



Title	The Nevanlinna counting functions for Rudin's orthogonal functions
Author(s)	Nakazi, T.
Citation	Hokkaido University Preprint Series in Mathematics, 542, 1-7
Issue Date	2001-12
DOI	10.14943/83687
Doc URL	http://hdl.handle.net/2115/69291
Type	bulletin (article)
File Information	pre542.pdf



[Instructions for use](#)

The Nevanlinna Counting Functions
For Rudin's Orthogonal Functions

Takahiko Nakazi

Series #542. December 2001

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

- #514 S. Izumiya, D-H. Pei & T. Sano, Singularities of hyperbolic Gauss maps, 27 pages. 2001.
- #515 S. Izumiya, N. Takeuchi, Generic special curves, 12 pages. 2001.
- #516 S. Izumiya, D-H. Pei & T. Sano, Horospherical surfaces of curves in hyperbolic space, 9 pages. 2001.
- #517 R. Yoneda, The composition operators on weighted Bloch space, 8 pages. 2001.
- #518 M. Jinzenji, T. Sasaki, $N = 4$ supersymmetric Yang-Mills theory on orbifold- T^4/\mathbb{Z}_2 , 18 pages. 2001.
- #519 Y. Giga, Viscosity solutions with shocks, 58 pages. 2001.
- #520 A. Inoue, On the worst conditional expectation, 10 pages. 2001.
- #521 Yumiharu Nakano, Efficient hedging with coherent risk measure, 10 pages. 2001.
- #522 T. Nakazi, Toeplitz operators and weighted norm inequalities on the bidisc, 15 pages. 2001.
- #523 T. Mikami, Covariance kernel and the central limit theorem in the total variation distance, 80 pages. 2001.
- #524 K. Yamaguchi and T. Yatsui, Geometry of higher order differential equations of finite type associated with symmetric spaces, 43 pages. 2001.
- #525 T. Suwa, Residues of Chern classes, 20 pages. 2001.
- #526 V. Anh and A. Inoue, Dynamic models of asset prices with long memory, 21 pages. 2001.
- #527 T. Izawa and T. Suwa, Multiplicity of functions on singular varieties, 21 pages. 2001.
- #528 T. Nakazi and T. Yamamoto, Two dimensional commutative Banach algebras and von Neumann inequality, 18 pages. 2001.
- #529 Y. Giga, N. Ishimura and Y. Kohsaka, Spiral solutions for a weakly anisotropic curvature flow equation, 16 pages. 2001.
- #530 Y. Giga and P. Rybka, Quasi-static evolution of 3-D crystals grown from supersaturated vapor, 16 pages. 2001.
- #531 Y. Tonegawa, Remarks on convergence of the Allen-Cahn equation, 18 pages. 2001.
- #532 T. Suwa, Characteristic classes of singular varieties, 26 pages. 2001.
- #533 J. Escher, Y. Giga and K. Ito, On a limiting motion and self-intersections for the intermediate surface diffusion flow, 20 pages. 2001.
- #534 Y.-H. R. Tsai, Y. Giga and S. Osher, A level set approach for computing discontinuous solutions of a class of Hamilton-Jacobi equations, 30 pages. 2001.
- #535 A. Yamagami, On Gouvéa's conjecture in the unobstructed case, 19 pages. 2001.
- #536 A. Inoue, What does the partial autocorrelation function look like for large lags, 27 pages. 2001.
- #537 T. Nakazi and T. Yamamoto, Norm of a linear combination of two operators of a Hilbert space, 16 pages. 2001.
- #538 Y. Giga, On the two-dimensional nonstationary vorticity equations, 12 pages. 2001.
- #539 M. Jinzenji, Gauss-Manin system and the virtual structure constants, 25 pages. 2001.
- #540 H. Ishii and T. Mikami, Motion of a graph by R -curvature, 28 pages. 2001.
- #541 M. Jinzenji and T. Sasaki, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on orbifold- T^4/\mathbb{Z}_2 : higher rank case, 17 pages. 2001.

