A level set method for spiral crystal growth

Takeshi Ohtsuka

Series #551. May 2002

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#525 T. Suwa, Residues of Chern classes, 20 pages. 2001.

#526 V. Anh and A. Inoue, Dynamic models of asset prices with long memory, 21 pages. 2001.

#527 T. Izawa and T. Suwa, Multiplicity of functions on singular varieties, 21 pages. 2001.

#528 T. Nakazi and T. Yamamoto, Two dimensional commutative Banach algebras and von Neumann inequality, 18 pages. 2001.

#529 Y. Giga, N. Ishimura and Y. Kohsaka, Spiral solutions for a weakly anisotropic curvature flow equation, 16 pages. 2001.

#530 Y. Giga and P. Rybka, Quasi-static evolution of 3-D crystals grown from supersaturated vapor, 16 pages. 2001.

#531 Y. Tonegawa, Remarks on convergence of the Allen-Cahn equation, 18 pages. 2001.

#532 T. Suwa, Characteristic classes of singular varieties, 26 pages. 2001.

#533 J. Escher, Y. Giga and K. Ito, On a limiting motion and self-intersections for the intermediate surface diffusion flow, 20 pages. 2001.

#534 Y.-H. R. Tsai, Y. Giga and S. Osher, A level set approach for computing discontinuous solutions of a class of Hamilton-Jacobi equations, 30 pages. 2001.

#535 A. Yamagami, On Gouvêa's conjecture in the unobstructed case, 19 pages. 2001.

#536 A. Inoue, What does the partial autocorrelation function look like for large lags, 27 pages. 2001.

#537 T. Nakazi and T. Yamamoto, Norm of a linear combination of two operators of a Hilbert space, 16 pages. 2001.

#538 Y. Giga, On the two-dimensional nonstationary vorticity equations, 12 pages. 2001.

#539 M. Jinzenji, Gauss-Manin system and the virtual structure constants, 25 pages. 2001.

#540 H. Ishii and T.Mikami, Motion of a graph by R-curvature, 28 pages. 2001.

#541 M. Jinzenji and T. Sasaki, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbf{Z}_2$: higher rank case, 17 pages. 2001.

#542 T. Nakazi, The Nevanlinna counting functions for Rudin's orthogonal functions, 7 pages. 2001.

#543 K. Sugano, On H-separable extensions of QF-3 rings, 7pages. 2001.

#544 A. Arai, Non-relativistic limit of a Dirac-Maxwell operator in relativistic quantum electrodynamics, 27pages. 2001.

#545 O. Sawada, On time-local solvability of the Navier-Stokes equations in Besov spaces, 30 pages. 2001.

#546 C. M. Elliott, Y. Giga, and S. Goto, Dynamic boundary conditions for Hamilton-Jacobi equations, 27 pages. 2001.

#547 Y. Nakano, Minimizing coherent risk measures of shortfall in discrete-time models with cone constraints, 22 pages. 2002.

#548 K. tachizawa, A generalization of the Lieb-Thirring inequalities in low dimensions, 13 pages. 2002.

#549 T. Nakazi, Absolute values and real parts for functions in the Smirnov class, 8 pages. 2002.

#550 T. Nakazi and T. Watanabe, Properties of a Rubin's orthogonal function which is a linear combination of two inner functions, 9 pages. 2002.

A level set method for spiral crystal growth

Takeshi Ohtsuka

Abstract. In this paper we introduce a new level set model for the growth of spirals on the surface of a crystal. Since the conventional level set method cannot express a spiral curve having orientation, we modify the level set method by using a sheet structure function. Since the model equation we obtain is degenerate parabolic, we need to consider a notion of weak solution. Our goal is to prove the existence and the uniqueness of the solution for our model in the sence of viscosity solutions.

1 Introduction

The theory of spiral crystal growth was proposed by F. C. Frank in 1948(see [1]). He first pointed out that dislocations play an important role in the theory of crystal growth. He especially pointed out the importance of the role of a screw dislocation. In his theory, if a screw dislocation terminates in the exposed surface of a crystal, there is a permanently exposed *cliff* of atoms, say the *step*. The step can grow perpetually *up a spiral staircase*, When one observe the surface from above, one can find spirals drawn by exposed *edge of the step*. He proposed an evolution equation of curves which indicates the location of edges of steps. The equation he proposed is of the form

$$V = C - \kappa, \tag{1.1}$$

where V is a normal velocity of the steps, κ is a curvature of the curve corresponding to the edge of steps, and C is the driving force of steps (see [2]). The sign of the curvature is taken so that the problem is parabolic. The curvature term is interpreted as a result of the Gibbs-Thomson effect. We postulate that steps moves under (1.1), and we construct a new mathematical model based on (1.1). The formula (1.1), says the *geometric model*, performs the model of spiral crystal growth for only one screw dislocation. However, it is not enough to handle other situation when there are two or more screw dislocations on the surface of the crystal and curves generated from each screw dislocations may touch each other. We would like to handle such a situation by adjusting the model. There are at least two methods to realize our purpose. One is the Allen–Cahn equation model, and the other is a level set method for geometric model. In this paper we propose a model reflecting a level set method.

Let Ω be a bounded domain in $\mathbb{R}^2$, which denotes the surface of the crystal. For technical reasons we postulate that a screw dislocation is a close disk on the surface. We also assume that all screw dislocations do not touch each other nor the boundary of Ω. We denote by W the complement of all screw dislocations in the surface of the crystal. We denote by Γ_t the curve corresponding to edges of steps at time t.

In conventional level set approach to (1.1), we denote the evolving curve by the zero-level set of auxiliary function u, i.e.,

$$\Gamma_t = \left\{x \in \overline{W};\ u(t,x) = 0\right\},$$

In this way, however, we cannot distinguish the direction of moving steps. To overcome this difficulty, we recall *sheet structure function* due to R. Kobayashi(See [12]).

We postulate that there are n screw dislocations on the crystal surface. Let a_j denote the position of the center of j-th screw dislocation. Let ρ_j denotes the radius of j-th screw dislocation. We denote by W the complement of all screw dislocations in the surface of the crystal, i.e.,

$$W = \Omega \setminus (\bigcup_{j=1}^{n} \overline{B_{\rho_j}(a_j)}),$$

where $B_\rho(a)$ denotes an open disk of radius ρ centered at a. We recall the sheet structure function θ defined by

$$\theta(x) = \sum_{j=1}^{n} m_j \arg(x - a_j),$$

where $m_j \neq 0$ is an integer such that $|m_j|$ denotes the height of steps and the sign of m_j denotes the direction of steps. We remark that each arguments of $x - a_j$ is multi-valued. We consider an auxiliary function $u = u(t,x)$ defined on $[0,+\infty) \times \overline{W}$. We interpret Γ_t as a level set of $u - \theta$ instead of u itself, i.e.,

$$\Gamma_t = \left\{x \in \overline{W};\ u(t,x) - \theta(x) = 0 \mod 2\pi m\right\},$$

where m is the greatest common divisor of $\{|m_j|\}_{j=1}^{n}$.

By the definition of Γ_t we formally observe that

$$V = \frac{1}{|\nabla(u-\theta)|}\frac{\partial u}{\partial t},$$
$$\kappa = -\mathrm{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|}.$$

We remark that $\nabla\theta$ is single-valued, so this formula is well-defined. We now obtain the level set model which is consistent with the geometric model (1.1):

$$\frac{\partial u}{\partial t} - |\nabla(u-\theta)|\left(\mathrm{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C\right) = 0 \quad \text{in } (0,T) \times W. \tag{1.2}$$

In fact, a level set of $u-\theta$ moves by (1.1) if u solves (1.2) at least formally. To complete the problem we need some boundary condition on ∂W. Here we postulate the Neumann boundary condition at the edge of Γ_t touching ∂W of the form

$$\langle \vec{\nu}(x), \nabla(u-\theta)\rangle = 0 \quad \text{on } (0,T)\times\partial W. \tag{1.3}$$

where $\vec{\nu}$ denotes a unit normal vector field of ∂W, and $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{R}^2$. Since the equation (1.2) is degenerate parabolic, we need to consider the solution of these equation in weak sense. We consider the solution in viscosity sense.

