CURVES AND SURFACES IN HYPERBOLIC SPACE

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Abstract. In the first part (§2, §3), we give a survey of the recent results on application of singularity theory for curves and surfaces in Hyperbolic space. After that we define the hyperbolic canal surface of a hyperbolic space curve and apply the results of the first part to get some geometric relations between the hyperbolic canal surface and the center curve.

1. Introduction In [4, 5, 6] we have applied singularity theory to local differential geometry on curves and hypersurfaces in Hyperbolic space. For hypersurfaces, we have the notion of hyperbolic Gauss maps originally introduced by Epstein [3]. The original definition of hyperbolic Gauss maps has been given in the Poincaré ball model of Hyperbolic space. It is, however, very hard to proceed the calculation because it has been given in the intrinsic form. In [5] we adopted the model of Hyperbolic space in Minkowski space. Then the target of hyperbolic Gauss maps is the unit sphere in the lightcone. Moreover, we have introduced the notion of hyperbolic Gauss indicatrices which are (singular) hypersurfaces in the lightcone. Hyperbolic Gauss indicatrices are much easier to

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calculate comparing with hyperbolic Gauss maps and contain a lot of geometric information of hypersurfaces. For example, we have shown the singularities of hyperbolic Gauss indicatrices describe the contact between hypersurfaces and horospheres.

In [6] we have considered curves in Hyperbolic space and define the notion of horospherical surfaces of curves which are located in the lightcone. The singularities of horospherical surfaces describe the contact between curves and hyperhorospheres.

In both papers [5, 6] we have introduced the notion of horospherical height functions on curves (or, hypersurfaces) as basic tools for the study of those subjects. We have applied the singularity theory for families of function germs to such functions and studied the contact between curves (or, hypersurfaces) and horospheres. In §2 and §3, we give a survey of the results in [5, 6]. In §4 we study horospherical surfaces as an application of the theory of Legendrian singularities and show that the horospherical surface can be considered as a wavefront. In [5] we have shown that the hyperbolic indicatrix of a hypersurface can be also considered as a wavefront. We show that the Legendrian lift of the horospherical surface of a curve and the Legendrian lift of the hyperbolic Gauss indicatrix of the corresponding hyperbolic canal surface are Legendrian equivalent. In §5 we apply the results in §2, §3 and §4 to hyperbolic space curves and show that the contact between hyperbolic space curves and horospheres corresponds to the contact between hyperbolic canal surfaces and horospheres (cf., Corollary 5.3, Theorems 5.6 and 5.7). We give a quick survey on the theory of Legendrian singularities in §6 as Appendix which are used in §4 and §5.

All maps considered here are of class C^∞ unless otherwise stated.

2. Horospherical surfaces of curves in Hyperbolic space In this section we give a survey on the explicit differential geometry for curves in $H^3_+(-1)$ due to [6].

We start to describe basic notions of Hyperbolic 3-space. Here we adopt the model of Hyperbolic 3-space in Minkowski space. Let $\mathbb{R}^4$ be a 4-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i.$$

We call $(\mathbb{R}^4, \langle , \rangle)$ *Minkowski space* and denote $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{ \boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *Hyperbolic 3-space* by

$$H^3_+(-1) = \{ \boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1 \}.$$

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\mathbb{R}^4_1$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}\ \boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$.

We also define a set $LC_a = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\ \}$, which is called *a closed lightcone* with the vertex $\boldsymbol{a}$. We denote that

$$LC^*_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in LC_0 \mid x_0 > 0\ \}$$

and we call it *the future lightcone* at the origin. We have three kinds of totally umbilic surfaces in $H^3_+(-1)$ which are given by intersections of $H^3_+(-1)$ and hyperplanes in $\mathbb{R}^4_1$. A surface $H^3_+(-1) \cap HP(\boldsymbol{v}, c)$ is called a *sphere*, a *equidistant plane* or a *horosphere* if $HP(\boldsymbol{v}, c)$ is spacelike, timelike or lightlike respectively. Especially we write a horosphere as $HS^2(\boldsymbol{v}, c) = H^3_+(-1) \cap HP(\boldsymbol{v}, c)$. If we consider a lightlike vector $\boldsymbol{v}_0 = (-1/c)\boldsymbol{v}$, we have $HS^2(\boldsymbol{v}, c) = HS^2(\boldsymbol{v}_0, -1)$. We call $\boldsymbol{v}_0$ the *polar vector* of $HS^2(\boldsymbol{v}_0, -1)$.

