$\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory on Orbifold-T^4/Z_2: Higher Rank Case

Masao Jinzenji, Toru Sasaki

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$\mathcal{N}=4$ Supersymmetric Yang-Mills Theory on Orbifold-$T^4/\mathbf{Z}_2$: Higher Rank Case

Masao Jinzenji[†], Toru Sasaki[*]

[†] *Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan*
[*] *Department of Physics, Hokkaido University, Sapporo 060-0810, Japan*
[†] *jin@math.sci.hokudai.ac.jp*
[*] *sasaki@particle.sci.hokudai.ac.jp*

Abstract

We derive the partition function of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbf{Z}_2$ for $SU(N)$. We generalize our previous work for $SU(2)$ to the $SU(N)$ case. These partition functions can be factorized into product of bulk contribution of quotient space $T^4/\mathbf{Z}_2$ and of blow-up formula including A_{N-1} theta functions with level N.

1 Introduction

$\mathcal{N}=4$ supersymmetric Yang-Mills theory on 4 dimensional manifold has the largest number of supersymmetry if it does not couples to gravity. This theory is conformal invariant and finite. Furthermore it is believed to have Montonen-Olive duality [14] (a type of S-duality). In [19], Vafa and Witten considered the twisted $\mathcal{N}=4$ supersymmetric Yang-Mills theory and derived its exact partition function. Using this exactly calculated partition function, they confirmed Montonen-Olive duality. In their derivation they separated its partition function into the following two factors (the same process was also used in [20]). One part is the contribution from bulk (non-zero locus of the section of the canonical bundle), and is expressed by Göttsche formula [16, 15, 19]. The other is the contribution from cosmic string (zero locus of the section of the canonical bundle), and it is expressed by blow-up formula [21, 19]. The both factors are multiplied and summed up so that the total contribution has the modular property (S-duality). After their work [19], deeper analysis and further applications were done such as in [2, 3, 5, 7, 8, 9, 10, 12, 18, 22, 24, 25, 26, 27]. Especially in [18], Sako and T.S. (one of the authors of this paper) revealed that Euler number of instanton moduli space and Seiberg-Witten invariants are connected in the framework of Vafa-Witten theory.

In our previous work [5], we derived the partition function of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbf{Z}_2$ for gauge group $SU(2)$ from the point of view of orbifold construction. Precisely speaking, we expressed the $SU(2)$ partition function of $K3$ surface as the the product of the bulk contribution from the quotient space $T^4/\mathbf{Z}_2=S_0$ and blow-up formula coming from the sixteen singularities of S_0 . In this paper, we try to generalize it to $SU(N)$ (N is odd prime) case. For $SU(N)$ theory it is well-known that the partition function on X ($X=K3, T^4$ or $\frac{1}{2}K3$) is almost a Hecke transformation of order N of $1/\eta^{\chi}(\tau)$ (χ is Euler number of X). This fact was derived and used from physical side in [19, 12, 9], and was confirmed mathematically in [25]. From this starting point, we ask whether $K3$ partition function for $SU(N)$ (or $SU(N)/\mathbf{Z}_N$) can be expressed by the product of the following two factors: the contribution on $T^4/\mathbf{Z}_2=S_0$ from bulk and $\mathcal{O}(-2)$ curve blow-up formula from 16 orbifold singularities. Our answer is no if we use the $\mathcal{O}(-2)$ curve blow-up formula. But our answer is yes if we use the $\mathcal{O}(-N)$ curve blow-up formula derived in [7] instead. The formula is given by

$$Z_k^{K3}=\sum_{j=0}^{N-1} Z_j^{S_0} Z_{k-j+1}^{B}, \tag{1.1}$$

$$Z_j^{B}=\sum_{\{\beta_l\}} a_{\{\beta_l\}}^{j} \prod_{l=1}^{16} \frac{\theta_{(N)}^{\beta_l}(\tau)}{\eta^N(\tau)}. \tag{1.2}$$

Here Z_j^X stands for the partition function on X with $v^2=2j \bmod N$. X is $K3, S_0$ or blow-up respectively. v is 't Hooft flux $v\in H^2(X,\mathbf{Z}_N)$. $\frac{\theta_{(N)}^{\beta^l}(\tau)}{\eta^N(\tau)}$ is $\mathcal{O}(-N)$ curve blow-up formula labeled by β^l [7]. We emphasize that $a_{\{\beta_l\}}$ is an integer. In our derivation of (1.2), we used the well-known "denominator identity" in the theory of affine Lie algebras. This fact suggests that $\mathcal{N}=4$ supersymmetric Yang-Mills theory and rational conformal field theory (affine Lie algebra) are closely related. This relation was already pointed out in [19].

$\mathcal{O}(-N)$ curve blow-up
In this paper, we found the following two key relations. One is

$$\frac{1}{\eta(\frac{\tau}{N})} = \frac{\theta^2_{A_{N-1}}(\tau)}{\eta^N(\tau)}, \tag{1.3}$$

where $\theta^2_{A_{N-1}}(\tau)$ is a theta series related to affine Lie algebra. The other is

$$\theta_{A_{N-1}}(\tau) = \sum_{\beta} a_\beta \theta^{\beta}_{(N)}(\tau), \tag{1.4}$$

where $\theta^{\beta}_{(N)}(\tau)$ is Kapranov's theta function labeled by β associated with $\mathcal{O}(-N)$ curve blow-up [7]. We emphasize that a_β is an integer. Using these two key relations, we could rewrite $1/\eta(\frac{\tau}{N})$ by $\mathcal{O}(-N)$ curve blow-up formulas. Why did we face $\mathcal{O}(-N)$ curve blow-up formula instead of $\mathcal{O}(-2)$ curve blow-up formula ? We want geometrical interpretation. The fact that a_β is an integer suggests the possibility of geometrical interpretation. Hint is in [19, 17]. We can easily see that $\mathcal{O}(-N)$ curve blow-up formula appearing in our formulas comes from level N $SU(N)$ characters. On the other hand level k characters of $SU(N)$ arise for $\mathcal{N}=4$ super Yang-Mills theory with gauge group $U(k)$ on A_{N-1} ALE space [19, 17]. Therefore, it is natural that $\mathcal{O}(-N)$ curve blow-up formula appears. For we consider $SU(N)$ gauge group [19, 17].

This paper is organized as follows. In Sec.2, we review our previous work for $SU(2)$ and overview the untwisted sector of S_0 for $SU(N)$. In Sec.3, we introduce the key identity and verify it using denominator identity of affine Lie algebra. In Sec.4, we give the key conjecture and rewrite $1/\eta(\frac{\tau}{N})$ by $\mathcal{O}(-N)$ curve blow-up formula. In Sec.5, we derive the partition function for $SU(N)/\mathbf{Z}_N$ using the results of Sec.3, 4. In Sec.6, we conclude and discuss the remaining problems.

