On Time-Local Solvability of the Navier-Stokes Equations in Besov Spaces

Okihiro Sawada

Series #545. December 2001

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#518 M. Jinzenji, T. Sasaki, $N = 4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbf{Z}_2$, 18 pages. 2001.

#519 Y. Giga, Viscosity solutions with shocks, 58 pages. 2001.

#520 A. Inoue, On the worst conditional expectation, 10 pages. 2001.

#521 Yumiharu Nakano, Efficient hedging with coherent risk measure, 10 pages. 2001.

#522 T. Nakazi, Toeplitz operators and weighted norm inequalities on the bidisc, 15 pages. 2001.

#523 T. Mikami, Covariance kernel and the central limit theorem in the total variation distance, 80 pages. 2001.

#524 K. Yamaguchi and T. Yatsui, Geometry of higher order differential equations of finite type associated with symmetric spaces, 43 pages. 2001.

#525 T. Suwa, Residues of Chern classes, 20 pages. 2001.

#526 V. Anh and A. Inoue, Dynamic models of asset prices with long memory, 21 pages. 2001.

#527 T. Izawa and T. Suwa, Multiplicity of functions on singular varieties, 21 pages. 2001.

#528 T. Nakazi and T. Yamamoto, Two dimensional commutative Banach algebras and von Neumann inequality, 18 pages. 2001.

#529 Y. Giga, N. Ishimura and Y. Kohsaka, Spiral solutions for a weakly anisotropic curvature flow equation, 16 pages. 2001.

#530 Y. Giga and P. Rybka, Quasi-static evolution of 3-D crystals grown from supersaturated vapor, 16 pages. 2001.

#531 Y. Tonegawa, Remarks on convergence of the Allen-Cahn equation, 18 pages. 2001.

#532 T. Suwa, Characteristic classes of singular varieties, 26 pages. 2001.

#533 J. Escher, Y. Giga and K. Ito, On a limiting motion and self-intersections for the intermediate surface diffusion flow, 20 pages. 2001.

#534 Y.-H. R. Tsai, Y. Giga and S. Osher, A level set approach for computing discontinuous solutions of a class of Hamilton-Jacobi equations, 30 pages. 2001.

#535 A. Yamagami, On Gouvêa's conjecture in the unobstructed case, 19 pages. 2001.

#536 A. Inoue, What does the partial autocorrelation function look like for large lags, 27 pages. 2001.

#537 T. Nakazi and T. Yamamoto, Norm of a linear combination of two operators of a Hilbert space, 16 pages. 2001.

#538 Y. Giga, On the two-dimensional nonstationary vorticity equations, 12 pages. 2001.

#539 M. Jinzenji, Gauss-Manin system and the virtual structure constants, 25 pages. 2001.

#540 H. Ishii and T.Mikami, Motion of a graph by R-curvature, 28 pages. 2001.

#541 M. Jinzenji and T. Sasaki, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbf{Z}_2$: higher rank case, 17 pages. 2001.

#542 T. Nakazi, The Nevanlinna counting functions for Rudin's orthogonal functions, 7 pages. 2001.

#543 K. Sugano, On H-separable extensions of QF-3 rings, 7pages. 2001.

#544 A. Arai, Non-relativistic limit of a Dirac-Maxwell operator in relativistic quantum electrodynamics, 27pages. 2001.

On Time-Local Solvability of the Navier-Stokes Equations in Besov Spaces

OKIHIRO SAWADA
Department of Mathematics
Hokkaido University,
Sapporo 060-0810, Japan

Abstract

A time-local solution is constructed for the Cauchy problem of the n-dimensional Navier-Stokes equations when the initial velocity belongs to Besov spaces of nonpositive order. The space contains L^∞ in some exponents, so our solution may not decay at space infinity. In order to use iteration scheme we have to establish the Hölder type inequality for estimating bilinear term by dividing the sum of Besov norm with respect to levels of frequency. Moreover, by regularizing effect our solutions belongs to L^∞ for any positive time.

1 Introduction and Main Results

In this paper we consider the initial value problem for the incompressible Navier-Stokes equations in $\mathbb{R}^n \quad (n \geq 2)$:

$$
\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = u_0 \qquad \text{in } \mathbb{R}^n, \quad (\text{with } \operatorname{div} u_0 = 0). \end{cases}
$$

Here $u = u(x,t) = (u^1(x,t), u^2(x,t), \dots, u^n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity and unknown scalar function respectively; $x = (x_1, x_2, \dots, x_n)$ denotes a point of the space $\mathbb{R}^n$ and t denotes the positive time. $u_0 = u_0(x) = (u_0^1(x), u_0^2(x), \dots, u_0^n(x))$ is a given initial velocity.

Before stating our main results we recall several Besov type function spaces used in this paper; see [Tr1].

Let ϕ_j be the Littlewood-Paley dyadic decomposition of unity i.e. $\hat{\phi}_0 \in C_0^\infty$, $\mathrm{supp}\hat{\phi}_0 \subset \{\xi;\ 1/2 \le |\xi| \le 2\}$, $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) = 1$ except $\xi = 0$, and let $\psi = \mathcal{F}^{-1}(1 - \sum_{j=1}^{\infty} \hat{\phi}_j)$, where $\mathcal{F}f = \hat{f}$ is denoted by the Fourier transform and $\mathcal{F}^{-1}$ is its inverse. Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, i.e. the topological dual of $\mathcal{S}(\mathbb{R}^n)$ which is the space of rapidly decreasing functions in the sense of L. Schwartz.

Definition 1 (inhomogeneous Besov spaces). *Let $n \ge 1$, $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$. An inhomogeneous Besov space is defined by*

$$B^s_{p,q}(\mathbb{R}^n) \equiv \{f \in \mathcal{S}';\ ||f; B^s_{p,q}|| < \infty\},$$
$$||f; B^s_{p,q}|| \equiv \begin{cases} ||\psi * f; L^p|| + \left[\sum_{j=1}^{\infty} 2^{jsq}||\phi_j * f; L^p||^q\right]^{1/q} & if \quad q < \infty, \\ ||\psi * f; L^p|| + \sup_{j\ge 1} 2^{js}||\phi_j * f; L^p|| & if \quad q = \infty. \end{cases}$$

Throughout this paper we suppress $n \ge 1$ and $\mathbb{R}^n$. Following J. Johnsen [Jo], we call s the differentiability-exponent, p the integral-exponent and q the sum-exponent.

Definition 2 (homogeneous Besov spaces). *Let $s \in \mathbb{R}$ and $1 \le p \le \infty$ and $1 \le q \le \infty$. A homogeneous Besov space is defined by*

$$\dot{B}^s_{p,q} \equiv \{f \in \mathcal{Z}';\ ||f; \dot{B}^s_{p,q}|| < \infty\},$$
$$||f; \dot{B}^s_{p,q}|| \equiv \begin{cases} \left[\sum_{j=-\infty}^{\infty} 2^{jsq}||\phi_j * f; L^p||^q\right]^{1/q} & if \quad q < \infty, \\ \sup_{-\infty \le j \le \infty} 2^{js}||\phi_j * f; L^p|| & if \quad q = \infty, \end{cases}$$

where $\mathcal{Z}'$ is the topological dual space of

$$\mathcal{Z} \equiv \{f \in \mathcal{S};\ D^\alpha \hat{f}(0) = 0,\ \forall \alpha \in \mathbb{N}_0^n\}.$$

Here we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers.

Remark 1. *It is well-known that the homogeneous Besov space can be regarded as subspace of $\mathcal{S}'$ if either $s < n/p$ or $s = n/p$ and $q = 1$, see* [Bou] *or* [KY]*. We hereafter only treat these spaces with exponents satisfying this condition.*

We also define several associated spaces. We set that $e^{t\Delta} = G_t *$ denotes the solution-operator of the heat equation; G_t is Gauss kernel denoted by $G_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$. One extends $e^{t\Delta}$ from $\mathcal{S}$ to $\mathcal{S}'$ in usual way. Unfortunately $e^{t\Delta}$ is not a continuous (C_0) semigroup in Besov spaces if integral-exponent or sum-exponent is infinity, so $e^{t\Delta} f \to f$ in $B^s_{p,q}$ need not hold for general element of $B^s_{p,q}$.

Definition 3 (small inhomogeneous Besov spaces). *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. A small inhomogeneous Besov space is the subspace defined by*

$$b^s_{p,q} \equiv \{f \in B^s_{p,q};\ e^{t\Delta} f \to f \text{ in } B^s_{p,q} \text{ as } t \downarrow 0\}.$$

Definition 4 (small homogeneous Besov spaces). *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Assume in addition that (in order to operate $e^{t\Delta}$) these exponents satisfy the condition of either $s < n/p$ or $s = n/p$ and $q = 1$. A small homogeneous Besov space is defined by*

$$\dot{b}^s_{p,q} \equiv \{f \in \dot{B}^s_{p,q};\ e^{t\Delta} f \to f \text{ in } \dot{B}^s_{p,q} \text{ as } t \downarrow 0\}.$$

It is easy to see that the small Besov space is a closed subspace of Besov space, so it is Banach space. We denote by $n^s_{p,q}$ is the closure of $\mathcal{S}$ with respect to the norm of $B^s_{p,q}$. Then we have

$$n^s_{p,q} \subset b^s_{p,q} \subset B^s_{p,q}.$$

Of course, these three spaces agree each other if p and q are finite. But otherwise these spaces are different from each other, for example, if $s < 0$, $p = \infty$ and $q < \infty$, then

$$n^s_{\infty,q} \subsetneqq b^s_{\infty,q} = B^s_{\infty,q}.$$

Indeed, non-zero constant function belongs to $b^s_{\infty,q}$, however, it does not belongs to $n^s_{\infty,q}$.

One can prove that

$$b^s_{\infty,\infty} = \{f \in B^s_{\infty,\infty};\ f \text{ is uniformly continuous}\};$$

the proof is not difficult (see [GIM; Lemma 5]).

Our goal is to prove the existence and uniqueness of time-local smooth solution of (NS) when the initial velocity u_0 belongs to $b^s_{p,q}$ or $\dot{b}^s_{p,q}$ with $s \leq 0$. It is interesting to construct the solution of (NS) in Besov space with the differential of negative order. The first reason is that this space is very large containing distributions other than Radon measures. The second reason is that it is useful to understand inequalities of classical Hardy-Littlewood-Sobolev type and estimates of the heat kernel reflecting its smoothing property (see Lemma 2).