Let $\gamma : I \longrightarrow H^3_+(-1)$ be a regular curve. Since $H^3_+(-1)$ is a Riemannian manifold, we can reparametrise γ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So we have the tangent vector $\boldsymbol{t}(s) = \gamma'(s)$ with $\|\boldsymbol{t}(s)\| = 1$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we have a unit vector $\boldsymbol{n}(s) = \dfrac{\boldsymbol{t}'(s) - \gamma(s)}{\|\boldsymbol{t}'(s) - \gamma(s)\|}$. Moreover, define $\boldsymbol{e}(s) = \gamma(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_1$ along γ. By standard arguments, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we have the following *Frenet-Serre type formulae*:

$$\begin{cases} \gamma'(s) &= \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) &= \kappa_h(s)\boldsymbol{n}(s) + \gamma(s) \\ \boldsymbol{n}'(s) &= -\kappa_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) &= -\tau_h(s)\boldsymbol{n}(s), \end{cases}$$

where $\kappa_h(s) = \|\boldsymbol{t}'(s) - \gamma(s)\|$ and $\tau_h(s) = -\dfrac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_h(s))^2}$.

We can easily show that the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$ is equivalent to the condition $\kappa_h(s) \neq 0$. Moreover, we can show that the curve $\gamma(s)$ satisfies the condition $\kappa_h(s) \equiv 0$ if and only if there exists a lightlike vector $\boldsymbol{c}$ such that $\gamma(s) - \boldsymbol{c}$ is a geodesic. Such a curve is called *an equidistant line.* We can study many properties of hyperbolic space curves by using this fundamental equation.

Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed curve. We now define a map

$$HS_\gamma : I \times J \longrightarrow LC^*_+$$

by $HS_\gamma(s, \theta) = \gamma(s) + \cos\theta \boldsymbol{n}(s) + \sin\theta \boldsymbol{e}(s)$. We call HS_γ the *horospherical surface* of γ. We also introduce a hyperbolic invariant

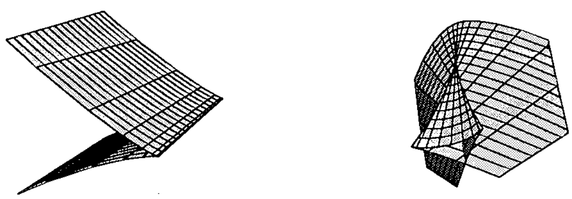
$$\sigma_h(s) = ((\kappa'_h)^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1))(s).$$

In [6] we have shown the following theorem:

THEOREM 2.1. *Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following:*

(1) *The horospherical surface HS_γ of γ is singular at (s_0, θ_0) if and only if* $\cos\theta_0 = 1/\kappa_h(s_0)$.
(2) *The horospherical surface HS_γ of γ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if* $\cos\theta_0 = 1/\kappa_h(s_0)$ *and* $\sigma_h(s_0) \neq 0$.
(3) *The horospherical surface HS_γ of γ is locally diffeomorphic to the swallowtail SW at (s_0, θ_0) if* $\cos\theta_0 = 1/\kappa_h(s_0)$, $\sigma_h(s_0) = 0$ *and* $\sigma_h'(s_0) \neq 0$.

Here, $C = \{(x_1, x_2) | x_1{}^2 = x_2{}^3\}$ *is the ordinary cusp and* $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *is the swallowtail (cf., Fig.1).*



cuspidaledge swallowtail

Fig. 1.

By using a kind of transversality theorem, we have shown the following genericity theorem:

THEOREM 2.2. *There exists an open and dense subset $\mathcal{O} \subset$* $\mathrm{Emb}\,(I, H^3_+(-1))$ *such that for any $\gamma \in \mathcal{O}$, the horospherical surface HS_γ of γ is locally diffeomorphic to the cuspidal edge or the swallowtail at any singular point.*

Here, $\mathrm{Emb}\,(I, H^3_+(-1))$ *is the space of embeddings $\gamma : I \longrightarrow H^3_+(-1)$ equipped with Whitney C^∞-topology.*

We now consider the geometric meaning of the invariant $\sigma_h(s)$. Let $\boldsymbol{v}$ be a lightlike vector and $\boldsymbol{w}$ be a spacelike vector. A hyperbolic space curve given by $HS^2(\boldsymbol{v}, -1) \cap HP(\boldsymbol{w}, 0)$ is called a *horocycle*. We have shown the following proposition.

PROPOSITION 2.3. *Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \geq 1$. We consider the vector field along γ given by* $\boldsymbol{v}(s) = \gamma(s) + \cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{e}(s)$ *with* $\cos\theta = 1/\kappa_h(s)$.
(1) *Suppose that $\kappa_h(s) \equiv 1$. Then the following conditions are equivalent:*
 (a) *$\boldsymbol{v}(s)$ is a constant vector.*
 (b) $\tau_h(s) \equiv 0$.
 (c) *γ is a part of horocycle.*
(2) *Suppose that the set $\{s \in I \mid \kappa_h(s) = 1\}$ consists of isolated points. Then the following conditions are equivalent:*
 (a) *$\boldsymbol{v}(s)$ is a constant vector.*
 (b) $\sigma_h(s) \equiv 0$.
 (c) *γ is located on a horosphere.*

Let $F : H^3_+(-1) \longrightarrow \mathbb{R}$ be a submersion and $\gamma : I \longrightarrow H^3_+(-1)$ be a regular curve. We say that γ and $F^{-1}(0)$ have *at least k-point contact* for $t = t_0$ if the function $g(t) = F\circ\gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$. If γ and $F^{-1}(0)$ have at least k-point contact for $t = t_0$ and satisfy the condition that $g^{(k)}(t_0) \neq 0$, then we say that γ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$. If a horosphere $HS^2(\boldsymbol{v}_0, -1)$ and a hyperbolic space curve γ have at least 3-point contact for a point t_0, we call $HS^2(\boldsymbol{v}_0, -1)$ the *osculating horosphere of* γ at $\gamma(t_0)$. Then we have shown the following proposition.

