

for all $\rho \in B_R(\Sigma)$, $h \in C^\alpha(\Sigma)$, $\delta \in (0, 1]$.

Proof. Given $h \in C^\alpha(\Sigma)$, we have

$$\delta(\delta - \Delta_\rho)^{-1} h = \delta(\delta - \Delta_\rho)^{-1}(h - \overline{h}) + \delta(\delta - \Delta_\rho)^{-1}\overline{h}.$$

Observing that $\overline{h - \overline{h}} = 0$ and $(\delta - \Delta_\rho)\overline{h} = \delta\overline{h}$, thus $\delta(\delta - \Delta_\rho)^{-1}\overline{h} = \overline{h}$, the conclusion follows from Lemma 3.4, since $\delta \le 1$. $\square$

Proof of Theorem 3.1. (i) Fix $\rho_0 \in W_{a/2}(\Sigma)$. Given $T \in (0, 1]$, $N > 0$, define

$$X_{T,N} := \{\rho \in W_a(\Sigma_T)\,;\, \|\rho\|_{C^{1,2+\alpha}(\Sigma_T)} \le N,\ \rho(0) = \rho_0\},$$

and, given $\delta \in [0, 1]$, let

$$S^\delta\,:\, X_{T,N} \to C^{1,2+\alpha}(\Sigma_T)$$

be the solution operator for (19), i.e., given $\rho \in X_{T,N}$, $S^\delta(\rho)$ is the unique solution to (19), cf. Lemma 3.3. Then, given $\beta \in (0, \alpha)$, we shall prove that there are positive constants C_0, C_1, C_2 independent of $\delta \in [0, 1]$ such that

$$\|S^\delta(\rho)\|_{C^{1,2+\alpha}(\Sigma_T)} \le C_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C_1(N)T^{(\alpha-\beta)/2} \tag{28}$$

and

$$\|S^\delta(\rho) - S^\delta(\sigma)\|_{C^{1,2+\alpha}(\Sigma_T)} \le C_2(N)T^{(\alpha-\beta)/2}\|\rho - \sigma\|_{C^{1,2+\alpha}(\Sigma_T)} \tag{29}$$

for all ρ, $\sigma \in X_{T,N}$.

(ii) Assume that (28), (29) hold true. Letting $N_0 := 2C_0(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$ and choosing $T_0 > 0$ such that

$$C_1(N_0)T_0^{(\alpha-\beta)/2} \leq N_0/2, \quad C_2(N_0)T_0^{(\alpha-\beta)/2} < 1, \quad T_0 N_0 \leq a/4,$$

it is easily verified that S^δ maps X_{T_0, N_0} as a contraction into itself. Since X_{T_0, N_0} is a closed subspace o the Banach space $C^{1,2+\alpha}(\Sigma_{T_0})$ we obtain the assertion from the contraction principle.

(iii) We only prove (28). The proof of (29) can be done similarly. Throughout the following we denote by $C(q)$ a universal constant depending on $q \geq 0$, but being independent of $\delta \in [0,1]$. Due to Lemma 3.3, it suffices to show that

$$\|\hat{F}^\delta(\rho)\|_{C^{0,\alpha}(\Sigma_T)} \leq C(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C(N)T^{(\alpha-\beta)/2}, \quad \rho \in X_{T,N}. \tag{30}$$

From (15) and (17) we conclude that

$$\begin{aligned} \|\hat{F}_1(\rho) + \hat{F}_2(\rho)\|_{C^{0,\alpha}(\Sigma_T)} &\leq \|\hat{F}_1(\rho_0)\|_{C^{0,\alpha}(\Sigma_T)} + \|\hat{F}_1(\rho) - \hat{F}_1(\rho_0)\|_{C^{0,\alpha}(\Sigma_T)} \\ &\quad + \|\hat{F}_2(\rho)\|_{C^{0,\alpha}(\Sigma_T)} \\ &\leq C(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C(N)\|\rho - \rho_0\|_{C^{0,2}(\Sigma_T)} \end{aligned} \tag{31}$$

Invoking the interpolation inequality

$$\|\rho - \rho_0\|_{C^{0,2}(\Sigma_T)} \leq C(N)T^{\alpha/2}, \quad \rho \in X_{T,N}, \tag{32}$$

cf. Lemma 5.1.1 in [9], we conclude from (31) that

$$\|\hat{F}_1(\rho) + \hat{F}_2(\rho)\|_{C^{0,\alpha}(\Sigma_T)} \leq C(\|\rho_0\|_{C^{2+\alpha}(\Sigma)}) + C(N)T^{\alpha/2}, \tag{33}$$

for all $\rho \in X_{T,N}$.