For a Banach space X and an interval $I \subset \mathbb{R}$ let $C(I;X)$ be the space of all continuous functions on I with value in X. We are now in position to state our main results.

Theorem . *Assume that $n \geq 2$, $n < p \leq \infty$, $1 \leq q \leq \infty$ and $0 \leq \varepsilon < 1 - n/p$, and assume that the initial data $u_0 \in b^{-\varepsilon}_{p,q}(\mathbb{R}^n)$ satisfies $\operatorname{div} u_0 = 0$. Then there exists a positive constant T_0 and a unique u satisfying*

$$u \in C([0,T_0]; b^{-\varepsilon}_{p,q}(\mathbb{R}^n)), \tag{1.1}$$

$$t^{\gamma/2} u \in C([0,T_0]; b^{\gamma-\varepsilon}_{p,q}(\mathbb{R}^n)) \quad \text{for all} \quad 0 < \gamma \leq 1, \tag{1.2}$$

$$t^{\delta/2} u \in C([0,T_0]; L^p(\mathbb{R}^n)) \quad \text{for all} \quad \varepsilon < \delta < 1, \tag{1.3}$$

such that $(u(t), \nabla p(t))$ is a unique classical solution of (NS) with

$$\nabla p(t) = \sum_{i,j=1}^{n} \nabla R_i R_j u^i(t) u^j(t), \tag{1.4}$$

where $R_i = \partial_i(-\Delta)^{-1/2}$ is the Riesz transform.

Note that $q = \infty$ is included, the space $b^{-\varepsilon}_{p,\infty}$ includes L^p spaces for $p < \infty$ and BUC for $p = \infty$ for any $\varepsilon \geq 0$. Here BUC represents the set of all bounded and uniformly continuous functions.

Remark 2. *Similarly one can construct the time-local solution in $\dot{b}^{-\varepsilon}_{p,q}(\mathbb{R}^n)$ with assumption of $n < p \leq \infty$, $1 \leq q \leq \infty$ and $0 < \varepsilon < 1 - n/p$. Of course we get the properties like (1.1)–(1.3) of the solution by replacing function space by their homogeneous version.*

Remark 3. *We still have a time-local solution if we replace small Besov spave by $B^{-\varepsilon}_{p,q}$ for initial data. However, we have to replace (1.1) and (1.2) by*

$$u \in C_w([0,T_0]; B^{-\varepsilon}_{p,q}),$$
$$t^{\gamma/2}u \in C_w([0,T_0]; B^{\gamma-\varepsilon}_{p,q}) \quad \text{for all} \quad 0 < \gamma \leq 1.$$

Here C_w denotes the space of all weakly continuous functions. The proof parallels that of Theorem 1. Similar remark applies to Remark 2.

We mention several known results on the time-local solvability for the Navier-Stokes equations in L^p. In whole spaces T.Kato [Kt] shows the time-local existence with initial data in $L^n(\mathbb{R}^n)$, and Y.Giga [Gi] obtains the time-local existence with initial data in $L^p(\mathbb{R}^n)$; $n \leq p < \infty$. The time-local existence for L^∞ initial data (or BUC initial data) is also constructed by [CK], [Ca] and [GIM] in general dimension. Our results include of theirs, in the sense that the space of initial data contains their.

Although there have been several results on solvability in Besov spaces [Am], [CM], [KM] and [KY], there seem to be no results when the space of initial data is not to decay at space infinity. Our results is the first results handling nondecaying Besov space as the space of initial data.

Recent work of Koch and Tataru [KT] consider the space of $BMO^{-1}(\mathbb{R}^n)$ of derivatives of BMO and related localized space BMO^{-1}_T. They show the existence of time-local solution of (NS) in this space, and they also construct the time-global solution with small data. This work is inspired by their work.

The time-local existence of the solutions for this type is proved for the method called *iteration*. For instance, by using this method L^∞-solutions are also constructed by [GIM].

We consider the integral equation:

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)ds,$$

where $u \otimes u$ is a tensor whose ij-component is $u^i u^j$; $\mathbf{P}$ denoted by (orthogonal) projection on solenoidal subspace, its ij-component is $\delta_{ij} + R_i R_j$. Here δ_{ij}

is Kronecker's delta. We call the solution of (INT) *mild solution.* Once we get the mild solution, it is easy to see that the mild solution satisfies (NS) in classical sense. To make matters better, it has the uniqueness. Then we can construct a strong solution of (NS).

By using iteration scheme much difficulty is to estimate for bilinear terms. To begin with, we may not multiply two tempered distributions in generally. We have to do it to make suitable sense, firstly we have to establish the regularizing effect for the mild solution. Actually the initial velocity belongs to $b_{p,q}^{-\varepsilon}$ or $\dot{b}_{p,q}^{-\varepsilon}$ when these exponents are assumption of Theorem 1 or Remark 2, then the solution belongs to L^p for any positive time.

Secondary we calculate the Besov norm of multiplication of two functions directly. We divide the sum of Besov norm by three parts with respect to frequencies. In order to get the Hölder type inequality in Besov spaces we use the paraproduct lemma. We know the Bony's paraproduct lemma (see [Bon]), and also one can see this type lemma in [RS]. While we establish in this paper to calculate the support of $\hat{\phi}_k * (\hat{\phi}_i \cdot \hat{\phi}_j)$, so this is different from Bony's. Consequencely obtained inequality seems like the differentiability-exponents in both-hand-side do not coincide, but it holds truly.

In applying the Hölder type inequality for nonlinear terms we have to take care of the property as follows: the Riesz transform may not be the bounded operator on inhomogeneous Besov space when integral-exponent is infinity. But we know the result of [GIM], they show that the operator $(\nabla \cdot e^{t\Delta}\mathbf{P})$ is bounded operator on L^∞ for any $t > 0$. Similarly we can also verify this type operator is bounded on $B_{\infty,q}^{s}$ easily. See Lemma 1.

We are firmly determined to get the solution continuously at initial time, so it is necessary to show the solution belongs to Besov space which contains the initial velocity. We have to verify it also belongs to small Besov space. The uniqueness is obtained by Gronwall's inequality easily.

The plan of this paper is as follows. In section 2 we recall various properties of Besov spaces. In section 3 we construct the solution of (INT) in $b_{\infty,q}^{-\varepsilon}(\mathbb{R}^n)$; in general dimension $n \geq 2$ with $\varepsilon \in (0, 1/2)$ and $q < \infty$. Lastly we mention the uniqueness of obtained solution and the case of another exponents.

After this work was completed, the author was informed of a recent work of H.Kozono, T.Ogawa and Y.Taniuchi [KOT] closely related to ours. They also proved the existence of a unique solution and studied various properties of Besov spaces, especially, generalized Beale-Kato-Majda type estimate for extension of time-interval when the solution exists. However, they only show time-local existence with initial data in $B^0_{\infty,\infty}$. We show the existence in $B^{-\varepsilon}_{p,q}$, $(n < p \leq \infty, 1 \leq q \leq \infty, 0 \leq \varepsilon < 1 - n/p)$, so our space is bigger than theirs. We also learned of books of e.g. [RS] including several important properties of Besov spaces. They mentioned various interpolation theory for concerning with multiplies, but we directly estimate the multiplication of functions by Hölder type inequality.

(Acknowledgment).
The author is grateful to Professor Yoshikazu Giga and Professor Shin'ya Matsui for letting him know many interesting literatures concerned with this paper (e.g. [Kt], [GIM]). He is also grateful to Professor Tohru Ozawa for letting him know literature [Bou]. He is also grateful to Professor Takayuki Kobayashi for letting him correct the error of the original proof of Lemma 1. He is also grateful to Professor Marco Cannone for letting him know literatures [Bon] and [Jo]. He is also grateful to Professor Herbert Amann for letting him know the book [RS].

2 Estimates of products

In this section we recall various properties of Besov spaces. Throughout this paper we denote positive constants by C the value of which may differ from one occasion to another.

We first examine whether the Riesz transforms are bounded in Besov spaces. Of course such an estimate is well-known if the space is homogeneous or of finite integral-exponent.

Lemma 1. *Let $n \geq 1$, $0 < \alpha < 1$ and $s \in \mathbb{R}$. Then there exists a positive constant $C = C(n, \alpha, s)$ such that*

$$||(-\Delta)^{\alpha/2} R_i f; B^s_{\infty,q}|| \leq C||f; B^{s+\alpha}_{\infty,q}||,$$

for any $1 \leq q \leq \infty$, $i = 1, 2, \ldots, n$ and $f \in B^{s+\alpha}_{\infty,q}$.

Proof. We note two estimates for ϕ_j and ψ in Definition 1.

Let $0 < \alpha < 1$. Then there exists a positive constant C (independent of j, ϕ_j and ψ) such that

$$||(-\Delta)^{\alpha/2} R_i \psi; L^1|| \leq C \tag{2.1}$$

$$||(-\Delta)^{\alpha/2} R_i \phi_j; L^1|| \leq C 2^{\alpha j}, \tag{2.2}$$

for any $i = 1, 2, \ldots, n$ and $j \in \mathbb{Z}$. The proof of (2.1) is essentially the same as [GIM; Appendix 2]. By a dilation and translation the estimate (2.1) for ϕ_j instead of ψ yields (2.2).