2 Review and Untwisted Sector

In this section, we briefly review the geometrical background of orbifold $T^4/\mathbf{Z}_2$ [1, 4] and overview untwisted sector of $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory on orbifold-$T^4/\mathbf{Z}_2$ for $SU(N)$ [19, 3, 9, 24, 10]. Then we review the reconstruction process of $K3$ partition function for $SU(2)$ using $\mathcal{O}(-2)$ curve blow-up formula [21, 5].

2.1 Orbifold $T^4/\mathbf{Z}_2$

In this subsection, we reconstruct $K3$ surface from T^4 surface, keeping trivial canonical bundle $K_X=0$ [1, 4].

First we consider T^4 surface. T^4 surface has trivial canonical bundle $K_{T^4}=0$, but has nontrivial Picard groups. After the $\mathbf{Z}_2$-identification, we have $T^4/\mathbf{Z}_2 = S_0$, which has $K_{S_0}=0$ and trivial Picard groups. However S_0 has sixteen orbifold singularity points. Thus we want to resolve these singularities. For this purpose, we replace these singularity points with $\mathcal{O}(-2)$ curves (we call this process $\mathcal{O}(-2)$ curve blow-up). After this process we have $\tilde{S}_0 = K3$, which has $K_{K3}=0$ and no singularity.

To review these processes, we write down the change of the Euler numbers.

$$\chi(T^4) = 0 \to \chi(S_0) = 8 \to \chi(K3) = 24. \tag{2.1}$$

2.2 Untwisted Sector

General Structure of Vafa-Witten Conjecture

Following [19, 9, 10, 22], we review the general structure of Vafa-Witten conjecture. For $SU(N)/\mathbf{Z}_N$ theory with 't Hooft flux v on X, partition function is defined by,

$$Z_v^X(\tau) \equiv q^{-\frac{N\chi(X)}{24}} \sum_k \chi(\mathcal{N}(v,k))q^k \quad (q := \exp(2\pi i\tau)), \tag{2.2}$$

where $\mathcal{N}(v,k)$ is the moduli space of anti-self-dual connections associated to $SU(N)/\mathbf{Z}_N$-principal bundle with 't Hooft flux $v \in H^2(X, Z_N)$ and fractional instanton number $k \in Z/2N$. $\chi(X)$ is Euler number of X. For this partition function, Vafa and Witten conjectured

$$Z_v^X\left(-\frac{1}{\tau}\right) = N^{-\frac{b_2(X)}{2}}\left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} \cdot \sum_{u\in H^2(X,Z_N)} \zeta_N^{u\cdot v} Z_u^X(\tau), \tag{2.3}$$

where $\zeta_N = \exp(\frac{2\pi i}{N})$.

For later use, we introduce

$$Z_{SU(N)}^X(\tau) \equiv \frac{1}{N} Z_0^X(\tau),$$
$$Z_{SU(N)/\mathbf{Z}_N}^X(\tau) \equiv \sum_{u\in H^2(X,Z_N)} Z_u^X(\tau). \tag{2.4}$$

For these partition functions, the conjecture is reduced to the following formula:

$$Z_0^X\left(-\frac{1}{\tau}\right) = N^{-\frac{b_2(X)}{2}}\left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} Z_{SU(N)/\mathbf{Z}_N}^X(\tau). \tag{2.5}$$

Partition Function of the Untwisted Sector of S_0

In this part, we derive the partition function of the untwisted sector of S_0 as a concrete example. First we think of the moduli space of $X = T^4$. In general, one treats the moduli space $\mathcal{M}_H(N, c_1, c_2)$ of rank N stable sheaves E with Chern classes c_1, c_2 [22, 24], but we want to treat the moduli space $\mathcal{N}(v,k)$ of $SU(N)/\mathbf{Z}_N$ or $SU(N)$ vector bundles with v, k. Thus following our previous work [5], we identify $\mathcal{M}_H(N, c_1, c_2) \equiv \mathcal{N}(v,k)$ with $c_1 = v \bmod N$ and $k = c_2 - \frac{(N-1)v^2}{2N}$ in the following. Throughout this paper, we restrict N to prime numbers. For this restriction we follow [27]. This restriction makes our problem simpler. According to [24], the moduli space of rank N stable sheaves E of V is given below. We introduce Mukai vector

$$V = ch(E)\sqrt{td_X} = N + c_1 + \frac{c_1^2 - 2c_2}{2}. \tag{2.6}$$

Note that in our case $X = T^4$, $td_X = 1$. We remark here that the condition "N: prime" makes the Mukai vector primitive. We also introduce the inner product

$$\begin{aligned} < V^2 > &= -\int_X \left(N + c_1 + \frac{c_1^2 - 2c_2}{2}\right) \vee \left(N + c_1 + \frac{c_1^2 - 2c_2}{2}\right) \\ &= 2Nc_2 - (N-1)c_1^2. \end{aligned} \tag{2.7}$$

Here we use a symmetric bilinear form on $\oplus_j H^{2j}(X,\mathbf{Z})$:

$$<x,y> \;=\; -\int_X (x\vee y) \;=\; \int_X (x_1y_1 - x_0y_2 - x_2y_0), \tag{2.8}$$

where $x = x_0 + x_1 + x_2, x_j \in H^{2j}(X,\mathbf{Z})$ and $x\vee = x_0 - x_1 + x_2$.

With these preparation, we can describe the moduli space labeled by V

$$M_H^X(V) \cong \hat{X} \times (X)^{[\frac{<V^2>}{2}]} = \hat{X} \times (X)^{[Nc_2 - \frac{(N-1)c_1^2}{2}]}, \tag{2.9}$$

where H is some lime bundle on X and $\hat{X}$ is the dual of X [24].
Since S_0 has the trivial canonical bundle like $K3$ and T^4, it doesn't have cosmic strings, which are given by zero locus of the section of the canonical bundle. Hence we assume the moduli space of V in S_0 case,

$$M_H(V) \cong (S_0)^{[\frac{<V^2>}{2}]} = (S_0)^{[Nn - \frac{(N-1)v^2}{2}]}, \tag{2.10}$$

where we used the vanishing Picard group to drop the $\hat{X}$ part [1]. Using this, we define the partition function of S_0 of the untwisted sector with $v^2 = 2j$ (mod N) type,

$$\begin{aligned} Z_j^{S_0}(\tau) &= q^{-\frac{1}{3N}} \sum_{v^2\equiv j (mod\ N), n} e(M_H(V)) q^{n - \frac{N-1}{N}j} \\ &= q^{-\frac{1}{3N}} \sum_n e((S_0)^{[Nn-(N-1)j]}) q^{n-\frac{N-1}{N}j} \\ &= q^{-\frac{1}{3N}} \sum_m e((S_0)^{[m]}) \frac{(q^{\frac{1}{N}})^m + \zeta_N^{-j}(\zeta_N q^{\frac{1}{N}})^m + \cdots + \zeta_N^{-j(N-1)}(\zeta_N^{N-1} q^{\frac{1}{N}})^m}{N} \\ &= \frac{1}{N}\left(\frac{1}{\eta^8(\frac{\tau}{N})} + \zeta_N^{\frac{1}{3}}\zeta_N^{-j}\frac{1}{\eta^8(\frac{\tau+1}{N})} + \cdots + \zeta_N^{\frac{N-1}{3}}\zeta_N^{-j(N-1)}\frac{1}{\eta^8(\frac{\tau+N-1}{N})}\right). \end{aligned} \tag{2.11}$$

