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ON GOUVÊA'S CONJECTURE IN THE UNOBSTRUCTED CASE

ATSUSHI YAMAGAMI

Abstract. In this article, for a residual Galois representation defined over an arbitrary finite field, Gouvêa's conjecture which says that the universal deformation ring is isomorphic to a certain Hecke algebra is proven in the unobstructed case.

0. Introduction

Let p be an odd prime number. We denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. We fix once and for all an isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$. Given a classical modular form f, we transport the Fourier coefficients of f in $\mathbb{C}$ to $\mathbb{C}_p$ via our fixed isomorphism. In this way we regard f as being defined over $\mathbb{C}_p$. Let $\boldsymbol{k}$ be a finite field of characteristic p and N a positive integer which is prime to p. Let

$$\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\boldsymbol{k})$$

be an absolutely irreducible residual representation, where $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$. We assume that $\bar{\rho}$ is associated to a classical eigenform f of tame level N defined over the ring of integers $\mathcal{O}$ of a totally ramified finite extension K over the fraction field of the Witt ring $W(\boldsymbol{k})$ over $\boldsymbol{k}$. Let $S = \{\text{the prime divisors } l \text{ of } Np\} \cup \{\infty\}$. Then we know that $\bar{\rho}$ factors through the Galois group G_S of the maximal Galois extension of $\mathbb{Q}$ unramified outside S, and we consider the deformation problem of the residual representation

$$\bar{\rho} : G_S \to \mathrm{GL}_2(\boldsymbol{k}).$$

In [9], Mazur showed that there exist a complete Noetherian local ring $\boldsymbol{R}(\bar{\rho}, S)$ with residue field $\boldsymbol{k}$ and a deformation of $\bar{\rho}$

$$\boldsymbol{\rho}^{\mathrm{univ}} : G_S \to \mathrm{GL}_2(\boldsymbol{R}(\bar{\rho}, S))$$

such that any deformation of $\bar{\rho}$ to a complete Noetherian local ring A with residue field $\boldsymbol{k}$

$$\rho : G_S \to \mathrm{GL}_2(A)$$

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is obtained from $\boldsymbol{\rho}^{\mathrm{univ}}$ via a unique homomorphism $\boldsymbol{R}(\bar{\rho}, S) \to A$.

On the other hand, we regard f as being in

$$\boldsymbol{V}_{\mathrm{par}}(\mathcal{O}, N) = \boldsymbol{V}_{\mathrm{par}}(W(\boldsymbol{k}), N)\hat{\otimes}\mathcal{O}$$

which is the ring of Katz' parabolic p-adic modular functions of tame level N defined over $\mathcal{O}$. Let $\boldsymbol{T}_0^{\star}(W(\boldsymbol{k}), N)$ be the restricted Hecke algebra on $\boldsymbol{V}_{\mathrm{par}}(W(\boldsymbol{k}), N)$ (cf. [5]). We denote by $\boldsymbol{T}(\bar{\rho}, N)$ the completion of $\boldsymbol{T}_0^{\star}(W(\boldsymbol{k}), N)$ at the maximal ideal $\mathbf{m}_f$ which is the kernel of the map

$$\boldsymbol{T}_0^{\star}(W(\boldsymbol{k}), N) \to \mathcal{O} \xrightarrow{\mathrm{red.}} \boldsymbol{k}$$

defined by f. Then it was shown in [5] that there exists a deformation of $\bar{\rho}$ to $\boldsymbol{T}(\bar{\rho}, N)$

$$\boldsymbol{\rho}^{\mathrm{mod}} : G_S \to \mathrm{GL}_2(\boldsymbol{T}(\bar{\rho}, N))$$

which is universal among deformations of $\bar{\rho}$ associated to Katz' p-adic modular functions. Thus we have a natural surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N).$$

In [5], Gouvêa gave the following

Conjecture ([5, Question III.5])**.** The natural homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism.

Remark 0.1. Although Gouvêa constructed the universal modular deformation $\boldsymbol{\rho}^{\mathrm{mod}}$ on the assumption $p \geq 7$, we can construct it for $p = 3, 5$. For the details, see [18, Remark 0.1].

The conjecture above was solved by Gouvêa and Mazur in [7] for a residual modular representation $\bar{\rho}$ for which the deformation problem is "unobstructed" and which is associated to a classical eigenform f of level p defined over $\mathbb{Z}_p$ with "non-critical slope" ([7, Proposition 2]). They used Mazur's theory of "infinite ferns" of level p and showed the density of modular representations in the universal deformation space. This result was generalized to the case of level Np in [18, Theorem 2.2] keeping the assumption that the deformation problem for $\bar{\rho}$ is unobstructed and that f is defined over $\mathbb{Z}_p$ with non-critical slope.

In this paper, we shall prove the conjecture above under the assumption that the residual representation $\bar{\rho}$ is defined over an arbitrary finite field $\boldsymbol{k}$ in the "unobstructed and non-critical slope" case. Namely, we show the following

Main Theorem. *Let $p > 2$ be a prime number, $\boldsymbol{k}$ a finite field of characteristic p, $k \geq 2$ an integer and N a positive integer prime to p. Let $S = \{$the prime divisors l of $Np\} \cup \{\infty\}$ and G_S the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S. We assume that an absolutely irreducible residual representation*

$$\bar{\rho} : G_S \to \mathrm{GL}_2(\boldsymbol{k})$$

is associated to a classical eigenform f of type $(Np, k, 1)_{\mathcal{O}}$ which is new away from p, where $\mathcal{O}$ is the ring of integers of a totally ramified finite extension K over the fraction field of the Witt ring $W(\boldsymbol{k})$ over $\boldsymbol{k}$. Further, assume that

$$0 < \mathrm{ord}_p(\lambda_p) < k - 1 \quad \text{and} \quad \lambda_p^2 \neq p^{k-1},$$

where λ_p is the U_p-eigenvalue of f. If the deformation problem for $\bar{\rho}$ is unobstructed, then the natural surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N)$$

is an isomorphism.