Based on (2.1) and (2.2) we shall complete the proof of Lemma 1. By using $f = \psi * f + \sum_{j \geq 1} \phi_j * f$ we have

$$\begin{aligned}
&||(-\Delta)^{\alpha/2} R_i f; B^s_{\infty,q}|| \\
&= ||\psi * \{(-\Delta)^{\alpha/2} R_i f\}; L^\infty|| + \Big[\sum_{k=1}^{\infty} 2^{ksq} ||\phi_k * \{(-\Delta)^{\alpha/2} R_i f\}; L^\infty||^q\Big]^{1/q} \\
&= ||\psi * \{(-\Delta)^{\alpha/2} R_i (\psi * f + \sum_{l=1}^{\infty} \phi_l * f)\}; L^\infty|| \\
&\quad + \Big[\sum_{k=1}^{\infty} 2^{ksq} ||\phi_k * \{(-\Delta)^{\alpha/2} R_i (\psi * f + \sum_{l=1}^{\infty} \phi_l * f)\}; L^\infty||^q\Big]^{1/q}.
\end{aligned}$$

By Young's inequality we have

$$\leq ||(-\Delta)^{\alpha/2}R_i\psi; L^1||\ ||\psi * f; L^\infty|| + \sum_{k=1}^{2} ||(-\Delta)^{\alpha/2}R_i\phi_k; L^1||\ ||\psi * f; L^\infty||$$
$$+ \Big[\sum_{k=1}^{2} 2^{ksq}||(-\Delta)^{\alpha/2}R_i\phi_k; L^1||^q\Big]^{1/q}||\psi * f; L^\infty||$$
$$+ \Big[\sum_{k=1}^{\infty} 2^{ksq}\sum_{l=k-1}^{k+1}||(-\Delta)^{\alpha/2}R_i\phi_k; L^1||^q||\phi_k * f; L^\infty||^q\Big]^{1/q}.$$

By (2.1) and (2.2) we have

$$\leq C||\psi * f; L^\infty|| + C\Big[\sum_{k=1}^{\infty} 2^{k(s+\alpha)q}||\phi_k * f; L^\infty||^q\Big]^{1/q}$$
$$\leq C||f; B^{s+\alpha}_{\infty,q}||.$$

Here C depends only on α, s and n. $\square$

Combining Lemma 1 with Proposition (stated later), we are able to justify our formula of (1.4) and bilinear term of (INT).

We next establish the fundamental estimate for the heat semigroup $e^{t\Delta}$ in Besov spaces.

Lemma 2. *Let $n \geq 1$, $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 < \alpha < 2$ if $n \geq 2$ and $1 < \alpha < 2$ if $n = 1$. Then there exists a positive constant C (depending only on α) such that*

$$||e^{t\Delta}f; B^s_{p,q}|| \leq Ct^{-\alpha/2}||f; B^{s-\alpha}_{p,q}||,$$

for all $f \in B^{s-\alpha}_{p,q}(\mathbb{R}^n)$ and $0 < t \leq 1$.

Proof. We assume that q is finite; the proof for $q = \infty$ is obtained by standard modification of that for finite q. By the form $e^{t\Delta} = G_t *$ it is sufficient to show that

$$2^{j\alpha}||G_t * \phi_j * f; L^p|| \leq C(1 + t^{-\alpha/2})||\phi_j * f; L^p||,$$

where $C = C(\alpha, n)$ is independent of $j \in \mathbb{Z}$, $t > 0$ and f. By Young's inequality we have

$$\begin{aligned}
||G_t * \phi_j * f; L^p|| &\leq ||G_t * (\sum_{k=-1}^{1} \phi_{j+k}) * \phi_j * f; L^p|| \\
&\leq ||G_t * (\sum_{k=-1}^{1} \phi_{j+k}); L^1|| \, ||\phi_j * f; L^p||.
\end{aligned}$$

We now obtain

$$\begin{aligned}
2^{j\alpha}||G_t * (\sum_{k=-1}^{1} \phi_{j+k}); L^1|| &\leq C \sup_{j-1 \leq l \leq j+1} \{2^{l\alpha}||\phi_l * G_t; L^1||\} \\
&\leq C||G_t; B^{\alpha}_{1,\infty}|| \\
&\leq C||(I - \Delta)^{\alpha/2} G_t; L^1||.
\end{aligned}$$

We now apply [St; p133, Lemma 2] to get

$$\begin{aligned}
&\leq C\{||(-\Delta)^{\alpha/2} G_t; L^1|| + ||G_t; L^1||\} \\
&\leq C\{||(-\Delta)^{\alpha/2} G_t; \mathcal{H}^1|| + 1\}
\end{aligned}$$

where $\mathcal{H}^1$ is the Hardy space of degree 1. Here we have defined $(-\Delta)^{\alpha/2} G_t$ by

$$\begin{aligned}
(-\Delta)^{\alpha/2} G_t(x) &= (-\Delta)(-\Delta)^{\alpha/2-1} G_t(x) \\
&= K(\alpha, n)(-\Delta) \int_{\mathbb{R}^n} |x - y|^{-n-\alpha+2} G_t(y) dy,
\end{aligned}$$

where $K(\alpha, n) = \pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)/\Gamma(n/2 - \alpha/2)$ as in [St]. This integral is well-defined since $G_t \in \mathcal{S}$. We recall an inequality

$$||(-\Delta)^{\alpha/2} G_t; \mathcal{H}^1|| \leq C t^{-\alpha/2}. \tag{2.3}$$

which is for example found in [GIM]. Since we assume that $t \leq 1$, the proof is now complete. $\square$

Remark 4. *In the homogeneous Besov spaces we obtain similar inequality. Let exponents be the same as in Lemma 2. Then there exists $C > 0$ such that*

$$||e^{t\Delta} f; \dot{B}^s_{p,q}|| \leq Ct^{-\alpha/2}||f; \dot{B}^{s-\alpha}_{p,q}||$$

for all $f \in \dot{B}^{s-\alpha}_{p,q}$ and $t > 0$.

It is unnecessary to assume the condition of $t \leq 1$ in the case of homogeneous spaces. The proof is similar to that of Lemma 2. It is also found in [CX], however, inhomogeneous version was not available so we have given a full proof.

We now establish Hölder type inequality in Besov spaces. We state it in the next proposition. This is crucial step in this paper.

It is standard to use the interpolation theory for estimating nonlinear terms. These are much literatures in this direction. However, here we do not use the interpolation theory, but we directly calculate the multiplication.

Proposition . *Let $\alpha > 0$, $1 \leq p, q \leq \infty$, and let $1 \leq r, s \leq \infty$ satisfying $1/p = 1/r + 1/s$. Let $\sigma > 0$, $\theta \geq 0$. Then there exists a positive constant $C = C(\alpha, p, q, r, s, \sigma, \theta)$ such that*

$$\begin{aligned}||fg; B^\alpha_{p,q}|| \leq & C\Big[(N^2+1)\big\{||f; B^{\theta+\alpha}_{r,q}||\ ||g; B^{-\theta}_{s,q}|| + ||f; B^{-\theta}_{r,q}||\ ||g; B^{\theta+\alpha}_{s,q}||\big\} \\ & + 2^{-N\delta}(N+1)\big\{||f; B^{\sigma+\alpha+\delta}_{r,q}||\ ||g; B^{-\sigma}_{s,q}|| + ||f; B^{-\sigma}_{r,q}||\ ||g; B^{\sigma+\alpha+\delta}_{s,q}||\big\}\Big]\end{aligned}$$

for all $N \in \mathbb{N}_0$, $0 < \delta \leq \alpha$, f and g belong to intersection of all inhomogeneous Besov spaces in right-hand-side respectively.

Remark 5. *We mention two remarkable facts.*
(1) If q is infinite, the inequality in Proposition holds with $(N^2 + 1)$ and $(N + 1)$ replaced by 1.
(2) One can proof similar inequality in the homogeneous Besov spaces. Let exponents be the same as in Proposition. Then

$$\begin{aligned}||fg; \dot{B}^\alpha_{p,q}|| \leq & C\Big[(N^2+1)\big\{||f; \dot{B}^{\theta+\alpha}_{r,q}||\ ||g; \dot{B}^{-\theta}_{s,q}|| + ||f; \dot{B}^{-\theta}_{r,q}||\ ||g; \dot{B}^{\theta+\alpha}_{s,q}||\big\} \\ & + 2^{-N\delta}(N+1)\big\{||f; \dot{B}^{\sigma+\alpha+\delta}_{r,q}||\ ||g; \dot{B}^{-\sigma}_{s,q}|| + ||f; \dot{B}^{-\sigma}_{r,q}||\ ||g; \dot{B}^{\sigma+\alpha+\delta}_{s,q}||\big\} \\ & + 2^{-N\delta}(N+1)\big\{||f; \dot{B}^{\sigma+\alpha-\delta}_{r,q}||\ ||g; \dot{B}^{-\sigma}_{s,q}|| + ||f; \dot{B}^{-\sigma}_{r,q}||\ ||g; \dot{B}^{\sigma+\alpha-\delta}_{s,q}||\big\}\Big].\end{aligned}$$

For the proof we prepare two lemmas. We shall use the convention that $f_k = \phi_k * f$, $g_l = \phi_l * g$, $f_\natural = \psi * f$ and $g_\natural = \psi * g$ as well as $a \vee b = \max(a, b)$.

Lemma 3 (paraproduct lemma — inhomogeneous version). *Let $j \in \mathbb{N}$. Let $f, g, fg \in \mathcal{S}'$. Then*

$$\psi * \Big\{(f_\natural + \sum_{k=1}^{\infty} f_k) \cdot (g_\natural + \sum_{l=1}^{\infty} g_l)\Big\}$$

$$= \psi * \Big\{ \sum_{\{k,l \geq 1;\ |k-l| \leq 2\}} f_k g_l \Big\} + \psi * \Big\{ \sum_{k=1}^{2} f_k g_\natural \Big\} + \psi * \Big\{ \sum_{l=1}^{2} f_\natural g_l \Big\} + \psi * \Big\{ f_\natural g_\natural \Big\},$$

and then

$$\phi_j * \Big\{(f_\natural + \sum_{k=1}^{\infty} f_k) \cdot (g_\natural + \sum_{l=1}^{\infty} g_l)\Big\}$$

$$= \phi_j * \Big\{ \sum_{(k,l) \in S_j} f_k g_l \Big\} + \phi_j * \Big\{ \sum_{k=1 \vee (j-2)}^{j+2} f_k g_\natural \Big\} + \phi_j * \Big\{ \sum_{l=1 \vee (j-2)}^{j+2} f_\natural g_l \Big\}$$
$$+ (\delta_{j1} + \delta_{j2}) \phi_j * \Big\{ f_\natural g_\natural \Big\},$$

where $S_j = S_j^1 + S_j^2 + S_j^3$;

$$S_j^1 = \Big\{(k,l) \in \mathbb{N}^2;\ k, l \geq j,\ |k-l| \leq 2\Big\},$$
$$S_j^2 = \Big\{(k,l) \in \mathbb{N}^2;\ k \leq j,\ |l-j| \leq 2\Big\},$$
$$S_j^3 = \Big\{(k,l) \in \mathbb{N}^2;\ l \leq j,\ |k-j| \leq 2\Big\}.$$