For trivial type $v = 0$, we follow [19] and set,

$$Z_t^{S_0}(\tau) = C\frac{1}{\eta^8(N\tau)} + \frac{1}{N}\frac{1}{\eta^8(\frac{\tau}{N})} + \frac{1}{N}\zeta_N^{\frac{1}{3}}\frac{1}{\eta^8(\frac{\tau+1}{N})} + \cdots + \frac{1}{N}\zeta_N^{\frac{N-1}{3}}\frac{1}{\eta^8(\frac{\tau+N-1}{N})}, \tag{2.12}$$

where C is not determined. Note that $\frac{1}{\eta^8(N\tau)}$ is obtained from $\frac{1}{\eta^8(\frac{\tau}{N})}$ using the modular transformation $\tau \to -\frac{1}{\tau}$.

[1] Precisely speaking, we have to take care of non-trivial Todd class of S_0. But we don't know the precise definition of Todd class of S_0. Moreover, this correction does not affect the results of computation severely. Therefore, we neglect here this correction.

2.3 $\mathcal{O}(-2)$ Curve Blow-up Formula and $K3$ Partition Function: Review of the $SU(2)$ case

In this subsection, we review the previous work for $SU(2)$ [5]. Especially, we reconstruct $K3$ partition function from the contribution from the untwisted sector for the $SU(2)$ case. In [5], we introduce $\mathcal{O}(-2)$ curve blow-up formulas as the contribution from the twisted sector:

$$\frac{\theta_2(\tau)}{\eta(\tau)^2}, \quad \frac{\theta_3(\tau)}{\eta(\tau)^2}, \quad \frac{\theta_4(\tau)}{\eta(\tau)^2}. \tag{2.13}$$

These formulas were speculated from the proof of $\mathcal{O}(-1)$ curve blow-up formula [21, 10]. Furthermore we introduce the blow-up formulas that correspond to blowing up 16 orbifold singularities:

$$\tilde{Z}_1(\tau) = \frac{\theta_2^8(\tau)(\theta_3^8(\tau)+\theta_4^8(\tau))}{\eta^{32}(\tau)}, \tag{2.14}$$

$$\tilde{Z}_2(\tau) = \frac{\theta_2^8(\tau)(\theta_3^8(\tau)-\theta_4^8(\tau))}{\eta^{32}(\tau)}, \tag{2.15}$$

$$\tilde{Z}_3(\tau) = \frac{\theta_3^8(\tau)\theta_4^8(\tau)}{\eta^{32}(\tau)}, \tag{2.16}$$

so that these are consistent with Vafa-Witten conjecture. Using these formulas, we reconstruct $K3$ partition function:

$$\begin{aligned} Z^{K3}_{even}(\tau) &= 2^{-7}\left(Z^{S_0}_{odd}(\tau)\tilde{Z}_2(\tau) + Z^{S_0}_{even}(\tau)\tilde{Z}_1(\tau)\right), \\ Z^{K3}_{odd}(\tau) &= 2^{-7}\left(Z^{S_0}_{odd}(\tau)\tilde{Z}_1(\tau) + Z^{S_0}_{even}(\tau)\tilde{Z}_2(\tau)\right). \end{aligned} \tag{2.17}$$

$$Z^{K3}_t(\tau) = \frac{1}{4}\frac{1}{\eta^8(2\tau)}\tilde{Z}_3(\tau) + 2^{-7}\left(Z^{S_0}_{even}(\tau)\tilde{Z}_2(\tau) + Z^{S_0}_{odd}(\tau)\tilde{Z}_1(\tau)\right). \tag{2.18}$$

Note that combinations and coefficients are also consistent with Vafa-Witten conjecture.

To compare these results with those of Vafa-Witten, we rewrite these partition function using identities between eta and theta functions.

$$\begin{aligned} Z^{K3}_t(\tau) &= \frac{1}{4}\frac{1}{\eta^{24}(2\tau)} + \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2})} + \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2}+\frac{1}{2})}, \\ Z^{K3}_{even}(\tau) &= \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2})} + \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2}+\frac{1}{2})}, \\ Z^{K3}_{odd}(\tau) &= \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2})} - \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2}+\frac{1}{2})}. \end{aligned} \tag{2.19}$$

3 From Bulk to Blow-up

In the rest of this paper, we generalize our previous work for $SU(2)$ [5] to the $SU(N)$ case. Especially in the following two sections, we introduce the blow-up formula appeared in our theory. What we want is the generalization of (2.13). In this section, we first introduce the key identity and then we briefly verify this identity [11].We also remark the link to the affine Lie algebra [6].

3.1 Key Identity

We will see in Sec.5 that the identities between eta and theta functions play the key role. For $SU(2)$ case this identity is simply given by

$$\frac{1}{\eta(2\tau)}=\frac{\theta_4(2\tau)}{\eta^2(\tau)}, \tag{3.1}$$

or

$$\frac{1}{\eta(\frac{\tau}{2})}=\frac{1}{2}\frac{\theta_2(\frac{\tau}{2})}{\eta^2(\tau)}. \tag{3.2}$$

The right hand sides of these identities have the form $\Theta/\eta^2(\tau)$. These forms are general for the blow-up formula of the compactified moduli space [21, 10]. Especially the part of eta function comes from Gieseker-Maruyama compactification [10, 21]. We expect the same situation for $SU(N)$. For $SU(N)$ case the corresponding form is $\Theta/\eta^N(\tau)$. In this principle, we searched and found the following key identities,

$$\frac{1}{\eta(N\tau)}=\frac{\theta^1_{A_{N-1}}(\tau)}{\eta^N(\tau)}, \tag{3.3}$$

or alternatively

$$\frac{1}{\eta(\frac{\tau}{N})}=\frac{\theta^2_{A_{N-1}}(\tau)}{\eta^N(\tau)}, \tag{3.4}$$

where we introduce A_{N-1} theta functions,

$$\begin{aligned}
\theta^0_{A_{N-1}}(\tau) &\equiv \sum_{m\in\mathbf{Z}^{N-1}} q^{\frac{1}{2}{}^t m A_{N-1} m}, &(3.5)\\
\theta^1_{A_{N-1}}(\tau) &\equiv \sum_{m\in\mathbf{Z}^{N-1}} q^{\frac{1}{2}{}^t m A_{N-1} m} e^{2\pi i m\cdot\delta}, &(3.6)\\
\theta^2_{A_{N-1}}(\tau) &\equiv \sum_{m\in\mathbf{Z}^{N-1}} q^{\frac{1}{2}{}^t (m+\frac{\rho}{N}) A_{N-1} (m+\frac{\rho}{N})}. &(3.7)
\end{aligned}$$

Here A_{N-1} stands for Cartan matrix for $SU(N)$. We also introduced

$$\delta\equiv\frac{1}{N}(1,\ldots,1),\ \rho=NA^{-1}_{N-1}\delta. \tag{3.8}$$

The vector ρ is the same as the usual ρ (half the sum of the positive root of Lie algebra A_{N-1}). Note that we can obtain (3.3) if we apply the modular transformation $\tau\to-\frac{1}{\tau}$ to (3.4).