The Nevanlinna Counting Functions For Rudin's Orthogonal Functions

by

Takahiko Nakazi*

* This research was partially supported by Grant-in-Aid for Scientific Research,
Ministry of Education

2000 Mathematics Subject Classification : 30 D 50

Abstract. H^∞ and H^2 denote the Hardy spaces on the open unit disc D . Let ϕ be a function in H^∞ and $\|\phi\|_\infty = 1$. If ϕ is an inner function and $\phi(0) = 0$, then $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 . W.Rudin asked if the converse is true and C.Sundberg and C.Bishop showed that the converse is not true. Therefore there exists a function ϕ such that ϕ is not an inner function and $\{\phi^n\}$ is orthogonal in H^2 . In this paper, the following is shown : $\{\phi^n\}$ is orthogonal in H^2 if and only if there exists a unique probability measure ν_0 on $[0,1]$ with $1 \in \text{supp } \nu_0$ such that $N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$ for nearly all z in D where N_ϕ is the Nevanlinna counting function of ϕ . If ϕ is an inner function, then ν_0 is a Dirac measure at $r = 1$.

§1. Introduction

Let D be the open unit disc in the complex plane \mathbf{C} and H^p , $1 \leq p \leq \infty$, the usual Hardy spaces on D . Any function f in H^p has a boundary value on ∂D almost everywhere with respect to the Lebesgue measure $d\theta/2\pi$. It may be assumed that H^p is a closed subspace in the usual Lebesgue space $L^p = L^p(\partial D, d\theta/2\pi)$. Hence H^2 is a Hilbert space.

If ϕ is an inner function in H^∞ , that is, $|\phi| = 1$ a.e. $d\theta/2\pi$ on ∂D and $\phi(0) = 0$, then $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 . The following problem was posed by W.Rudin in 1988 (at an MSRI conference) (see [2]) : If ϕ is a function in H^∞ with $\|\phi\|_\infty = 1$ such that $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 , must ϕ be an inner function ? C.Sundberg [8] and C.Bishop [1] solved independently the problem. In fact, they shows that there exists a function ϕ such that ϕ is not an inner function and $\{\phi^n\}$ is orthogonal in H^2 . We are still interested in a function ϕ in H^∞ such that $\{\phi^n\}$ is orthogonal in H^2 . Because such a function which is not an inner function is mysterious. For a function ϕ in H^∞ with $\|\phi\|_\infty = 1$, we call it a Rudin's orthogonal function when $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 .

For a function ϕ in H^∞ with $\|\phi\|_\infty = 1$, the Nevanlinna counting function of ϕ , N_ϕ , is defined on $D \setminus \{\phi(0)\}$ by

$$N_\phi(w) = \sum_{\phi(z)=w} \log \frac{1}{|z|}$$

where multiplicities are counted and $N_\phi(w)$ is taken to be zero if w is not in the range of ϕ . P.S.Bourdon [2] used the Nevanlinna counting function to attack the Rudin's orthogonality problem. In fact, he showed the following are equivalent : (a) $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 ; (b) There is a non-negative function g on $[0,1]$ such that for almost every $r \in [0,1]$, $N_\phi(re^{i\theta}) = g(r)$ for almost every $\theta \in [0, 2\pi]$.

In this paper, using a representing measure of a moment sequence we show the following : $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 if and only if there exists a unique positive Borel measure ν_0 such that $1 \in \text{supp } \nu_0$, $\nu_0([0, 1]) = 1$ and

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D .

§2. Result

Let ϕ be a function in H^∞ with $\|\phi\|_\infty = 1$. For an analytic function f on D , $C_\phi f(z) = f(\phi(z))$ ($z \in D$). When ν is a positive Borel measure on \bar{D} and $\nu(\bar{D}) = 1$, $H^2(\nu)$ denotes the closure of all analytic polynomials in $L^2(\nu)$.

Lemma 1. *Suppose ϕ is a function in H^∞ with $\|\phi\|_\infty = 1$. If $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 , then there exists a unique radial Borel measure ν such that C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp } \nu_0$.*

Proof. Suppose $\{\phi^n\}$ is orthogonal in H^2 . Let \mathcal{H}_ϕ^2 be the closure of all polynomials of ϕ in H^2 . Put

$$\beta(n) = \left(\int_0^{2\pi} |\phi|^{2n} d\theta/2\pi \right)^{1/2} \quad (n = 0, 1, 2, \dots).$$

$\mathcal{H}^2(\beta)$ denotes the set of all power series : $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ with $\|f\|_\beta = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2 < \infty$.