Proof. We shall verify whether $\phi_j * (f_k g_l) \equiv 0$ for given j, k and l. We consider its Fourier transforms and obtain

$$\mathcal{F}[\phi_j * \{(\phi_k * f) \cdot (\phi_l * g)\}] = \hat{\phi}_j \cdot \{(\hat{\phi}_k \hat{f}) * (\hat{\phi}_l \hat{g})\}.$$

Then it is enough to estimate the support of $\hat{\phi}_j \cdot (\hat{\phi}_k * \hat{\phi}_l)$. We have

$$\Phi_{jkl} = \Big(\hat{\phi}_j \cdot (\hat{\phi}_k * \hat{\phi}_l)\Big)(\xi) = \hat{\phi}_j(\xi) \int_{\mathbb{R}^n} \hat{\phi}_k(\xi - \eta) \hat{\phi}_l(\eta) d\eta,$$

and observe that Φ_{jkl} equals zero if (j, k, l) satisfies the following conditions:

$$\text{either} \quad 2^{l+1} + 2^{j+1} \leq 2^{k-1}, \tag{2.4}$$

$$\text{or} \quad 2^{j+1} + 2^{k+1} \leq 2^{l-1}, \tag{2.5}$$

$$\text{or} \quad 2^{k+1} + 2^{l+1} \leq 2^{j-1}. \tag{2.6}$$

The proof is now complete. □

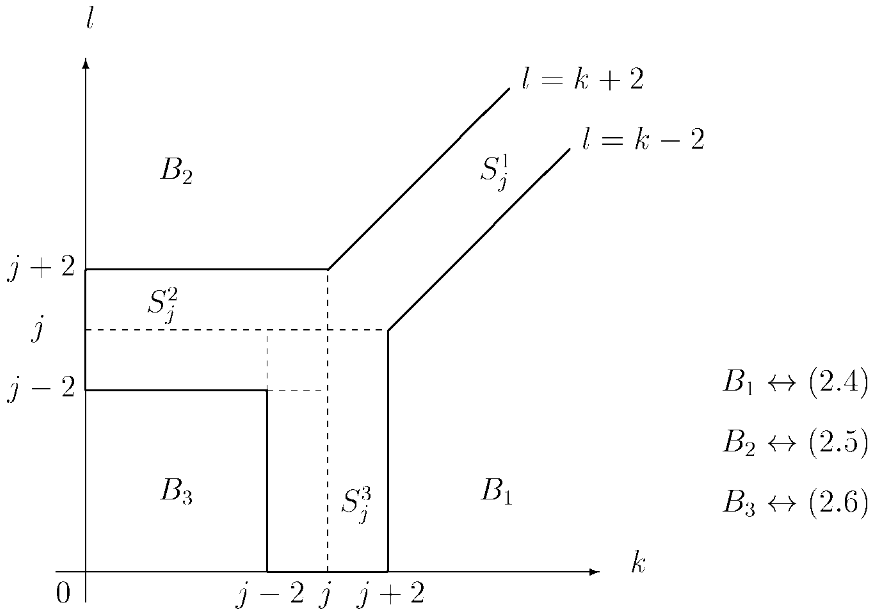

Figure 1:

Similar paraproduct lemma is found in [Bon] or [RS]. They show that Φ_{jkl} equals zero for the indices in B_1 and B_2. However, they do not study the case when indices in B_3. So our lemma is different from theirs. In order to state the next lemma it is necessary to study the part corresponding B_3.

The next lemma yields Proposition. This is one of the most general form of Hölder type inequality in inhomogeneous Besov spaces.

Lemma 4 (Hölder type inequality in inhomogeneous Besov spaces). *Let $1 \leq p, q \leq \infty$ and $\alpha > 0$. Let $i = 1, 2, \ldots, 12$; $\quad 1 \leq r_i, s_i \leq \infty$ satisfying $1/p = 1/r_i + 1/s_i$, and let $\rho_i \in \mathbb{R}$, $\theta_i \geq 0$ and $\sigma_i > 0$. Then there exists a positive constant $C = C(\alpha, p, q, r_i, s_i, \rho_i, \theta_i, \sigma_i)$ such that*

$$||fg; B^{\alpha}_{p,q}|| \leq C(N^2 + 1)\Pi_1(f, g) + C(N + 1)2^{-N\delta}\Pi_2(f, g),$$

where

$$
\begin{aligned}
\Pi_1(f,g) =& ||f; B^{\rho_1+\alpha}_{r_1,\infty}||\ ||g; B^{-\rho_1}_{s_1,\infty}|| + ||f; B^{-\theta_2}_{r_2,\infty}||\ ||g; B^{\theta_2+\alpha}_{s_2,\infty}|| \\
&+ ||f; B^{\theta_3+\alpha}_{r_3,\infty}||\ ||g; B^{-\theta_3}_{s_3,\infty}|| \\
\Pi_2(f,g) =& ||f; B^{\rho_4+\alpha+\delta}_{r_4,\infty}||\ ||g; B^{-\rho_4}_{s_4,\infty}|| + ||f; B^{-\sigma_5}_{r_5,\infty}||\ ||g; B^{\sigma_5+\alpha+\delta}_{s_5,\infty}|| \\
&+ ||f; B^{\sigma_6+\alpha+\delta}_{r_6,\infty}||\ ||g; B^{-\sigma_6}_{s_6,\infty}|| + ||f; B^{\rho_7+\sigma_7}_{r_7,\infty}||\ ||g; B^{-\rho_7}_{s_7,\infty}|| \\
&+ ||f; B^{\sigma_8+\alpha}_{r_8,\infty}||\ ||g; B^{\mu_8}_{s_8,\infty}|| + ||f; B^{\mu_9}_{r_9,\infty}||\ ||g; B^{\sigma_9+\alpha}_{s_9,\infty}|| \\
&+ ||f; B^{\sigma_{10}}_{r_{10},\infty}||\ ||g; B^{\mu_{10}}_{s_{10},\infty}|| + ||f; B^{\mu_{11}}_{r_{11},\infty}||\ ||g; B^{\sigma_{11}+\alpha}_{s_{11},\infty}|| \\
&+ ||f; B^{\mu_{12}}_{r_{12},\infty}||\ ||g; B^{\tilde{\mu}_{12}+\alpha}_{s_{12},\infty}||
\end{aligned}
$$

for all $N \in \mathbb{N}_0$, $0 < \delta \le \alpha$, $\mu_i, \tilde{\mu}_i \in \mathbb{R}$, f *and* g *belong to intersection of all inhomogeneous Besov spaces in right-hand-side respectively.*

Proof. We may assume that q is finite without loss of generality, since we give the proof for the case $q = \infty$ is obtained by a standard modification of that for finite q.

By the definition we have

$$
\begin{aligned}
||fg; B^{\alpha}_{p,q}|| =& \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} ||\phi_j * (fg); L^p||^q\Big]^{1/q} + ||\psi * (fg); L^p|| \\
=& \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} ||\phi_j * \big\{(\sum_{k=1}^{\infty} f_k + f_\sharp)\cdot(\sum_{l=1}^{\infty} g_l + g_\sharp)\big\}; L^p||^q\Big]^{1/q} \\
&+ ||\psi * \big\{(\sum_{k=1}^{\infty} f_k + f_\sharp)\cdot(\sum_{l=1}^{\infty} g_l + g_\sharp)\big\}; L^p||.
\end{aligned}
$$

Applying Lemma 3, we observe that

$$
\begin{aligned}
\le& \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} ||\phi_j * \big\{ \sum_{(k,l)\in S_j} f_k g_l + \sum_{k=1\vee(j-2)}^{j+2} f_k g_\sharp + \sum_{l=1\vee(j-2)}^{j+2} f_\sharp g_l \\
&+ (\delta_{j1} + \delta_{j2}) f_\sharp g_\sharp \big\}; L^p||^q\Big]^{1/q} \\
&+ ||\psi * \big\{ \sum_{\{k,l\in\mathbb{N}; |k-l|\le 2\}} f_k g_l + \sum_{k=1}^{2} f_k g_\sharp + \sum_{l=1}^{2} f_\sharp g_l + f_\sharp g_\sharp \big\}; L^p||.
\end{aligned}
$$

We set $||\phi_j; L^1|| = C_0$ (independent of j) and $||\psi; L^1|| = C_1$. By using l^q-Minkowski and L^p-Young inequalities, we get

$$
\begin{aligned}
&||fg; B^{\alpha}_{p,q}|| \\
&\leq C_0 \Big\{ \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} || \sum_{(k,l)\in S_j} f_k g_l; L^p||^q \Big]^{1/q} + \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} || \sum_{k=1\vee(j-2)}^{j+2} f_k g_{\sharp}; L^p||^q \Big]^{1/q} \\
&\quad + \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} || \sum_{l=1\vee(j-2)}^{j+2} f_k g_{\natural}; L^p||^q \Big]^{1/q} + \Big[\sum_{j=1}^{2} 2^{j\alpha q} ||f_{\sharp} g_{\sharp}; L^p||^q \Big]^{1/q} \Big\} \\
&\quad + C_1 \Big\{ || \sum_{\{k,l\in\mathbb{N}; |k-l|\leq 2\}} f_k g_l; L^p|| + || \sum_{k=1}^{2} f_k g_{\natural}; L^p|| + || \sum_{l=1}^{2} f_{\sharp} g_l; L^p|| + ||f_{\natural} g_{\natural}; L^p|| \Big\} \\
&\equiv C_0(\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4) + C_1(\mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4).
\end{aligned}
$$

We shall estimate each term.

We present estimates for I_1 and I_2 only, since other terms can be estimated in a similar (and easier) way. First we estimate I_1. We divide S_j into three sets, we have $\mathrm{I}_1 \leq \sum_{m=1}^{3} \mathrm{J}_m$ with $\mathrm{J}_m = \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} || \sum_{(k,l)\in S_j^m} f_k g_l; L^p||^q \Big]^{1/q}$.

