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On H-separable extensions of QF-3 rings

Kozo Sugano

(Received)

Introduction.

A ring A is said to be an H-separable extension of a subring B of it in the case where $A \otimes_B A$ is isomorphic to an A - A -direct summand of a finite direct sum of copies of A as A - A -module. H-separable extension is a special type of separable extensions. It was introduced by K. Hirata in [4] to generalize the notion of Azumaya algebra. The structure of H-separable extension was researched by Hirata himself in [5], [6] and the author in [10], [11], [12] and so forth. The structure of H-separable extension of simple ring was completely determined by the author in [13]. That is, in the case where B is a simple ring, A is a left (or right) projective H-separable extension of B , if and only if A is simple, $V_A(B)$ is a finite dimensional simple C -algebra and $B = V_A(V_A(B))$, where $V_A(B)$ is the centralizer of B in A and C is the center of A (Theorem 1 [13]). This theorem gives the generalization of classical inner Galois theory of simple artinian rings researched by Noether, Brauer and Artin and others. The definitions and characterizations of H-separable extension, separable extension and semisimple extension are always concerned with tensor products and Hom functors. Our desire is to find the characterizations of them without using tensor products or Hom functors, but using only inner structure of the ring, in the case where B has some special property. The above theorem is one of the successful case. On the other hand in [15] K. Hirata and the author found characterizations of semisimple extension in the case where both A and B are local serial rings using only the lengths of composition series (See Theorem 6 [15]). But when we omit the condition that A is local, we can not know anything even in the case where A is H-separable over B . The author expect that these results will be generalized to the case of H-separable extensions of indecomposable uniserial rings

In this small paper the author will show that under some additional conditions an H-separable extension of a left QF-3 ring is also a left QF-3 ring (Theorems 2 and 3). As a consequence we can show that a left projective H-separable extension of a serial ring is also a serial ring (Theorem 7), and that any H-separable extensions of uniserial rings are uniserial (Theorem 8).

H-separable extensions of QF-3 rings.

Throughout this paper A is a ring with the identity element 1, B a subring of A containing 1 and C is the center of A . Furthermore we write $V_A(X) = \{a \in A \mid xa = ax \text{ for } \forall x \in X\}$ for any subset X of A , and $D = V_A(B)$. First we will introduce some fundamental properties of H-separable extension briefly. Always we have the following A - A -homomorphism

$$\eta : A \otimes_B A \rightarrow \text{Hom}({}_c D, {}_c A) \quad \eta(a \otimes b)(d) = adb \quad (\forall a, b \in A \quad \forall d \in D)$$

A is an H-separable extension of B if and only if D is C -finitely generated projective and η is an isomorphism. We also have the following A - A - and D - D -homomorphism

$$\zeta : D \otimes_C A \rightarrow \text{Hom}({}_B A, {}_B A) \quad \zeta(d \otimes a)(x) = dxa \quad (\forall a, x \in A, \forall d \in D)$$

If A is H-separable over B , ζ is an isomorphism. Therefore in this case $\text{End}({}_B A)$ is left and right A -finitely generated projective, and contains A as an A - A -direct summand. The map ζ of $D \otimes_C A$

to $\text{End}(A_B)$ is similarly defined, and the same results hold. In the case where A is H -separable over B , we consider the following conditions;

- (a) B is a left B -direct summand of A .
- (a-2) B is a left B -direct summand of A , and A is left B -flat.
- (a-3) B is a left B -direct summand of A , and A is right B -flat.
- (b) A is left B -finitely generated projective.
- (b-2) A is left B -finitely generated projective, and $B = V_A(V_A(B))$.

In the case where B is commutative and A is an Azumaya B -algebra all of the above conditions are satisfied. Therefore it is not unnatural to add these conditions. In fact in the case where A is a B -algebra, A is an Azumaya B -algebra, if and only if A is H -separable over B and (a) is satisfied, if and only if A is H -separable over B and A is B -finitely generated (See Corollaries 1.1 and 1.2 [10]).

If A is an H -separable extension of B , and the condition (a) (resp. (b)) is satisfied, we have $I = A(I \cap B)$ (resp. $I = (I \cap B)A$) for any two-sided ideal I of A by Theorem 4.1 (resp. Theorem 3.1) [11]. Therefore in either case we have $I \cap B \neq 0$ for any non-zero two-sided ideal I of A . Then we can apply the next easy lemma,

Lemma 1. Suppose $I \cap B \neq 0$ holds for each non-zero two-sided ideal I of A . Then for each faithful injective left B -module M , $M^* = \text{Hom}({}_B A, {}_B M)$ is a faithful injective left A -module.

