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$\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory
on Orbifold- T^4/\mathbb{Z}_2 : Higher Rank Case

Masao Jinzenji, Toru Sasaki

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$\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory on Orbifold- T^4/\mathbf{Z}_2 : Higher Rank Case

Masao Jinzenji[†], Toru Sasaki^{*}

[†] *Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo
060-0810, Japan*

^{*} *Department of Physics, Hokkaido University, Sapporo 060-0810, Japan*

[†] *jin@math.sci.hokudai.ac.jp*

^{*} *sasaki@particle.sci.hokudai.ac.jp*

Abstract

We derive the partition function of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on orbifold- T^4/\mathbf{Z}_2 for $SU(N)$. We generalize our previous work for $SU(2)$ to the $SU(N)$ case. These partition functions can be factorized into product of bulk contribution of quotient space T^4/\mathbf{Z}_2 and of blow-up formula including A_{N-1} theta functions with level N .

1 Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory on 4 dimensional manifold has the largest number of supersymmetry if it does not couple to gravity. This theory is conformal invariant and finite. Furthermore it is believed to have Montonen-Olive duality [14] (a type of S -duality). In [19], Vafa and Witten considered the twisted $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and derived its exact partition function. Using this exactly calculated partition function, they confirmed Montonen-Olive duality. In their derivation they separated its partition function into the following two factors (the same process was also used in [20]). One part is the contribution from bulk (non-zero locus of the section of the canonical bundle), and is expressed by Göttsche formula [16, 15, 19]. The other is the contribution from cosmic string (zero locus of the section of the canonical bundle), and it is expressed by blow-up formula [21, 19]. The both factors are multiplied and summed up so that the total contribution has the modular property (S -duality). After their work [19], deeper analysis and further applications were done such as in [2, 3, 5, 7, 8, 9, 10, 12, 18, 22, 24, 25, 26, 27]. Especially in [18], Sako and T.S. (one of the authors of this paper) revealed that Euler number of instanton moduli space and Seiberg-Witten invariants are connected in the framework of Vafa-Witten theory.

In our previous work [5], we derived the partition function of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on orbifold T^4/\mathbf{Z}_2 for gauge group $SU(2)$ from the point of view of orbifold construction. Precisely speaking, we expressed the $SU(2)$ partition function of $K3$ surface as the product of the bulk contribution from the quotient space $T^4/\mathbf{Z}_2 = S_0$ and blow-up formula coming from the sixteen singularities of S_0 . In this paper, we try to generalize it to $SU(N)$ (N is odd prime) case. For $SU(N)$ theory it is well-known that the partition function on X ($X = K3, T^4$ or $\frac{1}{2}K3$) is almost a Hecke transformation of order N of $1/\eta^\chi(\tau)$ (χ is Euler number of X). This fact was derived and used from physical side in [19, 12, 9], and was confirmed mathematically in [25]. From this starting point, we ask whether $K3$ partition function for $SU(N)$ (or $SU(N)/\mathbf{Z}_N$) can be expressed by the product of the following two factors: the contribution on $T^4/\mathbf{Z}_2 = S_0$ from bulk and $\mathcal{O}(-2)$ curve blow-up formula from 16 orbifold singularities. Our answer is no if we use the $\mathcal{O}(-2)$ curve blow-up formula. But our answer is yes if we use the $\mathcal{O}(-N)$ curve blow-up formula derived in [7] instead. The formula is given by

$$Z_k^{K3} = \sum_{j=0}^{N-1} Z_j^{S_0} Z_{k-j+1}^B, \quad (1.1)$$

$$Z_j^B = \sum_{\{\beta_i\}} a_{\{\beta_i\}}^j \prod_{l=1}^{16} \frac{\theta_{(N)}^{\beta_l}(\tau)}{\eta^N(\tau)}. \quad (1.2)$$

Here Z_j^X stands for the partition function on X with $v^2 = 2j \pmod N$. X is $K3, S_0$ or blow-up respectively. v is 't Hooft flux $v \in H^2(X, \mathbf{Z}_N)$. $\frac{\theta_{(N)}^{\beta_l}(\tau)}{\eta^N(\tau)}$ is $\mathcal{O}(-N)$ curve blow-up formula labeled by β^l [7]. We emphasize that $a_{\{\beta_i\}}$ is an integer. In our derivation of (1.2), we used the well-known ‘‘denominator identity’’ in the theory of affine Lie algebras. This fact suggests that $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and rational conformal field theory (affine Lie algebra) are closely related. This relation was already pointed out in [19].

$\mathcal{O}(-N)$ curve blow-up

In this paper, we found the following two key relations. One is

$$\frac{1}{\eta(\frac{\tau}{N})} = \frac{\theta_{A_{N-1}}^2(\tau)}{\eta^N(\tau)}, \quad (1.3)$$

where $\theta_{A_{N-1}}^2(\tau)$ is a theta series related to affine Lie algebra. The other is

$$\theta_{A_{N-1}}(\tau) = \sum_{\beta} a_{\beta} \theta_{(N)}^{\beta}(\tau), \quad (1.4)$$

where $\theta_{(N)}^{\beta}(\tau)$ is Kapranov's theta function labeled by β associated with $\mathcal{O}(-N)$ curve blow-up [7]. We emphasize that a_{β} is an integer. Using these two key relations, we could rewrite $1/\eta(\frac{\tau}{N})$ by $\mathcal{O}(-N)$ curve blow-up formulas. Why did we face $\mathcal{O}(-N)$ curve blow-up formula instead of $\mathcal{O}(-2)$ curve blow-up formula? We want geometrical interpretation. The fact that a_{β} is an integer suggests the possibility of geometrical interpretation. Hint is in [19, 17]. We can easily see that $\mathcal{O}(-N)$ curve blow-up formula appearing in our formulas comes from level N $SU(N)$ characters. On the other hand level k characters of $SU(N)$ arise for $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(k)$ on A_{N-1} ALE space [19, 17]. Therefore, it is natural that $\mathcal{O}(-N)$ curve blow-up formula appears. For we consider $SU(N)$ gauge group [19, 17].

This paper is organized as follows. In Sec.2, we review our previous work for $SU(2)$ and overview the untwisted sector of S_0 for $SU(N)$. In Sec.3, we introduce the key identity and verify it using denominator identity of affine Lie algebra. In Sec.4, we give the key conjecture and rewrite $1/\eta(\frac{\tau}{N})$ by $\mathcal{O}(-N)$ curve blow-up formula. In Sec.5, we derive the partition function for $SU(N)/\mathbf{Z}_N$ using the results of Sec.3, 4. In Sec.6, we conclude and discuss the remaining problems.

2 Review and Untwisted Sector

In this section, we briefly review the geometrical background of orbifold T^4/\mathbf{Z}_2 [1, 4] and overview untwisted sector of $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory on orbifold- T^4/\mathbf{Z}_2 for $SU(N)$ [19, 3, 9, 24, 10]. Then we review the reconstruction process of $K3$ partition function for $SU(2)$ using $\mathcal{O}(-2)$ curve blow-up formula [21, 5].