PROPOSITION 2.4. *Let* $\gamma : I \longrightarrow H^3_+(-1)$ *be a unit speed hyperbolic space curve. Then we have the following:*
(1) *There exists the osculating horosphere of* γ *at a point* $\gamma(s_0)$ *if and only if* $\kappa_h(s_0) \geq 1$.
(2) *Suppose that* $\kappa_h(s_0) \geq 1$. *Then the osculating horosphere and* γ *have* 4*-point contact for* $s = s_0$ *if and only if* $\sigma_h(s_0) = 0$ *and* $\sigma'_h(s_0) \neq 0$.

By Theorem 2.1, the set of singular points of the horospherical surface of γ is the locus the polar vectors of osculating horospheres of γ. Moreover, the swallowtail of the horospherical surface of γ corresponds to the point $\gamma(s_0)$ at where the osculating horosphere and γ have 4-point contact.

On the other hand, we consider the horocycle $HS^2(\boldsymbol{v}(s_0), -1) \cap \langle \gamma(s_0), \boldsymbol{t}(s_0), \boldsymbol{n}(s_0)\rangle_{\mathbb{R}}$ at a point $s_0 \in I$ with $\kappa_h(s_0) \geq 1$. We call it the *osculating horocycle* of γ at $\gamma(s_0)$. The assertion (1) of Proposition 2.4, suggests that two invariants $\kappa_h(s_0)$ and $\tau_h(s_0)$ describe the contact between curves and horocycle. We do not, however, proceed to study about this topics here.

3. Hyperbolic Gauss indicatrices of surfaces In this section we give a survey on the explicit differential geometry on surfaces in $H^3_+(-1)$ due to our previous paper [5]. Let

$$\boldsymbol{x} : U \longrightarrow H^3_+(-1)$$

be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M with U by the embedding $\boldsymbol{x}$. Define a vector

$$\mathbb{E}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)\|},$$

then we have

$$\langle \boldsymbol{e}, \boldsymbol{x}_{u_i}\rangle \equiv \langle \boldsymbol{e}, \boldsymbol{x}\rangle \equiv 0, \quad \langle \boldsymbol{e}, \boldsymbol{e}\rangle \equiv 1.$$

Since $\boldsymbol{x}(u) \in H^3_+(-1)$ and $\langle \mathbb{E}(u), \mathbb{E}(u)\rangle = 1$ we can show that $\boldsymbol{x}(u) \pm \mathbb{E}(u) \in LC^*_+$. We define a map

$$\mathbb{L}^{\pm} : U \longrightarrow LC^*_+$$

by $\mathbb{L}^{\pm}(u) = \boldsymbol{x}(u) \pm \mathbb{E}(u)$ which is called the *hyperbolic Gauss indicatrix* (or the *lightcone dual*) of $\boldsymbol{x}$.

We have shown that $D_{\boldsymbol{v}}\mathbb{L}^{\pm} \in T_pM$ for any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, where $D_{\boldsymbol{v}}$ denotes the *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$.

We also have shown that the surface $\boldsymbol{x}(U) = M$ is a part of a horosphere if and only if the hyperbolic Gauss indicatrix $\mathbb{L}^{\pm}$ is constant. In Euclidean differential geometry, if the Gauss map of a surface is constant, then the surface is a part of a hyperplane. Therefore, we regard horospheres in our theory like as planes in Euclidean differential geometry.

In [5], we have established the "horospherical geometry" as an application of singularity theory.

Under the identification of U and M, the derivative $d\boldsymbol{x}(u_0)$ can be identified with the identity mapping id_{T_pM} on the tangent space T_pM, where $p = \boldsymbol{x}(u_0)$. This means that

$$d\mathbb{L}^{\pm}(u_0) = id_{T_pM} \pm d\mathbb{E}(u_0).$$

We call the linear transformation $S_p^{\pm} = -d\mathbb{L}(u_0) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We denote the eigenvalue of $S_p^{\pm}$ by $\bar{\kappa}_p^{\pm}$ and the eigenvalue of $-d\mathbb{E}(u_0)$ by κ_p, By the relation $S_p^{\pm} = -id_{T_pM} \mp d\mathbb{E}(u_0)$, $S_p^{\pm}$ and $-d\mathbb{E}(u_0)$ have same eigenvectors and we have a relation that $\bar{\kappa}_p^{\pm} = -1 \pm \kappa_p$.