Remark 0.2. (1) For the notion of "new away from p," see Section 3.

(2) We call $\mathrm{ord}_p(\lambda_p)$ the *slope* of f. If the weight of f is k, then we have

$$0 \leq \mathrm{ord}_p(\lambda_p) \leq k - 1.$$

We say that the eigenform f has *critical slope* when the slope of f is equal to 0 or $k-1$, and if not, we say that f has *non-critical slope.*

(3) We mean by "the deformation problem for $\bar{\rho}$ is unobstructed" that

$$\dim_{\boldsymbol{k}} H^2(G_S, \mathrm{Ad}(\bar{\rho})) = 0,$$

where $\mathrm{Ad}(\bar{\rho})$ is the $\boldsymbol{k}$-vector space of 2×2 matrices with entries in $\boldsymbol{k}$, on which G_S-action is given by

$$\sigma \cdot M = \bar{\rho}(\sigma) M \bar{\rho}(\sigma)^{-1} \quad (\sigma \in G_S,\ M \in \mathrm{Ad}(\bar{\rho})).$$

Mazur investigated that how often the deformation problem of a modular residual representation can be unobstructed in [10, Corollary 2 of Section 11].

By [1, Lemma 5.7], we know that an arbitrary residual modular representation must be associated to some eigenform which is new away from p and of properly large weight with non-critical slope. Therefore we can see easily the following

Corollary. *Gouvêa's conjecture is true under the assumption that the deformation problem is unobstructed.*

Remark 0.3. In [1], Böckle also showed Gouvêa's conjecture in the special case by using the isomorphisms between the "small rings" (which are quotients of the "big rings" $\boldsymbol{R}(\bar{\rho}, S)$ and $\boldsymbol{T}(\bar{\rho}, N)$) proven by Wiles [17] and Taylor-Wiles [16]. Gouvêa's conjecture should be proven under the conditions that (a) $\boldsymbol{R}(\bar{\rho}, S)/(p)$ is of dimension 3, (b) every irreducible component of $\boldsymbol{R}(\bar{\rho}, S)$ contains a modular point of finite slope and that (c) the modular point of (b) is smooth. Böckle showed these conditions under the assumptions of Wiles and Taylor-Wiles. His method is very remarkable for the point that it can be used in the "obstructed" case. Incidentally, the conditions (a)-(c) are trivial in the unobstructed case.

We prove the main theorem by similar arguments of Gouvêa and Mazur as in [7]. We need to generalize Mazur's theory of "infinite ferns" in [10] to the case that the universal deformation space is defined over K. In Section 1, we shall show some lemmas on K-analytic mappings. These lemmas are very essential to the study of the universal deformation space over K in Section 2. In Section 3, we shall construct an infinite fern of level Np over K and generalize the method of Gouvêa and Mazur of [7] to the K-analytic case. The main theorem is proven in Section 4, and we apply this theorem to Gouvêa's conjecture on "controlling the conductor" in Section 5.

1. Some lemmas on K-analytic mappings

In this section, we study some properties of K-analytic mappings. We use the term "K-analytic" in the sense of Serre [14]. The notion of K-analytic manifolds and K-analytic mappings are described in [14, Chapter II.III], and we omit the details about them.

Lemma 1.1. *Let $f : D \to K$ be a K-analytic function on a disc D and E a subset of D which has at least one accumulation point d in D. If $f|_E = 0$ then there exists $r > 0$ such that*

$$B_K(d; r)(= \{x \in K |\ |d - x| < r\}) \subset D$$

and $f|_{B_K(d;r)} = 0$, where $|\quad|$ is the standard absolute value on K which is normalized so that $|p| = 1/p$.

Proof. This lemma is what is called the "uniqueness theorem" in the theory of complex analysis, and we can prove it by the same way as in the complex version. □

Lemma 1.2. *If a function $f : D \to K$ on a disc D can be written as*

$$f(w) = \sum_{n\geq 0} a_n w^n \quad (a_n \in K, w \in D),$$

then f is K-analytic.

Proof. We fix an element $d \in D$. Since the power series $\sum_{n\geq 0} a_n d^n$ converges in K,

$$b_n = \sum_{m\geq n} \binom{m}{n} a_m d^{m-n}$$

also converges by [14, Lemma in p. 70]. Hence f can be written as

$$f(w) = \sum_{n\geq 0} a_n w^n = \sum_{n\geq 0} b_n (w-d)^n$$

and is K-analytic. $\square$

Next we consider "strictly analytic mappings" for the universal deformation space in the sense of Mazur ([10, Section 16]). For the residual Galois representation $\bar{\rho}$ given in the main theorem, we set

$$X = \mathrm{Hom}_{W(\boldsymbol{k})\text{-alg}}(\boldsymbol{R}(\bar{\rho}, S), \mathcal{O}),$$

and call it the *universal deformation space* for $\bar{\rho}$. On the assumption that the deformation problem for $\bar{\rho}$ is unobstructed, it was shown by Mazur in [9] that the universal deformation ring $\boldsymbol{R}(\bar{\rho}, S)$ is isomorphic to the power series ring of three variables over $W(\boldsymbol{k})$:

$$\boldsymbol{R}(\bar{\rho}, S) \cong W(\boldsymbol{k})[\![X_1, X_2, X_3]\!].$$

Then we have a bijection

$$X \xrightarrow{\sim} \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m}, \quad x \mapsto (x(X_1), x(X_2), x(X_3)),$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. This enables us to regard X as a K-analytic manifold of 3-dimension in the sense of Serre [14]. (We study some properties of X in the next section.)