3.2 Denominator (Macdonald) Identity and Affine Lie Algebra

In this subsection, we will give the proof of (3.4). First we think of the simplest case, that is (3.2). $\theta_2(\frac{\tau}{2})$ has the following product formula (which comes from Jacobi's triple product identity) [6]:

$$\theta_2(\frac{\tau}{2}) = 2q^{\frac{1}{16}} \prod_{n=1}^{\infty}(1-q^{\frac{n}{2}})(1+q^{\frac{n}{2}})^2 = 2\frac{\eta^2(\tau)}{\eta(\frac{\tau}{2})}. \tag{3.9}$$

One can easily find the justification of the last equality. This is the proof of (3.2) itself. Next we think of the general N case (3.4). We begin with the denominator identity (which corresponds to Jacobi's triple product identity for $N=2$ case) [6].

Theorem 1 (Denominator Identity)

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\ \alpha} = \sum_{w\in W}\epsilon(w)e(w(\rho)-\rho). \tag{3.10}$$

Here we introduce the notations of the affine Lie algebras.

Notation 1 *Notations [6, 11],*
Δ : *set of roots,* $\Delta_+ \subset \Delta$: *set of positive roots,* l : *rank of Cartan matrix*
mult α : *multiplicity of* $\alpha \in \Delta_+$,
W : *Weyl Group,* $w \in W$: *its element,* $\epsilon(w) = (-1)^{l(w)}$, $l(w)$: *length of* w,
ρ : *half the sum of the positive roots,*
$h^\vee$: *dual coxter number.*

From now on, the symbol "prime ($'$)" means restriction to classical Lie algebra associated with affine Lie algebra.

After several calculations and settings, one can rewrite (3.10) as

Corollary 1

$$q^{\frac{|\rho|^2}{2h^\vee}}\prod_{n\geq 1}((1-q^n)^l\prod_{\alpha\in\Delta'}(1-q^n e(\alpha))) = \sum_{\alpha\in M}\chi'(h^\vee\alpha)q^{\frac{|\rho+h^\vee\alpha|^2}{2h^\vee}}, \tag{3.11}$$

where

$$\chi'(\lambda) = \frac{\sum_{w\in W'}\epsilon(w)e(w(\lambda+\rho)-\rho)}{\prod_{\alpha\in\Delta'_+}(1-e(-\alpha))}. \tag{3.12}$$

Note that M is a root lattice spanned over $\mathbf{Z}$ by $\alpha_j \in \Delta'$, and $|u|^2$ is the square length of vector u in M. In our case, $|\sum_{j=1}^{N-1} m_j\alpha_j|^2 = {}^t m A_{N-1} m$.
We give the proof of (3.4).
proof) In our case, we take A_{N-1} as Lie algebra. Then we obtain

$$h^\vee = N,\ \ l = N-1,\ \ \frac{|\rho|^2}{2h^\vee} = \frac{N^2-1}{24}. \tag{3.13}$$

We also take the following specialization:

$$e(\alpha) = \exp(\frac{2\pi i\alpha\cdot\rho}{N}). \tag{3.14}$$

From Theorem 1 applied to classical Lie algebra A_{N-1}, we can see,

$$\chi'(N\alpha) = 1. \tag{3.15}$$

Applying (3.13), (3.14), (3.15) to Corollary 1, we obtain

$$\begin{aligned} &q^{\frac{N^2-1}{24}} \prod_{n=1}^{\infty} (1-q^n)^{N-1} \prod_{j=1}^{N-1} (1-q^n \zeta_N^j)^N \\ = &\sum_{m \in (N\mathbf{Z})^{N-1}} q^{\frac{1}{2N} {}^t(m+\rho) A_{N-1} (m+\rho)}. \end{aligned} \tag{3.16}$$

If we change q into $q^{\frac{1}{N}}$, we obtain (3.4). □

Note that this proof was first given by Macdonald (p.120,121) [11].

In this section, we have showed that the eta function $1/\eta(\frac{\tau}{N})$ can be described by $\Theta/\eta^N(\tau)$ using the key identity. But this identity is not enough to rewrite the eta function $1/\eta(\frac{\tau}{N})$ by the blow-up formula. In the next section, we complete our attempt to obtain the blow-up formula for $SU(N)$ in our theory. However, we have shown that this theory is closely related to the Affine Lie algebras. This relation was already pointed out in [19].

4 $\mathcal{O}(-N)$ Curve Blow-up

We continue the discussion of the blow-up formula. Especially in this section, we try to rewrite the theta function in the previous section as the sum of the $\mathcal{O}(-N)$ curve blow-up formulas [5, 7]. First, we introduce the key conjecture. Next, we show some examples of this conjecture.

4.1 Key Conjecture

In the previous section we introduced the similar expression to the blow-up formulas. However in the geometrical context, theta function $\theta^2_{A_{N-1}}(\tau)$ is not the theta function appeared in the blow-up formula. For $N = 2$ case fortunately there is the following duplication formula:

$$\theta^2_{A_1}(\tau)^2 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau). \tag{4.1}$$

That is why we have introduced (2.13) directly instead of $\frac{\theta^2_{A_1}(\tau)}{\eta^2(\tau)}$. As we have already mentioned, (2.13) are $\mathcal{O}(-2)$ curve blow-up formulas. $\mathcal{O}(-2)$ curve blow-up is consistent with geometrical context (recall Sec.2.1). In this principle we first wanted to find identities for general N case like (4.1), with which one can rewrite $\theta^2_{A_{N-1}}(\tau)$ into the theta function appearing in $\mathcal{O}(-2)$ curve blow-up formula. For this purpose we cite the theorem derived by Kapranov [7]:

Theorem 2 (Kapranov) *$\mathcal{O}(-d)$-curve blow-up formula for $SU(N)$ gauge theory is given by the A_{N-1} theta series with level d:*

$$\sum_{a \in L} q^{\Psi(a,f) - d\Psi(a,a)/2}, \tag{4.2}$$

where L is the A_{N-1} root lattice, f is an element of the weight lattice and $\Psi(a,b) = -{}^t a A_{N-1} b$.