We start to estimate J_1 by recalling definition of S_j^1:

$$
\mathrm{J}_1 \leq [\sum_{j=1}^{\infty} 2^{j\alpha q} \{ \sum_{k\geq j} \sum_{l=1\vee(k-2)}^{k+2} ||f_k g_l; L^p|| \}^q]^{1/q}.
$$

We divide the sum into three parts with respect to indices j and k of middle-middle, middle-high, high-high frequency. For all positive integer N

$$
\begin{aligned}
\mathrm{J}_1 \leq & [\sum_{1\leq j\leq N} 2^{j\alpha q} \{ \sum_{j\leq k\leq N} \sum_{l=1\vee(k-2)}^{k+2} ||f_k g_l; L^p|| \}^q]^{1/q} \\
& + [\sum_{1\leq j\leq N} 2^{j\alpha q} \{ \sum_{k\geq N} \sum_{l=1\vee(k-2)}^{k+2} ||f_k g_l; L^p|| \}^q]^{1/q} \\
& + [\sum_{j\geq N+1} 2^{j\alpha q} \{ \sum_{k\geq j} \sum_{l=1\vee(k-2)}^{k+2} ||f_k g_l; L^p|| \}^q]^{1/q}, \\
\equiv & \mathrm{J}_{MM} + \mathrm{J}_{MH} + \mathrm{J}_{HH}.
\end{aligned}
$$

[J_{MM} **estimate**]. We use exponents $1 \leq r, s \leq \infty$, $1/p = 1/r + 1/s$ and $\rho \in \mathbb{R}$ to get

$$\mathrm{J}_{MM} \leq \Big[\sum_{1 \leq j \leq N} 2^{j\alpha q} \big\{ \sum_{j \leq k \leq N} 2^{-k\rho} 2^{k\rho} ||f_k; L^r|| \sum_{l=1\vee(k-2)}^{k+2} ||g_l; L^s|| \big\}^q \Big]^{1/q}.$$

Since $j \leq k$ and $k-2 \leq l \leq k+2$, we obtain that $2^{j\alpha} \leq 2^{k\alpha}$ and $2^{-k\rho} \leq 2^{2|\rho|} \cdot 2^{-l\rho}$. We also observe that $2^{k(\alpha+\rho)}||f_k; L^r|| \leq \sup_k 2^{k(\alpha+\rho)}||f_k; L^r|| = ||f; B^{\rho+\alpha}_{r,\infty}||$ and similarly $2^{-l\rho}||g_l; L^s|| \leq ||g; B^{-\rho}_{s,\infty}||$. Combining these estimates yields

$$\begin{aligned} \mathrm{J}_{MM} \leq & C||f; B^{\rho+\alpha}_{r,\infty}|| \; ||g; B^{-\rho}_{s,\infty}|| \Big[\sum_{1 \leq j \leq N} 1 \Big]^{1/q} \cdot \big\{ \sum_{1 \leq k \leq N} 1 \big\} \\ \leq & C(N^2+1)||f; B^{\rho+\alpha}_{r,\infty}|| \; ||g; B^{-\rho}_{s,\infty}||. \end{aligned}$$

[J_{MH} **estimate**] Let r, s and ρ be as the same exponents as in J_{MM} estimate, and let $\delta > 0$. We obtain

$$\begin{aligned} \mathrm{J}_{MH} \leq & \Big[\sum_{1 \leq j \leq N} 2^{j\alpha q} \big\{ \sum_{k \geq N} 2^{-k(\alpha+\delta+\rho)} 2^{k(\alpha+\delta+\rho)} ||f_k; L^r|| \sum_{l=1\vee(k-2)}^{k+2} ||g_l; L^s|| \big\}^q \Big]^{1/q} \\ \leq & C \Big[\sum_{1 \leq j \leq N} 1 \Big] \big\{ \sum_{k \geq N} 2^{-k\delta} ||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \sum_{l=1\vee(k-2)}^{k+2} ||g; B^{-\rho}_{s,\infty}|| \big\} \\ \leq & C 2^{-N\delta}(N+1)||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \; ||g; B^{-\rho}_{s,\infty}||. \end{aligned}$$

[J_{HH} **estimate**] Let r, s, ρ, δ be as the same exponents as in J_{MH} estimate. We obtain

$$\begin{aligned} \mathrm{J}_{HH} \leq & \Big[\sum_{j \geq N} 2^{-j\delta q/2} 2^{j(\alpha+\delta/2)q} \big\{ \sum_{k \geq j} ||f_k; L^r|| \sum_{l=1\vee(k-2)}^{k+2} ||g_l; L^s|| \big\}^q \Big]^{1/q} \\ \leq & C \Big[\sum_{j \geq N} 2^{-j\delta q/2} \big\{ \sum_{k \geq j} 2^{-k\delta/2} ||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \sum_{l=1\vee(k-2)}^{k+2} ||g; B^{-\rho}_{s,\infty}|| \big\} \Big]^{1/q} \\ \leq & C||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \; ||g; B^{-\rho}_{s,\infty}|| \Big[\sum_{j \geq N} 2^{-j\delta q/2} \Big]^{1/q} \big\{ \sum_{k \geq N} 2^{-j\delta/2} \big\} \\ \leq & C 2^{-N\delta} ||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \; ||g; B^{-\rho}_{s,\infty}||. \end{aligned}$$

The estimates for J_2 and J_3 are basically the same as that for J_1, so we do not present the details.

We next estimate I_2. Let $1 \leq r, s \leq \infty$; $1/p = 1/r + 1/s$, $\sigma > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. We observe that

$$\mathrm{I}_2 \leq \Big[\sum_{j \geq 1} 2^{-j\sigma q} \Big\{ \sum_{k=1 \vee (j-2)}^{j+2} 2^{k(\sigma+\alpha)} ||f_k; L^r|| \, ||g_\sharp; L^s|| \Big\}^q \Big]^{1/q}.$$

Note that $||g_\sharp; L^s|| \leq ||g; B^\mu_{s,\infty}||$ for all $\mu \in \mathbb{R}$ to get

$$\leq C ||f; B^{\sigma+\alpha}_{r,\infty}|| \, ||g; B^\mu_{s,\infty}||.$$

Similarly one can estimate all of other terms. The proof is now complete.

$\square$

Our Proposition follows from Lemma 4 easily.

We mention the proof of Remark 5-(1). It is easy to see if q is infinite, we substitute the $\sum_{1 \leq j \leq N}$ by $\sup_{1 \leq j \leq N}$ and obtain the inequality with replaced by 1 in Lemma 4.

We next mention the proof of Remark 5-(2). One can obtain the Hölder type inequality similar to inhomogeneous one in the case of homogeneous Besov space. It follows from two lemmas below (corresponding Lemma 3 and Lemma 4).

Lemma 5 (paraproduct lemma — homogeneous version). *Let $j \in \mathbb{Z}$. Let $f, g, fg \in \mathcal{S}'$ with $f = \sum_{k=-\infty}^{\infty} f_k$ and $g = \sum_{l=-\infty}^{\infty} g_l$. Then*

$$\phi_j * \{fg\} = \phi_j * \Big\{ \sum_{(k,l) \in \tilde{S}_j} f_k g_l \Big\},$$

where $\tilde{S}_j = \tilde{S}_j^1 + \tilde{S}_j^2 + \tilde{S}_j^3$, and

$$\tilde{S}_j^1 = \big\{ (k,l) \in \mathbb{Z}^2;\ k, l \geq j,\ |k-l| \leq 2 \big\},$$
$$\tilde{S}_j^2 = \big\{ (k,l) \in \mathbb{Z}^2;\ k \leq j,\ |l-j| \leq 2 \big\},$$
$$\tilde{S}_j^3 = \big\{ (k,l) \in \mathbb{Z}^2;\ l \leq j,\ |k-j| \leq 2 \big\}.$$

The proof parallels that of Lemma 3. Lemma 5 yields:

Lemma 6 (Hölder type inequality in homogeneous Besov spaces). *Let $1 \le p, q \le \infty$ and $\alpha > 0$. Let $i = 1, 2, \dots, 9$; $\quad 1 \le r_i, s_i \le \infty$ satisfying $1/p = 1/r_i + 1/s_i$, and let $\rho_i \in \mathbb{R}$, $\theta_i \ge 0$ and $\sigma_i > 0$. Then there exists a positive constant $C = C(p, q, \alpha, r_i, s_i, \rho_i, \theta_i, \sigma_i)$ such that*

$$||fg; \dot{B}^{\alpha}_{p,q}|| \le C(N^2+1)\tilde{\Pi}_1(f,g) + C(N+1)2^{-N\delta}\{\tilde{\Pi}_2(f,g) + \tilde{\Pi}_3(f,g)\},$$

where

$$\begin{aligned}
\tilde{\Pi}_1(f,g) =& ||f; \dot{B}^{\rho_1+\alpha}_{r_1,\infty}||\ ||g; \dot{B}^{-\rho_1}_{s_1,\infty}|| + ||f; \dot{B}^{-\theta_2}_{r_2,\infty}||\ ||g; \dot{B}^{\theta_2+\alpha}_{s_2,\infty}|| \\
&+ ||f; \dot{B}^{\theta_3+\alpha}_{r_3,\infty}||\ ||g; \dot{B}^{-\theta_3}_{s_3,\infty}||, \\
\tilde{\Pi}_2(f,g) =& ||f; \dot{B}^{\rho_4+\alpha-\delta}_{r_4,\infty}||\ ||g; \dot{B}^{-\rho_4}_{s_4,\infty}|| + ||f; \dot{B}^{-\sigma_5}_{r_5,\infty}||\ ||g; \dot{B}^{\sigma_5+\alpha-\delta}_{s_5,\infty}|| \\
&+ ||f; \dot{B}^{\sigma_6+\alpha-\delta}_{r_6,\infty}||\ ||g; \dot{B}^{-\sigma_6}_{s_6,\infty}||, \\
\tilde{\Pi}_3(f,g) =& ||f; \dot{B}^{\rho_7+\alpha+\delta}_{r_7,\infty}||\ ||g; \dot{B}^{-\rho_7}_{s_7,\infty}|| + ||f; \dot{B}^{-\sigma_8}_{r_8,\infty}||\ ||g; \dot{B}^{\sigma_8+\alpha+\delta}_{s_8,\infty}|| \\
&+ ||f; \dot{B}^{\sigma_9+\alpha+\delta}_{r_9,\infty}||\ ||g; \dot{B}^{-\sigma_9}_{s_9,\infty}||,
\end{aligned}$$

for all $N \in \mathbb{N}_0$, $0 < \delta \le \min\{\alpha, \sigma_5, \sigma_6\}$, f and g belong to intersection of all homogeneous Besov spaces in right-hand-side respectively.

