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1 Introduction

It is our great honor to contribute to this special proceedings of the conference in celebration of the sixtieth birthday of Professor Stanley Osher. This is a continuation of recent works [11], [13] of the second author. It also provides an analytic verification of some numerical results obtained in [21].

In [11] we introduced the notion of proper viscosity solutions for a class of evolution equations of n space variables whose solutions may develop jump discontinuities in finite time. The class includes (scalar) conservation laws as special examples and a proper viscosity solution is essentially equivalent to an entropy solution for conservation laws. A proper viscosity solution may have jump discontinuities called shocks. Such an equation is also interpreted as a surface evolution equation in $\mathbf{R}^n \times \mathbf{R}$ where the graph of a solution lives at each time. In this formulation it is well-known that the problem is uniquely solvable globally-in-time for a given initial surface (curve if $n = 1$) at least for conservation laws. However, even if initial surface is the graph of a smooth function on $\mathbf{R}^n$, the evolving surface in $\mathbf{R}^n \times \mathbf{R}$ may develop 'overturning' in finite time in the sense that it cannot be viewed as the graph of any single-valued functions on $\mathbf{R}^n$. Such a surface evolution does not represent the graph of a proper viscosity solution. In [13] we proposed to interpret the graph evolution of a proper viscosity solution with shocks as a result of the vertical singular diffusion. By a formal argument we have noted in [13] that there is a threshold value of the strength of the vertical diffusion such that it prevents overturning of the graph surface (curve). Such a result is observed numerically by [21].

In this paper we give a rigorous proof for the fact that a solution develops overturning (when $n = 1$) if the strength $M > 0$ of the vertical diffusion is smaller than the critical value by studying a Riemann problem for the Burgers equation:

$$u_t + uu_x = 0, \quad x \in \mathbf{R}, \quad t > 0, \tag{1.1}$$

$$u(x,0) = -(\mathrm{sgn} x)d/2, \;\; x \in \mathbf{R} \tag{1.2}$$

for $u = u(x,t)$, where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x = \partial_x u$. The parameter d is taken positive so that the solution keeps a shock profile. If one views the graph of u as a level set of an auxiliary function $\psi(x,y,t)$, ψ must satisfy

$$\psi_t + y\psi_x = 0 \tag{1.3}$$

on the graph of u. If we consider (1.3) in $\mathbf{R}^2 \times (0,T)$, each level set of ψ moves by (1.1) if it is represented by the graph of a function $u = u(x,t)$. This kind of level-set formulation has been successful [15] to track discontinuous solutions for

$$u_t + H(u, u_x) = 0, \;\; x \in \mathbf{R}, \;\; t > 0$$

if $r \mapsto H(r,p)$ is nondecreasing so that a solution does not develop discontinuities if the initial data is continuous. However, for (1.1) the zero level set of the solution of (1.3) certainly overturns if initially

$$\begin{aligned} &\{(x,y) \in \mathbf{R} \times \mathbf{R}; \psi(x,y,0) > 0\} \\ &= \{(x,y); y < -d/2\} \cup \{(x,y); x < 0, -d/2 \leq y < d/2\}; \end{aligned} \tag{1.4}$$

in fact, the zero level set $\psi = 0$ for $t > 0$ cannot be viewed as the graph of a single-valued function in any sense.

In [13] we proposed to add the vertical diffusion term to (1.3) to get

$$\psi_t + y\psi_x = M|\nabla\psi|\partial_y(\psi_y/|\psi_y|), \tag{1.5}$$

where $\nabla\psi = (\psi_x, \psi_y)$. A formal argument [13, Theorem 2.1] reflecting [5] says that if M is large so that

$$V_I \geq V - 2M \quad \text{on} \quad I = (-d/2, d/2), \tag{1.6}$$

then the zero level set of ψ with initial condition (1.4) does not overturn and equals the graph of the entropy solution of (1.1), (1.2). Here $V(\eta) = -\eta^2/2$ which is the primitive of $-\eta$ and V_I denotes its convex hull in I. An elementary calculation shows that the minimum value M_0 of M satisfying (1.6) is $d^2/16$. In the numerical simulation [21] we also observe that the overturning occurs if and only if $M < M_0 = d^2/16$. (There, I is replaced by (a,b), so the value of M_0 equals $(b-a)^2/16$.)

In this paper we show analytically that M_0 is optimal in the sense that if $M < M_0$, the overturning is not prevented. It is also possible to prove that the overturning does not occur $M \geq M_0$ for more general equations but we shall discuss this problem in one of forthcoming papers.

Although the level set method (see e.g. [10], [12]) allowing the singular diffusivity is well-studied by [6], [7], [8], the equation handled there is spatially homogeneous and excludes (1.5). Instead of developing a general theory for (1.5) we rather study its approximation. In fact, we shall prove that there is a sequence of level set equations

$$\psi_t + y\psi_x = M|\nabla\psi\ |\mathrm{div}\ (\nabla\gamma_\varepsilon(-\nabla\psi)) \tag{1.7}$$

approximating (1.5) (as $\varepsilon \to 0$) such that the limit of zero level set of $\psi = \psi^\varepsilon$ develops 'overturning' if and only if $M < M_0$ provided that $M < d^2/8$. Here $\gamma_\varepsilon \in C^3(\mathbf{R}^2 \setminus \{0\})$ is convex and positively homogeneous of degree one.

The main idea of the proof is to convert the problem of evolution of $\{\psi^\varepsilon = 0\}$ to the evolution of $x = v^\varepsilon(y,t)$ starting with $v^\varepsilon(y,0) = 0$. (For this purpose we assume that $\nabla^2\gamma_\varepsilon(0,1) = 0$ so that the line segment on the line $y = \pm d/2$ does not move.) We study the equation for v^ε derived from (1.7) and prove that it converges to a function $v = v(y,t)$ which has strictly monotone increasing part in y if $M < M_0$. This means that 'overturning' occurs. Unfortunately, if ψ^ε solves (1.7), the boundary condition for v^ε at $y = \pm d/2$ is not conventional. It formally equals the Neumann condition

$$v_y^\varepsilon(\pm d/2, t) = \mp\infty. \tag{1.8}$$

This is hard to handle so we estimate from above and below by solutions of a homogeneous Neumann problem and an inhomogeneous Dirichlet problem. We prove that as $\varepsilon \to 0$ solutions of the latter two problems converge to the same function v which solves

$$v_t = M(\mathrm{sgn} v_y)_y + y \quad \text{in} \quad I \times (0,\infty), \tag{1.9}$$

$$v = 0 \quad \text{on} \quad \partial I \times (0,\infty), \quad v|_{t=0} = 0 \quad \text{in} \quad I \tag{1.10}$$

by using the theory of nonlinear semigroups [1]. The theory of viscosity solutions for spatially inhomogeneous problem with singular diffusivity is not yet studied, so we use the theory of nonlinear semigroups. This equation has a very singular diffusivity and a similar problem has been studied in [9]. Fortunately, it is not difficult to solve this problem explicitly. From the explicit shape of solution v one obtains the threshold value for M.