(iv) In order to estimate $\hat{F}_3^\delta(\rho)$, we first remark that, given $\beta \in (0,\alpha)$, we have

$$\|\rho - \rho_0\|_{C^{0,2+\beta}(\Sigma_T)} \leq C(N)T^{(\alpha-\beta)/2}, \quad \rho \in X_{T,N}. \tag{34}$$

This follows from (32) and from the interpolation inequality

$$\|\sigma\|_{C^\beta} \leq C\|\sigma\|_{C^\alpha}^{\beta/\alpha}\|\sigma\|_C^{1-\beta/\alpha} \quad \text{for} \quad \sigma \in C^\alpha(\Sigma),$$

applied to the second derivative $D^2(\rho - \rho_0)$ of ρ.

We next write $\hat{F}_3^\delta(\rho)$ in the following way:

$$\begin{aligned}\hat{F}_3^\delta(\rho)(t) &= \delta\Big(L(\rho(t)) - L(\rho_0)\Big)(\delta - \Delta_{\rho(t)})^{-1} f(\rho(t)) \\ &\quad + \delta L(\rho_0)\Big((\delta - \Delta_{\rho(t)})^{-1} - (\delta - \Delta_{\rho_0})^{-1}\Big) f(\rho(t)) \\ &\quad + \delta L(\rho_0)(\delta - \Delta_{\rho_0})^{-1}\Big(f(\rho(t)) - f(\rho_0)\Big) \\ &\quad + \delta L(\rho_0)(\delta - \Delta_{\rho_0})^{-1} f(\rho_0) \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t),\end{aligned}$$

with $f(\rho(t)) := \overline{K}(\rho(t)) - K(\rho(t))$. It follows from Corollary 3.5 and (7), (9) that

$$||I_4||_{C^{0,\alpha}(\Sigma_T)} \leq C(||\rho_0||_{C^{2+\alpha}(\Sigma)}) \tag{35}$$

To estimate the term I_3 we use Lemma 3.4 and (7) to obtain:

$$\begin{aligned}||I_3(t)||_{C^\alpha(\Sigma)} &\leq C||I_3(t)||_{C^{2+\beta}(\Sigma)} \\ &\leq C(||\rho_0||_{C^{2+\alpha}(\Sigma)})||f(\rho(t)) - f(\rho_0)||_{C^\beta(\Sigma)}.\end{aligned}$$

Similarly as in (15) and (17) we have

$$||f(\rho(t)) - f(\rho_0)||_{C^\beta(\Sigma)} \leq C(N)||\rho(t) - \rho_0||_{C^{2+\beta}(\Sigma)}$$

and thus, due to (34):

$$||I_3||_{C^{0,\alpha}(\Sigma_T)} \leq C(N)T^{(\alpha-\beta)/2}. \tag{36}$$

Using the identity

$$(\delta - \Delta_{\rho_0})^{-1} - (\delta - \Delta_\rho)^{-1} = (\delta - \Delta_{\rho_0})^{-1}(\Delta_{\rho_0} - \Delta_\rho)(\delta - \Delta_\rho)^{-1}$$

and Corollary 3.5 we find

$$||I_2(t)||_{C^\alpha(\Sigma)} \leq C(N)||(\Delta_{\rho_0} - \Delta_{\rho(t)})(\delta - \Delta_{\rho(t)})^{-1} f(\rho(t))||_{C^\beta(\Sigma)}.$$

Based on the representation (12) we can estimate $\Delta_{\rho_0} - \Delta_{\rho(t)}$ in the operator norm as follows

$$||\Delta_{\rho_0} - \Delta_{\rho(t)}||_{\mathcal{L}(C^{2+\beta}(\Sigma), C^\beta(\Sigma))} \leq C(N)||\rho(t) - \rho_0||_{C^{2+\beta}(\Sigma)}.$$

Finally, observing that $f(\rho(t))$ has vanishing average, i.e. that $\overline{f(\rho(t))} = 0$, Lemma 3.4 implies

$$\|(\delta - \Delta_{\rho(t)})^{-1} f(\rho(t))\|_{C^{2+\beta}(\Sigma)} \le C(N).$$