Definition 1.1 (Mazur [10, Section 16]). Let D be a disc in K. A mapping $f : D \to X$ is said to be *strictly analytic* if for each $\tau \in \boldsymbol{R}(\bar{\rho}, S)$, f has a convergent power series expansion

$$f(w)(\tau) = \sum_{n\geq 0} a_n(\tau) w^n \quad (a_n(\tau) \in K, w \in D).$$

Lemma 1.3. *A strictly analytic mapping $f : D \to X$ is a K-analytic mapping of K-analytic manifolds.*

Proof. Let $\tilde{f}$ be the composition

$$D \xrightarrow{f} X \xrightarrow{\sim} \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m}.$$

Since f is strictly analytic, we have

$$f(w)(X_i) = \sum_{n\geq 0} a_n(X_i)w^n \quad (w \in D,\ i = 1, 2, 3).$$

Hence we obtain

$$\tilde{f}(w) = (\sum_{n\geq 0} a_n(X_1)w^n, \sum_{n\geq 0} a_n(X_2)w^n, \sum_{n\geq 0} a_n(X_3)w^n),$$

and see that f is K-analytic by Lemma 1.2. □

2. The universal deformation space

Let $\bar{\rho}$ be the residual Galois representation given in the main theorem. As explained in the previous section, the universal deformation space

$$X = \mathrm{Hom}_{W(\boldsymbol{k})\text{-alg}}(\boldsymbol{R}(\bar{\rho}, S), \mathcal{O})$$

can be regarded as a K-analytic manifold of 3-dimension. In this section, we generalize some properties of X, which are shown by Mazur [10] in the $\mathbb{Q}_p$-analytic case, to the K-analytic case. These are very important for us when we show the density of modular representations in X in Section 4.

2.1. Hodge-Tate-Sen weights

Each point $x \in X$ corresponds to a deformation of $\bar{\rho}$ to $\mathcal{O}$

$$\rho_x : G_S \to \mathrm{GL}_2(\mathcal{O}).$$

Since $p \in S$, we can regard the absolute Galois group G_K of K as a subgroup of G_S via a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. We restrict ρ_x to G_K and consider the representation

$$\rho_x : G_K \to \mathrm{GL}_2(K).$$

In [12], Sen studied the "$\mathbb{C}_p$-representation" $W_x \otimes \mathbb{C}_p$ for the representation space W_x of ρ_x and defined the *Sen operator* φ_x on a certain space W_∞ associated to this $\mathbb{C}_p$-representation. If W_x is of Hodge-Tate type, then the pair $\{r_x, s_x\}$ of eigenvalues of the Sen operator φ_x are coincide with the Hodge-Tate weights of W_x. Sen called the pair $\{r_x, s_x\}$ the "generalized Hodge-Tate weights." In this article, we call them the *Hodge-Tate-Sen weights* following Mazur ([10, Section 6]).

Sen showed that we can take the representation matrix of ρ_x in K ([12, Theorem 5]), so we can define the following functions on X:

$$\delta : X \to K, \quad x \mapsto r_x + s_x,$$
$$\mathcal{S} : X \to K, \quad x \mapsto r_x \cdot s_x.$$

Theorem 2.1. *The functions δ and $\mathcal{S}$ are K-analytic.*

Proof. The result of Sen is applied to the case that we consider deformations of $\bar{\rho}$ not only to $W(\boldsymbol{k})$ but also to $\mathcal{O}$ (cf. [13, Corollary]). □

Definition 2.1. We set

$$X_0 = \{x \in X | \mathcal{S}(x) = 0\} = \{x \in X | r_x = 0 \text{ or } s_x = 0\},$$
$$X_{00} = \{x \in X | \delta(x) = \mathcal{S}(x) = 0\} = \{x \in X | r_x = s_x = 0\},$$

and we call X_0 the *Sen-null space.*

2.2. **Twists by wild characters**

For the ring of integers $\mathcal{O}$ of K, we put $\mathcal{U} = 1 + \mathfrak{m} \subset \mathcal{O}^\times$. It is well-known that $\mathcal{U}$ is a finitely generated pro-p abelian group, so we can decompose this group as

$$\mathcal{U} = (\text{pro-}p \text{ free}) \times (\text{finite})$$

(cf. [11, Theorem 4.3.4 (b)]). We denote the pro-p free part of $\mathcal{U}$ by $\mathcal{U}_{\text{free}}$ and let

$$\Psi = \mathrm{Hom}_{\text{cont}}(G_S, \mathcal{U}_{\text{free}})$$

be the group of continuous homomorphisms from G_S to $\mathcal{U}_{\text{free}}$. By Class Field Theory, we know that the maximal pro-p abelian torsion-free quotient of G_S is isomorphic to $\Gamma = 1 + p\mathbb{Z}_p$ via the p-adic cyclotomic character. Therefore we can identify Ψ with $\mathrm{Hom}_{\text{cont}}(\Gamma, \mathcal{U}_{\text{free}})$, and under the isomorphism

$$\mathrm{Hom}_{\text{cont}}(\Gamma, \mathcal{U}_{\text{free}}) \xrightarrow{\sim} \mathcal{U}_{\text{free}} (\hookrightarrow K),$$
$$\psi \mapsto \psi(\gamma),$$

we regard Ψ as a K-analytic group of 1-dimension. Here γ stands for a topological generator of Γ.

Now we shall see that the K-analytic group Ψ acts on the universal deformation space X. For $\psi \in \Psi$ and $x \in X$, the twist

$$\rho_x \otimes \psi : G_S \to \mathrm{GL}_2(\mathcal{O})$$

is also a deformation of $\bar{\rho}$ to $\mathcal{O}$, and we denote by $x \circ \psi$ the corresponding point of X to it.

Lemma 2.2. *The action of Ψ on X defined by*

$$\pi : \Psi \times X \to X, \quad (\psi, x) \mapsto x \circ \psi$$

is K-analytic.