2.1 Orbifold T^4/\mathbf{Z}_2

In this subsection, we reconstruct $K3$ surface from T^4 surface, keeping trivial canonical bundle $K_X = 0$ [1, 4].

First we consider T^4 surface. T^4 surface has trivial canonical bundle $K_{T^4} = 0$, but has nontrivial Picard groups. After the \mathbf{Z}_2 -identification, we have $T^4/\mathbf{Z}_2 = S_0$, which has $K_{S_0} = 0$ and trivial Picard groups. However S_0 has sixteen orbifold singularity points. Thus we want to resolve these singularities. For this purpose, we replace these singularity points with $\mathcal{O}(-2)$ curves (we call this process $\mathcal{O}(-2)$ curve blow-up). After this process we have $\hat{S}_0 = K3$, which has $K_{K3} = 0$ and no singularity.

To review these processes, we write down the change of the Euler numbers.

$$\chi(T^4) = 0 \rightarrow \chi(S_0) = 8 \rightarrow \chi(K3) = 24. \quad (2.1)$$

2.2 Untwisted Sector

General Structure of Vafa-Witten Conjecture

Following [19, 9, 10, 22], we review the general structure of Vafa-Witten conjecture. For $SU(N)/\mathbf{Z}_N$ theory with 't Hooft flux v on X , partition function is defined by,

$$Z_v^X(\tau) \equiv q^{-\frac{N\chi(X)}{24}} \sum_k \chi(\mathcal{N}(v, k)) q^k \quad (q := \exp(2\pi i\tau)), \quad (2.2)$$

where $\mathcal{N}(v, k)$ is the moduli space of anti-self-dual connections associated to $SU(N)/\mathbf{Z}_N$ -principal bundle with 't Hooft flux $v \in H^2(X, \mathbf{Z}_N)$ and fractional instanton number $k \in \mathbf{Z}/2N$. $\chi(X)$ is Euler number of X . For this partition function, Vafa and Witten conjectured

$$Z_v^X\left(-\frac{1}{\tau}\right) = N^{-\frac{b_2(X)}{2}} \left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} \cdot \sum_{u \in H^2(X, \mathbf{Z}_N)} \zeta_N^{u \cdot v} Z_u^X(\tau), \quad (2.3)$$

where $\zeta_N = \exp(\frac{2\pi i}{N})$.

For later use, we introduce

$$\begin{aligned} Z_{SU(N)}^X(\tau) &\equiv \frac{1}{N} Z_0^X(\tau), \\ Z_{SU(N)/\mathbf{Z}_N}^X(\tau) &\equiv \sum_{u \in H^2(X, \mathbf{Z}_N)} Z_u^X(\tau). \end{aligned} \quad (2.4)$$

For these partition functions, the conjecture is reduced to the following formula:

$$Z_0^X\left(-\frac{1}{\tau}\right) = N^{-\frac{b_2(X)}{2}} \left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} Z_{SU(N)/\mathbf{Z}_N}^X(\tau). \quad (2.5)$$

Partition Function of the Untwisted Sector of S_0

In this part, we derive the partition function of the untwisted sector of S_0 as a concrete example. First we think of the moduli space of $X = T^4$. In general, one treats the moduli space $\mathcal{M}_H(N, c_1, c_2)$ of rank N stable sheaves E with Chern classes c_1, c_2 [22, 24], but we want to treat the moduli space $\mathcal{N}(v, k)$ of $SU(N)/\mathbf{Z}_N$ or $SU(N)$ vector bundles with v, k . Thus following our previous work [5], we identify $\mathcal{M}_H(N, c_1, c_2) \equiv \mathcal{N}(v, k)$ with $c_1 = v \bmod N$ and $k = c_2 - \frac{(N-1)v^2}{2N}$ in the following. Throughout this paper, we restrict N to prime numbers. For this restriction we follow [27]. This restriction makes our problem simpler. According to [24], the moduli space of rank N stable sheaves E of V is given below. We introduce Mukai vector

$$V = ch(E)\sqrt{td_X} = N + c_1 + \frac{c_1^2 - 2c_2}{2}. \quad (2.6)$$

Note that in our case $X = T^4$, $td_X = 1$. We remark here that the condition “ N : prime” makes the Mukai vector primitive. We also introduce the inner product

$$\begin{aligned} \langle V^2 \rangle &= - \int_X \left(N + c_1 + \frac{c_1^2 - 2c_2}{2} \right) \vee \left(N + c_1 + \frac{c_1^2 - 2c_2}{2} \right) \\ &= 2Nc_2 - (N-1)c_1^2. \end{aligned} \quad (2.7)$$

Here we use a symmetric bilinear form on $\oplus_j H^{2j}(X, \mathbf{Z})$:

$$\begin{aligned} \langle x, y \rangle &= - \int_X (x \vee y) \\ &= \int_X (x_1 y_1 - x_0 y_2 - x_2 y_0), \end{aligned} \quad (2.8)$$

where $x = x_0 + x_1 + x_2, x_j \in H^{2j}(X, \mathbf{Z})$ and $x \vee = x_0 - x_1 + x_2$.

With these preparation, we can describe the moduli space labeled by V

$$M_H^X(V) \cong \hat{X} \times (X)^{[\leq \frac{v^2}{2}]} = \hat{X} \times (X)^{[Nc_2 - \frac{(N-1)c_1^2}{2}]}, \quad (2.9)$$

where H is some line bundle on X and \hat{X} is the dual of X [24].

Since S_0 has the trivial canonical bundle like $K3$ and T^4 , it doesn't have cosmic strings, which are given by zero locus of the section of the canonical bundle. Hence we assume the moduli space of V in S_0 case,

$$M_H(V) \cong (S_0)^{[\leq \frac{v^2}{2}]} = (S_0)^{[Nn - \frac{(N-1)v^2}{2}]}, \quad (2.10)$$

where we used the vanishing Picard group to drop the \hat{X} part ¹. Using this, we define the partition function of S_0 of the untwisted sector with $v^2 = 2j \pmod{N}$ type,

$$\begin{aligned} Z_j^{S_0}(\tau) &= q^{-\frac{1}{3N}} \sum_{v^2 \equiv j \pmod{N}, n} e(M_H(V)) q^{n - \frac{N-1}{N}j} \\ &= q^{-\frac{1}{3N}} \sum_n e((S_0)^{[Nn - (N-1)j]}) q^{n - \frac{N-1}{N}j} \\ &= q^{-\frac{1}{3N}} \sum_m e((S_0)^{[m]}) \frac{(q^{\frac{1}{N}})^m + \zeta_N^{-j} (\zeta_N q^{\frac{1}{N}})^m + \dots + \zeta_N^{-j(N-1)} (\zeta_N^{N-1} q^{\frac{1}{N}})^m}{N} \\ &= \frac{1}{N} \left(\frac{1}{\eta^8(\frac{\tau}{N})} + \zeta_N^{\frac{1}{3}} \zeta_N^{-j} \frac{1}{\eta^8(\frac{\tau+1}{N})} + \dots + \zeta_N^{\frac{N-1}{3}} \zeta_N^{-j(N-1)} \frac{1}{\eta^8(\frac{\tau+N-1}{N})} \right). \end{aligned} \quad (2.11)$$