Proof. We assume that $f \in \dot{B}^{-\sigma}_{r,\infty}$ with $\sigma > 0$ then we have $f = \sum_{k=-\infty}^{\infty} f_k$, of course the same property holds in g, see [BL]. We assume that q is finite without loss of generality. By the definition we have

$$||fg; \dot{B}^{\alpha}_{p,q}|| = \Big[\sum_{j=-\infty}^{\infty} 2^{j\alpha q} ||\phi_j * (fg); L^p||^q\Big]^{1/q}$$

$$= \Big[\sum_{j=-\infty}^{\infty} 2^{j\alpha q} ||\phi_j * \{(\sum_{k=-\infty}^{\infty} f_k) \cdot (\sum_{l=-\infty}^{\infty} g_l)\}; L^p||^q\Big]^{1/q}.$$

Using Lemma 5 and Young's inequality, we have

$$\le \Big[\sum_{j=-\infty}^{\infty} 2^{j\alpha q} ||\phi_j; L^1||^q \{\sum_{(k,l)\in\tilde{S}_j} ||f_k g_l; L^p||\}^q\Big]^{1/q}.$$

We set $||\phi_j;\ L^1|| = C_0$ (independent of j). By dividing the sum of k and l by three parts we get

$$\leq C_0\Big\{\Big[\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\{\sum_{(k,l)\in\tilde{S}_j^1} ||f_k g_l; L^p||\}^q\Big]^{1/q} + \Big[\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\{\sum_{(k,l)\in\tilde{S}_j^2} ||f_k g_l; L^p||\}^q\Big]^{1/q}$$
$$+ \Big[\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\{\sum_{(k,l)\in\tilde{S}_j^3} ||f_k g_l; L^p||\}^q\Big]^{1/q}\Big\},$$
$$\equiv C_0(\tilde{\mathrm{I}} + \tilde{\mathrm{II}} + \tilde{\mathrm{III}}).$$

Since the estimates for $\tilde{\mathrm{II}}$ and $\tilde{\mathrm{III}}$ are essentially similar to that of $\tilde{\mathrm{I}}$ we present the estimate for $\tilde{\mathrm{I}}$. We obtain that

$$\tilde{\mathrm{I}} \leq [\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\{\sum_{k\geq j}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q]^{1/q}.$$

We divide the sum into six parts with respect to frequencies of j and k, these are low-frequencies, middle-frequencies and high-frequencies. For all non-negative integer N, we have

$$\tilde{\mathrm{I}} \leq \Big[\sum_{j\leq -N} 2^{j\alpha q}\{\sum_{j\leq k\leq -N}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q\Big]^{1/q} + \Big[\sum_{j\leq -N} 2^{j\alpha q}\{\sum_{|k|\leq N}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q\Big]^{1/q}$$
$$+ \Big[\sum_{j\leq -N} 2^{j\alpha q}\{\sum_{k\geq N}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q\Big]^{1/q} + \Big[\sum_{|j|\leq N} 2^{j\alpha q}\{\sum_{j\leq k\leq N}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q\Big]^{1/q}$$
$$+ \Big[\sum_{|j|\leq N} 2^{j\alpha q}\{\sum_{k\geq N}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q\Big]^{1/q} + \Big[\sum_{j\geq N} 2^{j\alpha q}\{\sum_{k\geq j}\sum_{l=k-2}^{k+2} ||f_k g_l; L^p||\}^q\Big]^{1/q}.$$

One can estimate each term in a similar way in the proof of Lemma 4. Thus the proof is now complete. □

3 Time-local solvability of (INT)

In this section we construct a unique time-local solution of (INT) by using successive iteration. The method is standard when we construct an L^p-solution. Since we handle the small Besov space, we have to prove the continuity of approximate sequence in time, in particular continuity up to initial time with values in small Besov space.

We shall present the proof of Theorem and Remark 2 only in the case of $b^{-\varepsilon}_{\infty,q}$, $0 < \varepsilon < 1/2$ and $q < \infty$ since other cases can be proved by a similar argument (see §3.3).

3.1 Iteration scheme

We first construct the mild solutions by iteration scheme. We divide the proof into five steps.

Let $n \geq 2$, $0 < \varepsilon < 1/2$ and $1 \leq q < \infty$. Assume that an initial velocity u_0 belongs to $b^{-\varepsilon}_{\infty,q}$. We define the successive approximation by setting $u_1(t) = e^{t\Delta}u_0$ and

$$u_{j+1}(t) \equiv e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s)ds \quad \text{for} \quad j \geq 1. \tag{3.1}$$

(Step 1). In order to verify the meaning of nonlinear terms we have to show that successive approximation $u_j(t)$ $(j = 1, 2, \dots)$ belong to L^∞ for $t > 0$. That is to say, we should obtain some regularizing effect.

We show that there exists a positive constant T_1 such that

$$t^{\tau/2}u_j \in C([0, T_1]; L^\infty) \quad \text{with} \quad \sup_{0<t\leq T_1} t^{\tau/2}||u_j(t); L^\infty|| \leq 2CK_0 \tag{3.2}$$

for any $\tau \in (\varepsilon, 1)$ and for any $j \geq 1$. Here $K_0 = ||u_0; B^{-\varepsilon}_{\infty,q}||$ and C is a constant independent of j, u_0 and T_1.

Proof. Let $u_0 \in b^{-\varepsilon}_{\infty,q}$ and $\tau \in (\varepsilon, 1)$. For any $0 < t \leq T < 1$ and $j \geq 1$, we estimate the L^∞-norm of (3.1). Let $K_0 = ||u_0; B^0_{\infty,q}||$, and we put $A_j = A_j(T)$ defined by

$$A_j \equiv \sup_{0<t\leq T} t^{\tau/2}||u_j(t); L^\infty||.$$

In the case of $j = 1$ by Young's inequality we have

$$||e^{t\Delta}u_0; L^\infty|| \le ||e^{t\Delta}(\psi * u_0); L^\infty|| + ||e^{t\Delta}(\sum_{j=1}^\infty \phi_j * u_0); L^\infty||$$

$$\le ||G_t; L^1|| \, ||\psi * u_0; L^\infty|| + \sum_{j=1}^\infty ||G_t * \phi_j * u_0; L^\infty||$$

$$\le ||\psi * u_0; L^\infty|| + C\sum_{j=1}^\infty 2^{-j\tau}||(-\Delta)^{\tau/2}G_t * \phi_j * u_0; L^\infty||,$$

where the positive constant C independent of u_0 (see e.g. [BL]). Once we use Young's inequality so that

$$\le ||\psi * u_0; L^\infty|| + C\sum_{j=1}^\infty 2^{-j(\tau-\varepsilon)}||(-\Delta)^{\tau/2}G_t; L^1||2^{-j\varepsilon}||\phi_j * u_0; L^\infty||$$

$$\le ||\psi * u_0; L^\infty|| + Ct^{-\tau/2}||u_0; B^{-\varepsilon}_{\infty,\infty}||.$$

By embedding theorem with respect to sum-exponent we have

$$\le Ct^{-\tau/2}K_0,$$

since $t \le 1$. Thus we take the $\sup_t$ in both-hand-side of last inequality, we obtain $A_1 \le CK_0$.

In the case of $j + 1$ we have

$$||u_{j+1}(t); L^\infty|| \le \int_0^t ||\nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s); L^\infty||ds$$

$$\le C\int_0^t |||\nabla \cdot e^{(t-s)\Delta}\mathbf{P}||| \, ||u_j(s); L^\infty||^2 ds,$$

where $|||\cdot|||$ stands for a operator norm from L^∞ to L^∞, and $|||\nabla \cdot e^{t\Delta}\mathbb{P}|||$ is upper bound by $Ct^{-1/2}$ (see [GIM]). It follows from the definition of A_j so that we have

$$\le CA_j^2\int_0^t (t-s)^{-1/2}s^{-\tau}ds.$$

Since $\tau < 1$, the integral value of last term is finite. In taking $\sup_{0\le t\le T}$ we obtain $A_{j+1} \le CK_0 + C_1T^{1/2-\tau/2}A_j^2$. We take T_1 small so that $\min\{(4CC_1K_0)^{\frac{-2}{1-\tau}}, 1\}$. Hence we obtain (3.2). □

(Step 2). We make sure that u_j belongs to $B^{-\varepsilon}_{\infty,q}$.

We show that there exists a positive constant T_2 such that

$$t^{\gamma/2}u_j(t) \in B^{\gamma-\varepsilon}_{\infty,q} \quad \text{with} \quad \sup_{0\le t\le T_2} ||u_j(t); B^{-\varepsilon}_{\infty,q}|| \le 2K_0$$
$$\text{and} \quad \sup_{0\le t\le T_2} t^{\gamma/2}||u_j(t); B^{\gamma-\varepsilon}_{\infty,q}|| \le 2CK_0 \tag{3.3}$$

for any $t \in [0, T_2]$, $j \ge 1$, $\gamma \in (0, 1]$ and $T \le T_2$. Here C is a constant independent of j, u_0 and T_2.

Proof. Let $0 < t \le T \le T_1$, $\gamma \in [0, 1]$ and we put $K_j^\gamma = K_j^\gamma(T)$ defined by

$$K_j^\gamma \equiv \sup_{0\le t\le T} t^{\gamma/2}||u_j(t); B^{\gamma-\varepsilon}_{\infty,q}||.$$

We start to estimate of linear terms. By Young's inequality we have

$$||e^{t\Delta}u_0; B^{\gamma-\varepsilon}_{\infty,q}|| = ||\psi * (G_t * u_0); L^\infty|| + \Big[\sum_{j=1}^{\infty} 2^{j(\gamma-\varepsilon)q}||\phi_j * G_t * u_0; L^\infty||^q\Big]^{1/q}$$
$$\le ||G_t; L^1||\,||\psi * u_0; L^\infty|| + C\Big[\sum_{j=1}^{\infty} ||(-\Delta)^{\gamma/2}G_t; L^1||^q 2^{-j\varepsilon q}||\phi_j * u_0; L^\infty||^q\Big]^{1/q}.$$

By (2.3) we have

$$\le ||\psi * u_0; L^\infty|| + Ct^{-\gamma/2}\Big[\sum_{j=1}^{\infty} 2^{-j\varepsilon q}||\phi_j * u_0; L^\infty||^q\Big]^{1/q}$$
$$\le Ct^{-\gamma/2}K_0,$$

since $t \le 1$. In particular we note that if $\gamma = 0$ then we can choose these constants $C = 1$. Thus we obtain

$$K_1^0 \le K_0 \quad \text{and} \quad K_1^\gamma \le CK_0$$

for all $\gamma \in (0, 1]$.