Our goal is to prove the comparison principle, existence and uniqueness of a viscosity solution for (1.2) (1.3). The equation (1.2) has a moving singularity at $\nabla u(t,x) = \nabla\theta(x)$ so it seems to be hard to prove the comparison principle directly. To overcome this difficulty we introduce a *covering spaces* of $\overline{W}$ and $\overline{W}\times\overline{W}$ so that $u-\theta$ and $v-\theta$ respectively be a sub- and supersolution of

$$\frac{\partial u}{\partial t} - |\nabla u|\left\{\operatorname{div}\frac{\nabla u}{|\nabla u|} + C\right\} = 0 \tag{1.4}$$
$$\langle\vec{\nu}, \nabla u\rangle = 0 \tag{1.5}$$

if u and v respectively be a sub- and supersolution of (1.2) (1.3). We test $u(t,x)-\theta(x)-(v(t,y)-\theta(y))$ by standard test function by [8] but on the covering space. Then we apply the results for (1.4)–(1.5) in [8]. Once we obtain the comparison principle for (1.2)–(1.3), then it is easy to see a uniqueness of a viscosity solution for (1.2)–(1.3). It remains to prove the existence of a viscosity solution with a desired initial data. We construct a viscosity sub- and supersolution according to a Perron's method due to H. Ishii(see [10]). Perron's method for a second order equation with Neumann boundary condition is found by [15]. So we basically apply a result of [15]. In our problem, however, some difficulties lie in the term of θ since θ is not single valued. To overcome these difficulties, we first construct sub- and supersolutions on some small neighborhood of each points of W. Next we extend their domain of definition to $\overline{W}$ by using *Invariance Lemma*(see [9]). We apply the Perron's method.

We take this opportunity to mention somewhat related results. In [7] the uniqueness and existence of a *spiral solution* for a geometric model which includes a anisotropy is proved. In [13] a Allen Cahn model for spiral crystal growth is introduced. They also showed numerical computations. In [12] a Allen Cahn model including more generalized situations than that in [13] is introduced He also showed numerical computations. He introduced a *sheet structure function* in this model. We utilize his idea for expressing a edge of steps by level set method. In [14] an existence of *spiral traveling wave solution* for Kobayashi's model on a annulus is proved. A level set model different from ours are introduced by [16]. He expresses a location of edges of steps by using 2 auxiliary function, one denotes a existence of steps, and the other denotes a location of edges of steps. He also showed numerical computation. His model cannot treat a situation that, for examples, there are 2 screw dislocations and steps generated from each screw dislocations and heights of steps are different from each other. Our model includes such a situation.

Analytic foundation based on the theory of viscosity solution [4] has established by [3], [5]. It is extended to the Neumann boundary problem by [8] and [15]. From technical point of view we use the method developed by [8] and [15] although it does not apply to our settings directly.

The author would like to express my gratitude to Professors Yoshikazu Giga, Shun'ichi Goto, Hitoshi Ishii and Moto-Hiko Sato for giving me a valuable advice and helpful comments.

2 Main results

Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary $\partial\Omega$. We take $a_1, \ldots, a_n \in \Omega$ and $\rho_1, \ldots, \rho_n > 0$ satisfying

$$\overline{B_{\rho_j}(a_j)} \subset \Omega \quad \text{for} \quad j = 1, 2, \ldots, n, \tag{2.1}$$

$$\overline{B_{\rho_i}(a_i)} \cap \overline{B_{\rho_j}(a_j)} = \emptyset \quad \text{for} \quad i, j = 1, 2, \ldots, n, \ i \neq j, \tag{2.2}$$

where $B_{\rho_j}(a_j) = \{x \in \mathbb{R}^2; |x - a_j| < \rho_j\}$ and $\overline{D} \subset \mathbb{R}^k$ denotes the closure of D in $\mathbb{R}^k$. We set

$$W = \Omega \setminus \bigcup_{j=1}^{n} \overline{B_{\rho j}(a_j)}.$$

We introduce a multi valued function on $\mathbb{R}^2 \setminus \{a_1, \ldots, a_n\}$ defined by

$$\theta(x) = \sum_{j=1}^{n} m_j \arg(x - a_j),$$

where m_j is an integer and $\arg(x - a_j)$ is an argument of $x - a_j$, which is regarded as a multi–valued function.

We consider the equation of the form

$$\frac{\partial u}{\partial t} - |\nabla(u - \theta)| \left\{ \operatorname{div} \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} + C \right\} = 0 \quad \text{in} \quad (0, \infty) \times W, \tag{2.3}$$

$$\langle \vec{\nu}, \nabla(u - \theta) \rangle = 0 \quad \text{on} \quad (0, \infty) \times \partial W, \tag{2.4}$$

where C is a positive constant, and vector field $\vec{\nu}$ is a outer normal unit vector field of ∂W and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^2$. We remark that (2.3) is well–defined on W since $D\theta$ is single-valued.

We consider equations (2.3)–(2.4) in the viscosity sense. For $f\colon D(\subset \mathbb{R}^k) \to \mathbb{R}$ we denote respectively by f_*, f^* lower and upper semicontinuous envelope of f defined by

$$\begin{aligned} f_*\colon \quad & \overline{D} \to \mathbb{R} \cup \{\pm\infty\}, \\ & z \mapsto f_*(z) = \lim_{r \downarrow 0} \inf\{f(\omega); |z - \omega| < r\}, \\ f^*\colon \quad & \overline{D} \to \mathbb{R} \cup \{\pm\infty\}, \\ & z \mapsto f^*(z) = \lim_{r \downarrow 0} \sup\{f(\omega); |z - \omega| < r\}. \end{aligned}$$

We are now in position to state our main results.

Theorem 2.1 (Comparison Principle)
Let $u, v\colon (0,T)\times\overline{W}\to\mathbb{R}$ respectively be a viscosity sub- and supersolutions of (2.3) (2.4) in $(0,T)\times W$ for $T>0$. If

$$u^*(0,x)\le v_*(0,x) \quad \text{for} \quad x\in\overline{W},$$

then

$$u^*(t,x)\le v_*(t,x) \quad \text{for} \quad (t,x)\in(0,T)\times\overline{W}.$$

Theorem 2.2 (Existence and Uniqueness)
For a given $u_0\in C(\overline{W})$, there exist a unique global viscosity solution $u\in C([0,\infty)\times\overline{W})$ with initial data

$$u|_{t=0}=u_0 \quad \text{on} \quad \overline{W}.$$

Remark 2.3 (Generalization of the equation)
The equation (2.3) is written by

$$\frac{\partial u}{\partial t}+F(\nabla(u-\theta),\nabla^2(u-\theta))=0 \tag{2.5}$$

with $F\colon(\mathbb{R}^2\setminus\{0\})\times\mathbb{S}_2\to\mathbb{R}$ defined by

$$F(p,X)=-tr\left\{\left(I_2-\frac{p\otimes p}{|p|^2}\right)X\right\}-C\,|p|\,, \tag{2.6}$$

where $\mathbb{S}_2$ is the space of symmetric 2×2 matrices, I_k is an identity $k\times k$ matrix and $\otimes$ denotes a tensor product of vectors in $\mathbb{R}^2$. This function F satisfies the following property.

(F1) $F\colon(\mathbb{R}^2\setminus\{0\})\times\mathbb{S}_2\to\mathbb{R}$ is continuous.

(F2) (Degenerate elliptic) For all $\lambda>0$ and $\mu\in\mathbb{R}$,

$$F(\lambda p,\lambda X+\mu p\otimes p)=\lambda F(p,X)$$

holds for all $p\in\mathbb{R}^2\setminus\{0\}$ and $X\in\mathbb{S}_2$.

(F3) $-\infty<F_*(0,O)=F^*(0,O)<+\infty$.

(F4) There exists positive constants K_1,K_2,K_3 and K_4 such that the following holds; Suppose that $X,Y\in\mathbb{S}_2$ and non-negative constants ν_0,μ,ζ satisfy

$$\langle pX,p\rangle+\langle qY,q\rangle\le\nu_0|p-q|^2+\mu(|p|^2+|q|^2)+\zeta|p-q|(|p|+|q|)$$

for all $p,q\in\mathbb{R}^2$. Then the following holds;

$$F(p,X)-F(q,Y)\ge-K_1\nu_0|\bar p-\bar q|^2-K_2\mu-K_3\zeta|\bar p-\bar q|-K_4|p-q|$$
$$\text{for all} \quad p,q\in\mathbb{R}^2\setminus\{0\},$$

where $\bar p=\frac{p}{|p|}$.

Our results extend to general equation (2.5) *provided that F satisfies properties (F1)–(F4). In paticular it applies that anisotropic curvature flow motion of spilrals of the form*

$$b(\mathbf{n})V = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\frac{\partial H}{\partial p_j}(\mathbf{n}) + C \quad \text{on } \Gamma_t,$$

where $b \in C(\mathbb{R}^2 \setminus \{0\})$ is positive on S^1 and $H \in C^2(\mathbb{R}^2 \setminus \{0\})$ is positively homogeneous of degree 1. In fact, our results can extend to the equation (2.5) *for*

$$F(p, X) = -\mathrm{tr}\,\{A(\bar{p})X\} + B(p) \tag{2.7}$$
$$A(\bar{p}) = \frac{1}{b(-\bar{p})}\nabla^2 H(-\bar{p}), \quad B(p) = \frac{-c\,|p|}{b(-\bar{p})}, \quad \bar{p} = \frac{p}{|p|}.$$

It is easy to show that (2.7) *satisfies (F1)–(F4).*

3 Comparison principle

We shall prove Theorem 2.1.

