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Absolute Values And Real Parts For Functions In The Smirnov Class

by

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*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

2000 Mathematics Subject Classification : Primary 30 D 55, 47 B 35 ; Secondary 46 J 15

Keywords and phrases : Smirnov class, outer function, positive real part, absolute value

Abstract. Let N_+ denote the Smirnov class on the open unit disc D. It is easy to see that for any outer function g in N_+, there exists a function G in N_+ such that $|g| \le \mathrm{Re}G$ on ∂D. We describe such a G. In general, G may not be outer. In this paper, a necessary and sufficient condition on g is given for the existence of an outer function G such that $|g| \le \mathrm{Re}G$. When g belongs to the Hardy space H^1, G is trivially given as the Herglotz integral of $|g|$.

§1. Introduction

Let D be the open unit disc in the complex plane and let ∂D be the boundary of D. A holomorphic function f in D is said to be of the Nevanlinna class N if $\sup_{0\le r<1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta<\infty$. If f is in N, then $f(e^{i\theta})$ which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂D. If

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta=\int_{-\pi}^{\pi}\log^+|f(e^{i\theta})|d\theta,$$

then f is said to be the Smirnov class N_+. The set of all boundary functions in N or N_+ is also denoted by N or N_+, respectively. For $0<p\le\infty$, the Hardy space H^p, is defined by $N_+\cap L^p$.

If g is holomorphic in D and its absolute value $|g|$ has a harmonic majorant, then g belongs to H^1. The absolute value of a function in N_+ need not have a harmonic majorant. However, suppose that there exists an outer function h in H^1 such that g/h is non-negative on D. Let $G=G_0\ g/h$ where G_0 is the Herglotz integral of $|h|$. Then G belongs to N_+ and $|g|=\mathrm{Re}G$ on ∂D because $\mathrm{Re}G=|h|\ g/h$. If there exist a function G in N_+ such that $|g|\le\mathrm{Re}G$ on ∂D, then $\mathrm{Re}G$ is called a generalized harmonic majorant of $|g|$. In Section 2, we show that for every function g in N_+ the function $|g|$ admits a generalized harmonic majorant. Moreover we describe all generalized harmonic majorants of $|g|$.

A function g in N_+ is called outer if it is not divisible in N_+ by a non-constant inner function. A function g in H^1 is called strongly outer if the only functions f in H^1 such that f/g is non-negative are scalar multiples of g. It is easy to see that a strongly outer function is outer and an outer function may not be strongly outer. Like outer functions, strongly outer functions appear in many important areas. However the two definitions are very different. For example, so far strongly outer functions are only defined in H^1. The following definition of a strongly outer function extends this notion to N_+. Put $\tilde{N}=\{h/k\ ; h,k\in H^\infty\}$, then it is known that $\tilde{N}\underset{\neq}{\supset}N$ because $\tilde{N}\ni 1/z$ by its definition. Suppose g is a non-zero function in N_+. When s is a non-negative function in $\tilde{N}$ and sg belongs to H^1, g is called a **strongly outer function** if these conditions force s to be constant.

A lot of papers about strongly outer functions in H^1 have been published. For example, [3],[13],[8],[5],[12],[6],[7],[9],[10] and [11]. In Section 3, we give a necessary and sufficient condition for a function in N_+ to be strongly outer. Moreover we show that there exists an outer function G in N_+ such that $|g|\le\mathrm{Re}G$ on ∂D if and only if g belongs to H^1 or g is not a strongly outer function. In Section 4, we make clear the relation between an outer function and a strongly outer function.