For trivial type $v = 0$, we follow [19] and set,

$$Z_t^{S_0}(\tau) = C \frac{1}{\eta^8(N\tau)} + \frac{1}{N} \frac{1}{\eta^8(\frac{\tau}{N})} + \frac{1}{N} \zeta_N^{\frac{1}{3}} \frac{1}{\eta^8(\frac{\tau+1}{N})} + \dots + \frac{1}{N} \zeta_N^{\frac{N-1}{3}} \frac{1}{\eta^8(\frac{\tau+N-1}{N})}, \quad (2.12)$$

where C is not determined. Note that $\frac{1}{\eta^8(\frac{1}{N\tau})}$ is obtained from $\frac{1}{\eta^8(\frac{\tau}{N})}$ using the modular transformation $\tau \rightarrow -\frac{1}{\tau}$.

¹Precisely speaking, we have to take care of non-trivial Todd class of S_0 . But we don't know the precise definition of Todd class of S_0 . Moreover, this correction does not affect the results of computation severely. Therefore, we neglect here this correction.

2.3 $\mathcal{O}(-2)$ Curve Blow-up Formula and $K3$ Partition Function: Review of the $SU(2)$ case

In this subsection, we review the previous work for $SU(2)$ [5]. Especially, we reconstruct $K3$ partition function from the contribution from the untwisted sector for the $SU(2)$ case. In [5], we introduce $\mathcal{O}(-2)$ curve blow-up formulas as the contribution from the twisted sector:

$$\frac{\theta_2(\tau)}{\eta(\tau)^2}, \quad \frac{\theta_3(\tau)}{\eta(\tau)^2}, \quad \frac{\theta_4(\tau)}{\eta(\tau)^2}. \quad (2.13)$$

These formulas were speculated from the proof of $\mathcal{O}(-1)$ curve blow-up formula [21, 10]. Furthermore we introduce the blow-up formulas that correspond to blowing up 16 orbifold singularities:

$$\tilde{Z}_1(\tau) = \frac{\theta_2^8(\tau)(\theta_3^8(\tau) + \theta_4^8(\tau))}{\eta^{32}(\tau)}, \quad (2.14)$$

$$\tilde{Z}_2(\tau) = \frac{\theta_2^8(\tau)(\theta_3^8(\tau) - \theta_4^8(\tau))}{\eta^{32}(\tau)}, \quad (2.15)$$

$$\tilde{Z}_3(\tau) = \frac{\theta_3^8(\tau)\theta_4^8(\tau)}{\eta^{32}(\tau)}, \quad (2.16)$$

so that these are consistent with Vafa-Witten conjecture. Using these formulas, we reconstruct $K3$ partition function:

$$\begin{aligned} Z_{even}^{K3}(\tau) &= 2^{-7} \left(Z_{odd}^{S_0}(\tau)\tilde{Z}_2(\tau) + Z_{even}^{S_0}(\tau)\tilde{Z}_1(\tau) \right), \\ Z_{odd}^{K3}(\tau) &= 2^{-7} \left(Z_{odd}^{S_0}(\tau)\tilde{Z}_1(\tau) + Z_{even}^{S_0}(\tau)\tilde{Z}_2(\tau) \right). \end{aligned} \quad (2.17)$$

$$Z_t^{K3}(\tau) = \frac{1}{4} \frac{1}{\eta^8(2\tau)} \tilde{Z}_3(\tau) + 2^{-7} \left(Z_{even}^{S_0}(\tau)\tilde{Z}_2(\tau) + Z_{odd}^{S_0}(\tau)\tilde{Z}_1(\tau) \right). \quad (2.18)$$

Note that combinations and coefficients are also consistent with Vafa-Witten conjecture.

To compare these results with those of Vafa-Witten, we rewrite these partition function using identities between eta and theta functions.

$$\begin{aligned} Z_t^{K3}(\tau) &= \frac{1}{4} \frac{1}{\eta^{24}(2\tau)} + \frac{1}{2} \frac{1}{\eta^{24}(\frac{\tau}{2})} + \frac{1}{2} \frac{1}{\eta^{24}(\frac{\tau}{2} + \frac{1}{2})}, \\ Z_{even}^{K3}(\tau) &= \frac{1}{2} \frac{1}{\eta^{24}(\frac{\tau}{2})} + \frac{1}{2} \frac{1}{\eta^{24}(\frac{\tau}{2} + \frac{1}{2})}, \\ Z_{odd}^{K3}(\tau) &= \frac{1}{2} \frac{1}{\eta^{24}(\frac{\tau}{2})} - \frac{1}{2} \frac{1}{\eta^{24}(\frac{\tau}{2} + \frac{1}{2})}. \end{aligned} \quad (2.19)$$

3 From Bulk to Blow-up

In the rest of this paper, we generalize our previous work for $SU(2)$ [5] to the $SU(N)$ case. Especially in the following two sections, we introduce the blow-up formula appeared in our

theory. What we want is the generalization of (2.13). In this section, we first introduce the key identity and then we briefly verify this identity [11]. We also remark the link to the affine Lie algebra [6].

3.1 Key Identity

We will see in Sec.5 that the identities between eta and theta functions play the key role. For $SU(2)$ case this identity is simply given by

$$\frac{1}{\eta(2\tau)} = \frac{\theta_4(2\tau)}{\eta^2(\tau)}, \quad (3.1)$$

or

$$\frac{1}{\eta(\frac{\tau}{2})} = \frac{1}{2} \frac{\theta_2(\frac{\tau}{2})}{\eta^2(\tau)}. \quad (3.2)$$

The right hand sides of these identities have the form $\Theta/\eta^2(\tau)$. These forms are general for the blow-up formula of the compactified moduli space [21, 10]. Especially the part of eta function comes from Gieseker-Maruyama compactification [10, 21]. We expect the same situation for $SU(N)$. For $SU(N)$ case the corresponding form is $\Theta/\eta^N(\tau)$. In this principle, we searched and found the following key identities,

$$\frac{1}{\eta(N\tau)} = \frac{\theta_{A_{N-1}}^1(\tau)}{\eta^N(\tau)}, \quad (3.3)$$

or alternatively

$$\frac{1}{\eta(\frac{\tau}{N})} = \frac{\theta_{A_{N-1}}^2(\tau)}{\eta^N(\tau)}, \quad (3.4)$$

where we introduce A_{N-1} theta functions,

$$\theta_{A_{N-1}}^0(\tau) \equiv \sum_{m \in \mathbf{Z}^{N-1}} q^{\frac{1}{2} m A_{N-1} m}, \quad (3.5)$$

$$\theta_{A_{N-1}}^1(\tau) \equiv \sum_{m \in \mathbf{Z}^{N-1}} q^{\frac{1}{2} m A_{N-1} m} e^{2\pi i m \cdot \delta}, \quad (3.6)$$

$$\theta_{A_{N-1}}^2(\tau) \equiv \sum_{m \in \mathbf{Z}^{N-1}} q^{\frac{1}{2} (m + \frac{\rho}{N}) A_{N-1} (m + \frac{\rho}{N})}. \quad (3.7)$$