If we consider more general Riemann data than (1.2) of the form

$$u(x,0) = -(\mathrm{sgn} x)d/2 + \mu$$

with some constant $\mu \in \mathbf{R}$, our result still applies to find the threshold value $d^2/16$. Indeed, it suffices to replace variables by

$$\tilde{u}(x,t) = u(x + \mu t, t) - \mu$$

so that the problem is reduced to (1.1), (1.2). Also it is likely that our results extend to more general equations than the Burgers equation (1.1) of the form

$$u_t + f(u)_x = 0,$$

where f is a smooth, strictly convex function. In this case V in (1.6) should be replaced by $-f$ and the value M_0 may not be explicitly computable. We won't pursue this problem in the present paper.

Evolution of a curve in xy-plane by the heat equation $u_t = u_{xx}$ is studied in [4] by a level set method. Since the diffusion effect of the Laplace operator is so strong in vertical direction (y-direction), a solution starting from a curve with overturning instantaneously becomes the graph of a discontinuous function. Our vertical diffusion is weaker in the sense that a solution starting from data with overturning stays overturning at least for a short time although we do not prove it explicitly in the present paper (cf. [3]). This corresponds to the strength of singularities in the level set formulation of the diffusion term. The singularity of the right hand side of (1.5) at $\psi_y = 0$ is somewhat weaker than one appeared in the level set formulation of u_{xx} term. The weaker singularity is very helpful for numerical computation as presented in [21].

This paper is organized as follows. In section 2 we give an explict solution for (1.9)-(1.10). In section 3 we study the behaviour of an inhomogeneous Neumann problem for approximate equations to find a solution satisfying (1.8). In section 4 the solution satisfying (1.8) is shown to converge to the solution of (1.9)-(1.10) as approximation parameter $\varepsilon \to 0$. In section 5 we apply these results to level set solutions for (1.7) and study its limit as $\varepsilon \to 0$. A typical result is stated in Corollary 5.4.

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2 Explicit solutions for some inhomogeneous very singular diffusion equations

We consider a singular degenerate parabolic equation for $v = v(\eta, t)$ of the form

$$v_t = M(\mathrm{sgn} v_\eta)_\eta + \eta \quad \text{in} \quad I \times (0, \infty), \tag{2.1}$$

$$v = 0 \quad \text{on} \quad \partial I \times (0, \infty), \tag{2.2}$$

$$v|_{t=0} = 0 \quad \text{in} \quad I \tag{2.3}$$

with $I = (-d/2, d/2)$, where $M > 0$ is a parameter. Since $(\mathrm{sgn} v_\eta)_\eta$ formally equals $2\delta(v_\eta)v_{\eta\eta}$, the diffusion is degenerate for $v_\eta \neq 0$ and is very strong for $v_\eta = 0$. Naively, the meaning of a 'solution' is not clear. Fortunately, the theory of nonlinear semigroups [19] or subdifferential equations provides a suitable notion of a solution. We shall briefly review its notion and give an explicit representation formula of the solution.

We first give a subdifferential interpretation of the problem (2.1)-(2.3). For $w \in H =$

$L^2(I)$ we associate the energy $E(w)$ defined by

$$E(w) := \int_{\mathbf{R}} \{M|(\tilde{w})_\eta(\eta)| - \eta\tilde{w}(\eta)\}\mathrm{d}\eta \quad \text{if} \quad w \in BV(I)$$

and $E(w) := \infty$ if $w \notin BV(I)$. Here $BV(I)$ denotes the space of functions with bounded variation in I and $\tilde{w}$ denotes the extention of w to $\mathbf{R}$ such that $\tilde{w} = 0$ outside I. The integral $\int_{\mathbf{R}} |(\tilde{w})_\eta|\mathrm{d}\eta$ denotes the total variation of $\tilde{w}$ in $\mathbf{R}$. Then as in [9, the first lemma in §2] the functional E is convex, lower semicontinuous in the Hilbert space H equipped with the standard inner product $(f,g) = \int_I fg\mathrm{d}\eta$. Note that (2.1) is formally a gradient flow of E. Thus we formulate the problem (2.1)-(2.3) as

$$\frac{dv}{dt} \in -\partial E(v) \quad \text{for} \quad t > 0 \tag{2.4}$$

$$v(0) = 0, \tag{2.5}$$

where ∂E denotes the subdifferential of E in H; here and hereafter we sometimes regard a function $v = v(\eta, t)$ as a function of time with values in H. A general theory [19], [1] yields that there is a unique solution v of (2.4) and (2.5) in the sense that

(i) $v \in C([0,\infty), H)$ i.e., v is continuous from the time interval $[0,\infty)$ to H. Moreover, v satisfies (2.5);

(ii) v is absolutely continuous with values in H on each compact set in $(0,\infty)$;

(iii) v solves (2.4) for almost all $t \geq 0$.

As well-known (e.g. [1], see also [9, §2]) the solution $v(t)$ is right-differentiable at all $t > 0$ with values in H and its right derivative d^+v/dt satisfies

$$\frac{d^+v}{dt} = -\partial^0 E(v) \quad \text{for all} \quad t > 0, \tag{2.6}$$

where $\partial^0 E(v)$ is the canonical restriction (or minimal section) of $\partial E(v)$, *i.e.*, $\partial^0 E(v)$ is the unique element of the closed convex set $\partial E(v)$ which is closest to the origin of H. Moreover, we have another definition of solution equivalent to (i), (ii) and (iii). Namely, v is the solution of (2.4) and (2.5) if and only if v fulfills (i), (ii) and

(iii') v is right differentiable for all $t > 0$ with values in H and solves (2.6) for all $t > 0$.

Here and hereafter by a solution of (2.1)-(2.3) we mean that it satisfies (i), (ii), (iii) or (i), (ii), (iii'). Fortunately, the solution can be represented by an explicit formula.