The next is to estimate of bilinear terms. To begin with, we prepare as follows; there exists a positive constant C such that

$$||\nabla \cdot f; B^s_{p,q}|| \le C||f; B^{s+1}_{p,q}||,$$

for all $s \in \mathbb{R}$, $1 \le p \le \infty$, $1 \le q \le \infty$ and $f \in B^{s+1}_{p,q}$. Thus for all $\gamma \in [0,1]$ and $0 < s < t < T$ we have

$$||\nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s); B^{\gamma-\varepsilon}_{\infty,q}|| \le C||e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s); B^{1+\gamma-\varepsilon}_{\infty,q}||$$
$$\le C|||(I-\Delta)^{(\gamma+\varepsilon)/2}\mathbf{P}e^{(t-s)\Delta}|||\ ||(u_j \otimes u_j)(s); B^{1-2\varepsilon}_{\infty,q}||.$$

Using Lemma 1, Lemma 2 and Proposition, we get

$$\le C(t-s)^{-(\gamma+\varepsilon)/2}||(u_j \otimes u_j)(s); B^{1-2\varepsilon}_{\infty,q}||$$
$$\le C(t-s)^{-(\gamma+\varepsilon)/2}\Big[(N^2+1)||u_j(s); B^{1-\varepsilon}_{\infty,q}||\ ||u_j(s); B^{-\varepsilon}_{\infty,q}||$$
$$+ (N+1)2^{-N\varepsilon/2}||u_j(s); B^{1-\varepsilon}_{\infty,q}||\ ||u_j(s); B^{-\varepsilon/2}_{\infty,q}||\Big].$$

Here we may choose arbitrary number $N \sim \varepsilon^{-1}\log(||u_j(s); B^{1-\varepsilon}_{\infty,q}||+1)$, whose setting is similar to [BG] and [GMS], thus we obtain

$$\le C(t-s)^{-(\gamma+\varepsilon)/2}\Big[\Big(\big\{\log(||u_j(s); B^{1-\varepsilon}_{\infty,q}||+1)\big\}^2+1\Big)||u_j(s); B^{1-\varepsilon}_{\infty,q}||\ ||u_j(s); B^{-\varepsilon}_{\infty,q}||$$
$$+ \big\{\log(||u_j(s); B^{1-\varepsilon}_{\infty,q}||+1)+1\big\}||u_j; B^{-\varepsilon/2}_{\infty,q}||\Big]$$
$$\le C\tilde{K}_j(t-s)^{-(\gamma+\varepsilon)/2}s^{-1/2}(\log s^{-1})^2,$$

where $\tilde{K}_j = K^0_j K^1_j\big\{\log(K^1_j+1)+1\big\}^2 + K^{\varepsilon/2}_j\log(K^1_j+1)$. The last inequality is yielded by the definition of K^γ_j and the assumption of $T < 1$.

Therefore we obtain

$$K^\gamma_{j+1} \le CK_0 + C\tilde{K}_j \sup_{0\le t\le T} t^{\gamma/2}\int_0^t (t-s)^{-(\gamma+\varepsilon)/2}(\log s^{-1})^2 s^{-1/2}ds$$
$$\le CK_0 + C\tilde{K}_j(\log T^{-1})^2 T^{1/2-\varepsilon/2}.$$

Since $\varepsilon < 1$, we now take $T_2 \le T_1$ enough small, so we observe (3.3). $\square$

(Step 3). We now verify that u_j belongs to small Besov space. Of course $u(t)$ belongs to $b^{-\varepsilon}_{\infty,q}$ at $t = 0$. Moreover it is easy to see that u_1 belongs to small Besov space.

Let $t_0 \in (0, T_2]$, $t > 0$, $0 < \alpha < 1/2$ and $j \geq 2$. Then we have

$$\begin{aligned}
&||e^{t\Delta} u_j(t_0) - u_j(t_0); B^{-\varepsilon}_{\infty,q}|| \leq ||e^{(t_0+t)\Delta} u_0 - e^{t_0\Delta} u_0; B^{-\varepsilon}_{\infty,q}|| \\
&\quad + \int_0^t ||\{e^{t\Delta} - I\}\nabla \cdot e^{(t_0-s)\Delta}\mathbf{P}(u_{j-1} \otimes u_{j-1})(s); B^{-\varepsilon}_{\infty,q}||ds \\
&\leq Ct^\alpha t_0^{-\alpha} + Ct^\alpha \int_0^{t_0} (t_0 - s)^{-\alpha-\varepsilon} s^{-1/2} ds,
\end{aligned}$$

which tends to zero as $t \downarrow 0$ choosing $0 < \alpha < 1/2$. Therefore $u_j(t_0) \in b^{-\varepsilon}_{\infty,q}$ for $t_0 \in [0, T_2]$ and $j \geq 1$.

(Step 4). The next we should argue the continuity in time of successive iteration. We show that

$$t^{\gamma/2} u_j \in C([0, T_2]; b^{\gamma-\varepsilon}_{\infty,q}),$$

for all $\gamma \in [0, 1]$ and $j \geq 1$.

Proof. Let $t_0 \in [0, T_2)$, $0 < \alpha < 1/2$ and $t > 0$. Then for any $j \geq 2$,

$$\begin{aligned}
&||u_j(t_0 + t) - u_j(t_0); B^{-\varepsilon}_{\infty,q}|| \leq ||e^{(t_0+t)\Delta} u_0 - e^{t_0\Delta} u_0; B^{-\varepsilon}_{\infty,q}|| \\
&\quad + \int_0^{t_0} ||\{e^{t\Delta} - I\}\nabla \cdot e^{(t_0-s)\Delta}\mathbf{P}(u_{j-1} \otimes u_{j-1})(s); B^{-\varepsilon}_{\infty,q}||ds \\
&\quad + \int_{t_0}^{t_0+t} ||\nabla \cdot e^{(t_0+t-s)\Delta}\mathbf{P}(u_{j-1} \otimes u_{j-1})(s); B^{-\varepsilon}_{\infty,q}||ds \\
&\leq Ct^\alpha t_0^{-\alpha} + Ct^\alpha \int_0^{t_0} (t_0 - s)^{-\alpha-\varepsilon} s^{-1/2} (\log s^{-1})^2 ds \\
&\quad + C \int_{t_0}^{t_0+t} (t_0 + t - s)^{-\varepsilon} s^{-1/2} (\log s^{-1})^2 ds,
\end{aligned}$$

which tends to zero as $t \downarrow 0$ for $0 < \alpha < 1/2$.

On the other hand, let $t_0 \in (0, T_2]$, $0 < \alpha < 1/2$, $0 < t < t_0$ and $j \geq 2$. Then we can obtain similar estimate such as

$$\begin{aligned}
&||u_j(t_0) - u_j(t_0 - t); B^{-\varepsilon}_{\infty,q}|| \leq Ct^\alpha (t_0 - t)^{-\alpha} \\
&\quad + Ct^\alpha \int_0^{t_0-t} (t_0 - t - s)^{-\alpha-\varepsilon} s^{-1/2} (\log s^{-1})^2 ds + C \int_{t_0-t}^{t_0} (t_0 - s)^{-\varepsilon} s^{-1/2} (\log s^{-1})^2 ds,
\end{aligned}$$

which tends to zero as $t \downarrow 0$ for any $0 < \alpha < 1/2$. □

(Step 5). We set following quantities in order to conclude that the approximation $\{t^{\gamma/2}u_j(t)\}_{j\geq 1}$ for $\gamma \in [0,1]$ respectively has a unique limit function.

Let $T \leq T_2$, $j \geq 1$ and $\gamma \in [0,1]$, we put $D_j^\gamma = D_j^\gamma(T)$ denoted by

$$D_j^\gamma \equiv \sup_{0\leq t\leq T} t^{\gamma/2}||u_j(t) - u_{j-1}(t); B^{\gamma-\varepsilon}_{\infty,q}||.$$

We shall show that there exists a positive constant T_3 such that

$$\lim_{j\to\infty} D_j^\gamma(T) = 0$$

for any $T \leq T_3$ and $\gamma \in [0,1]$.

Proof. By using Lemma 2 we get

$$\begin{aligned}
&||u_{j+1}(t) - u_j(t); B^{\gamma-\varepsilon}_{\infty,q}|| \\
&\quad\leq C\int_0^t ||e^{(t-s)\Delta}\{(u_j\otimes u_j) - (u_{j-1}\otimes u_{j-1})\}(s); B^{1+\gamma-\varepsilon}_{\infty,q}||ds \\
&\quad\leq C\int_0^t (t-s)^{-(\gamma+\varepsilon)/2}||\{(u_j\otimes u_j) - (u_{j-1}\otimes u_{j-1})\}(s); B^{1-2\varepsilon}_{\infty,q}||ds \\
&\quad\leq C\int_0^t (t-s)^{-(\gamma+\varepsilon)/2}||\{(u_j\otimes u_j) - (u_j\otimes u_{j-1})\}(s); B^{1-2\varepsilon}_{\infty,q}||ds \\
&\qquad + C\int_0^t (t-s)^{-(\gamma+\varepsilon)/2}||\{(u_j\otimes u_{j-1}) - (u_{j-1}\otimes u_{j-1})\}(s); B^{1-2\varepsilon}_{\infty,q}||ds.
\end{aligned}$$

By using Proposition we get

$$\leq C\int_0^t (t-s)^{-(\gamma+\varepsilon)/2}\big\{(N^2+1)\Theta_1 + (N+1)2^{-N\varepsilon/2}\Theta_2\big\}ds,$$

where we denote that

$$\begin{aligned}
\Theta_1 &= \big\{||u_j(s); B^{1-\varepsilon}_{\infty,q}|| + ||u_{j-1}(s); B^{1-\varepsilon}_{\infty,q}||\big\}\ ||(u_j - u_{j-1})(s); B^{-\varepsilon}_{\infty,q}|| \\
&\quad + \big\{||u_j(s); B^{-\varepsilon}_{\infty,q}|| + ||u_{j-1}(s); B^{-\varepsilon}_{\infty,q}||\big\}\ ||(u_j - u_{j-1})(s); B^{1-\varepsilon}_{\infty,q}||, \\
\Theta_2 &= \big\{||u_j(s); B^{1-\varepsilon}_{\infty,q}|| + ||u_{j-1}(s); B^{1-\varepsilon}_{\infty,q}||\big\}\ ||(u_j - u_{j-1})(s); B^{-\varepsilon/2}_{\infty,q}|| \\
&\quad + \big\{||u_j(s); B^{-\varepsilon/2}_{\infty,q}|| + ||u_{j-1}(s); B^{-\varepsilon/2}_{\infty,q}||\big\}\ ||(u_j - u_{j-1})(s); B^{1-\varepsilon}_{\infty,q}||.
\end{aligned}$$