As usual, we suppose that

$$\sigma = \max\left\{u^*(t,x) - v_*(t,x); (t,x) \in [0,T] \times \overline{W}\right\} > 0, \tag{3.1}$$

and we lead a contradiction. To lead a contradiction, we recall a maximum principle for semicontinuous functions.

Lemma 3.1
Let u and v respectively be a viscosity sub- and supersolution of (2.3)–(2.4) on $(0,T)\times\overline{W}$. Let $\Psi(t,x,y)$ be a function defined on a neighborhood U of $(\hat{t},\hat{x},\hat{y}) \in [0,T) \times \overline{W} \times \overline{W}$, suppose that Ψ is once continuously differentiable in t and twice continuously differentiable in x and y. Suppose that

$$\Phi(t,x,y) = u^*(t,x) - v_*(t,x) - \Psi(t,x,y) \le \Phi(\hat{t},\hat{x},\hat{y}) \quad \text{for all} \ \ (t,x,y) \in U. \tag{3.2}$$

Suppose that

$$\langle \vec{\nu}(\hat{x}), \nabla_x \Psi(\hat{t},\hat{x},\hat{y})\rangle > 0 \quad \text{if} \ \ \hat{x} \in \partial W \ , \ \hat{y} \in \overline{W}, \tag{3.3}$$
$$\langle \vec{\nu}(\hat{y}), -\nabla_y \Psi(\hat{t},\hat{x},\hat{y})\rangle < 0 \quad \text{if} \ \ \hat{x} \in \overline{W} \ , \ \hat{y} \in \partial W. \tag{3.4}$$

Then , for all $\lambda > 0$, there exists $X,\ Y \in \mathbb{S}_2$ such that

$$\frac{\partial \Psi}{\partial t}(\hat{t},\hat{x},\hat{y}) + F_*(\nabla_x \Psi(\hat{t},\hat{x},\hat{y}), X) - F^*(-\nabla_y \Psi(\hat{t},\hat{x},\hat{y}), -Y) \le 0, \tag{3.5}$$

and

$$-\left(\frac{1}{\lambda} + \|A\|\right) I_4 \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le A + \lambda A^2 \tag{3.6}$$

where $A = \nabla^2_{x,y}\Psi(\hat{t},\hat{x},\hat{y})$ and $\|\cdot\|$ denotes the operator norm.

See [4, Theorem 8.3] for the proof of Lemma 3.1.

If we would use Lemma 3.1 directly to our problems, we would have to handle the problem with moving singularity in ∇u. In fact, the equation is singular at $\nabla u(t,x) = \nabla\theta(x)$ depending on x. We are tempting to consider $u-\theta$ instead of u, i.e., we are tempting to handle the function

$$\Phi(t,x,y) = u^*(t,x) - \theta(x) - (v_*(t,y) - \theta(y)) - \Psi(t,x,y) \tag{3.7}$$

instead of (3.2). However, this function is multi-valued. So we have to localize a domain of Φ so that Φ has a maximum value. To determine a domain of Φ in a suitable way, we introduce some *covering space* so that θ is single-valued.

To overcome the difficulty caused by the Neumann boundary condition we choose a *good* test function as in [8].

3.1 Test function

We shall define a *good* test function as in [8] to lead a contradiction.

Since ∂W is C^2, there is a positive constant C_0 such that

$$\langle \vec{\nu}(x), x-y\rangle \geq -C_0\,|x-y|^2 \quad \text{for } x\in\partial W,\ y\in\overline{W}. \tag{3.8}$$

Moreover, for all $\beta>0$, there exists $\varphi\in C^2(\overline{W})$ satisfying

$$-\frac{\beta}{2} < \varphi < 0 \quad \text{in } W,\ \varphi = 0 \quad \text{on } \partial W, \tag{3.9}$$

$$\vec{\nu} = \frac{\nabla\varphi}{|\nabla\varphi|} \quad \text{on } \partial W, \tag{3.10}$$

We fix $\beta>0$ and take $\varphi\in C^2(\overline{W})$ satisfying (3.9)–(3.10) and

$$|\nabla\varphi| \geq \max\{8C_0\beta, 1\} \quad \text{on } \partial W. \tag{3.11}$$

For $\varepsilon>0$, $\delta>0$ and $\gamma>0$, we define

$$\begin{aligned} \Psi(t,x,y) &= \frac{\Xi(x,y)}{\varepsilon} + \delta G(x,y) + \frac{\gamma}{T-t}, \\ \Xi(x,y) &= |x-y|^4\, G(x,y), \\ G(x,y) &= \varphi(x) + \varphi(y) + 2\beta. \end{aligned}$$

The next propositions are useful to prove Theorem 2.1. The proofs are the same as in [8] so we do not present them.

Proposition 3.2

$$\langle \vec{\nu}(x), \nabla_x\Psi(t,x,y)\rangle > \delta \quad \textit{for } x\in\partial W\ ,\ y\in\overline{W},$$
$$\langle \vec{\nu}(y), -\nabla_y\Psi(t,x,y)\rangle < -\delta \quad \textit{for } x\in\overline{W},\ y\in\partial W.$$

Proposition 3.2 says that Ψ satisfies (3.3) and (3.4), in other words, Ψ does not satisfy the boundary condition (2.4). So, we suffices to consider the equation (2.3) when we test a sub- and supersolution by Ψ.

Proposition 3.3
(i) There exists a positive constant L such that

$$\left\|\nabla^2_{x,y}\Psi(t,x,y)\right\| < L \quad \textit{for all } (t,x,y) \in [0,T) \times \overline{W} \times \overline{W}.$$

(ii) Suppose that, for all $\lambda \in \mathbb{R}$, there exists $X,\ Y \in \mathbb{S}_2$ such that

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \nabla^2_{x,y}\Psi(t,x,y) + \lambda(\nabla^2_{x,y}\Psi(t,x,y))^2.$$

Then

$$\langle pX, p\rangle + \langle qY, q\rangle \leq \nu_0 \left|p-q\right|^2 + \mu(\left|p\right|^2 - \left|q\right|^2) + \zeta\left|p-q\right|(\left|p\right| + \left|q\right|)$$

for all $p,\ q \in \mathbb{R}^2$, where

$$\begin{aligned}
\nu_0 &= \frac{24\beta}{\varepsilon}\left|x-y\right|^2, \\
\mu &= \left(\frac{\left|x-y\right|^4}{\varepsilon} + \delta\right) h_2 + \lambda L^2, \\
\zeta &= 8h_1\left|x-y\right|, \\
h_1 &= \sup_{\overline{W}}\left|\nabla\varphi\right|,\ h_2 = \sup_{\overline{W}}\left\|\nabla^2\varphi\right\|.
\end{aligned}$$

3.2 Covering space

We introduce a covering space so that $u - \theta$ is viewed as a single valued function. We set

$$\mathfrak{X} = \left\{ (x,\xi) \in \overline{W} \times \mathbb{R}^n;\ \begin{array}{l} \xi = (\xi_1, \xi_2, \ldots, \xi_n), \\ x - a_j = \left|x - a_j\right|(\cos\xi_j, \sin\xi_j)\ (j = 1,2,\ldots,n) \end{array} \right\}$$

We define $u_\theta,\ v_\theta \colon [0,T] \times \overline{W} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}
u_\theta(t,x,\xi) &= u^*(t,x) - \sum_{j=1}^n m_j\xi_j, \\
v_\theta(t,x,\xi) &= v_*(t,x) - \sum_{j=1}^n m_j\xi_j.
\end{aligned}$$

If we restrict the definition of u_0 on $[0,T] \times \overline{\mathfrak{X}}$, we can consider $\theta(x)$ formally

$$\theta(x) = u(t,x) - u_0(t,x,\xi).$$

We still denote by u_0 and v_0 their restriction in $[0,T] \times \overline{\mathfrak{X}}$.

We define $\tilde{\Phi}\colon [0,T) \times \overline{\mathfrak{X}} \times \overline{\mathfrak{X}} \to \mathbb{R}$ by

$$\tilde{\Phi}(t,x,\xi,y,\eta) = u_0(t,x,\xi) - v_0(t,y,\eta) - \Psi(t,x,y),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and Ψ is defined in the previous section. Since $\tilde{\Phi}$ is not bounded because of the term of arguments, we introduce a new covering space $\mathfrak{Y}$ instead of $\mathfrak{X} \times \mathfrak{X}$:

$$\mathfrak{Y} = \left\{(x,\xi,y,\eta) \in \overline{\mathfrak{X}} \times \overline{\mathfrak{X}};\ \xi_j - \pi \leq \eta_j \leq \xi_j + \pi\ (j = 1,2,\dots,n)\right\}.$$

We consider $\tilde{\Phi}$ on $[0,T) \times \overline{\mathfrak{Y}}$ rather than on $[0,T) \times \overline{\mathfrak{X}} \times \overline{\mathfrak{X}}$. On this set $\arg(x - a_j)$ and $\arg(y - a_j)$ take same branch of arguments.