Proof. Now we fix an isomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \cong W(\boldsymbol{k})[\![X_1, X_2, X_3]\!].$$

The chart map of the K-analytic manifold $\Psi \times X$ of 4-dimension is

$$\Psi \times X \xrightarrow{\sim} \mathcal{U}_{\mathrm{free}} \times \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m},$$
$$(\psi, x) \mapsto (\psi(\gamma), x(X_1), x(X_2), x(X_3)).$$

So we need to show that for each $i = 1, 2, 3$, $(x \circ \psi)(X_i)$ can be written as a convergent power series of $x(X_1), x(X_2), x(X_3)$ and $\psi(\gamma)$. The character

$$\psi : \Gamma \to \mathcal{U}_{\mathrm{free}} \hookrightarrow \mathcal{O}^{\times}$$

is a deformation of the trivial residual character

$$\bar{\varepsilon} : \Gamma \to \boldsymbol{k}^{\times}, \quad \gamma \mapsto 1$$

to $\mathcal{O}$. By [8, Proposition 2.22], we know that the universal deformation ring of $\bar{\varepsilon}$ is the formal power series $W(\boldsymbol{k})[\![T]\!]$ of one variable over $W(\boldsymbol{k})$ and the universal deformation is

$$\varepsilon^{\mathrm{univ}} : \Gamma \to W(\boldsymbol{k})[\![T]\!]^{\times}, \quad \gamma \mapsto 1 + T.$$

Hence we see that the $W(\boldsymbol{k})$-algebra homomorphism which corresponds to the deformation ψ is

$$W(\boldsymbol{k})[\![T]\!] \to \mathcal{O}, \quad T \mapsto \psi(\gamma) - 1.$$

In this way, we obtain the deformation $\boldsymbol{\rho}^{\mathrm{univ}} \otimes \varepsilon^{\mathrm{univ}}$ of $\bar{\rho}$ to $A = W(\boldsymbol{k})[\![X_1, X_2, X_3]\!] \hat{\otimes} W(\boldsymbol{k})[\![T]\!]$ and the corresponding homomorphism

$$\varphi : W(\boldsymbol{k})[\![X_1, X_2, X_3]\!] \to A.$$

Since the homomorphism $x \circ \psi$ corresponds to the deformation $\rho_x \otimes \psi$, we are able to write $(x \circ \psi)(X_i)$ as a convergent power series of $x(X_1), x(X_2), x(X_3)$ and $\psi(\gamma)$ by substituting them for the image of X_i under φ. $\square$

Next we consider how the Hodge-Tate-Sen weights associated to points of X vary under the action of Ψ. We restrict an element $\psi \in \Psi$ to the subgroup G_K of G_S. The Hodge-Tate-Sen weight of the character

$$\psi : G_K \to \mathcal{U}_{\mathrm{free}} \hookrightarrow K^{\times}$$

is nothing but the representation matrix $t_\psi \in K$ of the Sen operator associated to ψ.

Lemma 2.3. *We denote by (r_x, s_x) the Hodge-Tate-Sen weights for $x \in X$, then the Hodge-Tate-Sen weights of $x \circ \psi$ are $(r_x + t_\psi, s_x + t_\psi)$.*

Proof. Let φ_x and φ_ψ be the Sen operators associated to ρ_x and ψ, respectively. By [12, Remark (4) of p. 99], we see that the Sen operator associated to $\rho_x \otimes \psi$ is

$$\varphi_x \otimes 1 + 1 \otimes \varphi_\psi.$$

Denoting the representation matrix of φ_x by M_x, we see that there exists $A \in \mathrm{GL}_2(\overline{K})$ such that

$$A^{-1} M_x A = \begin{pmatrix} r_x & * \\ 0 & s_x \end{pmatrix}.$$

Since the representation matrix of $\varphi_x \otimes 1 + 1 \otimes \varphi_\psi$ is $M_x + \begin{pmatrix} t_\psi & 0 \\ 0 & t_\psi \end{pmatrix}$, and we have

$$A^{-1}(M_x + \begin{pmatrix} t_\psi & 0 \\ 0 & t_\psi \end{pmatrix})A = \begin{pmatrix} r_x + t_\psi & * \\ 0 & s_x + t_\psi \end{pmatrix},$$

the Hodge-Tate-Sen weights of $x \circ \psi$ is $(r_x + t_\psi, s_x + t_\psi)$. $\square$

Let

$$\pi_0 : \Psi \times X_0 \to X$$

be the restriction of K-analytic action π to the Sen-null space. We take an element (ψ, y) of $\pi_0^{-1}(x)$ for $x \in \mathrm{Im}(\pi_0)$. We note that $\mathcal{S}(y) = r_y \cdot s_y = 0$ since $y \in X_0$. By Lemma 2.3, we have

$$\mathcal{S}(x) = (r_y + t_\psi)(s_y + t_\psi) = \delta(x) t_\psi - t_\psi^2,$$

thus

$$t_\psi^2 - \delta(x) t_\psi + \mathcal{S}(x) = 0, \tag{1}$$

where t_ψ is the Hodge-Tate-Sen weight of ψ.