Here A_{N-1} stands for Cartan matrix for $SU(N)$. We also introduced

$$\delta \equiv \frac{1}{N}(1, \dots, 1), \quad \rho = N A_{N-1}^{-1} \delta. \quad (3.8)$$

The vector ρ is the same as the usual ρ (half the sum of the positive root of Lie algebra A_{N-1}). Note that we can obtain (3.3) if we apply the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ to (3.4).

3.2 Denominator (Macdonald) Identity and Affine Lie Algebra

In this subsection, we will give the proof of (3.4). First we think of the simplest case, that is (3.2). $\theta_2(\frac{\tau}{2})$ has the following product formula (which comes from Jacobi's triple product identity) [6]:

$$\theta_2\left(\frac{\tau}{2}\right) = 2q^{\frac{1}{16}} \prod_{n=1}^{\infty} (1 - q^{\frac{n}{2}})(1 + q^{\frac{n}{2}})^2 = 2 \frac{\eta^2(\tau)}{\eta(\frac{\tau}{2})}. \quad (3.9)$$

One can easily find the justification of the last equality. This is the proof of (3.2) itself. Next we think of the general N case (3.4). We begin with the denominator identity (which corresponds to Jacobi's triple product identity for $N = 2$ case) [6].

Theorem 1 (Denominator Identity)

$$\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha} = \sum_{w \in W} \epsilon(w) e(w(\rho) - \rho). \quad (3.10)$$

Here we introduce the notations of the affine Lie algebras.

Notation 1 *Notations [6, 11],*

Δ : set of roots, $\Delta_+ \subset \Delta$: set of positive roots, l : rank of Cartan matrix

$\text{mult } \alpha$: multiplicity of $\alpha \in \Delta_+$,

W : Weyl Group, $w \in W$: its element, $\epsilon(w) = (-1)^{l(w)}$, $l(w)$: length of w ,

ρ : half the sum of the positive roots,

h^\vee : dual coxeter number.

From now on, the symbol "prime (')" means restriction to classical Lie algebra associated with affine Lie algebra.

After several calculations and settings, one can rewrite (3.10) as

Corollary 1

$$q^{\frac{|\rho|^2}{2h^\vee}} \prod_{n \geq 1} ((1 - q^n)^l \prod_{\alpha \in \Delta'} (1 - q^n e(\alpha))) = \sum_{\alpha \in M} \chi'(h^\vee \alpha) q^{\frac{|\rho + h^\vee \alpha|^2}{2h^\vee}}, \quad (3.11)$$

where

$$\chi'(\lambda) = \frac{\sum_{w \in W'} \epsilon(w) e(w(\lambda + \rho) - \rho)}{\prod_{\alpha \in \Delta'_+} (1 - e(-\alpha))}. \quad (3.12)$$

Note that M is a root lattice spanned over \mathbf{Z} by $\alpha_j \in \Delta'$, and $|u|^2$ is the square length of vector u in M . In our case, $|\sum_{j=1}^{N-1} m_j \alpha_j|^2 = {}^t m A_{N-1} m$.

We give the proof of (3.4).

proof) In our case, we take A_{N-1} as Lie algebra. Then we obtain

$$h^\vee = N, \quad l = N - 1, \quad \frac{|\rho|^2}{2h^\vee} = \frac{N^2 - 1}{24}. \quad (3.13)$$

We also take the following specialization:

$$e(\alpha) = \exp\left(\frac{2\pi i \alpha \cdot \rho}{N}\right). \quad (3.14)$$

From Theorem 1 applied to classical Lie algebra A_{N-1} , we can see,

$$\chi'(N\alpha) = 1. \quad (3.15)$$

Applying (3.13), (3.14), (3.15) to Corollary 1, we obtain

$$\begin{aligned} & q^{\frac{N^2-1}{24}} \prod_{n=1}^{\infty} (1-q^n)^{N-1} \prod_{j=1}^{N-1} (1-q^n \zeta_N^j)^N \\ &= \sum_{m \in (N\mathbf{Z})^{N-1}} q^{\frac{1}{2N} t(m+\rho)_{A_{N-1}}(m+\rho)}. \end{aligned} \quad (3.16)$$

If we change q into $q^{\frac{1}{N}}$, we obtain (3.4). \square

Note that this proof was first given by Macdonald (p.120,121) [11].

In this section, we have showed that the eta function $1/\eta(\frac{\tau}{N})$ can be described by $\Theta/\eta^N(\tau)$ using the key identity. But this identity is not enough to rewrite the eta function $1/\eta(\frac{\tau}{N})$ by the blow-up formula. In the next section, we complete our attempt to obtain the blow-up formula for $SU(N)$ in our theory. However, we have shown that this theory is closely related to the Affine Lie algebras. This relation was already pointed out in [19].

4 $\mathcal{O}(-N)$ Curve Blow-up

We continue the discussion of the blow-up formula. Especially in this section, we try to rewrite the theta function in the previous section as the sum of the $\mathcal{O}(-N)$ curve blow-up formulas [5, 7]. First, we introduce the key conjecture. Next, we show some examples of this conjecture.

4.1 Key Conjecture

In the previous section we introduced the similar expression to the blow-up formulas. However in the geometrical context, theta function $\theta_{A_{N-1}}^2(\tau)$ is not the theta function appeared in the blow-up formula. For $N = 2$ case fortunately there is the following duplication formula:

$$\theta_{A_1}^2(\tau)^2 = \frac{1}{2} \theta_2(\tau) \theta_3(\tau). \quad (4.1)$$

That is why we have introduced (2.13) directly instead of $\frac{\theta_{A_1}^2(\tau)}{\eta^2(\tau)}$. As we have already mentioned, (2.13) are $\mathcal{O}(-2)$ curve blow-up formulas. $\mathcal{O}(-2)$ curve blow-up is consistent with geometrical context (recall Sec.2.1). In this principle we first wanted to find identities for general N case like (4.1), with which one can rewrite $\theta_{A_{N-1}}^2(\tau)$ into the theta function appearing in $\mathcal{O}(-2)$ curve blow-up formula. For this purpose we cite the theorem derived by Kapranov [7]:

Theorem 2 (Kapranov) $\mathcal{O}(-d)$ -curve blow-up formula for $SU(N)$ gauge theory is given by the A_{N-1} theta series with level d :

$$\sum_{a \in L} q^{\Psi(a,f) - d\Psi(a,a)/2}, \quad (4.2)$$

where L is the A_{N-1} root lattice, f is an element of the weight lattice and $\Psi(a, b) = -{}^t a A_{N-1} b$.