We shall prove a existence of maximum value of $\tilde{\Phi}$ on $[0,T) \times \mathfrak{Y}$. We need to consider a subset $\mathfrak{Z} \subset \mathfrak{Y}$ defined by

$$\mathfrak{Z} = \{(x,\xi,y,\eta) \in \mathfrak{Y};\ 0 \leq \xi_j < 2\pi\ (j = 1,2,\dots,n)\}.$$

Proposition 3.4

For each $(x,\xi,y,\eta) \in \mathfrak{Y}$, there exists a $(x,\tilde{\xi},y,\tilde{\eta}) \in \mathfrak{Z}$ such that

$$\tilde{\Phi}(t,x,\xi,y,\eta) = \tilde{\Phi}(t,x,\tilde{\xi},y,\tilde{\eta}).$$

Proof.

For $\xi_j \in \mathbb{R}^n$, there exists $k_j \in \mathbb{Z}$ and $\xi'_j \in [0,2\pi)$ satisfying

$$\xi_j = \xi'_j + 2\pi k_j.$$

We set

$$\eta'_j = \eta_j - 2\pi k_j$$

and let $\xi' = (\xi'_1, \xi'_2, \dots, \xi'_n)$ and $\eta' = (\eta'_1, \eta'_2, \dots, \eta'_n)$. If $(x,\xi,y,\eta) \in \mathfrak{Y}$ then we obtain $(x,\xi',y,\eta') \in \mathfrak{Z}$ and

$$\tilde{\Phi}(t,x,\xi,y,\eta) = \tilde{\Phi}(t,x,\xi',y,\eta')$$

for $0 \leq t < T$ and $x, y \in \overline{W}$ since $\xi_j - \eta_j = \xi'_j - \eta'_j$. $\square$

By Proposition 3.4 it suffices to consider $\tilde{\Phi}$ on $[0,T) \times \mathfrak{Z}$ if we are concerned with the maximum of $\tilde{\Phi}$ over $[0,T) \times \mathfrak{Y}$.

Proposition 3.5
The function $\tilde{\Phi}$ *has a maximum value on* $[0,T) \times \mathfrak{Y}$ *and*

$$\max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} = \max_{[0,T)\times\mathfrak{Z}} \tilde{\Phi}.$$

Proof.
By Proposition 3.4 it suffices to consider $\tilde{\Phi}$ on $[0,T) \times \overline{\mathfrak{Z}}$. Since $\Psi > 0$ we observe that

$$\begin{aligned}\tilde{\Phi}(t,x,\xi,y,\eta) &\leq u_0(t,x,\xi) - v_0(t,x,\eta)\\ &\leq \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} (v_*) + \pi \sum_{j=1}^{n} |m_j| < \infty.\end{aligned}$$

Thus $\tilde{\Phi}$ is bounded from above. Then there exists a sequence $\{(t_j, x_j, \xi^j, y_j, \eta^j)\} \subset [0,T) \times \mathfrak{Z}$ satisfying

$$\lim_{j\to\infty} \tilde{\Phi}(t_j, x_j, \xi^j, y_j, \eta^j) = \sup_{[0,T)\times\mathfrak{Z}} \tilde{\Phi}.$$

Since $(t_j, x_j, \xi^j, y_j, \eta^j) \in [0,T) \times \mathfrak{Z} \subset [0,T] \times \overline{\mathfrak{Z}}$, we may assume that

$$t_j \to \hat{t} \in [0,T],\ (x_j, \xi^j, y_j, \eta^j) \to (\hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) \in \overline{\mathfrak{Z}} \quad \text{as } j \to \infty$$

by taking a subsequence of $(t_j, x_j, \xi^j, y_j, \eta^j)$. If $\hat{\xi}_j = 2\pi$ for some j we can consider $\hat{\xi}_j = 0$ by Proposition 3.4. Therefore it suffices to prove $\hat{t} < T$.

Suppose that $\hat{t} = T$. Then we get

$$\tilde{\Phi}(t_j, x_j, \xi^j, y_j, \eta^j) \leq \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} v_* + \pi \sum_{j=1}^{n} |m_j| - \frac{\gamma}{T - t_j}.$$

Since $\frac{\gamma}{T-t_j} \to -\infty$ as $j \to \infty$, we obtain

$$\lim_{j\to\infty} \tilde{\Phi}(t_j, x_j, \xi^j, y_j, \eta^j) = -\infty.$$

This contradicts $\sup_{[0,T]\times\overline{\mathfrak{Z}}} \tilde{\Phi} > -\infty$. □

We denote by $(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) \in [0,T) \times \mathfrak{Y}$ the maximum point of $\tilde{\Phi}$ over $[0,T) \times \mathfrak{Y}$, i.e.,

$$\Phi(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) = \max_{[0,T)\times\mathfrak{Y}} \Phi. \tag{3.12}$$

The next proposition is standard once we know that $\tilde{\Phi}$ is taken its maximum on $[0,T)\times\mathfrak{Y}$. We give a proof for the idea's convenience and completeness.

Proposition 3.6
Assume that

$$\sigma = \max_{[0,T]\times\overline{W}} (u^* - v_*) > 0.$$

Let $(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) \in [0,T) \times \mathfrak{Y}$ *be taken as (3.12).*

(i) There exists constants $\delta_0 > 0$ and $\gamma_0 > 0$ such thatthe estimate of the form

$$\max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} > \frac{\sigma}{2}$$

holds for $0 < \varepsilon < 1$, $0 < \delta < \delta_0$ and $0 < \gamma < \gamma_0$

(ii) $|\hat{x} - \hat{y}| \to 0$ uniformly as $\varepsilon \to 0$ on $0 < \delta < \delta_0$ and $0 < \gamma < \gamma_0$.

(iii) $\Xi(\hat{x}, \hat{y})/\varepsilon \to 0$ uniformly as $\varepsilon \to 0$ on $0 < \delta < \delta_0$ and $0 < \gamma < \gamma_0$.

(iv) Suppose that $u^*(0,x) \le v_*(0,x)$ for $x \in \overline{W}$. Then there exists a constant $\varepsilon_0 > 0$ such that

$$\hat{t} > 0 \text{ for } 0 < \varepsilon < \varepsilon_0.$$

Proof.

(i) We first note that

$$\max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} \ge \tilde{\Phi}(t,x,\xi,x,\xi) \ge u^*(t,x) - v_*(t,x) - \delta G(x,x) - \frac{\gamma}{T-t} \tag{3.13}$$

for each $t \in [0,T)$, $(x,\xi) \in \mathfrak{X}$. By the definition of u^* and v_* there exists a $(s,z) \in (0,T) \times \overline{W}$ satisfying

$$u^*(s,z) - v_*(s,z) \ge \frac{3}{4}\sigma.$$

By (3.13) we get

$$\begin{aligned} \max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} &\ge u^*(s,z) - v_*(s,z) - \delta G(z,z) - \frac{\gamma}{T-s} \\ &> \frac{3}{4}\sigma - \delta G(z,z) - \frac{\gamma}{T-s}. \end{aligned} \tag{3.14}$$

Since

$$0 < \beta < G(x,y) < 2\beta \quad \text{for } x,y \in \overline{W}, \tag{3.15}$$

then there exists $\delta_0, \gamma_0 > 0$ such that

$$\delta G(z,z) + \frac{\gamma}{T-s} < \frac{\delta}{4} \quad \text{for } 0 < \delta < \delta_0, 0 < \gamma < \gamma_0. \tag{3.16}$$

Combining (3.14) and (3.16) we obtain

$$\max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} > \frac{\sigma}{2}$$

for $0 < \varepsilon < 1, 0 < \delta < \delta_0, 0 < \gamma < \gamma_0$.