Remark 1 *In Theorem 2, the factor coming from Gieseker-Maruyama compactification is neglected because Kapranov treated uncompactified case in [7]. Compactified version of $\mathcal{O}(-1)$ curve blow-up formula for $SU(N)$ case was first derived by Yoshioka [26].*

Remark 2 *The original version of Theorem 2 in [7] takes f as an element of the root lattice. But in [19] where $\mathcal{O}(-1)$-curve blow-up formula for $SU(2)$ gauge theory is considered, Vafa and Witten introduced the two types of blow-up formula:*

$$\Theta_0 := \sum_{n\in\mathbf{Z}} q^{n^2}, \quad \Theta_1 := \sum_{n\in\mathbf{Z}} q^{(n-\frac{1}{2})^2}. \tag{4.3}$$

Θ_0 and Θ_1 correspond to taking f as 0 and f as $\frac{1}{2}\alpha$ respectively. Therefore, it is natural to take f as an element of the weight lattice. This generalization is also compatible with our treatment of O(-2)-curve blow-up formula for $SU(2)$ gauge theory [5].

We applied Theorem 2 to the $d=2$ case and tried to describe $\theta_{A_{N-1}}(\tau)$ by this Kapranov's theta function. As a result our attempt failed. Instead, we have found the following key conjecture:

Conjecture 1 *For odd $N \geq 3$,*

$$\theta^2_{A_{N-1}}(\tau) = a_1\theta^{\beta_1}_{(N)}(\tau) + a_2\theta^{\beta_2}_{(N)}(\tau) + \cdots + a_k\theta^{\beta_k}_{(N)}(\tau), \tag{4.4}$$

where

$$\theta^{\beta_j}_{(N)}(\tau) \equiv \sum_{m\in\mathbf{Z}^{N-1}} q^{\frac{N}{2}{}^t(m+v(\beta_j))A_{N-1}(m+v(\beta_j))}, \tag{4.5}$$

$$v(\beta_j) \equiv \frac{1}{N}A^{-1}_{N-1}\begin{pmatrix} \beta^{(1)}_j \\ \beta^{(2)}_j \\ \vdots \\ \beta^{(N-1)}_j \end{pmatrix}, \quad \beta^{(n)}_j \in \mathbf{Z}. \tag{4.6}$$

a_j is a non-negative integer. k is an appropriate finite positive integer.

Using this conjecture, one can describe $\theta^2_{A_{N-1}}(\tau)$ by Kapranov-type theta function associated with $\mathcal{O}(-N)$ curve blow-up formula [7]. The reason why we call $\theta^{\beta_j}_{(N)}(\tau)$ Kapranov-type is given below. Using the notation in [7], $\theta^{\beta_j}_{(N)}(\tau)$ can be expressed as follows:

$$\theta^{\beta_j}_{(N)}(\tau) = \sum_{a\in L} q^{-\Psi(f,f)/2N+\Psi(a,f)-N\Psi(a,a)/2}. \tag{4.7}$$

Apart from the top factor $q^{-\Psi(f,f)/2N}$, (4.7) is the same theta function as (4.2) for $d=N$. Kapranov-type theta function is level N theta function [6].

We have found that (3.4) is the sum of the $\mathcal{O}(-N)$ curve blow-up formulas. Appearance of $\mathcal{O}(-N)$ curve blow-up for general N case instead of $\mathcal{O}(-2)$ is natural because level N representation of affine Lie algebra appears in $\mathcal{N}=4$ SYM theory with gauge group $U(N)$ on ALE spaces [17, 19].

4.2 Examples: $N = 3, 5, 7$ Case

In this subsection we give some explicit examples of Conjecture 1. We confirmed these examples using Maple V.

$N = 3$ **Case**

$$\theta^2_{A_2}(\tau) = \theta^{(1,0)}_{(3)}(\tau) + \theta^{(2,0)}_{(3)}(\tau) + \theta^{(4,0)}_{(3)}(\tau), \tag{4.8}$$

where

$$\theta^{(k,l)}_{(3)}(\tau) = \sum_{m \in \mathbf{Z}^2} q^{\frac{3}{2}{}^t(m+v(k,l))A_2(m+v(k,l))}, \tag{4.9}$$

$$v(k,l) = \frac{1}{3}A_2^{-1}\begin{pmatrix} k \\ l \end{pmatrix}. \tag{4.10}$$

Here, we give concrete expressions of $\theta^{(1,0)}_{(3)}(\tau), \theta^{(2,0)}_{(3)}(\tau), \theta^{(4,0)}_{(3)}(\tau)$.

$$\theta^{(1,0)}_{(3)}(\tau) = \sum_{(m_1,m_2)\in \mathbf{Z}^2} q^{\frac{1}{9}+m_1+3(m_1^2+m_2^2-m_1m_2)} \equiv q^{\frac{1}{9}}\tilde{\theta}^0_{(3)}(\tau), \tag{4.11}$$

$$\theta^{(2,0)}_{(3)}(\tau) = \sum_{(m_1,m_2)\in \mathbf{Z}^2} q^{\frac{4}{9}+2m_1+3(m_1^2+m_2^2-m_1m_2)} \equiv q^{\frac{4}{9}}\tilde{\theta}^1_{(3)}(\tau), \tag{4.12}$$

$$\theta^{(4,0)}_{(3)}(\tau) = \sum_{(m_1,m_2)\in \mathbf{Z}^2} q^{\frac{16}{9}+4m_1+3(m_1^2+m_2^2-m_1m_2)} \equiv q^{\frac{7}{9}}\tilde{\theta}^2_{(3)}(\tau). \tag{4.13}$$

We introduced $\tilde{\theta}^j_{(3)}(\tau), (j = 0, 1, 2)$, which have integral q expansions. One can easily see that the right hand side of (4.8) consists of 3 types of theta functions, which are classified by top q powers. Then we expect that this is the general property of Conjecture 1 for odd N. We will surely confirm that it is true for $N = 5, 7$.