The *hyperbolic Gauss curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$K_h^{\pm}(u_0) = \det S_p^{\pm}.$$

We have shown that the following explicit expression of the hyperbolic Gauss curvature by Riemannian metric and the hyperbolic second fundamental invariant:

$$K_h^{\pm} = \frac{\det\left(\bar{h}_{ij}^{\pm}\right)}{\det\left(g_{\alpha\beta}\right)},$$

where we have Riemannian metric (the *hyperbolic first fundamental form*) $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ and the *hyperbolic second fundamental invariant*

$$\bar{h}_{ij}^{\pm}(u) = \langle -\mathbb{L}_{u_i}^{\pm}(u), \boldsymbol{x}_{u_j}(u)\rangle$$

for any $u \in U$.

We say that a point $p = \boldsymbol{x}(u_0)$ is a (*positive or negative*) *horospherical parabolic point* (or, briefly a *$H^{\pm}$-parabolic point*) of $\boldsymbol{x} : U \longrightarrow H_+^n(-1)$ if $K_h^{\pm}(u_0) = 0$. We have shown the following results:

THEOREM 3.1. *There exists an open dense subset $\mathcal{O} \subset$ Emb$(U, H_+^3(-1))$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the following conditions hold:*

(1) *The $H^{\pm}$-parabolic set $K_h^{-1}(0)$ is a regular curve. We call such a curve the $H^{\pm}$-parabolic curve.*

(2) *The hyperbolic Gauss indicatrix $\mathbb{L}^{\pm}$ along the $H^{\pm}$-parabolic curve is a cuspidal edge except at isolated points. At such isolated points, $\mathbb{L}^{\pm}$ is the swallowtail.*

PROPOSITION 3.2. *Let $\mathcal{O} \subset$ Emb$(U, H_+^3(-1))$ be the same open dense subset as in Theorem 3.1. For any $\boldsymbol{x} \in \mathcal{O}$, the followings hold:*

(1) *An $H^{\pm}$-parabolic point $u_0 \in U$ is a fold of the hyperbolic Gauss map if and only if it is a cuspidal edge of the hyperbolic Gauss indicatrix.*

(2) *An $H^{\pm}$-parabolic point $u_0 \in U$ is a cusp of the hyperbolic Gauss map if and only if it is a swallowtail of the hyperbolic Gauss indicatrix.*

Here, a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^2, \boldsymbol{b})$ is called a fold if it is $\mathcal{A}$-equivalent to the germ (u_1, u_2^2) and a cusp if it is $\mathcal{A}$-equivalent to the germ $(u_1, u_2^3 + u_1u_2)$. We say that two map germs $f_i : (\mathbb{R}^n, \boldsymbol{a}_i) \longrightarrow (\mathbb{R}^p, \boldsymbol{b}_i)$ $(i = 1, 2)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (\mathbb{R}^n, \boldsymbol{a}_1) \longrightarrow (\mathbb{R}, \boldsymbol{a}_2)$ and $\psi : (\mathbb{R}^p, \boldsymbol{b}_1) \longrightarrow (\mathbb{R}^p, \boldsymbol{b}_2)$ such that $f_2 \circ \phi = \psi \circ f_1$.

The basic tool for the proof of the above results is also the *horospherical height function* of a surface $\boldsymbol{x}$. We define a function $\mathcal{H} : U \times LC_+^* \longrightarrow \mathbb{R}$ by $\mathcal{H}(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v}\rangle + 1$, where $\boldsymbol{x} : U \longrightarrow H_+^3(-1)$ is a surface in Hyperbolic space. We call $\mathcal{H}$ a *horospherical height* *function* on $\boldsymbol{x}(U) = M$. We denote that $h(u) = \mathcal{H}_{v_0}(u) = \mathcal{H}(u, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in LC^*_+$. Then we have shown the following simple lemma which is the base of our theory on hyperbolic Gauss indicatrices of surfaces.

LEMMA 3.3. *Let $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ be a surface in Hyperbolic space. Then we have the following:*

(1) $\mathcal{H}(u, \boldsymbol{v}) = 0$ *if and only if there exist real numbers μ, ξ_1, ξ_2 such that*

$$\boldsymbol{v} = \boldsymbol{x} + \mu \boldsymbol{e} + \xi_1 \boldsymbol{x}_{u_1} + \xi_2 \boldsymbol{x}_{u_2}.$$

(2) $\mathcal{H}(u, \boldsymbol{v}) = \dfrac{\partial \mathcal{H}}{\partial u_1}(u, \boldsymbol{v}) = \dfrac{\partial \mathcal{H}}{\partial u_2}(u, \boldsymbol{v}) = 0$ *if and only if* $\boldsymbol{v} = \boldsymbol{x}(u) \pm \boldsymbol{e}(u) = \mathbb{L}^{\pm}(u)$.