Proof. Since A is right A -flat and M is B -injective, M^* is left A -injective. Let I be the annihilator of M^* in A , and suppose $I \neq 0$. Then by the assumption we have $I \cap B \neq 0$. Let $0 \neq a \in I \cap B$. Since M is B -faithful, there exists an m in M with $am \neq 0$. Then the left B -homomorphism f of B to M such that $f(x) = xm$ ($\forall x \in B$) satisfies $f(a) \neq 0$. f is extended to an $f^* \in M^*$, since M is B -injective, and $af^* = 0$ since $a \in I$. Then we have $0 = af^*(1) = f^*(a) = f(a) \neq 0$, a contradiction. Thus we have $I = 0$.

Lemma 2. Let A be an H -separable extension of B . Then for each finitely generated projective and injective left B -module M , $M^* = \text{Hom}({}_B A, {}_B M)$ is finitely generated projective and injective left A -module.

Proof. Obviously M^* is left A -injective. Since M is B -finitely generated projective, M^* is an A -direct summand of a finite direct sum of copies of $\text{Hom}({}_B A, {}_B B)$, which is contained in $\text{Hom}({}_B A, {}_B A)$. Hence M^* is contained in $P = \bigoplus \text{Hom}({}_B A, {}_B A)$ (finite direct sum), which is left A -finitely generated projective. But M^* is A -injective. Hence M^* is an A -direct summand of P . Thus M^* is A -finitely generated projective.

As an immediate consequence of the above lemmas we have

Theorem 1. Let A be an H -separable extension of B , and assume that the condition (a) or (b) is satisfied. Then if M is a finitely generated faithful projective and injective left (resp. right) B -module, $M^* = \text{Hom}({}_B A, {}_B M)$ (resp. $\text{Hom}(A_B, M_B)$) is also a finitely generated faithful projective and injective left (resp. right) A -module.

Proof. By Lemma 2 M^* is finitely generated projective and injective left (resp. right) A -module. On the other hand, in the case of (a) we have $I = A(I \cap B)$, while in the case of (b) we have $I = (I \cap B)A$ for each two-sided ideal I of A . In either case we can apply Lemma 1 to

obtain that M^* is faithful left (resp. right) A -module.

A is said to be a left semisimple extension of B in the case where every left A module M is (A, B) -projective, that is, the map p of $A \otimes_B M$ to M such that $p(a \otimes m) = am$ ($a \in A, m \in M$) is an A -split epimorphism. Separable extensions, and consequently H -separable extensions are always semisimple extensions. We will now consider H -separable extensions of perfect QF-3 rings. Left perfect ring is characterized as a ring whose flat left A -module is always projective.

The next lemma is a small generalization of Proposition 5.3 (g) [6]

Lemma 3. Let A be a left semisimple extension of B , and assume that A is left B -flat. If B is left perfect, A is also left perfect.

Proof. Let M be a flat left A -module. Since A is left B -flat, M is B -flat. Then M is B -projective, since B is left perfect. Then M is A -projective, since A is left semisimple over B . Thus every flat left A -module is A -projective.

Theorem 2. Let A be an H -separable extension of B , and assume that the condition (a-2) or (b) is satisfied. If B is a left perfect right QF-3 ring, A is also a left perfect right QF-3 ring.

Proof. By the above remark we see that A is left perfect. Now by Theorem 1 A has a finitely generated faithful projective and injective right module M^* . Since A is semi-perfect, we have $M^* = (e_1 A)^{n_1} \oplus (e_2 A)^{n_2} \oplus \cdots \oplus (e_k A)^{n_k}$, where $\{e_i \mid 1 \leq i \leq k\}$ is a subset of the set of basic idempotents of A , and $n_i > 0$. Then eA ($e = e_1 + e_2 + \cdots + e_k$) is also a faithful projective and injective right A -module, and each $e_i A$ is the injective hull of a minimal right ideal r_i of A , since A is left perfect. Obviously $r_i \not\cong r_j$ for $i \neq j$. Then eA is the minimal faithful right A -module.

Proposition 1. Let A be an H -separable extension of B with the condition (a). Then if B is a right artinian QF-3 ring, A is also a right artinian QF-3 ring.

Proof. By Theorem 4.1 [11] A is right B -finitely generated, and hence is a right artinian ring. On the other hand A has a finitely generated faithful projective and injective left A -module M^* by Theorem 1. Then for the same reason as Theorem 2 M^* contains the minimal faithful left A -module as a left A -direct summand.