$N = 5$ **Case**

$$\begin{aligned}
\theta^2_{A_4}(\tau) &= 5\theta^{(1,1,1,1)}_{(5)}(\tau)\big\} \equiv q\tilde{\theta}^4_{(5)}(\tau) \\
&+ \theta^{(1,0,0,1)}_{(5)}(\tau) + 2\theta^{(3,1,0,0)}_{(5)}(\tau) + 2\theta^{(1,3,1,0)}_{(5)}(\tau)\big\} \equiv q^{\frac{1}{5}}\tilde{\theta}^0_{(5)}(\tau) \\
&+ \theta^{(2,0,0,2)}_{(5)}(\tau) + 2\theta^{(6,2,0,0)}_{(5)}(\tau) + 2\theta^{(2,6,2,0)}_{(5)}(\tau)\big\} \equiv q^{\frac{4}{5}}\tilde{\theta}^3_{(5)}(\tau) \\
&+ \theta^{(0,1,1,0)}_{(5)}(\tau) + 2\theta^{(3,0,1,1)}_{(5)}(\tau) + 2\theta^{(1,0,3,0)}_{(5)}(\tau)\big\} \equiv q^{\frac{2}{5}}\tilde{\theta}^1_{(5)}(\tau) \\
&+ \theta^{(0,2,2,0)}_{(5)}(\tau) + 2\theta^{(6,0,2,2)}_{(5)}(\tau) + 2\theta^{(2,0,6,0)}_{(5)}(\tau)\big\} \equiv q^{\frac{3}{5}}\tilde{\theta}^2_{(5)}(\tau),
\end{aligned} \tag{4.14}$$

where

$$\theta^{(n,k,l,p)}_{(5)}(\tau) = \sum_m q^{\frac{5}{2}{}^t(m+v(n,k,l,p))A_4(m+v(n,k,l,p))}, \tag{4.15}$$

$$v(n,k,l,p) = \frac{1}{5} A_4^{-1} \begin{pmatrix} n \\ k \\ l \\ p \end{pmatrix}. \tag{4.16}$$

Here we will not give concrete expressions of $\theta_{(5)}^{(1,1,1,1)}(\tau)$ etc. As we have already mentioned, the right hand side of (4.14) has 5 types of theta functions, which are classified by top q powers. In contrast to the $N=3$ case, $\tilde{\theta}^j_{(5)}(\tau)$ $(j=0,\cdots,4)$ is expressed by the sum of several theta functions. Then we assume the following common properties:

Conjecture 2 *For odd* $N \geq 3$,

$$\theta^2_{A_{N-1}}(\tau) = q^{t_N}\tilde{\theta}^0_{(N)}(\tau) + q^{t_N+\frac{1}{N}}\tilde{\theta}^1_{(N)}(\tau) + \cdots + q^{t_N+\frac{N-1}{N}}\tilde{\theta}^{N-1}_{(N)}(\tau), \tag{4.17}$$

where

$$t_N = \frac{1}{2N^2}{}^t\rho A_{N-1}\rho = \frac{1}{24}\frac{N^2-1}{N}. \tag{4.18}$$

For $j=0,\cdots,N-1$,

$$q^{t_N+\frac{j}{N}}\tilde{\theta}^j_{(N)}(\tau) = a^j_1\theta^{\beta^j_1}_{(N)}(\tau) + a^j_2\theta^{\beta^j_2}_{(N)}(\tau) + \cdots + a^j_{k_j}\theta^{\beta^j_{k_j}}_{(N)}(\tau). \tag{4.19}$$

$\{a^j_p\}$, $(j=0,\cdots,N-1)$ *satisfy*

$$\sum_{p=1}^{k_0} a^0_p = \sum_{p=1}^{k_1} a^1_p = \cdots = \sum_{p=1}^{k_{N-1}} a^{N-1}_p \equiv S_N. \tag{4.20}$$

The above conjecture means that each type of theta function consists of sets of theta functions satisfying the property that the sum of coefficients is equal to fixed S_N. For $N=3$, $t_3=\frac{1}{9}$ and $S_3=1$. For $N=5$, $t_5=\frac{1}{5}$ and $S_5=5$. Even for $N=7$, Conjecture 2 is true.

$N=7$ Case

$$\begin{aligned}
\theta^2_{A_6}(\tau) \;=\; & \left.\begin{array}{l} 7\theta^{(2,1,1,0,0,0)}_{(7)}(\tau) + 7\theta^{(4,2,2,0,0,0)}_{(7)}(\tau) + 7\theta^{(6,3,3,0,0,0)}_{(7)}(\tau) \\ +21\theta^{(1,1,1,1,1,1)}_{(7)}(\tau) + 7\theta^{(0,3,1,0,1,0)}_{(7)}(\tau) \end{array}\right\} \equiv q\tilde{\theta}^5_{(7)}(\tau) \\
& \left.\begin{array}{l} +\theta^{(0,1,0,0,1,0)}_{(7)}(\tau) + 2\theta^{(2,2,0,2,0,0)}_{(7)}(\tau) + 2\theta^{(5,0,1,0,0,1)}_{(7)}(\tau) + 2\theta^{(0,5,0,1,0,0)}_{(7)}(\tau) \\ +2\theta^{(1,0,5,0,1,0)}_{(7)}(\tau) + 4\theta^{(0,2,2,1,0,0)}_{(7)}(\tau) + 4\theta^{(2,0,2,2,1,0)}_{(7)}(\tau) + 4\theta^{(1,2,2,0,2,0)}_{(7)}(\tau) \\ +6\theta^{(1,1,1,0,3,0)}_{(7)}(\tau) + 8\theta^{(0,1,1,1,1,0)}_{(7)}(\tau) + 14\theta^{(1,1,1,1,0,3)}_{(7)}(\tau) \end{array}\right\} \equiv q^{\frac{2}{7}}\tilde{\theta}^0_{(7)}(\tau) \\
& \left.\begin{array}{l} +\theta^{(0,2,0,0,2,0)}_{(7)}(\tau) + 2\theta^{(4,4,0,4,0,0)}_{(7)}(\tau) + 2\theta^{(10,0,2,0,0,2)}_{(7)}(\tau) + 2\theta^{(0,10,0,2,0,0)}_{(7)}(\tau) \\ +2\theta^{(2,0,10,0,2,0)}_{(7)}(\tau) + 4\theta^{(0,4,4,2,0,0)}_{(7)}(\tau) + 4\theta^{(4,0,4,4,2,0)}_{(7)}(\tau) + 4\theta^{(2,4,4,0,4,0)}_{(7)}(\tau) \\ +6\theta^{(2,2,2,0,6,0)}_{(7)}(\tau) + 8\theta^{(0,2,2,2,2,0)}_{(7)}(\tau) + 14\theta^{(2,2,2,2,0,6)}_{(7)}(\tau) \end{array}\right\} \equiv q^{\frac{8}{7}}\tilde{\theta}^6_{(7)}(\tau) \\
& \left.\begin{array}{l} +\theta^{(0,3,0,0,3,0)}_{(7)}(\tau) + 2\theta^{(6,6,0,6,0,0)}_{(7)}(\tau) + 2\theta^{(15,0,3,0,0,3)}_{(7)}(\tau) + 2\theta^{(0,15,0,3,0,0)}_{(7)}(\tau) \\ +2\theta^{(3,0,15,0,3,0)}_{(7)}(\tau) + 4\theta^{(0,6,6,3,0,0)}_{(7)}(\tau) + 4\theta^{(6,0,6,6,3,0)}_{(7)}(\tau) + 4\theta^{(3,6,6,0,6,0)}_{(7)}(\tau) \\ +6\theta^{(3,3,3,0,9,0)}_{(7)}(\tau) + 8\theta^{(0,3,3,3,3,0)}_{(7)}(\tau) + 14\theta^{(3,3,3,3,0,9)}_{(7)}(\tau) \end{array}\right\} \equiv q^{\frac{4}{7}}\tilde{\theta}^2_{(7)}(\tau)
\end{aligned}$$