Following the terminology of Whitney [9], we say that a surface $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ has the *excellent hyperbolic Gauss indicatrix* $\mathbb{L}^{\pm}$ if the hyperbolic Gauss indicatrix $\mathbb{L}^{\pm}$ has only cuspidaledges and swallowtails as singularities. Theorem 3.1 asserts that a surface with the excellent hyperbolic Gauss indicatrix is generic in the space of all surfaces in $H^3_+(-1)$.

We now consider the geometric meanings of cuspidaledges and swallowtails of the hyperbolic Gauss indicatrix. Define a function $\mathfrak{H} : H^3_+(-1) \times LC^*_+ \longrightarrow \mathbb{R}$ by $\mathfrak{H}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle + 1$. For any $\boldsymbol{v}_0 \in LC^*_+$, we denote that $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have a horosphere $\mathfrak{h}^{-1}_{v_0}(0) = HP(\boldsymbol{v}_0, -1) \cap H^3_+(-1) = HS^2(\boldsymbol{v}_0, -1)$. For any $u_0 \in U$, we consider the lightlike vector $\boldsymbol{v}_0^{\pm} = \mathbb{L}^{\pm}(u_0)$, then we have

$$\mathfrak{h}_{v_0^{\pm}} \circ \boldsymbol{x}(u_0) = \mathfrak{H} \circ (\boldsymbol{x} \times id_{LC^*_+})(u_0, \boldsymbol{v}_0^{\pm}) = \mathcal{H}(u_0, \mathbb{L}^{\pm}(u_0)) = 0.$$

We also have relations that

$$\frac{\partial \mathfrak{h}_{v_0^{\pm}} \circ \boldsymbol{x}}{\partial u_i}(u_0) = \frac{\partial \mathcal{H}}{\partial u_i}(u_0, \mathbb{L}^{\pm}(u_0)) = 0,$$

for $i = 1, 2$. This means that the horosphere $\mathfrak{h}^{-1}_{v_0^{\pm}}(0) = HS^2(\boldsymbol{v}_0^{\pm}, -1)$ is tangent to $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. In this case, we call $HS^2(\boldsymbol{v}_0^{\pm}, -1)$ the *tangent horosphere* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ (or, u_0). If lightlike vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent, then corresponding lightlike hyperplanes $HP(\boldsymbol{v}_1, -1), HP(\boldsymbol{v}_2, -1)$ are parallel. Therefore, we say that two horospheres $HS^2(\boldsymbol{v}_1, -1), HS^2(\boldsymbol{v}_2, -1)$ are *parallel* if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent. For a surface germ $\boldsymbol{x} : (U, u_0) \longrightarrow (H^3_+(-1), \boldsymbol{x}(u_0))$, we call $(\boldsymbol{x}^{-1}(HS^2(\mathbb{L}^{\pm}(u_0), -1)), u_0)$ the *tangent horospherical indicatrix germ* of $\boldsymbol{x}$. We can borrow some basic invariants from the singularity theory on function germs. We denote that

$$\text{H-ord}^{\pm}(\boldsymbol{x}, u_0) = \dim \frac{C^{\infty}_{u_0}(U)}{\langle \langle \boldsymbol{x}(u), \mathbb{L}^{\pm}(u_0) \rangle + 1, \langle \boldsymbol{x}_{u_i}(u), \mathbb{L}^{\pm}(u_0) \rangle \rangle_{C^{\infty}_{u_0}}},$$

where $C^{\infty}_{u_0}(U)$ is the ring of function germs $(U, u_0) \longrightarrow \mathbb{R}$. Usually H-ord$^{\pm}(\boldsymbol{x}, u_0)$ is called the *$\mathcal{K}$-codimension of $\widetilde{h}_{v_0^{\pm}}$* (cf., [7]), where $\widetilde{h}_{v_0^{\pm}}(u) = \mathcal{H}(u, \boldsymbol{v}_0^{\pm})$. However, we call it the *order of contact with the tangent horosphere* at $\boldsymbol{x}(u_0)$. We also have the notion of corank of function germs.

$$\text{H-corank}^{\pm}(\boldsymbol{x}, u_0) = 2 - \operatorname{rank} \operatorname{Hess}(\widetilde{h}_{v_0^{\pm}}(u_0)),$$

where $v_0 = \mathbb{L}^{\pm}(u_0)$. We have shown the following results analogous to the results in Banchoff et al [2].