Now we prove a lemma on the Hodge-Tate-Sen weights of characters in Ψ:

Lemma 2.4. *For $\psi_1, \psi_2 \in \Psi$, $\psi_1 = \psi_2$ if and only if $t_{\psi_1} = t_{\psi_2}$.*

Proof. We assume that $t_{\psi_1} = t_{\psi_2}$. We denote by W_1 and W_2 the $\mathbb{C}_p$-representations associated to ψ_1 and ψ_2 (restricted to G_K), respectively. Then we know that there exists an isomorphism $f : W_1 \xrightarrow{\sim} W_2$ of $\mathbb{C}_p$-representations by [12, Proposition 9 (a)]. Hence we have $\psi_1 = \psi_2$ on G_K. Let Γ' be the image of G_K under the natural projection $G_S \to \Gamma$ and γ^s ($s \in \mathbb{Z}_p$) a topological generator of Γ', where γ is a topological generator of Γ. Then we have

$$\psi_1(\gamma)^s = \psi_2(\gamma)^s \quad \text{in} \quad \mathcal{U}_{\text{free}},$$

since ψ_1 and ψ_2 are continuous. Because $\mathcal{U}_{\text{free}}$ is pro-p free, we see that $\psi_1(\gamma) = \psi_2(\gamma)$. Hence $\psi_1 = \psi_2$ on G_S. $\square$

We return to the equation (1). The weight t_ψ is one of the roots of this equation, so the number of characters which can be taken as ψ is at most 2 by Lemma 2.4. On the other hand, in the pair (ψ, y), y should be determined by ψ, since we have $y = x \circ \psi^{-1}$. Therefore we see that the cardinality of the set $\pi_0^{-1}(x)$ is 1 or 2 for $x \in \mathrm{Im}(\pi_0)$. If this consists of the one element (ψ, y), then we have

$$\delta(x)^2 - 4\mathcal{S}(x) = 0.$$

Since

$$\delta(x) = \delta(y) + 2t_\psi \quad \text{and} \quad \mathcal{S}(x) = t_\psi^2 + \delta(y)t_\psi,$$

we get that $\delta(y)^2 = 0$, i.e., $y \in X_{00}$. Summing up, we obtain the following

Proposition 2.5. *The K-analytic mapping $\pi_0 : \Psi \times X_0 \to X$ is unramified off the locus $\Psi \times X_{00}$.*

3. Infinite ferns

In this section, we construct an infinite fern of level Np over K in the universal deformation space X following Mazur [10]. Let $\bar{\rho}$ be the residual representation given in the main theorem as in the previous section.

We recall that the classical eigenform f associated to $\bar{\rho}$ is of type $(Np, k, 1)_{\mathcal{O}}$, new away from p, of non-critical slope $\alpha = \mathrm{ord}_p(\lambda_p)$ with $(\lambda_p)^2 \neq p^{k-1}$, where λ_p is the U_p-eigenvalue of f. Here we mean by the term "of type $(Np, k, 1)_{\mathcal{O}}$" that f is a classical cusp form of level Np, weight k with trivial character defined over $\mathcal{O}$ whose first Fourier coefficient is equal to 1 and which is an eigenform for the Hecke operators T_l for all prime numbers $l \nmid Np$ and for the Atkin operators U_q for all prime numbers $q \mid Np$.

To construct an infinite fern, the following theorem by Coleman is very essential:

Theorem 3.1 ([3, Corollary B5.7.1]). *Let g be a classical eigenform of type $(Np, k, 1)_{\mathcal{O}}$, new away from p, of slope $\beta < k-1$ and $\lambda^2 \neq p^{k-1}$, where λ is the U_p-eigenvalue of g. Then there exist a disc D in K and K-analytic functions*

$$a_n : D \to \mathcal{O}$$

for each $n \geq 1$ such that for any $k' \in D \cap \mathbb{Z}$, $k' > \beta + 1$, the formal power series over $\mathcal{O}$

$$f_{k'}(q) = \sum_{n \geq 1} a_n(k') q^n$$

is the Fourier expansion of a classical eigenform of type $(Np, k', 1)_{\mathcal{O}}$, *new away from* p *and of slope* β, *and* $f_k = g$.

In particular, our eigenform f is a member of a *Coleman family* $\{f_d\}_{d\in D}$ which is parametrized by some disc D. Using this family, we need to draw a curve of modular representations in X which is called "modular arc" in order to construct an infinite fern.

Definition 3.1 (cf. [10, Section 16]). We say that a quadruplet $(D, E; z, u)$ is a *K-analytic family of modular representations of level Np, trivial character and slope α* if D is a disc, E is a subset of $D \cap \mathbb{Z}$ which has an accumulation point in D,

$$z : D \to X$$

is a strictly analytic mapping and

$$u : D \to \mathcal{O}$$

is a K-analytic function such that for each $k' \in E$, the deformation $\rho_{k'}$ corresponding to the point $x_{k'} = z(k')$ is associated to some eigenform of level Np, weight k' and trivial character with U_p-eigenvalue $u(k')$, and the slope $\mathrm{ord}_p(u(k'))$ is the constant α. We call the image $C = z(D)$ the *modular arc*.

Lemma 3.2. *If we shrink* D *enough, then we have* $C \subset X_0$ *and* $(\delta \circ z)(d) = d - 1$ *for any* $d \in D$.

Proof. By a result of Faltings [4], the Hodge-Tate-Sen weights of $x_{k'}$ are $(0, k' - 1)$ for each $k' \in E$. Therefore we see that for $k' \in E$,

$$\mathcal{S}(z(k')) = 0 \quad \text{and} \quad \delta(z(k')) = k' - 1.$$

Since $\mathcal{S}, \delta$ and z are K-analytic by Theorem 2.1 and Lemma 1.3, we know that if we shrink D enough, then $(\mathcal{S} \circ z)(d) = 0$ and $(\delta \circ z)(d) = d - 1$ for $d \in D$ by Lemma 1.1. Because $\mathcal{S} \circ z = 0$ implies that $C \subset X_0$, the lemma is proven. □

By this lemma, we see that the surjective K-analytic mapping

$$z : D \to C$$

has the inverse mapping defined by

$$C \ni z(d) \to \delta(z(d)) + 1 (= d) \in D,$$

and this inverse is also K-analytic. Hence $C = z(D)$ is a K-analytic manifold of 1-dimension, since so is D (cf. [14, Section II.III.10]).