Here we put $N \sim \varepsilon^{-1}\log(||u_j(s); B^{1-\varepsilon}_{\infty,q}|| + ||u_{j-1}(s); B^{1-\varepsilon}_{\infty,q}|| + ||u_j(s); B^{-\varepsilon/2}_{\infty,q}|| + ||u_{j-1}(s); B^{-\varepsilon/2}_{\infty,q}|| + 1)$. By (3.3) there exists a positive constant C such that

$$
\begin{aligned}
&||u_{j+1}(t) - u_j(t); B^{\gamma-\varepsilon}_{\infty,q}|| \\
&\leq C\int (t-s)^{-(\gamma+\varepsilon)/2}\Big[s^{-1/2}(\log s^{-1})^2\big\{||(u_j - u_{j-1})(s); B^{-\varepsilon}_{\infty,q}|| \\
&\quad + ||(u_j - u_{j-1})(s); B^{1-\varepsilon}_{\infty,q}||\big\} + (\log s^{-1})\big\{||(u_j - u_{j-1})(s); B^{-\varepsilon/2}_{\infty,q}|| \\
&\quad + ||(u_j - u_{j-1})(s); B^{1-\varepsilon}_{\infty,q}||\big\}\Big] ds \\
&\leq C\tilde{D}_j \int_0^t (t-s)^{-(\gamma+\varepsilon)/2} s^{-1/2}(\log s^{-1})^2 ds,
\end{aligned}
$$

where $\tilde{D}_j = D^0_j + D^{\varepsilon/2}_j + D^1_j$.

According to scaling argument we obtain

$$D^{\gamma}_{j+1} \leq C\tilde{D}_j T^{(1-\varepsilon)/2}(\log T^{-1})^2,$$

for all $j \geq 1$. We now take $T_3 \leq T_2$ enough small, we observe as what follows; $\{D^{\gamma}_j(T)\}_{j\geq 1}$ converge to 0 as $j \to \infty$ for any $T \leq T_3$ and $\gamma \in [0,1]$. $\square$

Thus we conclude that the approximation $\{t^{\gamma/2}(I-\Delta)^{\gamma/2}u_j(t)\}_{j\geq 1}$ $(\gamma = 0, \varepsilon/2, 1)$ have a unique limit functions $u(t), v(t), w(t) \in C([0,T_0]; B^{-\varepsilon}_{\infty,q})$ respectively. It is easy to see that $v(t) = t^{\varepsilon/4}(I-\Delta)^{\varepsilon/4}u(t)$ and $w(t) = t^{1/2}(I-\Delta)^{1/2}u(t)$. Moreover one can see that $u(t)$ satisfies (INT) for $t \in [0, T_0]$.

Therefore we obtain the time-local solution of (INT).

3.2 Uniqueness of mild solution

Secondly, the uniqueness of mild solution can be proved by estimating the difference of two solutions u and v. We estimate $\sup_{0\leq t\leq T} t^{\gamma/2}||u(t)-v(t);\ B^{\gamma-\varepsilon}_{\infty,q}||$, and we conclude this value is equivalent to 0 for any $\gamma \in [0,1]$ when we choice of $T \leq T_3$. Using Proposition and (3.3) we obtain

$$
\begin{aligned}
&||u(t) - v(t); B^{\gamma-\varepsilon}_{\infty,q}|| \leq \int_0^t ||e^{(t-s)\Delta}\mathbf{P}\{(u\otimes u) - (v \otimes v)\}(s); B^{1+\gamma-\varepsilon}_{\infty,q}|| \\
&\leq C\int_0^t (t-s)^{-(\gamma+\varepsilon)/2} s^{-1/2}(\log s^{-1})^2 F(s) ds,
\end{aligned}
$$

since $t \leq 1$. We now put $F(s) = \{||u - v; B^{-\varepsilon}_{\infty,q}|| + ||u - v; B^{-\varepsilon/2}_{\infty,q}|| + ||u - v; B^{1-\varepsilon}_{\infty,q}||\}(s)$. To sum up values with respect to γ, then we have

$$F(t) \leq C \int_0^t (t-s)^{-(1+\varepsilon)/2} s^{-1/2} (\log s^{-1})^2 F(s) ds.$$

We now use the Gronwall inequality with singularity, which is shown by [GMS]. Therefore we obtain $F \equiv 0$.

3.3 In the case of another exponents

Finally we refer to proofs with another exponents. If $q = \infty$ we modify in usual way. If $1/2 \leq \varepsilon < 1$ we modify for some parts, for example, replacing with [estimate of bilinear terms] in Step 2 as follows:

$$\begin{aligned} &||\nabla \cdot e^{(t-s)\Delta} \mathbf{P}(u_j \otimes u_j)(s); B^{\gamma-\varepsilon}_{\infty,q}|| \\ &\leq C|||(-\Delta)^{\gamma/2+(1-\varepsilon)/4} \mathbf{P} e^{(t-s)\Delta}|||\ ||(u_j \otimes u_j)(s); B^{(1-\varepsilon)/2}_{\infty,q}||, \end{aligned}$$

and we select $\delta = (1-\varepsilon)/4$ and $N \sim \delta^{-1} \log(||u_j(s); B^{1-\varepsilon}_{\infty,q}|| + 1)$. If $\varepsilon = 0$, by embedding theorem with respect to differentiability-exponents (inhomogeneous Besov spaces only), it is easy to give the proof by similar way.

In the case of general $p \in (n, \infty)$ we can obtain a mild solution in $b^{-\varepsilon}_{p,q}$ with $\varepsilon \in [0, 1 - n/p)$. Moreover to consider homogeneous Besov space we may expect same argument excepting for $\varepsilon = 0$.

References

[Am] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. fluid Mech., **2** (2000), 16-98.

[BKM] J. T. Beale, T. Kato and A. Majda *Remarks on the breakdown of smooth solutions for the 3-D Euler equations* Comm. Math. Phys., **94** (1984), 61-66.

[BL] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer, Berlin, (1976).

[Bon] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*. Annales Sci. de l'École Normale Sup. **14** (1981), 209-246.

[Bou] G. Bourdaud, *Réalisations des espaces de Besov homogènes*, Ark. Mat., **26** (1988), 41-54.

[BG] H. Brezis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Analysis, TMA **4** (1980), 677-681.

[CK] J. R. Cannon and G. H. Knightly, *A note on the Cauchy problem for the Navier Stokes equations*, SIAM J. Appl. Math., **18** (1970), 641-644.

[Ca] M. Cannone, *Ondelettes, Paraproduits et Navier Stokes, Diderot Editeur*, Arts et Sciences Paris–New York–Amsterdam, (1995).

[CM] M. Cannone and Y. Meyer, *Littlewood Paley decomposition and Navier-Stokes equations*, Methods and Applications of Analysis, **2** (1995), 307-319.

[CP] M. Cannone and F. Planchon, *Self-similar solutions for Navier-Stokes equations in* $\mathbb{R}^3$, Comm. PDE., **21** (1996), 179-193.

[CX] Z. M. Chen and Z. Xin, *Homogeneity Criterion for the Navier-Stokes equations in the whole spaces*, J. Math. fluid Mech., **3** (2001), 152-182.

[FK] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I*, Arch. Rat. Mech. Anal., **16** (1964), 269-315.

[Gi] Y. Giga, *Solutions for semilinear parabolic equations in* L^p *and regularity of weak solutions of the Navier Stokes system*, J. Differential Equations, **61** (1986), 186-212.

[GIM] Y. Giga, K. Inui and S. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*, Quaderni di Matematica, **4** (1999), 28-68.

[GIKM] Y. Giga, K. Inui, J. Kato and S. Matsui, *Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations*, Hokkaido Univ. Preprint Series in Math., Series ♯487, (2000).

[GMS] Y. Giga, S. Matsui and O. Sawada, *Global existence for the two-dimensional Navier-Stokes flows with nondecaying initial velocity,* J. Math. fluid Mech., (to appear).

[GOV] J. Ginibre, T. Ozawa and G. Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations,* Ann. Inst. Henri Poincaré, **60** (1994), 211-239.

[GV] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of non-linear wave equations,* Comm. Math. Phys., **123** (1989), 535-573.

[Jo] J. Johnsen, *Pointwise multiplication of Besov and Triebel-Lizorkin spaces*, Math. Nachr., **175** (1995), 85-133.

[Kj] J. Kato, *On the uniqueness of nondecaying solutions of the Navier–Stokes equations*, (preprint).

[Kt] T. Kato, *Strong L^p-solutions of Navier Stokes equations in $\mathbb{R}^n$ with applications to weak solutions*, Math. Z., **187** (1984), 471-480.

[KP] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure and Appl. Math., **41** (1988), 891-907.

[Kn] G. H. Knightly, *A Cauchy problem for the Navier-Soteks equations*, SIAM J. Math. Anal., **3** (1972), 506-511.

[KM] T. Kobayashi and T. Muramatu, *Abstract Besov space approach to the nonstationary Navier-Stokes equations*, Math. Methods Appl. Sci., **15** (1992), 599-620.

[KT] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., (to appear).

[KOT] H. Kozono, T. Ogawa and Y. Taniuchi, *Navier-Stokes equations in Besov spaces near L^∞ or BMO*, (preprint).

[KY] H. Kozono and M. Yamazaki, *Semilenear heat equations and the Navier-Stokes equation with disributions in new function spaces as initial data*, Comm. PDE., **19** (1994), 959-1014.

[St] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, New Jersey, (1970).

[Tr1] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel-Boston-Stuttgart, (1983).

[Tr2] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Basel-Boston-Stuttgart, (1992).

O. Sawada

Department of Mathematics
Hokkaido University
Sapporo 060–0810, Japan