$$
\left.\begin{array}{l}
+\theta_{(7)}^{(0,0,1,1,0,0)}(\tau)+2\theta_{(7)}^{(4,0,1,0,0,0)}(\tau)+2\theta_{(7)}^{(2,4,0,1,0,0)}(\tau)+2\theta_{(7)}^{(1,0,0,5,0,0)}(\tau)\\
+3\theta_{(7)}^{(2,1,0,0,1,2)}(\tau)+4\theta_{(7)}^{(2,0,1,1,1,0)}(\tau)+4\theta_{(7)}^{(2,1,2,1,0,0)}(\tau)+4\theta_{(7)}^{(1,2,1,2,1,0)}(\tau)\\
+5\theta_{(7)}^{(1,0,2,2,0,1)}(\tau)+6\theta_{(7)}^{(1,2,1,0,0,1)}(\tau)+8\theta_{(7)}^{(1,1,0,2,2,0)}(\tau)+8\theta_{(7)}^{(2,2,0,1,1,1)}(\tau)
\end{array}\right\}\equiv q^{\frac{3}{7}}\tilde{\theta}_{(7)}^{1}(\tau)
$$

$$
\left.\begin{array}{l}
+\theta_{(7)}^{(0,0,2,2,0,0)}(\tau)+2\theta_{(7)}^{(8,0,2,0,0,0)}(\tau)+2\theta_{(7)}^{(4,8,0,2,0,0)}(\tau)+2\theta_{(7)}^{(2,0,0,10,0,0)}(\tau)\\
+3\theta_{(7)}^{(4,2,0,0,2,4)}(\tau)+4\theta_{(7)}^{(4,0,2,2,2,0)}(\tau)+4\theta_{(7)}^{(4,2,4,2,0,0)}(\tau)+4\theta_{(7)}^{(2,4,2,4,2,0)}(\tau)\\
+5\theta_{(7)}^{(2,0,4,4,0,2)}(\tau)+6\theta_{(7)}^{(2,4,2,0,0,2)}(\tau)+8\theta_{(7)}^{(2,2,0,4,4,0)}(\tau)+8\theta_{(7)}^{(4,4,0,2,2,2)}(\tau)
\end{array}\right\}\equiv q^{\frac{5}{7}}\tilde{\theta}_{(7)}^{3}(\tau)
$$

$$
\left.\begin{array}{l}
+\theta_{(7)}^{(0,0,3,3,0,0)}(\tau)+2\theta_{(7)}^{(12,0,3,0,0,0)}(\tau)+2\theta_{(7)}^{(6,12,0,3,0,0)}(\tau)+2\theta_{(7)}^{(3,0,0,15,0,0)}(\tau)\\
+3\theta_{(7)}^{(6,3,0,0,3,6)}(\tau)+4\theta_{(7)}^{(6,0,3,3,3,0)}(\tau)+4\theta_{(7)}^{(6,3,6,3,0,0)}(\tau)+4\theta_{(7)}^{(3,6,3,6,3,0)}(\tau)\\
+5\theta_{(7)}^{(3,0,6,6,0,3)}(\tau)+6\theta_{(7)}^{(3,6,3,0,0,3)}(\tau)+8\theta_{(7)}^{(3,3,0,6,6,0)}(\tau)+8\theta_{(7)}^{(6,6,0,3,3,3)}(\tau)
\end{array}\right\}\equiv q^{\frac{6}{7}}\tilde{\theta}_{(7)}^{4}(\tau),
\tag{4.21}
$$

where

$$
\theta_{(7)}^{(n,k,l,p,r,s)}(\tau)=\sum_{m}q^{\frac{7}{2}{}^t(m+v(n,k,l,p,r,s))A_6(m+v(n,k,l,p,r,s))},
\tag{4.22}
$$

$$
v(n,k,l,p,r,s)=\frac{1}{7}A_6^{-1}\begin{pmatrix} n\\ k\\ l\\ p\\ r\\ s\end{pmatrix}.
\tag{4.23}
$$

This calculation was very hard. However, Conjecture 2 is also true and $t_7=\frac{2}{7}$, $S_7=49$.

For $N=7$, there are some remarks. $\tilde{\theta}_{(7)}^{j}(\tau)$ has several equivalent expressions using $\theta_{(7)}^{\beta_j}(\tau)$'s. This ambiguity comes from several identities between $\theta_{(7)}^{\beta_j}(\tau)$'s. Up to now $N=7$ is a critical bound for power of our computer. Odd $N>7$ is the remaining problem. But we believe that Conjecture 2 is true for odd $N>7$.

5 Formulas

In this section we generalize our previous work on $SU(2)/\mathbf{Z}_2$ partition function on orbifold-$T^4/\mathbf{Z}_2$ [5] to $SU(N)/\mathbf{Z}_N$ case, using tools prepared in Sec 3,4. We choose the following processes. First we introduce $K3$ partition function Z^{K3}. Next we separate Z^{K3} into Z^{S_0} (the contribution from S_0) and Z^B (the blow-up formula). As a concrete example we choose $v^2=0$ mod N on $K3$ case. One will soon understand that the other cases have the same structure as $v^2=0$ mod N case. $SU(N)/\mathbf{Z}_N$ partition function with $v^2=0$ mod N on K3 [19, 9] is given by

$$
\begin{aligned}
Z_0^{K3}(\tau) &= \frac{1}{N}\left(\frac{1}{\eta^{24}(\frac{\tau}{N})}+\frac{1}{\eta^{24}(\frac{\tau+1}{N})}+\cdots+\frac{1}{\eta^{24}(\frac{\tau+N-1}{N})}\right)\\
&= \frac{1}{N}\left(\frac{1}{\eta^{8}(\frac{\tau}{N})}\frac{1}{\eta^{16}(\frac{\tau}{N})}+\frac{1}{\eta^{8}(\frac{\tau+1}{N})}\frac{1}{\eta^{16}(\frac{\tau+1}{N})}+\cdots+\frac{1}{\eta^{8}(\frac{\tau+N-1}{N})}\frac{1}{\eta^{16}(\frac{\tau+N-1}{N})}\right)
\end{aligned}
$$

$$= \quad Z_0^{S_0}(\tau)Z_1^B(\tau) + Z_1^{S_0}(\tau)Z_0^B(\tau) + \cdots + Z_{N-1}^{S_0}(\tau)Z_2^B(\tau). \tag{5.1}$$