Remark 1 In Theorem 2, the factor coming from Gieseker-Maruyama compactification is neglected because Kapranov treated uncompactified case in [7]. Compactified version of $\mathcal{O}(-1)$ curve blow-up formula for $SU(N)$ case was first derived by Yoshioka [26].

Remark 2 The original version of Theorem 2 in [7] takes f as an element of the root lattice. But in [19] where $\mathcal{O}(-1)$ -curve blow-up formula for $SU(2)$ gauge theory is considered, Vafa and Witten introduced the two types of blow-up formula:

$$\Theta_0 := \sum_{n \in \mathbf{Z}} q^{n^2}, \quad \Theta_1 := \sum_{n \in \mathbf{Z}} q^{(n-\frac{1}{2})^2}. \quad (4.3)$$

Θ_0 and Θ_1 correspond to taking f as 0 and f as $\frac{1}{2}\alpha$ respectively. Therefore, it is natural to take f as an element of the weight lattice. This generalization is also compatible with our treatment of $\mathcal{O}(-2)$ -curve blow-up formula for $SU(2)$ gauge theory [5].

We applied Theorem 2 to the $d = 2$ case and tried to describe $\theta_{A_{N-1}}(\tau)$ by this Kapranov's theta function. As a result our attempt failed. Instead, we have found the following key conjecture:

Conjecture 1 For odd $N \geq 3$,

$$\theta_{A_{N-1}}^2(\tau) = a_1 \theta_{(N)}^{\beta_1}(\tau) + a_2 \theta_{(N)}^{\beta_2}(\tau) + \cdots + a_k \theta_{(N)}^{\beta_k}(\tau), \quad (4.4)$$

where

$$\theta_{(N)}^{\beta_j}(\tau) \equiv \sum_{m \in \mathbf{Z}^{N-1}} q^{\frac{N}{2} {}^t (m+v(\beta_j)) A_{N-1} (m+v(\beta_j))}, \quad (4.5)$$

$$v(\beta_j) \equiv \frac{1}{N} A_{N-1}^{-1} \begin{pmatrix} \beta_j^{(1)} \\ \beta_j^{(2)} \\ \vdots \\ \beta_j^{(N-1)} \end{pmatrix}, \quad \beta_j^{(n)} \in \mathbf{Z}. \quad (4.6)$$

a_j is a non-negative integer. k is an appropriate finite positive integer.

Using this conjecture, one can describe $\theta_{A_{N-1}}^2(\tau)$ by Kapranov-type theta function associated with $\mathcal{O}(-N)$ curve blow-up formula [7]. The reason why we call $\theta_{(N)}^{\beta_j}(\tau)$ Kapranov-type is given below. Using the notation in [7], $\theta_{(N)}^{\beta_j}(\tau)$ can be expressed as follows:

$$\theta_{(N)}^{\beta_j}(\tau) = \sum_{a \in L} q^{-\Psi(f, f)/2N + \Psi(a, f) - N\Psi(a, a)/2}. \quad (4.7)$$

Apart from the top factor $q^{-\Psi(f, f)/2N}$, (4.7) is the same theta function as (4.2) for $d = N$. Kapranov-type theta function is level N theta function [6].

We have found that (3.4) is the sum of the $\mathcal{O}(-N)$ curve blow-up formulas. Appearance of $\mathcal{O}(-N)$ curve blow-up for general N case instead of $\mathcal{O}(-2)$ is natural because level N representation of affine Lie algebra appears in $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ on ALE spaces [17, 19].

4.2 Examples: $N = 3, 5, 7$ Case

In this subsection we give some explicit examples of Conjecture 1. We confirmed these examples using Maple V.

$N = 3$ Case

$$\theta_{A_2}^2(\tau) = \theta_{(3)}^{(1,0)}(\tau) + \theta_{(3)}^{(2,0)}(\tau) + \theta_{(3)}^{(4,0)}(\tau), \quad (4.8)$$

where

$$\theta_{(3)}^{(k,l)}(\tau) = \sum_{m \in \mathbf{Z}^2} q^{\frac{3}{2}t(m+v(k,l))A_2(m+v(k,l))}, \quad (4.9)$$

$$v(k,l) = \frac{1}{3}A_2^{-1} \begin{pmatrix} k \\ l \end{pmatrix}. \quad (4.10)$$

Here, we give concrete expressions of $\theta_{(3)}^{(1,0)}(\tau)$, $\theta_{(3)}^{(2,0)}(\tau)$, $\theta_{(3)}^{(4,0)}(\tau)$.

$$\theta_{(3)}^{(1,0)}(\tau) = \sum_{(m_1, m_2) \in \mathbf{Z}^2} q^{\frac{1}{9} + m_1 + 3(m_1^2 + m_2^2 - m_1 m_2)} \equiv q^{\frac{1}{9}} \tilde{\theta}_{(3)}^0(\tau), \quad (4.11)$$

$$\theta_{(3)}^{(2,0)}(\tau) = \sum_{(m_1, m_2) \in \mathbf{Z}^2} q^{\frac{4}{9} + 2m_1 + 3(m_1^2 + m_2^2 - m_1 m_2)} \equiv q^{\frac{4}{9}} \tilde{\theta}_{(3)}^1(\tau), \quad (4.12)$$

$$\theta_{(3)}^{(4,0)}(\tau) = \sum_{(m_1, m_2) \in \mathbf{Z}^2} q^{\frac{16}{9} + 4m_1 + 3(m_1^2 + m_2^2 - m_1 m_2)} \equiv q^{\frac{7}{9}} \tilde{\theta}_{(3)}^2(\tau). \quad (4.13)$$

We introduced $\tilde{\theta}_{(3)}^j(\tau)$, ($j = 0, 1, 2$), which have integral q expansions. One can easily see that the right hand side of (4.8) consists of 3 types of theta functions, which are classified by top q powers. Then we expect that this is the general property of Conjecture 1 for odd N . We will surely confirm that it is true for $N = 5, 7$.