(ii) By (i) and the definition of Ψ and $\mathfrak{Y}$ we observe that

$$\begin{aligned}\frac{\sigma}{2} &< \tilde{\Phi}(\hat{t},\hat{x},\hat{\xi},\hat{y},\hat{\eta}) \\ &= u_0(\hat{t},\hat{x},\hat{\xi}) - v_0(\hat{t},\hat{x},\hat{\eta}) - \Psi(\hat{t},\hat{x},\hat{y}) \\ &= u^*(\hat{t},\hat{x}) - v_*(\hat{t},\hat{y}) - \sum_{j=1}^{n} m_j(\hat{\xi}_j - \hat{\eta}_j) - \frac{\Xi(\hat{x},\hat{y})}{\varepsilon} - \delta G(\hat{x},\hat{y}) - \frac{\gamma}{T-\hat{t}}.\end{aligned}$$

We thus obtain that

$$\begin{aligned}\frac{\Xi(\hat{x},\hat{y})}{\varepsilon} &\leq \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} v_* - \sum_{j=1}^{n} m_j(\hat{\xi}_j - \hat{\eta}_j) - \delta G(\hat{x},\hat{y}) - \frac{\gamma}{T-\hat{t}} - \frac{\sigma}{2} \\ &< \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} v_* + \pi \sum_{j=1}^{n} |m_j|.\end{aligned}$$

We now set $C_1 = \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} v_* + \pi\sum_{j=1}^{n}|m_j|$. Since $\Xi(x,y) = |x-y|^4 G(x,y) > |x-y|^4 \beta$, the estimate of Ξ/ε yields

$$|\hat{x}-\hat{y}|^4 \leq \frac{\varepsilon}{\beta} C_1.$$

Therefore we observe that $|\hat{x}-\hat{y}| \to 0$ as $\varepsilon \to 0$.

(iii) By (3.15) we observe that

$$\frac{\beta|\hat{x}-\hat{y}|^4}{\varepsilon} < \frac{\Xi(\hat{x},\hat{y})}{\varepsilon} < \frac{\sigma}{2} + \pi\sum_{j=1}^{n}|m_j| < \infty.$$

Then there exists $\Lambda \geq 0$ such that

$$\overline{\lim_{\varepsilon\to 0}} \frac{\Xi(\hat{x},\hat{y})}{\varepsilon} = \Lambda.$$

We may assume that there exists $\overline{t} \in [0,T]$ and $\overline{x} \in \overline{W}$ such that

$$\hat{t} \to \overline{t},\ \hat{x},\hat{y} \to \overline{x} \quad \text{as } \varepsilon \to 0$$

by taking a subsequence of ε. If $\overline{t} = T$ we get

$$\lim_{j\to\infty} \tilde{\Phi}(\hat{t},\hat{x},\hat{\xi},\hat{y},\hat{\eta}) = -\infty$$

by the same argument of Proposition 3.5. This contradicts (i) so we see that $\overline{t} \in [0,T)$.

We next set

$$\hat{t}_k = \hat{t}(\varepsilon_k,\delta),\ \hat{x}_k = \hat{x}(\varepsilon_k,\delta),\ \hat{y}_k = \hat{y}(\varepsilon_k,\delta),\ \hat{\xi}^k = \hat{\xi}(\varepsilon_k,\delta),\ \hat{\eta}^k = \hat{\eta}(\varepsilon_k,\delta).$$

Since $\left|\hat{\xi}_j^k - \hat{\eta}_j^k\right| < \pi$ and (ii), we get

$$\lim_{k\to\infty} \left|\hat{\xi}_j^k - \hat{\eta}_j^k\right| = 0,$$

where $\hat{\xi}_j^k$ and $\hat{\eta}_j^k$ respectively are j-th element of $\hat{\xi}^k$ and $\hat{\eta}^k$. By the definition of $\tilde{\Phi}$ we get

$$\begin{aligned}
\tilde{\Phi}(\bar{t}, \bar{x}, \bar{\xi}, \bar{x}, \bar{\xi}) &= u^*(\bar{t}, \bar{x}) - v_*(\bar{t}, \bar{x}) - \delta G(\bar{x}, \bar{x}) - \frac{\gamma}{T - \bar{t}} \\
&\leq \tilde{\Phi}(\hat{t}_k, \hat{x}_k, \hat{\xi}^k, \hat{y}_k, \hat{\eta}^k) \\
&= u^*(\hat{t}_k, \hat{x}_k) - v_*(\hat{t}_k, \hat{y}_k) - \sum_{j=1}^n m_j(\hat{\xi}_j^k - \hat{\eta}_j^k) \\
&\qquad - \frac{\Xi(\hat{x}_k, \hat{y}_k)}{\varepsilon_k} - \delta G(\hat{x}_k, \hat{y}_k) - \frac{\gamma}{T - \hat{t}_k}
\end{aligned}$$

where $\bar{\xi}$ satisfies $(\bar{x}, \bar{\xi}) \in \mathfrak{X}$. Since $\tilde{\Phi}$ is upper semicontinuous, we obtain

$$\begin{aligned}
&u^*(\bar{t}, \bar{x}) - v_*(\bar{t}, \bar{x}) - \delta G(\bar{x}, \bar{x}) - \frac{\gamma}{T - \bar{t}} \\
&\leq \overline{\lim_{k\to\infty}} \left\{ u^*(\hat{t}_k, \hat{x}_k) - v_*(\hat{t}_k, \hat{y}_k) - \sum_{j=1}^n m_j(\hat{\xi}_j^k - \hat{\eta}_j^k) \right. \\
&\qquad \left. - \frac{\Xi(\hat{x}_k, \hat{y}_k)}{\varepsilon_k} - \delta G(\hat{x}_k, \hat{y}_k) - \frac{\gamma}{T - \hat{t}_k} \right\} \\
&\leq u^*(\bar{t}, \bar{x}) - v_*(\bar{t}, \bar{y}) - \Lambda - \delta G(\bar{x}, \bar{x}) - \frac{\gamma}{T - \bar{t}}.
\end{aligned}$$

We thus observe that $\Lambda = 0$.

(iv) By (i) we see that

$$\tilde{\Phi}(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) > \frac{\sigma}{2}.$$

By (iii) and the upper semicontinuity of $\tilde{\Phi}$ we get

$$u^*(\overline{t}, \overline{x}) - v_*(\overline{t}, \overline{y}) - \delta G(\overline{x}, \overline{x}) - \frac{\gamma}{T - \overline{t}} \geq \frac{\sigma}{2}$$

if we take $\delta, \gamma > 0$ small enough. Since $u^* \leq v_*$ at $t = 0$, we get $\overline{t} > 0$. Since $\hat{t} \to \overline{t}$ as $\varepsilon \to 0$ there exists $\varepsilon_0 > 0$ such that $\hat{t} > 0$ for $0 < \varepsilon < \varepsilon_0$. $\square$

By Proposition 3.6 (ii) and the compactness of $\overline{W}$ we may assume that

$$\hat{x}(\varepsilon, \delta), \hat{y}(\varepsilon, \delta) \to \bar{x}(\delta) \quad \text{as } \varepsilon \to 0$$

by taking a subsequence of ε. Moreover, we may assume that

$$\bar{x}(\delta) \to x_0 \in \overline{W} \quad \text{as } \delta \to 0$$

by taking subsequence δ. We set

$$\rho_0 = \min\{\rho_1, \rho_2, \dots, \rho_n\}$$

and

$$U_{\rho_0}(x_0) = B_{\rho_0}(x_0) \cap \overline{W},$$

where $B_{\rho_0}(x_0) = \{x \in \mathbb{R}^2; |x - x_0| < \rho_0\}$. We are now in position to define $\theta(x)$. We now fix

$$\alpha_j \in \{\xi_j + 2k\pi; k \in \mathbb{Z},\ 0 \le \xi < 2\pi,\ x_0 - a_j = |x_0 - a_j| (\cos\xi_j, \sin\xi_j)\},$$

and we define $\psi_j \colon [\alpha_j - \frac{\pi}{2}, \alpha_j + \frac{\pi}{2}] \to \mathbb{S}^1$ by

$$\psi_j(\alpha) = (\cos\alpha, \sin\alpha).$$

We define $\theta_j \colon U_{\rho_0}(x_0) \to [\alpha_j - \frac{\pi}{2}, \alpha_j + \frac{\pi}{2}]$ by

$$\theta_j(x) = \psi^{-1}\left(\frac{x - a_j}{|x - a_j|}\right),$$

We note that θ_j is single-valued and $\theta_j \in C^2(U_{\rho_0}(x_0))$. We define $\theta \colon U_{\rho_0}(x_0) \to \mathbb{R}$ by

$$\theta(x) = \sum_{j=1}^{n} \theta_j(x).$$

We define $\Phi \colon [0, T) \times U_{\rho_0}(x_0) \times U_{\rho_0}(x_0) \to \mathbb{R}$ by

$$\Phi(t, x, y) = u^*(t, x) - \theta(x) - (v_*(t, x) - \theta(x)) - \Psi(t, x, y)$$

for $0 < \varepsilon < \varepsilon_1$, $0 < \delta < \delta_1$ and $0 < \gamma < \gamma_0$, where $\varepsilon_1,\ \delta_1 > 0$ satisfy the following:

$$\hat{x}(\varepsilon, \delta), \hat{y}(\varepsilon, \delta) \in U_{\rho_0}(x_0)$$

for $0 < \varepsilon < \varepsilon_1$ and $0 < \delta < \delta_1$.

Proposition 3.7
The function Φ attains its maximum on $[0, T) \times U_{\rho_0}(x_0) \times U_{\rho_0}(x_0)$ at $(\hat{t}, \hat{x}, \hat{y})$.