§2. Generalized harmonic majorants

Let $L^0(\partial D)$ be the space of all Lebesgue measurable functions. Put

$$L_+ = \{f \in L^0(\partial D) \ ; \ \int_0^{2\pi} \log(1+|f(e^{i\theta})|)d\theta/2\pi < \infty\}.$$

Then $L_+ = N_+ + \bar{N}_+$ (see [1, §2]). Hence Proposition 1 is clear. We give a proof of that not using $L_+ = N_+ + \bar{N}_+$. For a function h in H^1, put

$$\frac{1+Q_h(z)}{1-Q_h(z)} = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}|h(e^{it})|dt/2\pi \quad (z \in D).$$

Then Q_h is a contractive function in H^∞, that is, $\|Q_h\| < 1$.

Proposition 1. *For arbitrary function g in N_+, there exists a function G in N_+ such that $|g| \le \mathrm{Re}G$ on ∂D.*

Proof. We may assume that g is outer in N_+ Since $g/|g|$ is a unimodular function, by a theorem of R.G.Douglas and W.Rudin [4, Theorem 2.1 in Chapter V] there exist two inner functions q and q' such that

$$\left|\frac{(1+q)^2 g}{|(1+q)^2 g|} - q'\right| = \left|\frac{g}{|g|} - \bar{q}q'\right| \le 1$$

because $(1+q)^2/|1+q|^2 = q$. By the proof of Lemma 5.4 in [4, Chapter IV], there exists an outer function h in H^1 such that $(1+q)^2 g/|(1+q)^2 g| = \dfrac{qg}{|qg|} = h/|h|$. Put

$$G = \frac{qg}{h}\,\frac{2}{1-Q_h}\ ,$$

then

$$\begin{aligned} \mathrm{Re}G &= \frac{qg}{h}\,\mathrm{Re}\left(\frac{1+Q_h}{1-Q_h}+1\right) \\ &\ge \frac{qg}{h}\,\mathrm{Re}\,\frac{1+Q_h}{1-Q_h} = \frac{qg}{h}\,|h| = |qg| = |g|. \end{aligned}$$

Theorem 1. *Let g be an outer function in N_+. Then G is a function in N_+ such that*

$$|g| \le \mathrm{Re}G \quad \text{on } \partial D$$

if and only if there exist an inner function q and an outer function h such that qg/h is non-negative on ∂D and

$$G = \frac{qg}{h}\,\frac{2(1-Q_h w)}{(1-Q_h w)(1-w)}$$

for some contractive function w in H^∞. Hence $|g| = \mathrm{Re}G$ on ∂D if and only if w is an inner function.

Proof. If $G = \dfrac{qg}{h} \dfrac{2(1-Q_h w)}{(1-Q_h)(1-w)}$, then

$$\begin{aligned} \mathrm{Re}G &= \frac{qg}{h}\mathrm{Re}\left(\frac{1+Q_h}{1-Q_h} + \frac{1+w}{1-w}\right) \\ &\geq \frac{qg}{h}\mathrm{Re}\left(\frac{1+Q_h}{1-Q_h}\right) = \frac{qg}{h}|h| = |g|. \end{aligned}$$

Next we will show the 'only if' part. Suppose $|g| \leq \mathrm{Re}G$ on ∂D and $G = qG_0$ where q is inner and G_0 is outer. Put $k = 2g/G_0$, then k belongs to H^∞ and

$$\begin{aligned} \left|\frac{qg}{|g|} - k\right|^2 &= 1 - 2\mathrm{Re}\frac{qg}{|g|}\bar{k} + |k|^2 \\ &= 1 - 4\mathrm{Re}\frac{|g|}{\bar{q}\bar{G}_0} + 4\frac{|g|^2}{|G_0|^2} \\ &= 1 - \frac{4|g|\mathrm{Re}G}{|G|^2} + \frac{4|g|^2}{|G|^2} \leq 1 \end{aligned}$$

because $|g| \leq \mathrm{Re}G \leq |G|$. By [9, Lemma 6]

$$k = \frac{h(1-Q_h)(1-w)}{1-Q_h w}$$

where h is an outer function in H^1, $qg/h \geq 0$ on ∂D, and w is a contractive function in H^∞. Since $G = qG_0 = 2q\ g/k$,

$$G = \frac{qg}{h} \frac{2(1-Q_h w)}{(1-Q_h)(1-w)}$$

§3. Strongly outer functions

In this section, we study a strongly outer function in N_+ which is not necessarily in H^1. (1) and (2) of Proposition 2 were proved by K.Yabuta [13] when g is in H^1. Theorem 3 was proved by T.Nakazi [10] when g is in H^1. Propsition 4 was proved by E.Hayashi [5] when g is in H^1.