$N = 5$ Case

$$\begin{aligned} \theta_{A_4}^2(\tau) &= 5\theta_{(5)}^{(1,1,1,1)}(\tau) \} \equiv q\tilde{\theta}_{(5)}^4(\tau) \\ &+ \theta_{(5)}^{(1,0,0,1)}(\tau) + 2\theta_{(5)}^{(3,1,0,0)}(\tau) + 2\theta_{(5)}^{(1,3,1,0)}(\tau) \} \equiv q^{\frac{1}{5}}\tilde{\theta}_{(5)}^0(\tau) \\ &+ \theta_{(5)}^{(2,0,0,2)}(\tau) + 2\theta_{(5)}^{(6,2,0,0)}(\tau) + 2\theta_{(5)}^{(2,6,2,0)}(\tau) \} \equiv q^{\frac{4}{5}}\tilde{\theta}_{(5)}^3(\tau) \\ &+ \theta_{(5)}^{(0,1,1,0)}(\tau) + 2\theta_{(5)}^{(3,0,1,1)}(\tau) + 2\theta_{(5)}^{(1,0,3,0)}(\tau) \} \equiv q^{\frac{2}{5}}\tilde{\theta}_{(5)}^1(\tau) \\ &+ \theta_{(5)}^{(0,2,2,0)}(\tau) + 2\theta_{(5)}^{(6,0,2,2)}(\tau) + 2\theta_{(5)}^{(2,0,6,0)}(\tau) \} \equiv q^{\frac{3}{5}}\tilde{\theta}_{(5)}^2(\tau), \end{aligned} \quad (4.14)$$

where

$$\theta_{(5)}^{(n,k,l,p)}(\tau) = \sum_m q^{\frac{5}{2}t(m+v(n,k,l,p))A_4(m+v(n,k,l,p))}, \quad (4.15)$$

$$v(n, k, l, p) = \frac{1}{5} A_4^{-1} \begin{pmatrix} n \\ k \\ l \\ p \end{pmatrix}. \quad (4.16)$$

Here we will not give concrete expressions of $\theta_{(5)}^{(1,1,1,1)}(\tau)$ etc. As we have already mentioned, the right hand side of (4.14) has 5 types of theta functions, which are classified by top q powers. In contrast to the $N = 3$ case, $\tilde{\theta}_{(5)}^j(\tau)$ ($j = 0, \dots, 4$) is expressed by the sum of several theta functions. Then we assume the following common properties:

Conjecture 2 For odd $N \geq 3$,

$$\theta_{A_{N-1}}^2(\tau) = q^{t_N} \tilde{\theta}_{(N)}^0(\tau) + q^{t_N + \frac{1}{N}} \tilde{\theta}_{(N)}^1(\tau) + \dots + q^{t_N + \frac{N-1}{N}} \tilde{\theta}_{(N)}^{N-1}(\tau), \quad (4.17)$$

where

$$t_N = \frac{1}{2N^2} {}^t \rho A_{N-1} \rho = \frac{1}{24} \frac{N^2 - 1}{N}. \quad (4.18)$$

For $j = 0, \dots, N - 1$,

$$q^{t_N + \frac{j}{N}} \tilde{\theta}_{(N)}^j(\tau) = a_1^j \theta_{(N)}^{\beta_1^j}(\tau) + a_2^j \theta_{(N)}^{\beta_2^j}(\tau) + \dots + a_{k_j}^j \theta_{(N)}^{\beta_{k_j}^j}(\tau). \quad (4.19)$$

$\{a_p^j\}$, ($j = 0, \dots, N - 1$) satisfy

$$\sum_{p=1}^{k_0} a_p^0 = \sum_{p=1}^{k_1} a_p^1 = \dots = \sum_{p=1}^{k_{N-1}} a_p^{N-1} \equiv S_N. \quad (4.20)$$

The above conjecture means that each type of theta function consists of sets of theta functions satisfying the property that the sum of coefficients is equal to fixed S_N . For $N = 3$, $t_3 = \frac{1}{9}$ and $S_3 = 1$. For $N = 5$, $t_5 = \frac{1}{5}$ and $S_5 = 5$. Even for $N = 7$, Conjecture 2 is true.

$N = 7$ Case

$$\begin{aligned} \theta_{A_6}^2(\tau) &= \left. \begin{aligned} &7\theta_{(7)}^{(2,1,1,0,0,0)}(\tau) + 7\theta_{(7)}^{(4,2,2,0,0,0)}(\tau) + 7\theta_{(7)}^{(6,3,3,0,0,0)}(\tau) \\ &+ 21\theta_{(7)}^{(1,1,1,1,1,1)}(\tau) + 7\theta_{(7)}^{(0,3,1,0,1,0)}(\tau) \end{aligned} \right\} \equiv q \tilde{\theta}_{(7)}^5(\tau) \\ &+ \left. \begin{aligned} &\theta_{(7)}^{(0,1,0,0,1,0)}(\tau) + 2\theta_{(7)}^{(2,2,0,2,0,0)}(\tau) + 2\theta_{(7)}^{(5,0,1,0,0,1)}(\tau) + 2\theta_{(7)}^{(0,5,0,1,0,0)}(\tau) \\ &+ 2\theta_{(7)}^{(1,0,5,0,1,0)}(\tau) + 4\theta_{(7)}^{(0,2,2,1,0,0)}(\tau) + 4\theta_{(7)}^{(2,0,2,2,1,0)}(\tau) + 4\theta_{(7)}^{(1,2,2,0,2,0)}(\tau) \\ &+ 6\theta_{(7)}^{(1,1,1,0,3,0)}(\tau) + 8\theta_{(7)}^{(0,1,1,1,1,0)}(\tau) + 14\theta_{(7)}^{(1,1,1,1,0,3)}(\tau) \end{aligned} \right\} \equiv q^{\frac{2}{7}} \tilde{\theta}_{(7)}^0(\tau) \\ &+ \left. \begin{aligned} &\theta_{(7)}^{(0,2,0,0,2,0)}(\tau) + 2\theta_{(7)}^{(4,4,0,4,0,0)}(\tau) + 2\theta_{(7)}^{(10,0,2,0,0,2)}(\tau) + 2\theta_{(7)}^{(0,10,0,2,0,0)}(\tau) \\ &+ 2\theta_{(7)}^{(2,0,10,0,2,0)}(\tau) + 4\theta_{(7)}^{(0,4,4,2,0,0)}(\tau) + 4\theta_{(7)}^{(4,0,4,4,2,0)}(\tau) + 4\theta_{(7)}^{(2,4,4,0,4,0)}(\tau) \\ &+ 6\theta_{(7)}^{(2,2,2,0,6,0)}(\tau) + 8\theta_{(7)}^{(0,2,2,2,2,0)}(\tau) + 14\theta_{(7)}^{(2,2,2,2,0,6)}(\tau) \end{aligned} \right\} \equiv q^{\frac{8}{7}} \tilde{\theta}_{(7)}^6(\tau) \\ &+ \left. \begin{aligned} &\theta_{(7)}^{(0,3,0,0,3,0)}(\tau) + 2\theta_{(7)}^{(6,6,0,6,0,0)}(\tau) + 2\theta_{(7)}^{(15,0,3,0,0,3)}(\tau) + 2\theta_{(7)}^{(0,15,0,3,0,0)}(\tau) \\ &+ 2\theta_{(7)}^{(3,0,15,0,3,0)}(\tau) + 4\theta_{(7)}^{(0,6,6,3,0,0)}(\tau) + 4\theta_{(7)}^{(6,0,6,6,3,0)}(\tau) + 4\theta_{(7)}^{(3,6,6,0,6,0)}(\tau) \\ &+ 6\theta_{(7)}^{(3,3,3,0,9,0)}(\tau) + 8\theta_{(7)}^{(0,3,3,3,3,0)}(\tau) + 14\theta_{(7)}^{(3,3,3,3,0,9)}(\tau) \end{aligned} \right\} \equiv q^{\frac{4}{7}} \tilde{\theta}_{(7)}^2(\tau) \end{aligned}$$