Proof.
This follows from

$$\tilde{\Phi}(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) = \Phi(\hat{t}, \hat{x}, \hat{y}). \square$$

Proof of Theorem 2.1.
The proof is similar to that given in [8]. But their proof has a small flaw(p. 1224, line 6). So we take this opportunity to indicate the way to correct it.

In [8], they argued that $A \le B$ for $A, B > 0$ implies $A^2 \le B^2$, but it is not true for matrices. One should replace the righthand side of matrix inequality by

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le A + A^2,$$

where $A = \nabla^2_{x,y} \Psi(\hat{t}, \hat{x}, \hat{y})$. Fortunately, the remaining arguments is similar.

4 Construction of a solution

In this section, we prove the existence of a viscosity solution for the initial-boundary value problem applying Perron's method. For that purpose, we construct a subsolution (denoted by $f(t,x)$) and a supersolution (denoted by $g(t,x)$) satisfying

$$f(t,x) \leq g(t,x) \quad \text{for } (t,x) \in (0,T) \times \overline{W}, \tag{4.1}$$

with some positive T independent of $u_0 \in C(\overline{W})$ and satisfying the initial condition, i.e.,

$$f(0,x) = g(0,x) = u_0(x) \in C(\overline{W}) \quad \text{for } x \in \overline{W}, \tag{4.2}$$

with the continuity at time zero:

$$f \text{ and } g \text{ are continuous at } t = 0. \tag{4.3}$$

The solution constructed by Perron's method satisfies the initial condition.

We recall the Perron's method for our problem and some properties in §4.1. We construct f and g satisfying (4.1), (4.2) and (4.3) in §4.2.

4.1 Perron's method and some properties

First we recall Perron's method due to H. Ishii ([10]). We present its version for the Neumann boundary problem (2.3)–(2.4) which is standard [15].

Proposition 4.1
Let T be a positive constant.

(i) Let $\mathcal{F}$ be a non-empty family of subsolutions of (2.3)–(2.4) on $(0,T) \times \overline{W}$. Then $f\colon (0,T) \times \overline{W} \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f(t,x) = \sup\{u(t,x); u \in \mathcal{F}\}$$

is a subsolution of (2.3)–(2.4) on $(0,T) \times \overline{W}$ provided that $f^* < +\infty$ on $[0,T] \times \overline{W}$.

(ii) Let $\mathcal{G}$ be a non-empty family of supersolutions of (2.3)–(2.4) on $(0,T) \times \overline{W}$. Then $g : (0,T) \times \overline{W} \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$g(t,x) = \inf\{v(t,x); v \in \mathcal{G}\}$$

is a supersolution of (2.3)–(2.4) on $(0,T) \times \overline{W}$ provided that $g_* > -\infty$ on $[0,T] \times \overline{W}$.

Proposition 4.2 (Perron's method)
Let f and g respectively be a sub- and a supersolution of (2.3)–(2.4) on $(0,T) \times \overline{W}$. Suppose that f and g satisfy the relation (4.1) and satisfy $-\infty f_*,\ \ g^* < +\infty$.

(i) *Let $\mathcal{F}$ be a family of subsolutions of (2.3)–(2.4) on $(0,T)\times\overline{W}$ satisfying $v \geq f$ on $(0,T)\times\overline{W}$ for $v\in\mathcal{F}$. Then $u\colon (0,T)\times\overline{W}\to\mathbb{R}\cup\{\pm\infty\}$ defined by*

$$u(t,x) = \sup\{v(t,x); v\in\mathcal{F},\ v\leq g \text{ on } (0,T)\times\overline{W}\} \tag{4.4}$$

is a solution of (2.3)–(2.4) on $(0,T)\times\overline{W}$.

(ii) *Let $\mathcal{G}$ be a family of supersolutions of (2.3)–(2.4) on $(0,T)\times\overline{W}$ satisfying $v\leq g$ on $(0,T)\times\overline{W}$ for $v\in\mathcal{G}$. Then $u\colon (0,T)\times\overline{W}\to\mathbb{R}\cup\{\pm\infty\}$ defined by*

$$u(t,x) = \inf\{v(t,x); v\in\mathcal{G}, f\leq v \quad \text{on } (0,T)\times\overline{W}\} \tag{4.5}$$

is a solution of (2.3)–(2.4) on $(0,T)\times\overline{W}$. □

Proof.

We define $G:\overline{W}\times\mathbb{R}^2\times\mathbb{S}_2\to\mathbb{R}$ by

$$G(x,p,X) = F(p-\nabla\theta(x), X-\nabla^2\theta(x)),$$

and applies [Sa, Proposition 3.4] and [Sa, Proposition 3.5]. □

In §4.2 we shall construct f and g on $(0,T)\times\overline{W}$ for some $T>0$ satisfying a initial condition (4.2) and a continuity condition at $t=0$ (4.3). By comparison principle(Theorem 2.1) the relation (4.1) clearly holds. By Proposition 4.2 we get the solution of (2.3)–(2.4) defined by (4.4) or (4.5). The uniqueness of solutions follows easily from Theorem 2.1, so we may choose either (4.4) or (4.5). By the inequalities

$$\begin{aligned} f(0,x) = f_*(0,x) &\leq u_*(0,x),\\ u^*(0,x) \leq g^*(0,x) &= g(0,x) \end{aligned}$$

and (4.2) we get

$$u^*(0,x)\leq u_*(0,x) \quad \text{for } x\in\overline{W}.$$

By comparison principle(Theorem 2.1) we have

$$u^*(t,x)\leq u_*(t,x) \quad \text{for } (t,x)\in(0,T)\times\overline{W}.$$

Then, by defining $u(0,x)=u^*(0,x)$, we conclude that

$$u\in C([0,T)\times\overline{W}).$$

If $T>0$ can be taken independent of initial data, it is clear that one can extend to the global solution u in $[0,\infty)\times\overline{W}$.

The next proposition studies the relation between a solution of (2.3)–(2.4) and that of

$$\frac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0,\infty)\times W, \tag{4.6}$$

$$\langle\vec{\nu},\nabla u\rangle = 0 \quad \text{on } (0,\infty)\times\partial W \tag{4.7}$$

with F defined as in (2.6). It easily follows from the definition of viscosity sub- and supersolution.

Proposition 4.3
Let D be a simply connected open subset of $\overline{W}$ (so that θ is single-valued on D). The function u is a subsolution (resp. supersolution, solution) of (2.3)–(2.4) on $(0,T)\times D\cap\overline{W}$ for any simply connected subdomain D of $\overline{W}$ if and only if $u-\theta$ is a subsolution (resp. supersolution, solution) of (4.6)–(4.7) on $(0,T)\times D\cap\overline{W}$.

Proof.
Since the proof is symmetric, we prove the case of supersolution only. Let u be a supersolution of (2.3)–(2.4) on $(0,T)\times\overline{W}$. Let φ be a test function of $u-\theta$ for (4.6)–(4.7). Then $-(\theta+\varphi)$ is a test function of u for (2.3)–(2.4). So $u-\theta$ is a supersolution of (4.6)–(4.7).
Let $u-\theta$ be a supersolution of (4.6)–(4.7). For a test function φ for (2.3)–(2.4), $-\theta+\varphi$ is a test function of $u-\theta$ for (4.6)–(4.7). Then we obtain that u is a supersolution of (4.6)–(4.7). □

We next remark that F is geometric in the sense of [3]. See [15].

Proposition 4.4
For $\lambda>0$, $\mu\in\mathbb{R}$ we have

$$F(\lambda p, \lambda X+\mu p\otimes p)=\lambda F(p,X).$$

Proposition 4.4 will be important when we apply *Invariance Lemma* (Lemma 4.8).

Remark 4.5
Argument hear also applies to general equation in Remark 2.3.