THEOREM 3.4. *Let $\mathbb{L}^\pm : (U, u_0) \longrightarrow (H^3_+(-1), \boldsymbol{v}_0)$ be the excellent hyperbolic Gauss indicatrix of a surface $\boldsymbol{x}$ and $h_{v_0^\pm} : (U, u_0) \longrightarrow \mathbb{R}$ be the horospherical height function germ at $\boldsymbol{v}_0^\pm = \mathbb{L}^\pm(u_0)$. Then we have the following:*

(1) *u_0 is an $H^\pm$-parabolic point of $\boldsymbol{x}$ if and only if* H-corank$^\pm(\boldsymbol{x}, u_0) = 1$ *(i.e.,u_0 is not a horospherical point of $\boldsymbol{x}$).*

(2) *If u_0 is an $H^\pm$-parabolic point of $\boldsymbol{x}$, then $\widetilde{h}_{v_0^\pm}$ has the A_k-type singularity for $k = 2, 3$.*

(3) *Suppose that u_0 is an $H^\pm$-parabolic point of $\boldsymbol{x}$. Then the following conditions are equivalent:*

(a) *$\mathbb{L}^\pm$ has a cuspidaledge at u_0*

(b) *$\widetilde{h}_{v_0^\pm}$ has the A_2-type singularity.*

(c) H-ord$^\pm(\boldsymbol{x}, u_0) = 2$.

(d) *The tangent horospherical indicatrix is a ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called an ordinary cusp if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$.*

(e) *For each $\varepsilon > 0$, there exist two distinct points $u_1, u_2 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2$, both of u_1, u_2 are not $H^\pm$-parabolic points and the tangent horosphere to $M = \boldsymbol{x}(U)$ at u_1, u_2 are parallel.*

(4) *Suppose that u_0 is an $H^\pm$-parabolic point of $\boldsymbol{x}$. Then the following conditions are equivalent:*

(a) *$\mathbb{L}^\pm$ has a swallowtail at u_0*

(b) *$\widetilde{h}_{v_0^\pm}$ has the A_3-type singularity.*

(c) H-ord$^\pm(\boldsymbol{x}, u_0) = 3$.

(d) *The tangent horospherical indicatrix is a point or a tachnodal, where a curve $C \subset \mathbb{R}^2$ is called a tachnodal if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$.*

(e) *For each $\varepsilon > 0$, there exist three distinct points $u_1, u_2, u_3 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2, 3$ and the tangent horosphere to $M = \boldsymbol{x}(U)$ at u_1, u_2, u_3 are parallel.*

(f) *For each $\varepsilon > 0$, there exist two distinct points $u_1, u_2 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2$ and the tangent horosphere to $M = \boldsymbol{x}(U)$ at u_1, u_2 are equal.*

4. Horospherical surfaces as Wave fronts In this section we naturally interpret the horospherical surface of a space curve in Hyperbolic space as a wave front in the framework of contact geometry and consider the geometric meaning of singularities. In §6 Appendix we give a quick survey on the theory of Legendrian singularities. For notions and basic results on generating families, please refer to Appendix. For any lightlike vector $\boldsymbol{v} = (v_0, v_1, v_2, v_3) \in LC^*_+$, we have a relation $v_0 = \sqrt{v_1^2 + v_2^2 + v_3^2}$. So we adopt the coordinate system (v_1, v_2, v_3) of LC^*_+ as a manifold. Here, we consider the projective cotangent bundle $\pi : PT^*(LC^*_+) \longrightarrow LC^*_+$ with the canonical contact structure. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(LC^*_+) \to PT^*(LC^*_+)$ and the differential map $d\pi : TPT^*(LC^*_+) \to TLC^*_+$ of π. For any $X \in TPT^*(LC^*_+)$, there exists an element $\alpha \in T^*(LC^*_+)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(LC^*_+)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(LC^*_+)$ by

$$K = \{X \in TPT^*(LC^*_+) | \tau(X)(d\pi(X)) = 0\}.$$

In the coordinate system (v_1, v_2, v_3), we have the trivialisation

$$PT^*(LC^*_+) \cong LC^*_+ \times P(\mathbb{R}^2)^*$$

and we call

$$((v_1, v_2, v_3), [\xi_1 : \xi_2 : \xi_3])$$

homogeneous coordinates, where $[\xi_1 : \xi_2 : \xi_3]$ are homogeneous coordinates of the dual projective plane $P(\mathbb{R}^2)^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^3 \mu_i \xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=1}^3 \mu_i \frac{\partial}{\partial v_i}$. An immersion $i : L \to PT^*(LC_+^*)$ is said to be a *Legendrian immersion* if $\dim L = 2$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the *Legendrian map* and the set $W(i) = \text{image}\, \pi \circ i$ the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

The main tool for the proof of Theorem 2.1 has been the *horospherical height function* on γ. For a hyperbolic space curve $\gamma : I \longrightarrow H_+^3(-1)$, we define a function

$$H : I \times LC_+^* \longrightarrow \mathbb{R}$$

by $H(s, \boldsymbol{v}) = \langle \gamma(s), \boldsymbol{v} \rangle + 1$. We call H a *horospherical height function* on γ. We denote that $h(s) = H_{v_0}(s) = H(s, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in LC_+^*$. The proof for the following proposition is given by a direct calculation (cf., [5]) but it has induced the notion of the horospherical surface of a curve.