$$\begin{aligned}
& \left. \begin{aligned}
& +\theta_{(7)}^{(0,0,1,1,0,0)}(\tau) + 2\theta_{(7)}^{(4,0,1,0,0,0)}(\tau) + 2\theta_{(7)}^{(2,4,0,1,0,0)}(\tau) + 2\theta_{(7)}^{(1,0,0,5,0,0)}(\tau) \\
& + 3\theta_{(7)}^{(2,1,0,0,1,2)}(\tau) + 4\theta_{(7)}^{(2,0,1,1,1,0)}(\tau) + 4\theta_{(7)}^{(2,1,2,1,0,0)}(\tau) + 4\theta_{(7)}^{(1,2,1,2,1,0)}(\tau) \\
& + 5\theta_{(7)}^{(1,0,2,2,0,1)}(\tau) + 6\theta_{(7)}^{(1,2,1,0,0,1)}(\tau) + 8\theta_{(7)}^{(1,1,0,2,2,0)}(\tau) + 8\theta_{(7)}^{(2,2,0,1,1,1)}(\tau)
\end{aligned} \right\} \equiv q^{\frac{3}{7}} \tilde{\theta}_{(7)}^1(\tau) \\
& \left. \begin{aligned}
& +\theta_{(7)}^{(0,0,2,2,0,0)}(\tau) + 2\theta_{(7)}^{(8,0,2,0,0,0)}(\tau) + 2\theta_{(7)}^{(4,8,0,2,0,0)}(\tau) + 2\theta_{(7)}^{(2,0,0,10,0,0)}(\tau) \\
& + 3\theta_{(7)}^{(4,2,0,0,2,4)}(\tau) + 4\theta_{(7)}^{(4,0,2,2,2,0)}(\tau) + 4\theta_{(7)}^{(4,2,4,2,0,0)}(\tau) + 4\theta_{(7)}^{(2,4,2,4,2,0)}(\tau) \\
& + 5\theta_{(7)}^{(2,0,4,4,0,2)}(\tau) + 6\theta_{(7)}^{(2,4,2,0,0,2)}(\tau) + 8\theta_{(7)}^{(2,2,0,4,4,0)}(\tau) + 8\theta_{(7)}^{(4,4,0,2,2,2)}(\tau)
\end{aligned} \right\} \equiv q^{\frac{5}{7}} \tilde{\theta}_{(7)}^3(\tau) \\
& \left. \begin{aligned}
& +\theta_{(7)}^{(0,0,3,3,0,0)}(\tau) + 2\theta_{(7)}^{(12,0,3,0,0,0)}(\tau) + 2\theta_{(7)}^{(6,12,0,3,0,0)}(\tau) + 2\theta_{(7)}^{(3,0,0,15,0,0)}(\tau) \\
& + 3\theta_{(7)}^{(6,3,0,0,3,6)}(\tau) + 4\theta_{(7)}^{(6,0,3,3,3,0)}(\tau) + 4\theta_{(7)}^{(6,3,6,3,0,0)}(\tau) + 4\theta_{(7)}^{(3,6,3,6,3,0)}(\tau) \\
& + 5\theta_{(7)}^{(3,0,6,6,0,3)}(\tau) + 6\theta_{(7)}^{(3,6,3,0,0,3)}(\tau) + 8\theta_{(7)}^{(3,3,0,6,6,0)}(\tau) + 8\theta_{(7)}^{(6,6,0,3,3,3)}(\tau)
\end{aligned} \right\} \equiv q^{\frac{6}{7}} \tilde{\theta}_{(7)}^4(\tau),
\end{aligned} \tag{4.21}$$

where

$$\theta_{(7)}^{(n,k,l,p,r,s)}(\tau) = \sum_m q^{\frac{7}{2}t(m+v(n,k,l,p,r,s))} A_6(m+v(n,k,l,p,r,s)), \tag{4.22}$$

$$v(n,k,l,p,r,s) = \frac{1}{7} A_6^{-1} \begin{pmatrix} n \\ k \\ l \\ p \\ r \\ s \end{pmatrix}. \tag{4.23}$$

This calculation was very hard. However, Conjecture 2 is also true and $t_7 = \frac{2}{7}$, $S_7 = 49$.

For $N = 7$, there are some remarks. $\tilde{\theta}_{(7)}^j(\tau)$ has several equivalent expressions using $\theta_{(7)}^{\beta_j}(\tau)$'s. This ambiguity comes from several identities between $\theta_{(7)}^{\beta_j}(\tau)$'s. Up to now $N = 7$ is a critical bound for power of our computer. Odd $N > 7$ is the remaining problem. But we believe that Conjecture 2 is true for odd $N > 7$.

5 Formulas

In this section we generalize our previous work on $SU(2)/\mathbf{Z}_2$ partition function on orbifold- T^4/\mathbf{Z}_2 [5] to $SU(N)/\mathbf{Z}_N$ case, using tools prepared in Sec 3,4. We choose the following processes. First we introduce $K3$ partition function Z^{K3} . Next we separate Z^{K3} into Z^{S_0} (the contribution from S_0) and Z^B (the blow-up formula). As a concrete example we choose $v^2 = 0 \pmod N$ on $K3$ case. One will soon understand that the other cases have the same structure as $v^2 = 0 \pmod N$ case. $SU(N)/\mathbf{Z}_N$ partition function with $v^2 = 0 \pmod N$ on $K3$ [19, 9] is given by

$$\begin{aligned}
Z_0^{K3}(\tau) &= \frac{1}{N} \left(\frac{1}{\eta^{24}(\frac{\tau}{N})} + \frac{1}{\eta^{24}(\frac{\tau+1}{N})} + \cdots + \frac{1}{\eta^{24}(\frac{\tau+N-1}{N})} \right) \\
&= \frac{1}{N} \left(\frac{1}{\eta^8(\frac{\tau}{N})} \frac{1}{\eta^{16}(\frac{\tau}{N})} + \frac{1}{\eta^8(\frac{\tau+1}{N})} \frac{1}{\eta^{16}(\frac{\tau+1}{N})} + \cdots + \frac{1}{\eta^8(\frac{\tau+N-1}{N})} \frac{1}{\eta^{16}(\frac{\tau+N-1}{N})} \right)
\end{aligned}$$

$$= Z_0^{S_0}(\tau)Z_1^B(\tau) + Z_1^{S_0}(\tau)Z_0^B(\tau) + \cdots + Z_{N-1}^{S_0}(\tau)Z_2^B(\tau). \quad (5.1)$$