Lemma 2.1. *Let v be the solution of (2.1)-(2.3). Then v is represented by*

$$v(\eta, t) = tv_1(\eta), \ t \geq 0, \ \eta \in I \tag{2.7}$$

with v_1 satisfying

$$
\begin{aligned}
v_1(\eta) &= \min(\eta, (\frac{d}{2} - 2M^{1/2})_+) && \textit{for} \quad \eta \in [0, \frac{d}{2}), \\
v_1(\eta) &= -v_1(-\eta) && \textit{for } \eta \in (-\frac{d}{2}, 0],
\end{aligned}
$$

where $\alpha_+ = \max\,(\alpha, 0)$. *In particular,* $v_1 \equiv 0$ *if and only if* $M \geq d^2/16$ *and otherwise* v_1 *has a strictly increasing part.*

Remark 2.2. (i) If we replace the homogeneous Dirichlet condition (2.2) by the homogeneous Neumann condition

$$
v_\eta = 0 \quad \text{on} \quad \partial I \times (0, \infty), \tag{2.2$'$}
$$

the problem (2.1), (2.2′), (2.3) can be formulated by (2.4)-(2.5) with different E. Actually, if we replace the definition of E by

$$
E_N(w) := \int_I \{M|w_\eta| - \eta w\} \mathrm{d}\eta \quad \text{if} \quad w \in BV(I) \tag{2.8}
$$

and $E_N(w) := \infty$ if $w \notin BV(I)$, then (2.1), (2.2′), (2.3) is formulated by (2.4)-(2.5) with E replaced by E_N. By a solution of (2.1), (2.2′), (2.3) we mean that it is solution of (2.4)-(2.5) with $E = E_N$. Surprisingly, the solution is the same as in (2.7).

(ii) We may replace the homogeneous Dirichlet condition (2.2) by inhomogeneous Dirichlet condition

$$
v = \mp R \quad \text{at} \quad \eta = \pm d/2. \tag{2.2$''$}
$$

The solution of (2.1), (2.2″), (2.3) can be formulated by (2.4)-(2.5) with different E. In fact, it suffices to replace E by

$$
E_R(w) := \int_{\mathbf{R}} \{M|(\bar{w})_\eta| - \eta \bar{w}\} \mathrm{d}\eta \quad \text{if} \quad w \in BV(I) \tag{2.9}
$$

and $E_R(w) := \infty$ if $w \notin BV(I)$. The extention $\bar{w}$ of w equals $-R$ for $\eta \geq d/2$ and R for $\eta \leq -d/2$. The equation (2.1), (2.2″), (2.3) is now formulated by (2.4)-(2.5) with E replaced by E_R. By a solution of (2.1), (2.2″), (2.3) we mean that it is a solution of (2.4)-(2.5) with $E = E_R$. If $R > 0$, its solution is the same as in (2.7).

Proof of Lemma 2.1. Clearly, v in (2.7) satisfies (i) and is absolutely continuous in $[0, \infty)$ with values in H. So it suffices to prove that $-v_1 \in \partial E(tv_1)$ for each $t > 0, i.e.,$

$$
E(tv_1 + h) - E(tv_1) \geq (-v_1, h), \; t > 0 \tag{2.10}
$$

for all $h \in BV(I)$. We may assume that $t = 1$ by dividing both sides by t.

We define an even function $\xi = \xi(\eta)$ defined on I of the form

$$\xi(\eta) = 1 \qquad \text{for} \quad \eta \in [0, \rho) \quad \text{with} \quad \rho = \left(\frac{d}{2} - 2M^{1/2}\right)_+,$$
$$\xi(\eta) = 1-(x-\rho)^2/(2M) \quad \text{for} \quad \eta \in [\rho, d/2]$$

This function is C^1 in $\bar{I}$ and satisfies $|\xi| \leq 1$ on I. By definition [16] of total variation we see that

$$\int_{\mathbf{R}} |(\tilde{w})_\eta| \mathrm{d}\eta = \sup\{-\int_I w\zeta_\eta \mathrm{d}\eta; \zeta \in C^1(\bar{I}),\ \zeta \in C^1(\bar{I}),\ |\zeta| \leq 1 \quad \text{on} \quad I\},$$

so

$$E(v_1 + h) \geq -M \int_I (v_1 + h)\xi_\eta \mathrm{d}\eta - \int_I (v_1 + h)\eta \mathrm{d}\eta.$$

Since

$$\int_{\mathbf{R}} |(\tilde{v}_1)_\eta| \mathrm{d}\eta = 2\rho + 2\rho = -\int_I v_1 \xi_\eta \mathrm{d}\eta,$$

we now observe that

$$E(v_1 + h) - E(v_1) \geq -M \int_I h\xi_\eta \mathrm{d}\eta - \int_I h\eta \mathrm{d}\eta = -\int_I v_1 h \mathrm{d}\eta.$$

We have thus proved (2.10). We note that Remarks 2.2 and 2.3 can be proved similarly. $\square$

3 Neumann problems for some non-uniform parabolic equations

To study solutions of problems approximating (2.1)-(2.3) we consider the Neumann problem:

$$v_t = a(v_\eta)v_{\eta\eta} + \eta \quad \text{in} \quad I \times (0, \infty), \tag{3.1}$$

$$v_\eta = -\alpha \quad \text{on} \quad \partial I \times (0, \infty), \tag{3.2}$$

$$v|_{t=0} = 0 \quad \text{in} \quad I. \tag{3.3}$$

Here $a \in C^1(\mathbf{R})$ is assumed to be positive and α is a non-negative constant. Since v_η of (3.1) solves

$$v_{\eta t} = (a(v_\eta)v_{\eta\eta})_\eta + 1 \quad \text{in} \quad I \times (0, \infty) \tag{3.4}$$

by the maximum principle we have an a priori bound $|v_\eta(n, t)| \leq \max(t, \alpha)$ for v_η. So in $I \times (0, T)$ with $T > 0$ we may assume that equation is uniformly parabolic by restricting a on $[-\max(T, \alpha),\ \max(T, \alpha)]$. A general theory of parabolic equations [18] yields an unique global classical solution $v \in C^{2,1}(I \times [0, \infty)) \cap C^{2,1}(\bar{I} \times (0, \infty))$ of (3.1)-(3.3). Here

$C^{2,1}$ means twice continuously differentiable in space and once continuously differentiable in time.

Our main goal in this section is to prove several properties of the solution of (3.1)-(3.3).

Theorem 3.1. *Let v^α be the solution of (3.1)-(3.3) with $\alpha \geq 0$.*

(i) (Symmetry). $v^\alpha(\eta, t) = -v^\alpha(-\eta, t)$ for $\eta \in I,\ t \geq 0$. In particular, $v^\alpha(0, t) = 0$ for $t > 0$.

(ii) (Concavity). $v^\alpha(\eta, t) \leq \eta t,\ 0 \leq v_t^\alpha(\eta, t) \leq \eta$ for $\eta \in I_+,\ t \geq 0$ with $I_+ = (0, d/2)$. In particular, $v_{\eta\eta}^\alpha \leq 0$ in $I_+ \times (0, \infty)$.