From these considerations and (34) we conclude

$$\|I_2\|_{C^{0,\alpha}(\Sigma_T)} \le C(N) T^{(\alpha-\beta)/2}. \tag{37}$$

It remains to estimate I_1. From (7), (9) and Corollary 3.5 one easily gets

$$\|I_1(t)\|_{C^{\alpha}(\Sigma)} \le C(N) \|\rho(t) - \rho_0\|_{C^{2+\beta}(\Sigma)},$$

and hence, due to (34):

$$\|I_1\|_{C^{0,\alpha}(\Sigma_T)} \le C(N) T^{(\alpha-\beta)/2}. \tag{38}$$

Combing (33), (35)–(38) we get (30), and hence (28). $\square$

4 Convergence and loss of embeddedness

In this section we apply Theorem 3.1 to establish the $C^{1,2+\alpha}$- convergence of solutions Γ^{δ} to the intermediate surface diffusion flow (4)

$$V = \Delta_{\Gamma(t)} \left(\delta - \Delta_{\Gamma(t)}\right)^{-1} H(t) \quad t > 0\,; \quad \Gamma(0) = \Gamma_0 \tag{39}$$

toward the solution Γ^0 of the averaged mean curvature flow

$$V = \overline{H}(t) - H(t) \quad t > 0\,; \quad \Gamma(0) = \Gamma_0 \tag{40}$$

as $\delta \to 0$. Additionally, we show that there exists a smooth embedded surface Γ_0 such that the corresponding solution to (39) loses embeddedness in finite time.

Given $\rho_0 \in W_{a/2}(\Sigma)$ and $\delta \in [0,1]$, we denote in the following by

$$\rho^{\delta} := \rho^{\delta}(\cdot, \rho_0) \in C^{1,2+\alpha}(\Sigma_{T_0})$$

the unique solution to (6) in X_{T_0,N_0}, cf. Theorem 3.1. Recall that this means that the family $\{\Gamma^{\delta}(t)\,;\, t \in [0,T_0]\}$ with

$$\Gamma^{\delta}(t) := \Gamma_{\rho^{\delta}(t)} = \theta_{\rho^{\delta}(t)}(\Sigma) \quad \text{for} \quad t \in [0, T_0],$$

cf. Section 2, is a solution to (39) if $\delta \in (0,1]$ and to (40) in case $\delta = 0$.

Theorem 4.1 *Given $\rho_0 \in W(\Sigma)$, there exists a positive constant $K = K(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$, independent of $\delta \in [0,1]$, such that*

$$\|\rho^\delta - \rho^0\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \leq \delta\, K \quad \text{for } \delta \in [0,1].$$

Proof. In the following $K = K(\|\rho_0\|_{C^{2+\alpha}(\Sigma)})$ stands for a universal constant, which is independent of $\delta \in [0,1]$. Further we use the notation of Theorem 3.1. Then $\varepsilon := 1 - C_2(N_0)T_0^{(\alpha-\beta)/2}$ is positive and independent of $\delta \in [0,1]$. Moreover, writing for simplicity $E := C^{1,2+\alpha}(\Sigma_{T_0})$, we have

$$\begin{aligned}
\|\rho^\delta - \rho^0\|_E &= \|S^\delta(\rho^\delta) - S^0(\rho^0)\|_E \\
&\leq \|S^\delta(\rho^\delta) - S^\delta(\rho^0)\|_E + \|S^\delta(\rho^0) - S^0(\rho^0)\|_E \\
&\leq (1-\varepsilon)\|\rho^\delta - \rho^0\|_E + \|S^\delta(\rho^0) - S^0(\rho^0)\|_E.
\end{aligned}$$

Observing that $\sigma^\delta := S^\delta(\rho^0) - S^0(\rho^0)$ is a solution to

$$\frac{d\,\sigma^\delta}{dt} = A\sigma^\delta + \hat{F}_3^\delta(\rho^0) \quad \text{in} \quad \Sigma_{T_0}, \qquad \sigma^\delta(0) = 0 \quad \text{on} \quad \Sigma,$$

we obtain as in Lemma 3.3 the estimate

$$\|\sigma^\delta\|_E \leq K\|\hat{F}_3^\delta(\rho^0)\|_{C^{0,\alpha}(\Sigma_{T_0})}.$$

Moreover, Lemma 3.4 yields

$$\|\hat{F}_3^\delta(\rho^0)\|_{C^{0,\alpha}(\Sigma_{T_0})} \leq \delta\, K \quad \text{for} \quad \delta \in [0,1],$$

and we find

$$\|\rho^\delta - \rho^0\|_{C^{1,2+\alpha}(\Sigma_{T_0})} \leq \delta\, \frac{K}{\varepsilon} \quad \text{for} \quad \delta \in [0,1],$$

completing the proof. □

Corollary 4.2 *Given $\delta > 0$ there is a $T_1 := T_1(\delta) \in (0, T_0]$ and a closed compact embedded C^∞-hypersurface $\Gamma_0 = \Gamma_0(\delta)$ such that the solution $\{\Gamma^\delta(t)\,,\, t \in [0,T_1]\}$ to (39) with initial data Γ_0 loses its embeddedness in finite time although $M^\delta := \cup_{t\in(0,T_0]}(\{t\} \times \Gamma^\delta(t))$ stays a $C^{1,2+\alpha}$-manifold.*