Now we will consider the H -separable extension of semiprimary QF-3 rings. Let us denote the Jacobson radical of A by $J(A)$. In order to know the relation between $J(A)$ and $J(B)$ the following easy remark is very useful

Remark 1. If $B = V_A(V_A(B))$, then we have $J(A) \cap B \subseteq J(B)$

Proof. For each x in $J(A) \cap B$, $1 - x$ has the inverse in A . But since $1 - x \in V_A(D)$, its inverse is contained in $V_A(D) = B$. Hence $J(A) \cap B$ is a quasi-regular ideal of B , and is contained in $J(B)$.

Proposition 2. Let A be an H -separable extension of B , and assume that the condition (a-2) or (a-3) or (b-2) is satisfied. Then if B is semiprimary, A is also semiprimary.

Proof. Write $N = J(A)$. In the case of (a) we have $N = A(N \cap B)$ and $B = V_A(D)$. Then by the above remark $N \cap B$ is contained in $J(B)$, which is nilpotent. Hence N is nilpotent. In the case of (b-2) we have $N = (N \cap B)A$ and $B = V_A(D)$, and have the same result. On the other hand A is a separable extension of a perfect ring B and is left (or right) B -flat. Hence by Lemma 3 A is left (or right) perfect. Therefore A/N is a semisimple ring.

By Theorem 2 and Proposition 2 we have

Theorem 3. Let A be an H -separable extension of B , and suppose that the condition (a-2) or (b-2) is satisfied. If B is a semiprimary left (resp. right) QF-3 ring, A is also a semiprimary left (resp. right) QF-3 ring.

Theorem 4. Let A be an H -separable extension of B with the condition (a). Then if $E({}_B B)$ is B -projective, $E({}_A A)$ is A -projective, where $E({}_A A)$ is the injective hull of the left A -module A .

Proof. By the assumption and Lemma 6.1 [16] $M = E({}_B B)$ is finitely generated projective and injective as left B -module. Hence $M^* = \text{Hom}({}_B A, {}_B M)$ is a finitely generated projective and injective left A -module by Lemma 2. On the other hand by the condition (a) and Proposition 5 [12] $B^* = \text{Hom}({}_B A, {}_B B)$ is a left A -progenerator. Therefore we have

$$A \subseteq B^* \oplus B^* \oplus \cdots \oplus B^* \subseteq M^* \oplus M^* \oplus \cdots \oplus M^* = P$$

Since P is left A -injective, $E({}_A A)$ is contained in P . And $E({}_A A)$ is an A -direct summand of P , since $E({}_A A)$ is A -injective. Then $E({}_A A)$ is A -finitely generated projective since P is A -finitely generated projective.

In the case where the condition (b) is satisfied the results of the above theorem holds under a weaker condition. In their paper [9] E. McMahon and A. Mewborn introduced a new type of separable extensions. A is said to be a strongly separable extension of B in the case where D is C -finitely generated projective and the map η of $A \otimes_B A$ to $\text{Hom}({}_C D, {}_C A)$ defined above is an A - A -split epimorphism. In this case $D \otimes_C A$ is an A - A -direct summand of $\text{Hom}({}_B A, {}_B A)$ by Proposition 3.9 [9].

Theorem 5. Let A be a strongly separable extension of B with the condition (b). Then if $E({}_B B)$ is B -projective, $E({}_A A)$ is A -projective.

Proof. Let M, M^* and B^* be as in the proof of Theorem 4. Since $C \subseteq D$ and D is C -finitely generated projective, C is a C -direct summand of D . Hence we have

$$A \subseteq D \otimes_C A \subseteq \text{Hom}({}_B A, {}_B A) \subseteq B^* \oplus B^* \oplus \cdots \oplus B^* \subseteq M^* \oplus M^* \oplus \cdots \oplus M^*$$

Then for the same reason as above we have the results.

Corollary 1. Let A be a strongly separable extension of B with the condition (b). If B is a left artinian QF-3 ring, A is also a left artinian QF-3 ring.

Proof. This is clear by Theorem 5 and Theorem 31.6 [1].

Next we will consider the case where A is torsionless as left B -module, which is a weaker condition than (b).

Let M be a left B -module and $a \in A$ with $aM^* = 0$, where $M^* = \text{Hom}({}_B A, {}_B M)$. Then for each f in M^* we have $0 = (af)(1) = f(a)$. Thus if A is M -torsionless (i.e., $A \subseteq \Pi M$), M^* is faithful as left A -module. If A is left B -torsionless, and M is B -faithful, we have $A \subseteq \Pi B, B \subseteq \Pi M$ and consequently $A \subseteq \Pi M$. Thus M^* is A -faithful. Then by Theorem 1 we have

Proposition 1. Let A be an H -separable extension of B , and assume that A is left B -torsionless. If B has a finitely generated faithful projective and injective left B -module, A has also a finitely generated faithful projective and injective left A -module.