(iii) (Monotonicity). $v^\alpha \leq v^\beta$ in $I_+ \times (0, \infty)$ if $\alpha \geq \beta \geq 0$. Moreover, $v_\eta^\alpha \leq v_\eta^\beta$ in $I_+ \times (0, \infty)$ if $\alpha \geq \beta \geq 0$.

(iv) (Lower bound). Assume that

$$c_0 := (2c_*)^{-1/2} \quad \text{with} \quad c_* := \frac{d^2}{8} - \int_{-\infty}^{0} a(\tau)\mathrm{d}\tau > 0 \tag{3.5}$$

and

$$c_1 := \int_{-\infty}^{0} |\tau| a(\tau)\mathrm{d}\tau < \infty. \tag{3.6}$$

Then $v^\alpha(\eta, t) \geq -c_0 c_1$ for $\eta \in [0, d/2], t \geq 0$.

Proof. (i) Since $-v^\alpha(-\eta, t)$ solves (3.1)-(3.3), the uniqueness of a solution yields the symmetry.

(ii) By a standard approximation argument (based on a priori estimates in [18]) we may assume that a is smooth so that v is smooth in $\bar{I} \times (0, \infty)$ and $I \times [0, \infty)$. Clearly, ηt is a supersolution of (3.1)-(3.3) in $I_+ \times (0, \infty)$ with zero boundary condition at $\eta = 0$, so the standard comparison principle (e.g. [20]) yields $v^\alpha \leq \eta t$ in $I_+ \times (0, \infty)$. We differentiate (3.1), (3.2) in t to get

$$\begin{aligned} &w_t = a(v_\eta^\alpha) w_{\eta\eta} + a'(v_\eta^\alpha) a(v_\eta^\alpha)^{-1} w_\eta (w - \eta) \quad \text{in} \quad I \times (0, \infty) \\ &w_\eta(d/2, t) = 0 \quad \text{for} \quad t > 0,\ w(0, t) = 0 \quad \text{for} \quad t > 0 \text{ (by (i))} \end{aligned}$$

for $w = v_t^\alpha$. Since $v^\alpha \leq \eta t$, we observe that $v_t^\alpha \leq \eta$ at $t = 0$. Since η is a supersolution of this w-problem in $I_+ \times (0, \infty)$, the comparison principle implies that $w \leq \eta$ in $[0, d/2] \times [0, \infty)$. The concavity follows from $v_t \leq \eta$ and the equation (3.1) since $a > 0$. Since $w = 0$ is a solution of the w-problem and $v_t \geq 0$ at $t = 0$ in I_+, we conclude that $v_t \geq 0$ for $I_+ \times (0, \infty)$.

(iii) For $\beta \leq \alpha$ the solution v^β is a supersolution of (3.1)-(3.3) in $I_+ \times (0, \infty)$ with $v = 0$ at $\eta = 0$, the comparison principle yields $v^\alpha \leq v^\beta$ in $I_+ \times (0, \infty)$. Since $v^\alpha \leq v^\beta$ and

$v^\alpha = v^\beta = 0$ at $\eta = 0$, we observe that $v^\alpha_\eta \le v^\beta_\eta$ at $\eta = 0$. Since v^β_η solves (3.4) and $v^\alpha_\eta \le v^\beta_\eta$ at $\eta = d/2$, the comparison principle yields $v^\alpha_\eta \le v^\beta_\eta$ in $I_+ \times (0, \infty)$.
(iv) As in the next Lemma 3.2 we shall construct a time independent subsolution $f = f_\alpha$ for (3.1)-(3.3) in $I_+ \times (0, \infty)$ with the zero-boundary condition at $\eta = 0$ such that $f_\alpha \ge -c_0 c_1$. Once such a subsolution is constructed, the comparison principle yields the bound $v^\alpha \ge -c_0 c_1$.

Lemma 3.2. *Assume that (3.5) holds. Then there exists a unique $\sigma \in I_+ \cup \{d/2\}$ with $I_+ = (0, d/2)$ and a C^1 function $f = f_\alpha$ on $\bar{I}_+$ such that*

$$-(A(f'(\eta))' = \eta \quad \text{on} \quad I_+, \tag{3.7}$$

$$f'(d/2) = -\alpha, \ \ f'(\sigma) = f(\sigma) = 0, \tag{3.8}$$

where $A(q) = \int_0^q a(\tau)\mathrm{d}\tau$ and f' denotes the derivative of f. If moreover a satisfies (3.6), then

$$-c_0 c_1 \le \inf\{f_\alpha(\eta); \quad \eta \in [0, d/2], \alpha \ge 0\} = \inf\{f_\alpha(d/2); \alpha \ge 0\} \tag{3.9}$$

(The zero-extension of f_α to $[0, \sigma]$ is still denoted by f_α).

Proof. Integrating (3.7) from σ to η yields

$$-A(f'(\eta)) = (\eta^2 - \sigma^2)/2 \tag{3.10}$$

since $f'(\sigma) = 0$. Since $-A(p) < d^2/8$ for $p \le 0$ by (3.5), there is a unique $\sigma \in I_+ \cup \{d/2\}$ such that

$$-A(-\alpha) = \frac{1}{2}\left(\frac{d}{2}\right)^2 - \frac{\sigma^2}{2}.$$

We fix such a σ and then taking the inverse A^{-1} of (3.10) to get

$$f'(\eta) = A^{-1}((\sigma^2 - \eta^2)/2), \ \ \eta \in [\sigma, d/2]. \tag{3.11}$$

Integrating this with $f(\sigma) = 0$ we obtain the solution f and $\sigma \in I_+ \cup \{d/2\}$ satisfying (3.7) and (3.8).