4.2 Construction of sub- and supersolution

We construct the supersolution and subsolution such that they satisfies the assumption of Perron's method. The construction of supersolution and subsolution is symmetric, so we only construct the supersolution.
Suppose that $\partial\Omega$ is C^2. We recall the exterior ball condition (3.8), i.e., there exists a positive constant C_0 such that

$$\langle\vec{\nu}(x),\ x-y\rangle\geq -C_0|x-y|^2 \quad \text{for } x\in\partial W,\ y\in\overline{W}. \tag{4.8}$$

We also recall that there exists $\varphi\in C^2(\overline{W})$ satisfying (3.9)–(3.10) with $\beta=2C_0$ and $|\nabla\varphi|=2C_0$ on ∂W instead of (3.11), i.e.,

$$-C_0<\varphi<0 \quad \text{in } W, \quad \varphi=0 \quad \text{on} \quad \partial W, \tag{4.9}$$
$$\nabla\varphi=2C_0\vec{\nu} \quad \text{on} \quad \partial W. \tag{4.10}$$

Since the initial value u_0 is uniformly continuous on $\overline{W}$, for fixed $\varepsilon>0$ there exists a positive constant A_ε such that

$$|u_0(x)-u_0(y)|<A_\varepsilon e^{-C_0}|x-y|^2+\varepsilon \quad \text{for} \quad x,y\in\overline{W}. \tag{4.11}$$

Because the function θ is Lipschitz continuous if we choose a branch the value of θ, there exists $\delta = \delta(\varepsilon) > 0$ such that the following holds;

$$|\theta(x) - \theta(y)| < \varepsilon \quad \text{if} \quad |x - y| < \delta. \tag{4.12}$$

We now fix $y \in \overline{W}$ and set $U_\delta(y) = B_\delta(y) \cap \overline{W}$, and we consider the function θ on $\overline{U_\delta(y)}$. We fix a branch of the value of θ on $U_\delta(y)$. We define the function $v_{\varepsilon,y} \colon [0,\infty) \times U_\delta(y) \to \mathbb{R}$ by

$$v_{\varepsilon,y}(t,x) = B_t + A_\varepsilon e^{\varphi(x)}|x - y|^2 + 2\varepsilon + \theta(x) - \theta(y). \tag{4.13}$$

Proposition 4.6
(i) $v_{\varepsilon,y}$ satisfies the boundary condition, i.e.

$$\langle \vec{\nu}, \nabla(v_{\varepsilon,y} - \theta)\rangle \geq 0 \quad on \quad (0,\infty) \times (U_\delta(y) \cap \partial W).$$

(ii) There exists a constant B_ε such that the following holds: if $B \geq B_\varepsilon$, then

$$\frac{\partial v_{\varepsilon,y}}{\partial t}(t,x) + F^*(\nabla(v_{\varepsilon,y}(t,x) - \theta(x)), \nabla^2(v_{\varepsilon,y}(t,x) - \theta(x))) \leq 0$$

for $(t,x) \in (0,\infty) \times (U_\delta(y) \cap W)$.

Proof.
We calculate derivatives of $v_{\varepsilon,y}$:

$$\begin{aligned}
\frac{\partial v_{\varepsilon,y}}{\partial t}(t,x) &= B, && (4.14)\\
\nabla(v_{\varepsilon,y}(t,x) - \theta(x)) &= A_\varepsilon e^{\varphi(x)}(|x-y|^2\nabla\varphi(x) + 2(x-y)), && (4.15)\\
\nabla^2(v_{\varepsilon,y}(t,x) - \theta(x)) &= A_\varepsilon e^{\varphi(x)}(|x-y|^2\nabla\varphi(x)\otimes\nabla\varphi(x)\\
&\quad +2(\nabla\varphi(x)\otimes(x-y) + (x-y)\otimes\nabla\varphi(x))\\
&\quad +|x-y|^2\nabla^2\varphi(x) + 2I). && (4.16)
\end{aligned}$$

(i) By (4.8), (4.10) and (4.15) we get

$$\begin{aligned}
\langle \vec{\nu}(x), \nabla(v_{\varepsilon,y}(t-x) - \theta(x))\rangle &= A_\varepsilon e^{\varphi(x)}(|x-y|^2\langle\vec{\nu}, \nabla\varphi(x)\rangle + 2\langle\vec{\nu}, x-y\rangle)\\
&\geq A_\varepsilon e^{\varphi(x)}(2C_0|x-y|^2 - 2C_0|x-y|^2) = 0.
\end{aligned}$$

(ii) We set

$$\begin{aligned}
p = p(x,y) &= e^{\varphi(x)}(|x-y|^2\nabla\varphi(x) + 2(x-y)),\\
X = X(x,y) &= e^{\varphi(x)}(|x-y|^2\nabla\varphi(x)\otimes\nabla\varphi(x)\\
&\quad +2(\nabla\varphi(x)\otimes(x-y) + (x-y)\otimes\nabla\varphi(x))\\
&\quad +|x-y|^2\nabla^2\varphi(x) + 2I);
\end{aligned}$$

in other words,

$$\nabla(v_{\varepsilon,y}(t,x)-\theta(x)) = A_\varepsilon p,$$
$$\nabla^2(v_{\varepsilon,y}(t,x)-\theta(x)) = A_\varepsilon X.$$

By the definition of p and X $\left\{(p(x,y),X(x,y));\ (x,y)\in [0,\overline{W}\times\overline{W}\right\}$ is bounded in $\mathbb{R}^2\times\mathbb{S}_2$. So there exists a compact set K such that K is independent of u_0 satisfying

$$K\supset\{(p(x,y),X(x,y));(x,y)\in\overline{W}\times\overline{W}\}.$$

Since F_* is lower semicontinuous on a compact set K, F_* has a minimum value on K. We set

$$R=-\min\{F_*(p,X);(p,X)\in K\}.$$

By Proposition 4.4, we get

$$\begin{aligned}&\frac{\partial v_{\varepsilon,y}}{\partial t}(t,x)+F^*(\nabla(v_{\varepsilon,y}(t,x)-\theta(x)),\nabla^2(v_{\varepsilon,y}(t,x)-\theta(x))\\&\geq B+F_*(A_\varepsilon,A_\varepsilon X)\\&=B+A_\varepsilon F_*(p,X)\\&\geq B-A_\varepsilon R.\end{aligned}$$

So it is enough to see 2) that we set $B_\varepsilon = A_\varepsilon R$. $\square$

To construct a subsolution of (2.3) (2.4), we define $u_{\varepsilon,y}\colon [0,\infty)\times U_\delta(y)\to\mathbb{R}$ by

$$u_{\varepsilon,y}(t,x) = -B't - A_\varepsilon e^{\varphi(x)}|x-y|^2 - 2\varepsilon + \theta(x)-\theta(y), \tag{4.17}$$

where B' is a positive constant. We may assume that $B'=A_\varepsilon R$ by take

$$R=\max\left\{F^*(p,X);(p,X)\in K\right\}.$$

Proposition 4.7

(i) $u_{\varepsilon,y}$ satisfies that boundary condition, i.e.,

$$\langle\vec{\nu},\nabla(u_{\varepsilon,y}-\theta)\rangle\leq 0 \quad on \quad (0,\infty)\times(U_\delta(y)\cap\partial W).$$

(ii) There exists a constant B'_ε such that the following holds; if $B'\geq B'_\varepsilon$, the inequality

$$\frac{\partial u_{\varepsilon,y}}{\partial, t}(t,x)+F_*(\nabla(u_{\varepsilon,y}(t,x)-\theta(x)),\nabla^2(u_{\varepsilon,y}(t,x)-\theta(x))\leq 0$$

holds for $(t,x)\in(0,\infty)\times(U_\delta(y)\cap W)$.

The proof is symmetric to Proposition 4.6. We need to extend the function $v_{\varepsilon,y}$ (resp. $u_{\varepsilon,y}$) on $(0,T)\times\overline{W}$. For this purpose we prepare:

Lemma 4.8 (Invariance Lemma)
Let u be a supersolution (resp. subsolution) of (2.3) (2.4). Let σ be a function such that the following holds;

(i) $\sigma\colon \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is lower (resp. upper) semicontinuous,

(ii) $\sigma \not\equiv +\infty$ (resp. $\not\equiv -\infty$),

(iii) σ is non-decreasing

Then the function $\sigma(u)$ is a supersolution (resp. subsolution) of (2.3) (2.4).

See [9] for the proof of this lemma.

Proposition 4.9
Let $B = B_\varepsilon = A_\varepsilon R$, $T_\varepsilon = \frac{A_\varepsilon e^{-C_0\delta(\varepsilon)^2}}{B_\varepsilon} = \frac{e^{-C_0\delta(\varepsilon)^2}}{R}$, and $M_\pm = \pm(A_\varepsilon e^{-C_0}\delta^2 + 2\varepsilon) - \theta(y)$. Let $\sigma_\pm\colon \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\sigma_+(s) = \begin{cases} s & (s \le M_+), \\ +\infty & (s > M_+), \end{cases}$$
$$\sigma_-(s) = \begin{cases} -\infty & (s < M_-), \\ s & (s \ge M_-). \end{cases}$$

Let $f_{\varepsilon,y}$, $g_{\varepsilon,y}\colon [0, T_\varepsilon) \times \overline{W} \to \mathbb{R}$ be defined by

$$f_{\varepsilon,y}(t,x) = \begin{cases} \sigma_-(u_{\varepsilon,y}(t,x) - \theta(x)) + \theta(x) & x \in U_\delta(y), \\ -\infty & x \in \overline{W} \setminus U_\delta(y), \end{cases}$$
$$g_{\varepsilon,y}(t,x) = \begin{cases} \sigma_+(v_{\varepsilon,y}(t,x) - \theta(x)) + \theta(x) & x \in U_\delta(y), \\ +\infty & x \in \overline{W} \setminus U_\delta(y). \end{cases}$$