Proof. (i) Theorem 2 in [10] ensures the existence of a closed compact embedded smooth hypersurface Γ_0 and a number $T > 0$ such that there is smooth solution $\{\overline{\Gamma}(t)\,;\, t \in [0,T]\}$ to (40) with initial data Γ_0 losing its embeddedness in finifte time.
More precisely, there exist a smooth compact closed immersed hypersurface Σ and a smooth function σ_0 such that Γ_0 is the embedded graph of σ_0 in normal direction of over Σ with the following properties: Let $\overline{\rho}(t,\sigma_0)$, $t \in [0,T]$, denote the signed distance of $\overline{\Gamma}(t)$ to Σ. Then there are points $s_0^\pm \in \Sigma$ with $s_0^- \neq s_0^+$ and neighbourhoods $U^\pm$ of $s_0^\pm$ in Σ with $U^- \cap U^+ = \emptyset$ such that $\overline{\rho}(0,\sigma_0)(s_0^\pm) = \sigma_0(s_0^\pm) < 0$ and such that $\overline{\Gamma}(t)$ is not an embedded hypersurface, provided $\overline{\rho}(t,\sigma_0)(s_0^\pm) > 0$. With this notation it is shown in [10] that there is a $\mu > 0$ and $0 < t_0 < t_1 < T$ such that $\overline{\rho}(t,\sigma_0)(s_0^\pm) > \mu$ for $t \in (t_0,t_1)$.

(ii) It follows from Theorem 4.1 that there exist positive constants K and $T_* \leq T$, both independent of $\delta \in (0,1]$, such that

$$\|\rho^\delta(\cdot,\sigma_0) - \overline{\rho}(\cdot,\sigma_0)\|_{C^{1,2+\alpha}(\Sigma_{T^*})} \leq \delta\, K \quad \text{for} \quad \delta \in (0,1].$$

Clearly, choosing $\delta \in (0,1]$ small enough, we have that $\rho^\delta(t,\sigma_0)(s_0^\pm) > \mu/2$ for $t \in (t_0,t_1)$. Hence, given $t \in (t_0,t_1)$, the hypersurface $\Gamma^\delta(t)$ cannot be embedded. Observe that Theorem 3.1 implies that M^δ is for any $\delta \in (0,1]$ a $C^{1,2+\alpha}$-manifold.
(iii) To get the result for an arbitrary $\delta \in (0,\infty)$, we introduce the scaling $\tilde{t} = t/\lambda^2$, $\tilde{x} = x/\lambda$ for $\lambda > 0$. Then the normal velocity, the mean curvature, and the Laplace-Beltrami operator have the scaling properties $\widetilde{V} = \lambda V$, $\widetilde{H} = \lambda H$, and $\widetilde{\Delta} = \lambda^2 \Delta$, respectively, and hence (39) becomes

$$\widetilde{V}(\tilde{t}) = \widetilde{\Delta}_{\Gamma(\tilde{t})}\left(\lambda^2\delta - \widetilde{\Delta}_{\Gamma(\tilde{t})}\right)^{-1}\widetilde{H}(\tilde{t}) \quad \tilde{t} > 0; \quad \widetilde{\Gamma}(0) = \widetilde{\Gamma}_0. \tag{41}$$

More precisely, letting

$$\rho^\delta_\lambda(\tilde{t},\tilde{s}) := \rho^\delta(\lambda^2\tilde{t},\lambda\tilde{s}), \quad (\tilde{t},\tilde{s}) \in [0,\lambda^{-2}T_0] \times \Sigma^\lambda,$$

where Σ^λ stands for the scaled reference surface Σ, it is easily verified that ρ^δ_λ is the unique solution to

$$\frac{d\,\rho^\delta_\lambda}{d\tilde{t}} = F^{\lambda^2\delta}(\rho^\delta_\lambda)(\tilde{t},\tilde{s}), \quad (\tilde{t},\tilde{s}) \in [0,\lambda^{-2}T_0] \times \Sigma^\lambda.$$

Thus, given $\delta \in (0,1]$, we find $\lambda = \lambda(\delta) > 0$ sufficiently small, an embedded hypersurface $\widetilde{\Gamma}_0$, and $\tilde{t}_* \in (0, \lambda^{-2}T_0)$ such that the solution $\widetilde{\Gamma}^{\lambda^2\delta}$ to (41) is not embedded at $\tilde{t}_*$. Rescaling, this means that the solution Γ^δ to (39) with the embedded initial surface Γ_0, obtained by rescaling $\widetilde{\Gamma}_0$, is not embedded at $t_* := \lambda^2\tilde{t}_* \in (0, T_0)$. $\square$

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