By (3.11) we have $f'(\eta) \le 0$ in I_+, so $\inf_{I_+} f = f(d/2)$. Thus to prove (3.9) it suffices to prove that

$$\inf_\alpha f_\alpha(d/2) \ge -c_0 c_1. \tag{3.12}$$

Integrating (-1)$\times$(3.11) over $[\sigma, d/2]$ to get

$$\begin{aligned} - \ f_\alpha(d/2) &= -\int_\sigma^{d/2} A^{-1}((\sigma^2 - \eta^2)/2)\mathrm{d}\eta \\ &= -\int_{A(-\alpha)}^0 A^{-1}(\xi)(\sigma^2 - 2\xi)^{-1/2}\mathrm{d}\xi \le -\sigma^{-1}\int_{A(-\infty)}^0 A^{-1}(\xi)\mathrm{d}\xi. \end{aligned}$$

Since

$$-\int_{A(-\infty)}^{0} A^{-1}(\tau)\mathrm{d}\tau = \int_{-\infty}^{0} |\tau| a(\tau)\mathrm{d}\tau = c_1$$

and since $-A(-\infty) > -A(-\alpha) = d^2/8 - \sigma^2/2$ so that $\sigma > (2c_*)^{1/2}$, we now obtain that $-f_\alpha(d/2) \leq c_0 c_1$. $\square$

4 Approximate problems

Let v^α be the solution of (3.1)-(3.3). We define v^∞ by

$$v^\infty(\eta, t) = \inf_{\alpha>0} v^\alpha(\eta, t), \eta \in I_+ = (0, d/2)$$

$$v^\infty(\eta, t) = -v^\infty(-\eta, t), \eta \in (-d/2, 0)$$

$$v^\infty(0, t) = 0,\ t > 0.$$

It we assume (3.5), (3.6), the monotone properties and bounds (Theorem 3.1) v^∞ is well-defined and $\eta \mapsto v^\infty(\eta, t)$ is concave in I_+.

Our goal in this section is to prove the convergence of v^∞ to v defined by (2.7) when $\int^q a$ approximates $M\mathrm{sgn}\ q$.

Theorem 4.1. *Assume that $a = a^\varepsilon \in C^1(\mathbf{R}), a^\varepsilon > 0$ satisfies (3.5) and (3.6) with $\varepsilon > 0$. Assume that c_0^ε, c_1^ε defined by (3.5), (3.6) with $a = a^\varepsilon$ are bounded as $\varepsilon \to 0$. Assume that $A^\varepsilon(q) = \int_0^q a^\varepsilon(\tau)\mathrm{d}\tau$ converges to $M\mathrm{sgn} q + c$ with some constant c as $\varepsilon \to 0$ (as monotone graphs (cf. [7])). Let v_ε^α be the solution of (3.1), (3.2), (3.3) with $a = a^\varepsilon$. Let v_ε^∞ be $v_\varepsilon^\infty = \inf_{\alpha>0} v_\varepsilon^\alpha$ on $I_+ \times (0, \infty)$ and it is extended to an odd function of η for $\eta < 0$. Let v be the function defined in (2.7). Then v_ε^∞ converges to v as $\varepsilon \to 0$ uniformly in every compact subset of $I \times [0, \infty)$.*

We shall prove this result by estimating v_ε^∞ from above by the solution of the homogeneous Neumann problem and from below by that of a nonhomogeneous Dirichlet problem.

4.1 Convergence of the Neumann problem

Proposition 4.2. *Assume that $A^\varepsilon(q) = \int_0^q a^\varepsilon(\tau)\mathrm{d}\tau$ converges to $M\mathrm{sgn}\ q + c$ with some constant c as $\varepsilon \to 0$ (as monotone graphs), where $a^\varepsilon \in C^1(\mathbf{R})$ and $a^\varepsilon > 0$ with $\varepsilon > 0$. Let v_ε^0 be the solution of (3.1)-(3.3) with $\alpha = 0$ and $a = a^\varepsilon$. Then v_ε^0 converges to v (defined by (2.7)) as $\varepsilon \to 0$ uniformly in $\bar{I} \times [0, T]$ for any $T > 0$.*

Proof. We formulate the problem (3.1)-(3.3) by using a subdifferential equation

$$u_t \in -\partial E_N^\varepsilon(u) \quad \text{for} \quad t > 0,$$
$$u|_{t=0} = 0$$

with

$$E_N^\varepsilon(w) := \int_I \{W^\varepsilon(w_\eta) - \eta w\} \mathrm{d}\eta \quad \text{for} \quad w \in H,$$

where $W^\varepsilon(q) \to M|q|$ locally uniformly as $\varepsilon \to 0$ and $W^{\varepsilon''} = a^\varepsilon$. Evidently, the solution u of the above problem equals v_ε^0. By a stability theorem of J. Watanabe [22] based on [2] the solution v_ε^0 converges to the solution u of $u_t \in -\partial E_N(u),\ t > 0$ with $u|_{t=0} = 0$ in $C([0,T], L^2(I))$ for any $T > 0$, where E_N is defined by (2.8). Since the solution u of $u_t \in -\partial E_N(u),\ t > 0$ with $u|_{t=0} = 0$ equals v of (2.7) as in Remark 2.2, $v_\varepsilon^0 \to v$ in $C([0,T], L^2(I))$ as $\varepsilon \to 0$, *i.e.*,

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T} ||v_\varepsilon^0(t) - v(t)||_{L^2(I)} = 0,$$

where $||h||_{L^2(I)}$ denotes the norm of h in $H = L^2(I)$. By Theorem 3.1 $v_\varepsilon^0(\eta, t)$ is convave in $\eta \in I_+$. Moreover, $(v_\varepsilon^0)_\eta(0,t) \le t$ for $t > 0$ since $v(\eta, t) \le \eta t$ in $I_+ \times (0, \infty)$ by Theorem 3.1 and $v(0,t) = 0$ for $t > 0$. Clearly, $(v_\varepsilon^0)_\eta(d/2, t) = 0$. Thus, by Arzelà-Ascoli's theorem we see that $v_{\varepsilon_j}^0(\cdot,\ t_j)$ always contains a uniform convergent subsequence on $\bar{I}$ as $j \to \infty$ if $\varepsilon_j \to 0,\ t_j \in [0,T]$. Since $v_\varepsilon^0 \to v$ in $C([0,T], L^2(I))$, this implies the uniform convergence of v_ε^0 in $\bar{I} \times [0,T]$ as stated in the next lemma whose proof is elementary. □

Lemma 4.3. *Assume that u^ε is uniformly bounded in $\bar{\Omega} \times [0,T]$ for $\varepsilon \in (0,1)$. Assume that $u^\varepsilon \to u$ in $C([0,T], L^2(\Omega))$ as $\varepsilon \to 0$, where Ω is an open set in $\mathbf{R}^d$. Assume that $\{u^{\varepsilon_j}(\cdot, t_j)\}$ has a uniform convergent subsequence in $\bar{\Omega}$ provided that $\varepsilon_j \to 0,\ t_j \in [0,T]$. Then $u^\varepsilon \to u$ uniformly in $\bar{\Omega} \times [0,T]$ as $\varepsilon \to 0$.*