$\mathcal{H}^2(\beta)$ is a Hilbert space with the inner product : $(f, g) = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2$. Then \mathcal{H}_ϕ^2 is isometrically isomorphic to $\mathcal{H}^2(\beta)$ where

$$F = \sum_{n=0}^{\infty} \left(\int_0^{2\pi} F \bar{\phi}^n d\theta/2\pi \right) \phi^n \rightarrow f = \sum_{n=0}^{\infty} \hat{f}(n)z^n.$$

The multiplication operator M_z on $\mathcal{H}^2(\beta)$ by z is injective and subnormal because the multiplication operator M_ϕ on \mathcal{H}_ϕ^2 is subnormal. $\sup_n |\beta(n+1)/\beta(n)| = 1$ because $\beta(n) \geq \beta(n+1)$ and so $\|M_z\| = 1$ (see [7, p59]). It is known [7, Proposition 25] that $\mathcal{H}^2(\beta)$ is isometrically isomorphic to $H^2(\nu)$ where $d\nu = d\nu_0(r)d\theta/2\pi$ and ν_0 is a Borel probability measure on $[0,1]$ with $1 \in \text{supp } \nu_0$. Thus \mathcal{H}_ϕ^2 is isometrically isomorphic to $H^2(\nu)$ and so C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 . For the uniqueness of ν , if there exists another radial measure $d\lambda = d\lambda_0(r)d\theta/2\pi$ such that C_ϕ is isometric from $H^2(\lambda)$ into H^2 , then

$$\int_{\bar{D}} |z|^{2n} d\nu = \int_{\bar{D}} |z|^{2n} d\lambda \quad (n = 0, 1, 2, \dots)$$

and so $\int_0^1 r^{2n} d\nu_0 = \int_0^1 r^{2n} d\lambda_0$ for $n = 0, 1, 2, \dots$. By the Müntz-Szasz theorem [6, Theorem 15.26], $\nu_0 = \lambda_0$ because $\sum_{n=1}^{\infty} 1/2n = \infty$. Thus $\nu = \lambda$.

Lemma 2. *For nearly all w in D .*

$$N_\phi(w) = \int_0^{2\pi} \log \left| \frac{w - \phi(e^{i\theta})}{1 - \bar{w}\phi(e^{i\theta})} \right| d\theta/2\pi - \log \left| \frac{w - \phi(0)}{1 - \bar{w}\phi(0)} \right|.$$

Proof. By the proof of Littlewood's inequality [6, p187], for $w \in D \setminus \{\phi(0)\}$

$$N_\phi(w) = \lim_{r \rightarrow 1} \int_0^{2\pi} \log \left| \frac{w - \phi(re^{i\theta})}{1 - \bar{w}\phi(re^{i\theta})} \right| d\theta/2\pi - \log \left| \frac{w - \phi(0)}{1 - \bar{w}\phi(0)} \right|.$$

By a generalization of a theorem of Frostman [5], the lemma follows.

Lemma 3. Suppose ν is a radial measure, that is, $d\nu = d\nu_0(r)d\theta/2\pi$. If C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 , then

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D .

Proof. If C_ϕ is isometric, then for any $w \in D$

$$\int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right|^{2n} d\nu = \int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^{2n} d\theta/2\pi \quad (n = 0, 1, 2, \dots)$$

It is elementary to see that $x = \sqrt{1-(1-x^2)} = \sum_{n=0}^{\infty} a_n(1-x^2)^n$ and $\sum_{n=0}^{\infty} |a_n|(1-x^2)^n < \infty$ ($0 \leq x \leq 1$). Hence by Lebesgue's dominated convergence theorem

$$\begin{aligned} \int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right| d\theta/2\pi &= \int_0^{2\pi} \sum_{n=0}^{\infty} a_n \left(1 - \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^2 \right)^n d\theta/2\pi \\ &= \sum_{n=0}^{\infty} a_n \int_0^{2\pi} \left(1 - \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^2 \right)^n d\theta/2\pi \\ &= \sum_{n=0}^{\infty} a_n \int_{\bar{D}} \left(1 - \left| \frac{w-z}{1-\bar{w}z} \right|^2 \right)^n d\nu \\ &= \int_{\bar{D}} \sum_{n=0}^{\infty} a_n \left(1 - \left| \frac{w-z}{1-\bar{w}z} \right|^2 \right)^n d\nu = \int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right| d\nu \end{aligned}$$

because $\left| \sum_{n=0}^k a_n \left(1 - \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^2 \right)^n \right| \leq \sum_{n=0}^{\infty} |a_n| < \infty$ and $\left| \sum_{n=0}^k a_n \left(1 - \left| \frac{w-z}{1-\bar{w}z} \right|^2 \right)^n \right| \leq \sum_{n=0}^{\infty} |a_n| < \infty$. Similarly, as $x^{2\ell+1} = \sqrt{1-(1-x^{4\ell+2})}$ we can show that

$$\int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^{2n+1} d\theta/2\pi = \int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right|^{2n+1} d\nu \quad (n = 0, 1, 2, \dots).$$