Here $Z_j^{S_0}(\tau)$ is already introduced in Sec.2 as

$$\begin{aligned} Z_j^{S_0}(\tau) &= \frac{1}{N} \left(\frac{1}{\eta^{8(\frac{\tau}{N})}} + \zeta_N^{\frac{1}{3}} \zeta_N^{-j} \frac{1}{\eta^{8(\frac{\tau+1}{N})}} + \cdots + \zeta_N^{\frac{N-1}{3}} \zeta_N^{-j(N-1)} \frac{1}{\eta^{8(\frac{\tau+N-1}{N})}} \right) \\ &= q^{-\frac{1}{3N}} (q^{\frac{j}{N}} + \cdots). \end{aligned} \quad (5.2)$$

Form (5.1) and (5.2) one can read $Z_j^B(\tau)$ as

$$\begin{aligned} Z_j^B(\tau) &= \frac{1}{N} \left(\frac{1}{\eta^{16(\frac{\tau}{N})}} + \zeta_N^{\frac{2}{3}} \zeta_N^{-j} \frac{1}{\eta^{16(\frac{\tau+1}{N})}} + \cdots + \zeta_N^{\frac{2(N-1)}{3}} \zeta_N^{-j(N-1)} \frac{1}{\eta^{16(\frac{\tau+N-1}{N})}} \right) \\ &= q^{-\frac{2}{3N}} (q^{\frac{j}{N}} + \cdots). \end{aligned} \quad (5.3)$$

Using the key identities and conjecture 2, one can rewrite (5.3) as

$$\begin{aligned} Z_j^B(\tau) &= \frac{1}{N} \frac{1}{\eta^{16N}(\tau)} \left((\theta_{A_{N-1}}^2(\tau))^{16} + \zeta_N^{-j} (\theta_{A_{N-1}}^2(\tau+1))^{16} + \cdots + \zeta_N^{-j(N-1)} (\theta_{A_{N-1}}^2(\tau+N-1))^{16} \right) \\ &= \frac{1}{N} \frac{1}{\eta^{16N}(\tau)} q^{\frac{2}{3} \frac{N^2-1}{N}} \left[\left(\sum_{j=0}^{N-1} q^{\frac{j}{N}} \tilde{\theta}_{(N)}^j(\tau) \right)^{16} \right]_{q^{\frac{j}{N}}}, \end{aligned} \quad (5.4)$$

where $[\cdots]_{q^{\frac{j}{N}}}$ stands for projecting out $q^{\frac{j}{N}+n}$ ($n \in \mathbf{Z}_{\geq 0}$) powers. Here we will give explicit expression of (5.4) for $N = 3$:

$$\begin{aligned} Z_0^B(\tau) &= \frac{1}{3} \frac{1}{\eta^{48}(\tau)} \left((\theta_{A_2}^2(\tau))^{16} + (\theta_{A_2}^2(\tau+1))^{16} + (\theta_{A_2}^2(\tau+2))^{16} \right) \\ &= \frac{1}{3} \frac{1}{\eta^{48}(\tau)} q^{\frac{16}{9}} \left[\left(\tilde{\theta}_{(3)}^0(\tau) + q^{\frac{1}{3}} \tilde{\theta}_{(3)}^1(\tau) + q^{\frac{2}{3}} \tilde{\theta}_{(3)}^2(\tau) \right)^{16} \right]_{q^{\frac{0}{3}}} \\ &= \frac{1}{3} \frac{1}{\eta^{48}(\tau)} q^{\frac{16}{9}} \left((\tilde{\theta}_{(3)}^0(\tau))^{16} + 560q (\tilde{\theta}_{(3)}^0(\tau))^{13} (\tilde{\theta}_{(3)}^1(\tau))^3 + \cdots + 16q^{10} (\tilde{\theta}_{(3)}^0(\tau)) (\tilde{\theta}_{(3)}^2(\tau))^{15} \right). \end{aligned} \quad (5.5)$$

We have reconstructed $K3$ partition function which has the factor of the partition function on S_0 and that of the blow-up formula. Note that our processes in this section are opposite direction to our previous work [5]. In the previous work for $SU(2)$, we first prepared the partition function on S_0 and $\mathcal{O}(-2)$ curve blow-up formula. Next we multiplied and summed up them, so that the total partition function has S -duality. The final result was surely $K3$ partition function given by Vafa and Witten [19]. In both cases, the partition function has the same structure. It consists of the factor of the partition function on S_0 and that of the blow-up formula. The contribution from the blow-up formulas can be expressed by $\mathcal{O}(-N)$ blow-up formulas with $\frac{j}{N} + \mathbf{Z}_{\geq 0}$ powers.

6 Conclusion

We reconstructed $K3$ partition function for $SU(N)/\mathbf{Z}_N$ using orbifold construction T^4/\mathbf{Z}_2 . Especially we rewrote $1/\eta(\frac{\tau}{N})$ into the contribution from blow-up by $\mathcal{O}(-N)$ curve blow-up formula.

In this paper remaining most serious problem is geometrical interpretation of $\mathcal{O}(-N)$ curve blow-up in our theory. Since coefficients of $\mathcal{O}(-N)$ curve blow-up formulas are all integers, we hope that there may be some geometrical interpretation. For $SU(2)$ case we can show that the contribution from blow-up comes from two blocks of $\mathcal{O}(-2)$ curve obtained from blowing up 8 points simultaneously [28]. We want similar geometrical interpretation for general N case. Hint is in [19, 17]. From our computation, we have to say that for $SU(N)$ case the contribution from blow-up comes from $U(N)$ gauge theory on ALE A_{N-1} space. If we think so, level N $SU(N)$ characters naturally appear [19, 17]. If we believe these scenario, there are the following interesting applications. In Nakajima's work [17], there are already ADE gauge theory on ALE space associated with the same ADE Lie algebra. Then, we can speculate the form of "ADE blow-up" formula using the denominator identity of ADE affine Lie algebra. Moreover, we may determine the form of the partition function of $N = 4$ ADE gauge theory on $K3$ surfaces, tracing back the process of our computation in this paper.

The most crucial point in this paper is the process of rewriting $1/\eta(\frac{\tau}{N})$ by $\mathcal{O}(-N)$ curve blow-up formula. We may call this rewriting as "bulk-blow-up duality". Physical meaning of this duality is a remaining problem. But, if we apply "ADE blow-up formula" to "bulk-blow-up duality", we will get the partition function for ADE gauge theory on $K3$.

Finally, we mention level-rank duality in affine Lie algebra, that states duality between $\hat{sl}(l)_r$ and $\hat{sl}(r)_l$ [13]. In our case, we observed appearance of $\hat{sl}(N)_N$, which is self-dual in the context of level-rank duality. We expect that this special symmetry may have some physical meaning.

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