Proof of Lemma 4.3. If not, there would exist $t_j \in [0,T]$ and $\varepsilon_j \to 0$ such that

$$\sup_{x \in \bar{\Omega}} |u^{\varepsilon_j}(x, t_j) - u(x, t_j)| \ge \delta$$

with some $\delta > 0$. By the assumption $\{u^{\varepsilon_j}(\cdot, t_j)\}$ has a uniform convergent subsequence still denoted by $\{u^{\varepsilon_j}(\cdot, t_j)\}$. We may assume that $t_j \to \hat{t} \in [0,T]$. Since $u^\varepsilon \to u$ in $C([0,T]; L^2(\Omega))$ so that $\{u^{\varepsilon_j}(\cdot, t_j)\}$ converges to $u(\cdot, \hat{t})$ in $L^2(\Omega)$, $u^{\varepsilon_j}(\cdot,\ t_j) \to u(\cdot,\ \hat{t})$ uniformly in $\bar{\Omega}$. This would contradict $\delta > 0$. □

4.2 Dirichlet problem

We consider the Dirichlet problem for (3.1), (3.3) with $a = a^\varepsilon$ with the boundary condition

$$v(\pm d/2, t) = \mp R, \tag{4.1}$$

where R is a positive constant. Let $v_{R\varepsilon}$ be the solution of (3.1), (3.3) with (4.1). The solution may not satisfy (4.1). It can be understood as the limit of a uniformly parabolic problem which approximates (3.1), (3.3) and (4.1). Since we may assume that $v_{R\varepsilon}$ is a solution of an approximate uniform parabolic problem so that it is smooth in $\bar{I} \times (0, \infty)$ and $I \times [0, \infty)$, we argue in the same way as in the proof of Theorem 3.1 and conclude that $(v_{R\varepsilon})_{\eta\eta} \leq 0$ in $I_+ \times (0, \infty)$.

Proposition 4.4. *Assume the same hypotheses of Proposition 4.2 concerning a^ε. Let $v_{R\varepsilon}$ be the solution of (3.1), (3.3) and (4.1) with $a = a^\varepsilon$. Then $v_{R\varepsilon} \to v$ as $\varepsilon \to 0$ uniformly in each compact subset of $I \times [0, \infty)$, where v is defined by (2.7).*

Proof. As in the proof of Proposition 4.2 we observe that $v_{R\varepsilon} \to v$ in $C([0, T], L^2(I))$ for any $T > 0$. Again $v_{R\varepsilon}$ is concave in $\eta \in I_+$ and $(v_{R\varepsilon})_\eta(0, t) \leq t$ for $t > 0$. However, there is no control on $(v_{R\varepsilon})_\eta(d/2, t)$. All we expect is that $v_{R\varepsilon}$ is uniformly bounded in $I_+ \times [0, T]$ (in fact $v_{R\varepsilon} \geq -R$ there) and $v_{R\varepsilon}$ is concave in $\eta \in I_+$. From these facts we observe that

$$(v_{R\varepsilon})_\eta(\eta, t) \geq -C \quad \text{in} \quad [0, d/2 - \delta] \times [0, T]$$

for each $\delta > 0$ with some constant C. By Arzelà-Ascoli's theorem we see that $v_{R\varepsilon_j}(\cdot,\ t_j)$ has a uniform convergent subsequence in $[0, d/2 - \delta]$ for each $\delta > 0$ if $t_j \in [0, T]$ and $\varepsilon_j \to 0$. By Lemma 4.3 we now conclude that $v_{R\varepsilon} \to v$ in each compact subset of $I \times [0, T]$. Since $T > 0$ is arbitrary, the proof is now complete. □

Proof of Theorem 4.1. By Theorem 3.1 (iii) we see that $v_\varepsilon^\infty \leq v_\varepsilon^0$ in $I_+ \times (0, \infty)$. We take $R \geq c_0^\varepsilon c_1^\varepsilon$ for small $\varepsilon > 0$. Then by the comparison for the Dirichlet problem

$$v_{R\varepsilon} \leq v_\varepsilon^\alpha \quad \text{in} \quad I_+ \times (0, \infty),$$

since $v_{R\varepsilon} = v_\varepsilon^\alpha = 0$ at $\eta = 0$. This implies

$$v_{R\varepsilon} \leq v_\varepsilon^\infty \quad \text{in} \quad I_+ \times (0, \infty).$$

The convergence results (Propositions 4.2, 4.4) yield the convergence $v_\varepsilon^\infty \to v$. □

5 Level set solutions

We consider the level set equation of the form

$$\psi_t + y\psi_x = M|\nabla\psi| \mathrm{div}\{\nabla\gamma(-\nabla\psi/|\nabla\psi|)\} \quad \text{in} \quad \mathbf{R}^2 \times (0, \infty). \tag{5.1}$$

Here γ is a convex, positively homogeneous of degree one in $\mathbf{R}^2$. If $M = 0$, the zero level set

$$\{\psi = 0\} := \{(x, y, t)(\in \mathbf{R} \times \mathbf{R} \times [0, \infty)); \psi(x, y, t) = 0\}$$

formally represents the graph of a solution of the Burgers equation for $u = u(x, t)$:

$$u_t + uu_x = 0 \quad \text{in} \quad \mathbf{R} \times (0, \infty).$$

We shall use the convention that $\psi > 0$ below the graph of u. By the standard theory of the level set equation for each initial data $\psi_0 \in \mathrm{BUC}(\mathbf{R}^2)$ there is a unique viscosity solution $\psi \in \mathrm{BUC}(\mathbf{R}^2 \times [0,T])$ for any $T > 0$ of (5.1) satisfying $\psi(x,y,0) = \psi_0(x,y)$ provided that $\gamma \in C^2(\mathbf{R}^2 \setminus \{0\})$; see [12], [14], [17]. Let D_0 be an open set of the form

$$D_0 := \{(x,y); y < -d/2\} \cup \{(x,y); x < 0, y < d/2\}.$$

We consider the initial data ψ_0 satisfying

$$\{(x,y)(\in \mathbf{R} \times \mathbf{R}); \psi_0(x,y) > 0\} = D_0$$

and call the set

$$D = \{\psi > 0\} := \{(x,y,t)(\in \mathbf{R} \times \mathbf{R} \times [0,\infty)); \psi(x,y,t) > 0\}$$

the *level set solution* (of (5.1)) with the initial data D_0. The set D is independent of the choice of ψ_0 and is uniquely determined by D_0.

Our main goal is to show that if $M < d^2/16$, then for a large class of γ such that $\nabla\gamma(-\nabla\psi/|\nabla\psi|)$ approximating $\psi_y/|\psi_y|$, the limit of D develop 'overturning'.

Lemma 5.1. *Let $\gamma \in C^2(\mathbf{R}^2 \setminus \{0\})$ be convex and positively homogeneous of degree one. Then*

$$\nabla^2\gamma(0,1) = 0$$

if and only if $|q|^3 W''(q) \to 0$ as $q \to -\infty$ for $W(q) = \gamma(1,-q)$.