The following two essential lemmas on K-analytic families of modular representations are proven by similar arguments as in [10]:

Lemma 3.3. *Let $(D, E; z, u)$ and $(D', E'; z', u')$ be K-analytic families of slope α and α', respectively. Let C and C' be their respective images in X_0. If $C \cap C'$ is infinite, then the K-analytic families $(D \cap D', E \cap E'; z, u)$ and $(D \cap D', E \cap E'; z', u')$ are equal.*

Lemma 3.4. *Let $(D, E; z, u)$ and $(D', E'; z', u')$ be K-analytic families of slope α and α', respectively. Let C and C' be their respective images in X_0. If $\alpha \neq \alpha'$, then the cardinality of $z(E) \cap z'(E')$ is at most* 1.

It follows from Lemma 3.3 that any modular arc $C \subset X$ is the image of a unique K-analytic family. So we define the *slope* of a modular arc as the slope of the unique family of which it is the image.

We now construct a K-analytic family $(D, E; z, u)$ following Mazur [10, Section 16], using the disc D and K-analytic functions a_n which we obtain by Theorem 3.1 for the eigenform f given in the main theorem. For $d \in D$ and a prime number $l \nmid Np$, we put

$$z(d)(\tau_l) = a_l(d),$$

where $\tau_l = \mathrm{Trace}(\boldsymbol{\rho}^{\mathrm{univ}}(\mathrm{Frob}_l))$ and Frob_l is a Frobenius element at l. It is known by Carayol [2] that the universal deformation ring $\boldsymbol{R}(\bar{\rho}, S)$ is generated by $\mathrm{Trace}(\boldsymbol{\rho}^{\mathrm{univ}}(G_S))$ over $W(\boldsymbol{k})$. Therefore we can extend $z(d)$ to a $W(\boldsymbol{k})$-algebra homomorphism $\boldsymbol{R}(\bar{\rho}, S) \to \mathcal{O}$ and obtain a mapping of D to X

$$z : D \to X, \quad d \mapsto z(d)$$

by the Chebotarev density theorem. Since a_l is a K-analytic function on D for any l, we see that the mapping z is strictly analytic. Putting

$$E = D \cap \{k' \in \mathbb{Z} | k' > \alpha + 1\} \quad \text{and} \quad u = a_p,$$

we obtain a K-analytic family of modular representations $(D, E; z, u)$ containing the representation associated to f.

To construct an infinite fern, we now recall "twins" coming from a newform of level N following [10, Section 12]. Let φ be a newform of type $(N, k, 1)_{\mathbb{C}_p}$ and A_p the T_p-eigenvalue of φ. We now construct a basis which diagonalizes the operator U_p acting on the 2-dimensional space of oldforms of level Np spanned by φ and $\varphi|B_p$, where B_p is the standard "degeneracy operator" which acts on the Fourier expansion of φ as

$$(\varphi|B_p)(q) = \varphi(q^p).$$

From the relation

$$U_p = T_p - p^{k-1}B_p,$$

we know that the characteristic polynomial of U_p acting on that space is

$$X^2 - A_p X + p^{k-1}.$$

Therefore if the two roots λ_1 and λ_2 of this polynomial are distinct, then the following two eigenforms of type $(Np, k, 1)_{\mathbb{C}_p}$

$$g_1 = \varphi - \lambda_2 \cdot \varphi | B_p,$$
$$g_2 = \varphi - \lambda_1 \cdot \varphi | B_p$$

form a basis of that space. Their T_l-eigenvalues for prime numbers $l \nmid Np$ and U_q-eigenvalues for prime numbers $q \mid N$ are equal to those of φ, and U_p-eigenvalues are λ_1 and λ_2, respectively. In particular, from the equality of the T_l-eigenvalues for $l \nmid Np$, the associated Galois representations to g_1 and g_2 are both equivalent to the representation associated to φ. We call g_1 and g_2 the *twins* coming from φ.

Let α_1 and α_2 be the slopes of g_1 and g_2, respectively. Since we have

$$\lambda_1 \lambda_2 = p^{k-1},$$

the slopes satisfy the relation

$$\alpha_1 + \alpha_2 = k - 1.$$

We say that a classical eigenform g of level Np is *new away from p* when g is a newform of level Np or g belongs to a space of oldforms spanned by φ and $\varphi | B_p$ for a newform φ of level N. In particular, the twins coming from a newform of level N are new away from p. Conversely, we see easily the following

Proposition 3.5. *Let g be a classical eigenform of type $(Np, k, 1)_{\mathcal{O}}$. Suppose that the slope of g is equal to neither $(k-1)/2$ nor $(k-2)/2$ and that g is new away from p. Then g has a twin defined over $\mathcal{O}$.*

Now we shall sketch the construction of an infinite fern in X_0 containing the representation associated to the eigenform f given in the main theorem. We put

$$\tilde{E} = \{k' \in E | k' \neq 2\alpha + 1,\ 2\alpha + 2\}.$$

Let $(D, \tilde{E}; z, u)$ be the K-analytic family of modular representations which we obtained above from Coleman's family containing our eigenform f, and we draw the modular arc $C = z(D)$ in X_0 (see the figure in [10, p. 187]). We can see easily that for each $k' \in \tilde{E}$, the eigenform $f_{k'}$ satisfies the condition of Proposition 3.5, so $f_{k'}$ has a twin $f'_{k'}$. These twins correspond to the same point $x_{k'}$ in X_0 and the slope of $f'_{k'}$ is equal to $k' - 1 - \alpha \neq \alpha$. We can see easily that the eigenform $f'_{k'}$ satisfies the condition of Theorem 3.1, so we can draw a modular arc $C^{(k')}$

which intersects C at the point $x_{k'}$. We call $\{C^{(k')}\}_{k'\in\tilde{E}}$ the *needles* which intersect the *spine* C (see the figure in [10, p. 188]). Further regarding a needle as a spine, we can construct an infinite fern of level Np over K (see the figure in [10, p. 190]).