Proof. If M is a left B -module satisfying the condition of the theorem, M^* satisfies the same condition as A -module.

We use the same notation as [16] concerning with the notion of left (right) quotient ring and the maximal left (right) quotient ring. So see [16] for detail. The next remark, which is a generalization of the remark stated in page 46 [13], may be already well known. But it is very useful. Therefore we will state it as a remark.

Remark 2. Let M be a faithful torsionless right A -module. Then the double centralizer T of M_A is a left quotient ring of A .

Proof. Let $S = \text{End}(M_A)$. Then $T = \text{End}({}_S M)$. Since M is A -faithful, A is a subring of T . For any g in $\text{Hom}(M_A, A_A)$ and z in M , we define the map $g_{(\omega)}$ of M to M by $g_{(\omega)}(x) = zg(x)$ for each x in M . Clearly $g_{(\omega)} \in S$, and we have $g_{(\omega)}(xt) = (g_{(\omega)}(x))t$, $zg(xt) = (zg(x))t = z(g(x)t)$, regarding $g(x)$ as an element of T . This means $g(xt) = g(x)t$ for each x in M and t in T . Now let $t_1, t_2 \in T$ with $t_2 \neq 0$. Then there exists an m in M such that $mt_2 \neq 0$. Since M is A -torsionless there exists an f in $\text{Hom}(M_A, A_A)$ such that $f(mt_2) \neq 0$. But $f(mt_2) = f(m)t_2 \neq 0$, and $A \ni f(mt_1) = f(m)t_1$. Hence there exists an $r (= f(m))$ in A with $rt_1 \in A$, $rt_2 \neq 0$.

Theorem 6. Let A be an H -separable extension of B , and assume that A is left B -torsionless. Then we have;

- (1) $B'' = V_A(V_A(B))$ is a right quotient ring of B .
- (2) If B is right QF-3, B'' is a left and right quotient ring of B and right QF-3.
- (3) If B is QF-3, B'' is also QF-3.

Proof. (1). Since the map ζ_1 of $D_X cA$ to $\text{Hom}({}_B A, {}_B A)$ is an isomorphism, the double centralizer of the left B -module ${}_B A$ is isomorphic to $\text{Hom}({}_D A_A, {}_D A_A) = V_A(V_A(B))$. Hence we have (1) by the above remark. (2) follows from (4.3) and (4.5) [16] and (1). (3) is also immediate by (4.3) [16] and (2).

Lastly we will apply the above results to serial or uniserial rings. A left artinian ring is a serial ring if and only if all of its factor rings are QF-3. Let ϕ be a ring homomorphism of A to the other ring \bar{A} , and $\bar{B} = \phi(B)$. Then by Proposition 3.4 [10] \bar{A} is also an H -separable extension of \bar{B} with the condition (a). Therefore by Theorem 2 and Proposition 1 we have

Theorem 7. Let A be an H -separable extension of B , and assume that the condition (a) or (b) is satisfied. If B is a serial ring, A is also a serial ring.

Proof. All what we need is to prove the next lemma

Lemma 5. Let A be an H -separable extension of B with the condition (b). Then for any ring homomorphism ϕ of A onto \bar{A} , \bar{A} is an H -separable extension of $\bar{B} = \phi(B)$ with the condition (b)

Proof. Let $\{f_i, z_i\}$ be a dual basis of ${}_B A$. Then there exists $\sum e_{ij} x a_{ij}$ in $D \otimes cA$ such that $f_i(x) = \sum e_{ij} x a_{ij}$ for each x in A , since ζ_1 is an isomorphism. Write $\phi(x) = \bar{x}$ for each x in A . Since $\sum e_{ij} A a_{ij} = f_i(A) \subseteq B$, we see $\sum \bar{e}_{ij} \bar{A} \bar{a}_{ij} \subseteq B$. Then since $\phi(D) \subseteq V_A(B)$, if we define $\bar{f}_i(\bar{x}) = \sum \bar{e}_{ij} \bar{x} \bar{a}_{ij}$ for each \bar{x} in \bar{A} , \bar{f}_i is a left \bar{B} -homomorphism of \bar{A} to \bar{B} . And we have $\sum \bar{f}_i(\bar{x}) \bar{z}_i = \sum \overline{e_{ij} x a_{ij} z_i} = \overline{\sum f_i(x) z_i} = \bar{x}$. Thus $\{\bar{f}_i, \bar{z}_i\}$ is a dual basis for ${}_B A$.