Proof. By definition

$$\gamma_2(1,-q) = -W'(q) \quad \text{and} \quad \gamma_{22}(1,-q) = W''(q),$$

where $\gamma_i = \partial\gamma/\partial p_i$, $\gamma_{ij} = \partial^2\gamma/\partial p_i \partial p_j$. Since γ_i is positively homogeneous of degree zero, *i.e.*, $\gamma_i(\lambda p, \lambda q) = \gamma_i(p,q)$ for $\lambda > 0$ differentiating in λ we have

$$\gamma_{12}(1,-q) - q\gamma_{22}(1,-q) = 0,$$

$$\gamma_{11}(1,-q) - q\gamma_{12}(1,-q) = 0.$$

Thus

$$\gamma_{11}(1,-q) = q^2 W''(q), \quad \gamma_{12}(1,-q) = qW''(q).$$

Since γ_{ij} is positively homogeneous of degree -1,

$$\gamma_{ij}(1/(1+q^2)^{1/2}, -q/(1+q^2)^{1/2}) = (1+q^2)^{1/2}\gamma_{ij}(1,-q) \to \gamma_{ij}(0,1)$$

as $q \to -\infty$. Thus $|q|^3 W''(q) \to 0$ as $q \to -\infty$ is equivalent to $\gamma_{ij}(0,1) = 0$ for all $i,j \in \{1,2\}$. $\square$

The next lemma relates the level set solution D and a solution of (3.1), (3,3).

Lemma 5.2. *Let $\gamma \in C^3(\mathbf{R}^2\setminus\{0\})$ be convex and positively homogeneous of degree one. Assume that $|q|^3 W''(q) \to 0$ as $q \to -\infty$ for $W(q) = \gamma(1, -q)$. Assume that $W''(q) > 0$. For $a(q) = M(1+q^2)^{1/2} W''(q)$ assume (3.5) and (3.6). Let v^α be the solution of (3.1)-(3.3). Let v^∞ be as in the begining of section 4. Let D be the level set solution with initial data D_0. Then*

$$D = \{(x,y,t); y < -d/2\} \cup \{(x,y,t); x < v^\infty(y,t),\ -d/2 \le y < d/2\}. \tag{5.2}$$

Proof. Step 1. *The function v^∞ is a solution of (3.1), (3.3) and $v^\infty \in C^{2,1}(I \times [0,\infty)) \cap C(\bar{I} \times [0,\infty))$. Moreover,*

$$v^\infty_\eta(\eta, t) \to -\infty \quad \text{as} \quad \eta \uparrow d/2 \quad \text{for} \quad t > 0. \tag{5.3}$$

Indeed, by Theorem 3.1 $\{v^\alpha\}_{\alpha>0}$ is uniformly bounded in $I \times [0,T]$ for any $T > 0$ and v^α is concave in $\eta \in I_+$. Moreover, $v^\alpha \downarrow v^\infty$ by Theorem 3.1. Thus, if we note that

$$0 \le v^\alpha_t(\eta, t) \le \eta \quad \text{for} \quad \eta \in I_+$$

by Theorem 3.1, we see that $v^\infty \in C(\bar{I} \times [0,T])$. By Dini's theorem $v^\alpha \to v^\infty$ uniformly in $\bar{I} \times [0,T]$ as $\alpha \to \infty$ since $v^\alpha \downarrow v^\infty$ by Theorem 3.1. Moreover, $v^\alpha_\eta \downarrow v^\infty_\eta$ $a, e.$ in $I_+ \times (0,\infty)$ by Theorem 3.1. This implies $A(v^\alpha_\eta) \to A(v^\infty_\eta)$ in $L^1_{\mathrm{loc}}\,(I \times (0,\infty))$, where A is a primitive of $a > 0$ so that A is monotone increasing. From these results v^∞ solves (3.1) in $I \times (0,\infty)$ at least in a weak sense but the standard regularity theory [18] implies that v^∞ is a classical solution of (3.1) in $I \times (0,\infty)$ satisfying (3.3) and $v^\infty \in C^{2,1}(I \times [0,\infty))$. Since $v^\alpha_\eta \le v^\beta_\eta$ in $I_+ \times (0,\infty)$ for $\alpha \ge \beta \ge 0$ by Theorem 3.1, the property (5.3) follows.

Step 2. *The right hand side (denoted $\tilde{D}$) of (5.2) is a solution of (5.1) in the sense that the characteristic function $\chi_{\tilde{D}}$ of $\tilde{D}$ solves (5.1) in the viscosity sense.*

Indeed, as in [12, Theorem 5.1.2] to prove that $\chi_{\tilde{D}}$ is a viscosity subsolution it suffices to test an evolving smooth curve to $\partial\bar{\tilde{D}}$ if $\tilde{D}$ is left accessible (in time). Since $v^\infty \in C(\bar{I} \times [0,\infty))$, the left accessibility is clear. Assume that a smoothly evolving curve $\{S_t\}$ around $(x_0, y_0, t_0) \in \partial\tilde{D}$ has only intersection with $\tilde{D}$ at $(x_0, y_0) \in (\partial\tilde{D})(t_0)$ around (x_0, y_0, t_0). If $\{S_t\}$ is written as $x = h(y,t)$ near (x_0, y_0, t_0), then, by (5.3), $|y_0| < d/2$. We observe that h must satisfy

$$h_t \le a(h_y) h_{yy} + yt \quad \text{at} \quad (y_0, t_0) \tag{5.4}$$

since v^∞ solves (3.1), which is the graph version of (5.1) for a function of y *i.e.*, the equation of $w = w(y,t)$ if $\psi(x,y,t) = w(y,t) - x$ satisfies (5.1). If $\{S_t\}$ is written as

$y = k(x,t)$ near (x_0, y_0, t_0), then $y_0 = \pm d/2$. Since the argument for $y_0 = -d/2$ is easir, we shall study the case $y_0 = d/2$. If $x_0 < v^\infty(d/2, t)$, it is clear that $k_t = k_x = 0$ at (x_0, t_0) and $k_{xx}(x_0, t_0) \geq 0$ so k satisfies

$$k_t + kk_x - b(k_x)k_{xx} \leq 0 \quad \text{at} \quad (x_0, t_0), \tag{5.5}$$

where $b(p) = M(1 + |p|^2)^{1/2} Z''(p)$ with $Z(p) = \gamma(-p, 1)$ for $p \in \mathbf{R}$. The equation

$$u_t + uu_x - b(u_x)u_{xx} = 0$$

is the graph version of (5.1) for a function of x, *i.e.*, the equation of $u = u(x,t)$ if $\psi(x,y,t) = u(x,t) - y$ satisfies (5.1). If $x_0 = v^\infty(d/2, t)$, we still have $k_x(x_0, t_0) = 0$ by (5.3) and $k_t(x_0, t_0) = 0$. We do not expect that $k_{xx}(x_0, t_0) \leq 0$. However, thanks to Lemma 5.1 we have $b(k_x(x_0, t_0)) = 0$ so we still have (5.5). We now apply [12, Theorem 5.1.2] with (5.3) and (5.4) to $\tilde{D}$ to conclude that $\chi_{\tilde{D}}$ is a viscosity subsolution of (5.1). The proof for viscosity supersolution is similar so is omitted.