Thus for any $w \in D$

$$\int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^n d\theta/2\pi = \int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right|^n d\nu \quad (n = 0, 1, 2, \dots).$$

Since $\log x = \log(1-(1-x)) = -\sum_{n=0}^{\infty} \frac{(1-x)^n}{n}$ ($0 < x \leq 1$), $\sum_{n=0}^k \frac{(1-x)^n}{n}$ is increasing when $0 \leq x \leq 1$ and $\lim_{k \rightarrow \infty} \sum_{n=0}^k \frac{(1-x)^n}{n} = -\log x$. By Lebesgue monotone convergence

theorem, for any $w \in D$

$$\int_{\bar{D}} \log \left| \frac{w-z}{1-\bar{w}z} \right| d\nu = \int_0^{2\pi} \log \left| \frac{w-\phi}{1-\bar{w}\phi} \right| d\theta/2\pi.$$

Hence by Lemma 1, for nearly all $w \in D$

$$\begin{aligned} N_\phi(w) &= \int_{\bar{D}} \log \left| \frac{w-z}{1-\bar{w}z} \right| d\nu - \log |w| \\ &= \int_0^1 d\nu_0(r) \left(\int_0^{2\pi} \log \left| \frac{w-re^{i\theta}}{1-\bar{w}re^{i\theta}} \right| d\theta/2\pi - \log |w| \right) \end{aligned}$$

because $\phi(0) = 0$ and $\nu_0([0, 1]) = 1$. Then

$$N_\phi(w) = \int_{|w|}^1 \log \frac{r}{|w|} d\nu_0(r)$$

for nearly all $w \in D$ because

$$\int_0^{2\pi} \log \left| \frac{w-re^{i\theta}}{1-\bar{w}re^{i\theta}} \right| d\theta/2\pi - \log |w| = \begin{cases} \log \frac{r}{|w|} & (|w| \leq r) \\ 0 & (|w| > r). \end{cases}$$

Theorem 1. Suppose ϕ is a function in H^∞ with $\|\phi\|_\infty = 1$. Then the following (1) \sim (3) are equivalent.

(1) $\{\phi^n ; n = 0, 1, 2, \dots\}$ is orthogonal in H^2 .

(2) $N_\phi(z) = N_\phi(|z|)$ for nearly all z in D .

(3) There exists a unique Borel probability measure ν_0 on $[0, 1]$ with $1 \in \text{supp } \nu_0$ such that

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D .

Proof. (1) \Rightarrow (3). By Lemma 1, there exists a unique radial measure ν such that C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp } \nu_0$. By Lemma 3,

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all $z \in D$. (3) \Rightarrow (2) is clear. (2) \Rightarrow (1). By the Littlewood-Paley theorem (see [2]), for $n > m$

$$\begin{aligned}
\int_0^{2\pi} \phi^n \bar{\phi}^m d\theta / 2\pi &= 2nm \int_D z^{n-1} \bar{z}^{m-1} N_\phi(|z|) dA(z) \\
&= 4nm \int_0^1 r^{n+m-1} N_\phi(r) dr \int_0^{2\pi} e^{i(n-m)\theta} d\theta / 2\pi = 0.
\end{aligned}$$

References

1. C.Bishop, Orthogonal functions in H^∞ , preprint
2. P.S.Bourdon, Rudin's orthogonality problem and the Nevanlinna counting function, Proc.Amer.Math.Soc. 125(1997), 1187-1192.
3. J.A.Cima, B.Korenblum and M.Stessin, On Rudin's orthogonality and independence problem, preprint
4. W.Rudin, Real and Complex Analysis, Third Edition, McGraw-Hill,Inc.
5. W.Rudin, A generalization of a theorem of Frostman, Math.Scand. 21(1967), 136-143.
6. J.H.Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag
7. A.L.Shields, Weighted shift operators and analytic function theory, Math.Surveys Amer.Math.Soc. 13, 49-128.
8. C.Sundberg, Measures induced by analytic functions and a problem of Walter Rudin, preprint

Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

nakazi @ math.sci.hokudai.ac.jp