Uniserial ring is characterized as a ring all of whose factor rings are quasi-Frobenius (QF) rings. A ring is uniserial if and only if it is a finite direct product of matrix rings over local uniserial rings. On the other hand by Theorem 4.2 [11] an H -separable extension of a QF ring is

also a QF ring. In this case B is a left (as well as right) B -direct summand of A , since each QF ring is left (and right) self injective. Therefore we have

Theorem 8. Let B be a uniserial ring, and $B = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ be the ring decomposition with each B_i a matrix ring over a local serial ring R_i . Let A be an H -separable extension of B . Then we have

(1) A is a uniserial ring, and we have $B = V_A(V_A(B))$. A is finitely generated as a left (as well as right) B -module.

(2) If $N \cap B_i \neq 0$ for each i where $N = J(A)$, A is left (as well as right) B -finitely generated projective.

(3) If B is indecomposable as a ring, A is also indecomposable, and A is left (and right) B -finitely generated projective.

Proof. (1) is clear by the above remark and Theorem 4.1 [11]. Each central idempotent of A is contained in the center of B , since $B = V_A(D) \supseteq C$. Therefore in order to prove (2) we may assume that A is indecomposable as a ring. Then A is a finite direct sum of mutually isomorphic indecomposable left ideals, and each left A -direct summand of A is faithful. Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_m$, and $J(R_i) = J_i$. Since each B_i is H -separable over R_i , B is H -separable over R . Hence A is H -separable over R . Moreover we have $(N \cap R_i)B_i = N \cap B_i \neq 0$, since B_i is a matrix ring over R_i . Thus we have $N \cap R_i = J_i^{r_i} \neq 0$ for some r_i . Each indecomposable left R -module is isomorphic to some $R_i/J_i^{r_i}$, and each left R -module is a direct sum of indecomposable left R -modules. Therefore we can write $A = \bigoplus (R_i/J_i^{r_i})^{n_{ik}}$. Suppose there exist some i, k such that $J_i^{r_i} \neq 0$, and let, for example, $R_i/J_i^{r_i}$ is a member of the above indecomposable decomposition with $J_i^{r_i} \neq 0$. Then we have

$$\begin{array}{ccccccc} A \otimes_B J_i^{r_i} \oplus A \otimes_B R_2 \oplus \cdots \oplus A \otimes_B R_m & \rightarrow & A \otimes_B R & \rightarrow & A \otimes_B (R_i/J_i^{r_i}) & \rightarrow & 0 \quad (\text{exact}) \\ & & \downarrow \text{epi.} & & \downarrow \text{iso.} & \downarrow \text{iso.} & \\ 0 & \rightarrow & A J_i^{r_i} \oplus A R_2 \oplus \cdots \oplus A R_m & \rightarrow & A & \rightarrow & A R_i / A J_i^{r_i} \rightarrow 0 \quad (\text{exact}) \end{array}$$

But $A \otimes_B (R_i/J_i^{r_i})$ is left A -projective, since it is an A -direct summand of $A \otimes_B A$ which is isomorphic to $\text{Hom}({}_C V_A(R), {}_C A)$ and $V_A(R)$ is C -projective. Therefore the above exact sequences split as A -map, and we can write $A = A J_i^{r_i} \oplus A R_i / A J_i^{r_i} \oplus I$ as left A -module. On the other hand since R is a left (as well as right) R -direct summand of A , we have $N = A(N \cap R) = (N \cap R)A = A(J_i^{r_i} \oplus J_2^{r_2} \oplus \cdots \oplus J_m^{r_m})$ for some r_i . If $r = r_1 \leq 1$, $J_i^{r_i} \subseteq J_i^r$, and $A J_i^{r_i}$ is nilpotent. But it is impossible because $A J_i^{r_i}$ is generated by an idempotent. If $1 < r$, we have

$$J_i^r A R_i \subseteq (N \cap R) A R_i = A(N \cap R) R_i \subseteq A J_i^r \subseteq A J_i^{r_i}.$$

Thus $A R_i / J_i^{r_i}$ is annihilated by J_i^r . But since it is an A -direct summand of A , it must be faithful. This means that each $J_i^{r_i}$ in the above indecomposable decomposition must be 0, and A is left R -projective. Then since B is separable over R , A is B -projective. (3) is obvious by (2), and the fact that each central idempotent of A is contained in B .

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