4. The proof of the main theorem

From now on, we shall give the proof of the main theorem. Let $\bar{\rho}$ be the residual Galois representation given in the main theorem. We denote by $\mathcal{F}_0$ the set of all classical eigenforms of tame level N defined over $\mathcal{O}$ whose associated residual representation is equal to $\bar{\rho}$, and by $\mathcal{F}$ the set of all eigenforms which are twists of eigenforms in $\mathcal{F}_0$ by characters whose order is a power of p. It suffices to show that the subset $\mathcal{X}$ in $\mathrm{Spec}(\boldsymbol{R}(\bar{\rho}, S))$ of points corresponding to eigenforms of $\mathcal{F}$ is dense in $\mathrm{Spec}(\boldsymbol{R}(\bar{\rho}, S))$ in order to prove the isomorphism $\boldsymbol{R}(\bar{\rho}, S) \cong \boldsymbol{T}(\bar{\rho}, N)$. Now we assume that $\mathcal{X}$ is not dense in $\mathrm{Spec}(\boldsymbol{R}(\bar{\rho}, S))$. Then there is a non-zero element $\tau \in \boldsymbol{R}(\bar{\rho}, S)$ which vanishes on $\mathcal{X}$. We may assume that it is an irreducible element in $\boldsymbol{R}(\bar{\rho}, S)$ following the argument of the beginning of [7, Section 4]. (Note that $\boldsymbol{R}(\bar{\rho}, S) \cong W(\boldsymbol{k})[\![X_1, X_2, X_3]\!]$ is a UFD.)

For the eigenform f to which $\bar{\rho}$ is associated, we have constructed a K-analytic family of modular representations $(D, \tilde{E}; z, u)$ in the previous section. Then we have a modular arc $C = z(D)$ which is a K-analytic manifold of 1-dimension. If we shrink the disc D so that D does not contain 1, then C does not intersect X_{00}. Therefore we may assume that the restricted K-analytic action

$$\pi_0 : \Psi \times C \to X$$

is unramified by Proposition 2.5. Furthermore if we shrink D enough, then we can assume that π_0 is a K-analytic embedding of K-analytic manifolds (for the notion of "embedding," see [14, II.III.10]). Then the image $M = \pi_0(\Psi \times C)$ is a K-analytic submanifold of codimension one in X, and we shrink our disc D further so that $\tau = 0$ is a local defining equation for M as in the end of [7, Section 4].

To apply the argument of Gouvêa and Mazur, we must investigate the "conjugate arc" to C. In general, for a character $\chi : G_S \to \mathcal{O}^\times$, we denote by χ_{free} the composition of χ and the natural projection $\mathcal{O}^\times \to \mathcal{U}_{\mathrm{free}}$. For $x \in X$, the twist

$$\rho_x \otimes (\det \rho_x)^{-1}_{\mathrm{free}} : G_S \to \mathrm{GL}_2(\mathcal{O})$$

is also a deformation of $\bar{\rho}$ to $\mathcal{O}$, and we denote by $\tilde{x}$ the point in X corresponding to this deformation.

Proposition 4.1. *The map*

$$\iota : X \to X, \quad x \mapsto \tilde{x}$$

is a K-analytic involution of X.

Proof. It is proven in [7, Section 5] that ι is an involution of X. We now show that ι is K-analytic. Under the fixed isomorphism $\boldsymbol{R}(\bar{\rho}, S) \cong W(\boldsymbol{k})[\![X_1, X_2, X_3]\!]$, we regard the power series ring $W(\boldsymbol{k})[\![X_1, X_2, X_3]\!]$ as the universal deformation ring of $\bar{\rho}$. The character

$$(\det \boldsymbol{\rho}^{\mathrm{univ}})^{-1}_{\mathrm{free}} : G_S \to (\text{the pro-}p \text{ free part of } W(\boldsymbol{k})[\![X_1, X_2, X_3]\!]^\times)$$

factors through the maximal pro-p abelian torsion-free quotient $\Gamma = 1 + p\mathbb{Z}_p$ of G_S, and we put

$$F(X_1, X_2, X_3) = (\det \boldsymbol{\rho}^{\mathrm{univ}})^{-1}_{\mathrm{free}}(\gamma),$$

where γ is a topological generator of Γ. We then have

$$(\det \rho_x)^{-1}_{\mathrm{free}}(\gamma) = F(x(X_1), x(X_2), x(X_3))$$

for $x \in X$. Therefore we know that ι is K-analytic because this is the composition of the mapping

$$X \to \Psi \times X, \quad x \mapsto ((\det \rho_x)^{-1}_{\mathrm{free}}, x)$$

and $\pi : \Psi \times X \to X$, and π is K-analytic by Lemma 2.2. $\square$

By the proposition above, we see that the image of C under the involution ι is also a K-analytic manifold of 1-dimension. We put $\tilde{C} = \iota(C)$ and call this the *conjugate arc* to C. The following proposition implies that the involution ι preserves the Sen-null space X_0:

Proposition 4.2. *If we denote by $(0, r)$ the Hodge-Tate-Sen weights of a point $x \in X_0$, then the Hodge-Tate-Sen weights of the point $\iota(x)$ are $(-r, 0)$.*

Proof. For $x \in X_0$, let φ_x be the Sen operator associated to the (restricted) representation

$$\rho_x : G_K \to \mathrm{GL}_2(K).$$

Since the Hodge-Tate-Sen weights of x are $(0, r)$, we see that the trace of ρ_x is r. By Lemma 2.3, it suffices to show that the Hodge-Tate-Sen weight of the (restricted) character

$$(\det \rho_x)^{-1}_{\mathrm{free}} : G_K \to K^\times$$

is equal to $-r$. We are going to calculate the weight of $(\det \rho_x)^{-1}_{\mathrm{free}}$ following [13, Proposition 4], so we use the notation of the proposition from now on. (For the details of these notation, see [13].) First we take a triplet $(M, \mathcal{G}', \rho)$ of Proposition 4 of loc.cit. for ρ_x, where $M \in$