PROPOSITION 4.1. *Let $\gamma : I \longrightarrow H_+^3(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following:*
(1) $h(s_0) = 0$ *if and only if there exist real numbers λ, μ, η with $\lambda^2 + \mu^2 + \eta^2 = 1$ such that* $\boldsymbol{v}_0 = \gamma(s_0) + \lambda \boldsymbol{t}(s_0) + \mu \boldsymbol{n}(s_0) + \eta \boldsymbol{e}(s_0)$.
(2) $h(s_0) = h'(s_0) = 0$ *if and only if there exists $\theta_0 \in [0, 2\pi]$ such that* $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$.
(3) $h(s_0) = h'(s_0) = h''(s_0) = 0$ *if and only if* $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ *and* $\cos\theta_0 = 1/\kappa_h(s_0)$.
(4) $h(s_0) = h'(s_0) = h''(s_0) = h^{(3)}(s_0) = 0$ *if and only if* $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$, $\cos\theta_0 = 1/\kappa_h(s_0)$ *and* $\sigma_h(s_0) = ((\kappa_h')^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1)(s_0) = 0$.
(5)$h(s_0) = h'(s_0) = h''(s_0) = h^{(3)}(s_0) = h^{(4)}(s_0) = 0$ *if and only if* $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$, $\cos\theta_0 = 1/\kappa_h(s_0)$ *and* $\sigma_h(s_0) = \sigma_h'(s_0) = 0$

We have the following proposition:

PROPOSITION 4.2. *The horospherical height function $H : I \times LC_+^* \longrightarrow \mathbb{R}$ is a Morse family.*

Proof. For any $\boldsymbol{v} = (v_0, v_1, v_2, v_3) \in LC_+^*$, we have $v_0 = \sqrt{v_1^2 + v_2^2 + v_3^2}$, so that

$$H(s, \boldsymbol{v}) = -x_0(s)\sqrt{v_1^2 + v_2^2 + v_3^2} + x_1(s)v_1 + x_2(s)v_2 + x_n(s)v_3 + 1,$$

where $\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s))$. We have to prove that the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial s}\right)$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \gamma'(s), \boldsymbol{v} \rangle & -x_0(s)\frac{v_1}{v_0} + x_1(s) & -x_0(s)\frac{v_2}{v_0} + x_(s) & -x_0(s)\frac{v_3}{v_0} + x_3(s) \\ \langle \gamma''(s), \boldsymbol{v} \rangle & -x_0'(s)\frac{v_1}{v_0} + x_1'(s) & -x_0'(s)\frac{v_2}{v_0} + x_2'(s) & -x_0'(s)\frac{v_3}{v_0} + x_3'(s) \end{pmatrix}.$$

We now show that the rank of the matrix

$$A = \begin{pmatrix} -x_0(s)\frac{v_1}{v_0} + x_1(s) & -x_0(s)\frac{v_2}{v_0} + x_2(s) & -x_0(s)\frac{v_3}{v_0} + x_3(s) \\ -x_0'(s)\frac{v_1}{v_0} + x_1'(s) & -x_0'(s)\frac{v_2}{v_0} + x_2'(s) & -x_0'(s)\frac{v_3}{v_0} + x_3'(s) \end{pmatrix}.$$

is two at $(s, \boldsymbol{v}) \in \Sigma_*(H)$.

In this case we now calculate the Gram-Schimidt matrix of

$$B = v_0^2 A = \begin{pmatrix} -x_0(s)v_1 + x_1(s)v_0 & -x_0(s)v_2 + x_2(s)v_0 & -x_0(s)v_3 + x_3(s)v_0 \\ -x_0'(s)v_1 + x_1'(s)v_0 & -x_0'(s)v_2 + x_2'(s)v_0 & -x_0'(s)v_3 + x_3'(s)v_0 \end{pmatrix}.$$

We denote that

$$F = (-x_0(s)v_1 + x_1(s)v_0, -x_0(s)v_2 + x_2(s)v_0, -x_0(s)v_3 + x_3(s)v_0),$$

$$G = (-x_0'(s)v_1 + x_1'(s)v_0, -x_0'(s)v_2 + x_2'(s)v_0, -x_0'(s)v_3 + x_3'(s)v_0).$$

Then we have

$$F \cdot F = v_0^2 x_0^2(s) - 2x_0(s)v_0(v_1x_1(s) + v_2x_2(s) + v_3x_3(s) + v_0^2(x_1^2(s) + x_2^2(s) + x_3^2(s)).$$

Since $\langle \gamma(s), \boldsymbol{v}_0 \rangle = -1$, we have $F \cdot F = -v_0^2 + 2x_0(s)v_0$. We also have $G \cdot G = -v_0^2$. Moreover, we can show that

$$F \cdot G = (-x_0(s)x_0'(s) + x_1(s)x_1'(s) + x_2(s)x_2'(s) + x_3(s)x_3'(s))v_0^2 = 0.$$

Therefore the Gram-Schmit matrix of B is

$$\begin{pmatrix} -v_0^2 + 2x_0(s)v_0 & 0 \\ 0 & -v_0^2 \end{pmatrix}.$$

By a Lorentzian motion of the curve on $H_+^3(-1)$, we may assume that $x_0(s) \neq v_0/2$. Thus the rank of the matrix is equal to two. This completes the proof. □