Proposition 2. *If an outer function g in N_+ satisfies one of the following (1), (2) or (3), then g is a strongly outer function.*

(1) g^{-1} belongs to H^1.

(2) $\mathrm{Re}g \geq 0$ on D.

(3) $g = g_0/k$ where g_0 is a strongly outer function in H^1 and k is a non-zero function in H^∞.

Proof. (1) If there exists $s \in \tilde{N}$ such that $s \geq 0$ and $sg \in H^1$, then s belongs to $H^{1/2}$ because $g^{-1} \in H^1$. This implies that s is constant because $H^{1/2}$ does not contain any non-constant non-negative function. (2) If there exists $s \in \tilde{N}$ such that $s \geq 0$ and $sg \in H^1$, put $h = sg$. Then $\mathrm{Re}h \geq 0$ on ∂D. Since $h \in H^1$, $\mathrm{Re}h > 0$ on D and so h is outer. $g + h = (s+1)g$ and $\mathrm{Re}(g+h) > 0$ on $\mathrm{Re}g > 0$ on D. Hence $g + h$ is outer and so $s + 1$ is outer because g is outer. Therefore $(s+1)^{-1}$ belongs to H^∞ and so s is constant. (3) If there exists $s \in \tilde{N}$ such that $s \geq 0$ and $sg \in H^1$, put $h = sg$. Then $sg_0 = kh \in H^1$. Since g_0 is a strongly outer function, s is constant.

Theorem 2. *Suppose g is an outer function in N_+. Then, g is a strongly outer function if and only if there does not exist an outer function G such that $|g| \leq \mathrm{Re}G$, or if such a G exists then $ReG > 0$ on D.*

Proof. If there exists an outer function G in N_+ such that $|g| \leq \mathrm{Re}G$ on ∂D, then by hypothesis $\mathrm{Re}G > 0$ on D and so $\mathrm{Re}G \in L^1(\partial D)$. Hence g belongs to H^1. By [13], g is strongly outer. Suppose there does not exist any outer function G in N_+ such that $|g| \leq \mathrm{Re}G$ on ∂D. If there exists $s \in \tilde{N}$ such that $sg \in H^1$, put $h = sg$ and $H =$ the Herglotz integral of $|h|$, then $|h| = \mathrm{Re}H$ on ∂D. If $s \not\equiv 0$, then $H \not\equiv 0$ and so H is outer in N_+. Put $k = 2h/H$, then k belongs to H^∞ and $\left|\dfrac{g}{|g|} - k\right| = 1$. Hence $|k|^2 = 2\mathrm{Re}\dfrac{\bar{g}}{|g|}k$ and so $|g| = \mathrm{Re}2g/k$. Put $G = 2g/k$, then $|g| = \mathrm{Re}G$ and G is outer because g and k are outer. This contradicts the hypothesis and so $s \equiv 0$. This implies that g is strongly outer by definition. Conversely suppose g is a strongly outer function. If g is in H^1, then there exists an outer function G succh that $|g| \leq \mathrm{Re}G$. But then Theorem 6 in [10] implies that $\mathrm{Re}G > 0$ on D always. Suppose $g \notin H^1$. If there exists an outer function G in N_+ such that $|g| \leq \mathrm{Re}G$ on ∂D, put $k = 2g/G$, then k belongs to H^∞ and $\left|\dfrac{g}{|g|} - k\right| \leq 1$. Hence by the proof of Lemma 5.4 in [4, Chapter IV] there exists an outer function h in H^1 such that $g/|g| = h/|h|$ on ∂D. Put $s = h/g$ then $s \in \tilde{N}$ and $s \geq 0$, s is non-constant and $sg = h$ belongs to H^1. This contradiction implies that there does not exist any outer function G in N_+ such that $|g| \leq \mathrm{Re}G$ on ∂D.