$\mathrm{GL}_2(\mathbb{C}_p)$, $\mathcal{G}'$ is an open subgroup of G_K and $\rho : \Gamma' \to \mathrm{GL}_2(K')$ is a continuous representation such that

$$\rho(\sigma) = M^{-1} \cdot \rho_x(\sigma) \cdot \sigma(M). \tag{2}$$

Here Γ' is the image of $\mathcal{G}'$ under the natural projection $G_K \to G_K/\ker\chi$, where $\chi : G_K \to \mathbb{Z}_p^\times$ is a continuous character, and K' is the fixed field of $\mathcal{G}'$. Then Proposition 4 of loc.cit. says that the representation matrix of the Sen operator φ_x is given by

$$M_x = \frac{\log \rho(\sigma)}{\log \chi(\sigma)}$$

for some $\sigma \neq 1 \in \Gamma' \cap \Gamma_0$, $|\rho(\sigma) - 1| < 1$. We note that M_x is independent of the choice of σ.

Next we consider the character $(\det \rho_x)^{-1} : G_K \to K^\times$. By the equation (2), we have

$$\det \rho(\sigma)^{-1} = (\det M^{-1})^{-1} \cdot \det \rho_x(\sigma)^{-1} \cdot \sigma(\det M^{-1}), \tag{3}$$

and then we can take $(\det M^{-1}, \mathcal{G}', \det \rho^{-1})$ as the triplet of Proposition 4 of loc.cit. for the character $(\det \rho_x)^{-1}$. On the other hand, since we have the decomposition $\mathcal{O}^\times = \mathcal{U}_{\mathrm{free}} \times (\text{finite})$, there is an open subgroup $\mathcal{G}''$ of $\mathcal{G}'$ such that for any $\sigma \in \mathcal{G}''$,

$$(\det \rho_x)^{-1}(\sigma) = (\det \rho_x)^{-1}_{\mathrm{free}}(\sigma).$$

Then we can take $(\det M^{-1}, \mathcal{G}'', \det \rho^{-1})$ as the triplet for $(\det \rho_x)^{-1}_{\mathrm{free}}$ by the equation (3). Therefore we obtain that

$$\begin{aligned}
(\text{the weight of } (\det \rho_x)^{-1}_{\mathrm{free}}) &= \frac{\log(\det \rho^{-1}(\sigma))}{\log \chi(\sigma)} \\
&= -\operatorname{Trace}\left(\frac{\log \rho(\sigma)}{\log \chi(\sigma)}\right) \\
&= -r.
\end{aligned}$$

□

Now we have the conjugate arc $\tilde{C}$ to C in the Sen-null space X_0 over K, then $\tilde{C} \cup C = M \cap X_0$ as proven in [7, Proposition 1 of Section 5]. Therefore the existence of the irreducible element τ implies a contradiction by the same argument as in [7, pp. 135–137] using Lemmas 3.3, 3.4. We omit the details of the proof.

5. Application to controlling the conductor

In this section, we prove Gouvêa's conjecture on controlling the conductor in the unobstructed case by means of the main theorem. First, we recall this conjecture which was stated in [6]. For an absolutely irreducible residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\boldsymbol{k}),$$

we denote its conductor by $N(\bar{\rho})$. (For the definition of the conductor of a residual representation, see [15], [6].) We assume that $\bar{\rho}$ is associated to a newform f of level $N(\bar{\rho})$ defined over the ring of integers $\mathcal{O}$ of a totally ramified finite extension over the fraction field of the Witt ring $W(\boldsymbol{k})$ over $\boldsymbol{k}$. Gouvêa's conjecture on the controlling the conductor is stated as follows (for the details, see [6]). Let $S = \{\text{the prime divisors } l \text{ of } N(\bar{\rho})p\} \cup \{\infty\}$ and S^0 be the subset of S consisting of prime numbers $l \mid N(\bar{\rho})$ at which $\bar{\rho}$ is "l-ordinary" (for the definition of ordinariness, see [9], [6]). In [6], Gouvêa showed that the conductor of any "S^0-ordinary" deformation of $\bar{\rho}$ is $N(\bar{\rho})$. Then he defined the level $N(\bar{\rho})$ universal deformation ring $\boldsymbol{R}(\bar{\rho}, N(\bar{\rho}))$ to be the universal S^0-ordinary deformation ring. He also showed that the surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N(\bar{\rho}))$$

factors through the universal level $N(\bar{\rho})$ deformation ring:

$$\begin{array}{ccc} \boldsymbol{R}(\bar{\rho}, S) & \longrightarrow & \boldsymbol{T}(\bar{\rho}, N(\bar{\rho})) \\ & \searrow \quad \nearrow & \\ & \boldsymbol{R}(\bar{\rho}, N(\bar{\rho})), & \end{array}$$

and he gave the following

Conjecture ([6], Section 4)**.** The surjective homomorphism

$$\boldsymbol{R}(\bar{\rho}, N(\bar{\rho})) \to \boldsymbol{T}(\bar{\rho}, N(\bar{\rho}))$$

is an isomorphism.

Now we assume that the deformation problem for $\bar{\rho}$ is unobstructed. Then the main theorem of this article shows that the surjection

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{T}(\bar{\rho}, N(\bar{\rho}))$$

is an isomorphism. Further, we know that the homomorphism

$$\boldsymbol{R}(\bar{\rho}, S) \to \boldsymbol{R}(\bar{\rho}, N(\bar{\rho}))$$

is surjective by [18, Theorem 1.2]. Therefore by the commutative diagram above, we can see easily the following

Theorem 5.1. *Gouvêa's conjecture on controlling the conductor is true under the assumption that the deformation problem is unobstructed.*

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