Then the function $f_{\varepsilon,y}$ (resp. $g_{\varepsilon,y}$) is a subsolution (resp. supersolution) of (2.3) (2.4). In particular,

$$f_{\varepsilon,y}(t,y) > -\infty, \;\; g_{\varepsilon,y}(t,x) < +\infty \quad \text{for } t \in (0, T_\varepsilon).$$

Proof.
Since the proof is symmetric, we shall give a proof for $g_{\varepsilon,y}$.
To calculate $\liminf_{|x-y|\to\delta}(v_{\varepsilon,y}(t,x) - \theta(x))$, we get

$$\begin{aligned} \liminf_{|x-y|\to\delta}(v_{\varepsilon,y}(t,x) - \theta(x)) &\ge B_\varepsilon t + A_\varepsilon e^{-C_0}\delta^2 + 2\varepsilon - \theta(y) \\ &= B_\varepsilon t + M_+ > M_+. \end{aligned}$$

So, if $t > 0$,

$$u_{\varepsilon,y}(t,x) - \theta(x) > M_+$$

for x belongs to a neighborhood of $\partial U_\delta(y)$. So, $\sigma_+(u_{\varepsilon,y}(t,x) - \theta(x))$ has a minimum value in $(0,\infty) \times U_\delta(y)$. Because of Lemma 4.8 we observe that $g_{\varepsilon,y}$ is a supersolution of (2.3)–(2.4) by noting that $v_{\varepsilon,y} - \theta$ a supersolution of (4.6) (4.7) on $(0,T) \times U_{\delta(\varepsilon)}$(Proposition 4.3).

To estimate $v_{\varepsilon,y}(t,y)$ for $t \in [0,t_\varepsilon)$, we have

$$\begin{aligned} v_{\varepsilon,y}(t,y)-\theta(y) &= B_\varepsilon t + 2\varepsilon - \theta(y) \\ &\le B_\varepsilon T_\varepsilon + M_+ - A_\varepsilon e^{-C_0}\delta^2 \\ &= M_+. \end{aligned}$$

So we get $g_{\varepsilon,y}(t,y) < +\infty$. □

We have extended the domain of $f_{\varepsilon,y}$ and $g_{\varepsilon,y}$ to $[0,T_\varepsilon)\times\overline{W}$. To complete the proof of Theorem 2.2, we construct a sub– and supersolution of (2.3)–(2.4). We shall prepare some notes.

We set $T_1 = T_\varepsilon|_{\varepsilon=1}$, i.e.,

$$T_1 = \frac{e^{-C_0}\delta(1)^2}{R}.$$

We now define $f, g\colon [0,T_1)\times\overline{W}\to\mathbb{R}$ as a real-valued functions defined by

$$f(t,x) = \sup\{f_{\varepsilon,y}(t,x)+u_0(y);\ 0<\varepsilon\le 1,\ y\in\overline{W}\}, \tag{4.18}$$
$$g(t,x) = \inf\{g_{\varepsilon,y}(t,x)+u_0(y);\ 0<\varepsilon\le 1,\ y\in\overline{W}\}. \tag{4.19}$$

Proposition 4.10

(i) f (resp. g) is a subsolution (resp. supersolution) of (2.3) (2.4).

(ii) The initial value condition

$$f(0,x) = g(0,x) = u_0(x) \quad \textit{for} \ \ x\in\overline{W}$$

holds.

(iii) f and g are continuous at $t=0$.

Proof.

Since the proof is symmetric, we only give it for g.

(i) This has been proved in Proposition 4.1.

(ii) By the definition of g we get

$$g(0,x)\le g_{\varepsilon,x}(0,x)+u_0(x) = 2\varepsilon + u_0(x).$$

Since $\varepsilon>0$ in arbitrary, we see that

$$g(0,x)\le u_0(x) \quad \text{for} x\in\overline{W}. \tag{4.20}$$

For a fixed $y\in\overline{W}$, let x in $U_\delta(y)$. As in (4.11) we have

$$\begin{aligned} u_0(x)-u_0(y) &\le A_\varepsilon e^{-C_0}|x-y|^2+\varepsilon \\ &\le A_\varepsilon e^{\varphi(x)}|x-y|^2+\varepsilon. \end{aligned}$$

By (4.12) we obtain

$$u_0(x) - u_0(y) \leq A_\varepsilon |x-y|^2 + 2\varepsilon + \theta(x) - \theta(y) = g_{\varepsilon,y}(0,x)$$

or

$$u_0(x) \leq g_{\varepsilon,y}(0,x) + u_0(y) \quad \text{for } x \in \overline{W}.$$

Thus we obtain

$$u_0(x) \leq \inf\{g_{\varepsilon,y}(0,x) + u_0(y); \varepsilon > 0, y \in \overline{W}\} = g(0,x) \quad \text{for } x \in \overline{W}. \tag{4.21}$$

By (4.20) and (4.21) we get

$$g(0,x) = u_0(x).$$

(iii) We first verify that $g(t,x) \to g(0,x)$ uniformly on $\overline{W}$ as $t \to 0$. Since $g(t,x)$ is non-decreasing at t, we get

$$\begin{aligned} |g(t,x) - g(0,x)| &= g(t,x) - g(0,x) \\ &\leq g_{\varepsilon,y}(t,x) - g(0,x) \\ &= B_\varepsilon t + g_{\varepsilon,y}(0,x) - g(0,x) \end{aligned}$$

for a fixed number $\mu > 0$ and all $y \in \overline{W}$ and $\varepsilon > 0$. So, let $t \leq \frac{\mu}{2B_\varepsilon}$ and take $y \in \overline{W}$ satisfies

$$|g_{\varepsilon,y}(0,x) - g(0,x)| < \frac{\mu}{2},$$

then we get

$$|g(t,x) - g(0,x)| < \mu, \quad \text{i.e.,} \quad \lim_{t\to 0} g(t,x) = g(0,x).$$

We shall complete the proof. Let $t < \frac{\mu}{2B_\varepsilon}$ and take $r > 0$ so that

$$|u_0(x) - u_0(y)| < \frac{\mu}{2} \quad \text{if } |x-y| < r.$$

Combining these inequalities, we obtain

$$\begin{aligned} |g(t,x) - g(0,y)| &\leq |g(t,x) - g(0,x)| + |g(0,x) - g(0,y)| \\ &< \frac{\mu}{2} + |u_0(x) - u_0(y)| \\ &< \mu. \square \end{aligned}$$

By Proposition 4.10 we obtain a time local solution. Fortunately, T_1 depends only W and F, so we can extend the solution of (2.3)–(2.4) to a time global solution.

References

[1] W. K. Burton, N. Cabrera and F. C. Frank, *Role of dislocations in crystal growth*, Nature, **163**(1949), 398-399.

[2] W. K. Burton, N. Cabrera and F. C. Frank, *The growth of crystals and equilibrium structure of their surfaces*, Phi. Trans. Roy. Soc. Lond. A, **243**(1951), 299-358.

[3] Y.-G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geometry, **33**(1991), 749-786.

[4] M. G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., **27**(1992), 1-67.

[5] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Diff. Geometry, **33**(1991), 635-681.

[6] F. C. Frank, *The refluence of dislocations on crystal growth*, Disc. Faraday Soc., **5**(1949), 48-54.

[7] Y. Giga, N. Ishimura and Y. Kohsaka, *Spiral solutions for a weak anisotropic curvature flow equations*, Adv. Math. Sci. Appl., to appear.

[8] Y. Giga and M.-H. Sato, *Neumann problem for singular degenerate parabolic equations*, Diff. and Int. Eq., **6**(1993), 1217-1230.

[9] Y. Giga and M.-H. Sato, *A level set approach to semicontinuous viscosity solutions for Cauchy problems*, Hokkaido University Preprint Series in Mathematics #472, 1999.

[10] H. Ishii, *Perron's method for Hamilton-Jacobi equation*, Duke Math. J., **55**(1987), 369-384.

[12] R. Kobayashi, Private communication.

[13] A. Karma and M. Plapp, *Spiral surface growth without desorption*, Physical Review Letters, **81**(1998), 4444-4447.

[14] K.-I. Nakamura and T. Ogiwara, *Spiral traveling wave solutions of some parabolic equations on annuli*, Josai Mathematical Monographs, **2**(2000), 15-34.

[15] M.-H. Sato, *Interface evolution with Neumann boundary condition*, Advances in Mathematical Sciences and Applications, **4**(1994), 249-264.

[16] P. Smereka, *Spiral crystal growth*, Physica D: Nonlinear Phenomena, **138**(2000), 282-301.

Takeshi Ohtsuka
Division of Mathmatics,
Graduate School of Sciences,
Hokkaido University.
Sapporo, 060-0810 Japan.