Step 3. We shall prove that solutions of (5.1) with initial data D_0 is unique so that $\tilde{D}$ is the level set solution starting from D_0. In other works we shall prove that the fattening phenomena does not occur for D_0 or that the level set solution D is regular [10], [12].

For $\delta \in \mathbf{R}$ we set

$$D^\delta = \{(x,y,t); y < -d/2 + \delta\} \cup \{(x,y,t); x < v^\infty(y,t) + \delta t + \delta, -d/2 + \delta < y < d/2 + \delta\}.$$

As in Steps 1,2 it is not difficult to see that χ_{D^δ} is a solution of (5.1) with initial data $\chi_{{D_0}^\delta}$ with

$$D_0^\delta = \{(x,y); y < -d/2 + \delta\} \cup \{(x,y); x < \delta, -d/2 + \delta < y < d/2 + \delta\}.$$

Let E be the closed level set solution with initial data $\bar{D}_0$. Applying the comparison principle to χ_E and χ_{D^δ} [12, Theorem 3.1.4] for $\delta > 0$ to get $E \subset D^\delta$ since dist $(\bar{D}_0, (D_0^\delta)^c) > 0$. Since $\bar{\tilde{D}} = \cap_{\delta > 0} D^\delta$ we see that $E \subset \bar{\tilde{D}}$. Similarly, we observe that $\cup_{\delta < 0} \overline{D^\delta} \subset D$ to get $\tilde{D} \subset D$. We thus conclude that

$$\tilde{D} = D \quad \text{and} \quad \bar{\tilde{D}} = E. \ \Box$$

As an application of Theorem 4.1 with Lemma 5.2 we have a convergence result.

Theorem 5.3. *Let γ_ε fulfill the assumption of γ in Lemma 5.2 with $W^\varepsilon(q) = \gamma_\varepsilon(1, -q)$. for $\varepsilon \in (0,1)$ Assume that $W^{\varepsilon'}(q) \to \operatorname{sgn} q + c$ with some constant c as $\varepsilon \to 0$ in the sense of monotone graphs. Let D^ε be the level set solution of (5.1) with $\gamma = \gamma_\varepsilon$ starting from D_0. Assume that there is $r > 0$ such that*

$$\int_{-\infty}^{0} (1 + q^2)^{1/2} \, W^{\varepsilon''}(q) \mathrm{d}q \leq r \quad \text{for small} \quad \varepsilon \tag{5.6}$$

and

$$\sup_{0<\varepsilon<1} \int_{-\infty}^{0} |q|(1+q^2)^{1/2}\ W^{\varepsilon''}(q)\mathrm{d}q < \infty. \tag{5.7}$$

Then $\overline{D^\varepsilon}$ converges to

$$E = \{(x,y,t); y \le -d/2\} \cup \{(x,y,t); x < v(y,t), -d/2 \le y \le d/2\} \tag{5.8}$$

in the sense of Hausdorff distance topology in $\mathbf{R}^2 \times [0,T]$ for any $T > 0$ provided that $Mr < d^2/8$. Here v is defined by (2.7).

Remark. The assumptions (5.6) and (5.7) together with $Mr < d^2/8$ guarantee (3.5), (3.6) and moreover, imply a uniform bound for c_0^ε, c_1^ε for small $\varepsilon > 0$, when $a^\varepsilon(q) = M(1+q^2)^{1/2} W^{\varepsilon''}(q)$. So Theorem 4.1 is applicable.

Example. If $W^\varepsilon(q) = \int_0^q \tanh\,(\tau/\varepsilon)\mathrm{d}\tau + c$, then it is not difficult to find a γ_ε satisfying $W^\varepsilon(q) = \gamma(1,-q)$ and assumptions of Lemma 5.2. Moreover,

$$\int_{-\infty}^{0} (1+q^2)^{1/2}\ W^{\varepsilon''}(q)\mathrm{d}q \to 1,$$

so for each $\delta > 0$, there is $\varepsilon_0 > 0$ such that

$$\int_{-\infty}^{0} (1+q^2)^{1/2}\ W^{\varepsilon''}(q)\mathrm{d}q \le 1+\delta \quad \text{for} \quad \varepsilon \in (0,\varepsilon_0).$$

The condition

$$\sup_{0<\varepsilon<1} \int_{-\infty}^{0} |q|(1+q^2)^{1/2}\ W^{\varepsilon''}(q)\mathrm{d}q < \infty$$

is evidently fulfilled. Thus, the convergence results (Theorem 5.3) holds for $M(1+\delta) < d^2/8$. If $\delta > 0$ is taken small so that $(1+\delta)/16 < 8$, then we have a threshold value $M = d^2/16$ such that if $M < d^2/16$, then E experiences ‘overturning’ in the sense that there is a point (x_0, y_0, t_0) and (x_0, y_1, t_0) satisfying $y_1 < y_0$ such that

$$(x_0, y_0, t_0) \in E \quad \text{while} \quad (x_0, y_1, t_0) \notin E.$$

If $M \ge d^2/16$, $E = \bar{D}_0 \times [0,\infty)$ so no overturning occurs.

From this example we have:

Corollary 5.4. *Assume that $M < d^2/8$. Then, there exists a sequence of convex function $\gamma_\varepsilon = \gamma_\varepsilon(p,q)$ converging to $M|q|$ locally uniformly in $\mathbf{R}^2$ as $\varepsilon \to 0$ satisfying following properties.*

(i) Let D^ε be the level set solution of (5.1) starting from D_0. Then $\overline{D^\varepsilon}$ converges to E (defind by (5.8)) in the Hausdorff distance topology in $\mathbf{R}^2 \times [0,T]$ for any $T > 0$.

(ii) $\gamma_\varepsilon \in C^\infty(\mathbf{R}^2 \setminus \{0\})$ and γ_ε is positively homogeneous of degree one.

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