Here $Z_j^{S_0}(\tau)$ is already introduced in Sec.2 as

$$\begin{aligned} Z_j^{S_0}(\tau) &= \frac{1}{N}\left(\frac{1}{\eta^8(\frac{\tau}{N})} + \zeta_N^{\frac{1}{3}}\zeta_N^{-j}\frac{1}{\eta^8(\frac{\tau+1}{N})} + \cdots + \zeta_N^{\frac{N-1}{3}}\zeta_N^{-j(N-1)}\frac{1}{\eta^8(\frac{\tau+N-1}{N})}\right) \\ &= q^{-\frac{1}{3N}}(q^{\frac{j}{N}} + \cdots). \end{aligned} \tag{5.2}$$

Form (5.1) and (5.2) one can read $Z_j^B(\tau)$ as

$$\begin{aligned} Z_j^B(\tau) &= \frac{1}{N}\left(\frac{1}{\eta^{16}(\frac{\tau}{N})} + \zeta_N^{\frac{2}{3}}\zeta_N^{-j}\frac{1}{\eta^{16}(\frac{\tau+1}{N})} + \cdots + \zeta_N^{\frac{2(N-1)}{3}}\zeta_N^{-j(N-1)}\frac{1}{\eta^{16}(\frac{\tau+N-1}{N})}\right) \\ &= q^{-\frac{2}{3N}}(q^{\frac{j}{N}} + \cdots). \end{aligned} \tag{5.3}$$

Using the key identities and conjecture 2, one can rewrite (5.3) as

$$\begin{aligned} Z_j^B(\tau) &= \frac{1}{N}\frac{1}{\eta^{16N}(\tau)}\left((\theta^2_{A_{N-1}}(\tau))^{16} + \zeta_N^{-j}(\theta^2_{A_{N-1}}(\tau+1))^{16} + \cdots + \zeta^{-j(N-1)}(\theta^2_{A_{N-1}}(\tau+N-1))^{16}\right) \\ &= \frac{1}{N}\frac{1}{\eta^{16N}(\tau)}q^{\frac{2}{3}\frac{N^2-1}{N}}\left[\left(\sum_{j=0}^{N-1} q^{\frac{j}{N}}\tilde{\theta}^j_{(N)}(\tau)\right)^{16}\right]_{q^{\frac{j}{N}}}, \end{aligned} \tag{5.4}$$

where $[\cdots]_{q^{\frac{j}{N}}}$ stands for projecting out $q^{\frac{j}{N}+n}$ $(n \in \mathbf{Z}_{\geq 0})$ powers. Here we will give explicit expression of (5.4) for $N = 3$:

$$\begin{aligned} Z_0^B(\tau) &= \frac{1}{3}\frac{1}{\eta^{48}(\tau)}\left((\theta^2_{A_2}(\tau))^{16} + (\theta^2_{A_2}(\tau+1))^{16} + (\theta^2_{A_2}(\tau+2))^{16}\right) \\ &= \frac{1}{3}\frac{1}{\eta^{48}(\tau)}q^{\frac{16}{9}}\left[\left(\tilde{\theta}^0_{(3)}(\tau) + q^{\frac{1}{3}}\tilde{\theta}^1_{(3)}(\tau) + q^{\frac{2}{3}}\tilde{\theta}^2_{(3)}(\tau)\right)^{16}\right]_{q^{\frac{0}{3}}} \\ &= \frac{1}{3}\frac{1}{\eta^{48}(\tau)}q^{\frac{16}{9}}\left((\tilde{\theta}^0_{(3)}(\tau))^{16} + 560q(\tilde{\theta}^0_{(3)}(\tau))^{13}(\tilde{\theta}^1_{(3)}(\tau))^3 + \cdots + 16q^{10}(\tilde{\theta}^0_{(3)}(\tau))(\tilde{\theta}^2_{(3)}(\tau))^{15}\right). \end{aligned} \tag{5.5}$$

We have reconstructed $K3$ partition function which has the factor of the partition function on S_0 and that of the blow-up formula. Note that our processes in this section are opposite direction to our previous work [5]. In the previous work for $SU(2)$, we first prepared the partition function on S_0 and $\mathcal{O}(-2)$ curve blow-up formula. Next we multiplied and summed up them, so that the total partition function has S-duality. The final result was surely $K3$ partition function given by Vafa and Witten [19]. In both cases, the partition function has the same structure. It consists of the factor of the partition function on S_0 and that of the blow-up formula. The contribution from the blow-up formulas can be expressed by $\mathcal{O}(-N)$ blow-up formulas with $\frac{j}{N} + \mathbf{Z}_{\geq 0}$ powers.

6 Conclusion

We reconstructed $K3$ partition function for $SU(N)/\mathbf{Z}_N$ using orbifold construction $T^4/\mathbf{Z}_2$. Especially we rewrote $1/\eta(\frac{\tau}{N})$ into the contribution from blow-up by $\mathcal{O}(-N)$ curve blow-up formula.

In this paper remaining most serious problem is geometrical interpretation of $\mathcal{O}(-N)$ curve blow-up in our theory. Since coefficients of $\mathcal{O}(-N)$ curve blow-up formulas are all integers, we hope that there may be some geometrical interpretation. For $SU(2)$ case we can show that the contribution from blow-up comes from two blocks of $\mathcal{O}(-2)$ curve obtained from blowing up 8 points simultaneously [28]. We want similar geometrical interpretation for general N case. Hint is in [19, 17]. From our computation, we have to say that for $SU(N)$ case the contribution from blow-up comes from $U(N)$ gauge theory on ALE A_{N-1} space. If we think so, level N $SU(N)$ characters naturally appear [19, 17]. If we believe these scenario, there are the following interesting applications. In Nakajima's work [17], there are already ADE gauge theory on ALE space associated with the same ADE Lie algebra. Then, we can speculate the form of "ADE blow-up" formula using the denominator identity of ADE afiine Lie algebra. Moreover, we may determine the form of the partition function of $\mathcal{N}=4$ ADE gauge theory on $K3$ surfaces, tracing back the process of our computation in this paper.

The most crucial point in this paper is the process of rewriting $1/\eta(\frac{\tau}{N})$ by $\mathcal{O}(-N)$ curve blow-up formula. We may call this rewriting as "bulk-blow-up duality". Physical meaning of this duality is a remaining problem. But, if we apply "ADE blow-up formula" to "bulk-blow-up duality", we will get the partition function for ADE gauge theory on $K3$.

Finally, we mention level-rank duality in affine Lie algebra, that states duality between $\hat{sl}(l)_r$ and $\hat{sl}(r)_l$ [13]. In our case, we observed appearance of $\hat{sl}(N)_N$, which is self-dual in the context of level-rank duality. We expect that this special symmetry may have some physical meaning.

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