Corollary. *Suppose g is an outer function in H^1.*

(1) If g is in H^1, then $|g| = \mathrm{Re}G$ and G is the Herglotz integral of $|g|$.

(2) If g is not a strongly outer function, then $|g| = \mathrm{Re}G$ and $G = \dfrac{g}{h}G_0$ where h is an outer function in H^1, g/h is nonnegative on ∂D and G_0 is the Herglotz integral of $|h|$.

Proposition 3. *Let g be a non-zero function in N_+. Then there exist a strongly outer function g_0 in N_+ and an inner function q such that $g = tg_0$ where t is a function in N_+ and $\bar{q}t$ is a non-negative function on ∂D. When t is not constant, g_0 is in H^1.*

Proof. By definition, if there does not exist any non-constant s in $\tilde{N}$ such that s is non-negative on ∂D and $sg \in H^1$, then put $g = g_0$. Unless it is so, we may assume that g is outer. Put $h = sg$, then s is outer and $h \in H^1$. By a theorem of E.Hayashi [4] there exist a strongly outer function g_0 in H^1 and an inner function q such that $h = \ell g_0$ and ℓ is a function in H^1 with $\bar{q}\ell \geq 0$ on ∂D. Put $t = \ell/s$.

§4. Remarks

1. In order to make clear the relation between an outer function and a strongly outer function, we give a definition, $\mathcal{H}$ denotes a subset of N_+ such that (1) $k\mathcal{H} \subseteq \mathcal{H}$ if k is in H^∞ and (2) the outer part of each function in $\mathcal{H}$ belongs to $\mathcal{H}$. For example, $\mathcal{H} = N_+$, $\mathcal{H} = H^p$ and $\mathcal{H} = a$ weighted Hardy space. Suppose g is a non-zero function in $\mathcal{H}$. When s is a non-negative function in $\tilde{N}$ and sg is in $\mathcal{H}$, s is called $\mathcal{H}$-outer if these conditions force s to belong to $\mathcal{H}$.

(a) Any $\mathcal{H}$-outer function is an outer function : For if $g \in \mathcal{H}$ is $\mathcal{H}$-outer and not outer, by definition $h = g/q$ is in N_+ for some non-constant inner function q. Put $s = (1+q)^2/q$, then $s \in \tilde{N}$, $s \geq 0$, $s \notin \mathcal{H}$ and $sg = (1+q)^2 h \in \mathcal{H}$. This contradiction shows that g is outer.

(b) A function in N^+ is N^+-outer if and only if it is outer : The 'only if' part is clear by (a). Conversely, if g is outer, and $sg \in N_+$, $s \in \tilde{N}$ and $s \geq 0$, then s belongs to N_+ because g is invertible and N_+ is algebra.

(c) A function in H^1 is H^1-outer if and only if it is strongly outer : For since a constant function belongs to H^1, the 'if' part is clear. Since H^1 does not have any non-constant real functions, the 'only if' part follows.

2. Let $B(H^1)$ be the unit ball of H^1. There exist several kinds of important points in $B(H^1)$ for extremal problems in H^1. For example, extreme points, exposed points and strongly exposed points. Suppose a function f is in $\partial B(H^1)$. f is an extreme point if and only if f is an outer function [3]. f is an exposed point if and only if f is a strongly outer function [3]. When f is an outer function, f is a strongly exposed point if and only if $|f|$ is a Helson-Szegő weight (see [2], [14]). An outer function is defined in N_+. In this paper, we define a strongly outer function for N_+. It will be interesting to generalize a strongly exposed point for N_+.

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