By the method for constructing the Legendrian immersion germ from a Morse family, we can define a Legendrian immersion germ whose generating family is the horospherical height function on γ as follows: For a unit speed regular curve $\gamma : I \longrightarrow H_+^3(-1)$, we denote

$$\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s)), \quad HS_\gamma(s,\theta) = (v_0(s,\theta), v_1(s,\theta), v_2(s.\theta), v_3(s,\theta))$$

as coordinate representations. We define a smooth mapping

$$\mathcal{L}_\gamma : I \times J \longrightarrow PT^*(LC_+^*)$$

by

$$\mathcal{L}_\gamma(s,\theta) = (HS_\gamma(s,\theta), [\ell(s,\theta)]),$$

where

$$\ell(s,\theta) = \left(-x_0(s)\frac{v_1}{v_0}(s,\theta) + x_1(s), -x_0(s)\frac{v_2}{v_0}(s,\theta) + x_2(s), -x_0(s)\frac{v_3}{v_0}(s,\theta) + x_3(s)\right).$$

By definition, we have the following corollary of the above theorem:

COROLLARY 4.3. *For a unit speed regular curve $\gamma : I \longrightarrow H_+^3(-1)$, $\mathcal{L}_\gamma$ is a Legendrian immersion such that the horospherical height function $H : I \times LC_+^* \longrightarrow \mathbb{R}$ of γ is a global generating family of $\mathcal{L}_\gamma$.*

Therefore, we have the Legendrian immersion $\mathcal{L}_\gamma$ whose wave front set is the horospherical surface of γ.

On the other hand, we can also define a lift

$$\mathcal{L}^\pm : U \longrightarrow PT^*(LC^*_+)$$

of the hyperbolic Gauss indicatrix $\mathbb{L}^\pm$ of a surface $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ as follows: We denote $\boldsymbol{x}(u) = (x_0(u), x_1(u), x_2(u), x_3(u))$ and $\mathbb{L}^\pm(u) = (\ell^\pm_0(u), \ell^\pm_1(u), \ell^\pm_2(u), \ell^\pm_3(u))$ as coordinate representations and define

$$\mathcal{L}^\pm(u) = (\mathbb{L}^\pm(u), [\ell^\pm(u)]),$$

where

$$\ell^\pm(u) = (-\ell^\pm_1(u)x_0(u) + \ell^\pm_0(u)x_1(u), -\ell^\pm_2(u)x_0 + \ell^\pm_0(u)x_2(u), -\ell^\pm_3(u)x_0 + \ell^\pm_0(u)x_3(u)).$$

By the similar calculation as in the proof of Proposition 4.2, we can prove that the horospherical height function $\mathcal{H} : U \times LC^*_+ \longrightarrow \mathbb{R}$ of $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ is a Morse family and it is a global generating family of the Legendrian lift $\mathcal{L}^\pm$ of $\mathbb{L}^\pm$ (cf., [6]).

5. The canal surface of a hyperbolic space curve Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed curve. We now define a surface

$$HC\gamma_\phi(s,\theta) = \cosh\phi\gamma(s) + \sinh\phi(\cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{e}(s))$$

for a non zero real number ϕ. We call $HC\gamma_\phi$ the *hyperbolic canal surface of* γ. By a straightforward calculation, we have

$$\left(HC\gamma_\phi \wedge \frac{\partial HC\gamma_\phi}{\partial s} \wedge \frac{\partial HC\gamma_\phi}{\partial \theta}\right)(s,\theta)$$
$$= -\sinh\phi(\cosh\phi - \kappa_h(s)\cos\theta\sinh\phi)\left(\sinh\phi\gamma(s) + \cosh\phi(\sin\theta\boldsymbol{e}(s) + \cos\theta\boldsymbol{n}(s))\right).$$

Therefore, the hyperbolic canal surface of γ is singular at (s_0,θ_0) if and only if $A(s_0,\theta_0) = \cosh\phi - \kappa_h(s_0)\cos\theta_0\sinh\phi = 0$. For a sufficiently small $|\phi|$, $A(s,\theta) \neq 0$ for any $(s,\theta) \in I \times [0,2\pi]$ (under the assumption that $\bar{I}$ is compact). Therefore the hyperbolic canal surface of γ is a regular surface for sufficiently small $|\phi|$. If we fix ϕ as a negative real number, then $-\sinh\phi(\cosh\phi - \kappa_h(s)\cos\theta\sinh\phi)$ is positive. Therefore the unit normal of the canal surface is given by

$$\mathbb{E}(s,\theta) = \sinh\phi\gamma(s) + \cosh\phi(\sin\theta\boldsymbol{e}(s) + \cos\theta\boldsymbol{n}(s))).$$

It follows that the hyperbolic Gauss indicatrix of $HC\gamma_\phi$ is

$$\mathbb{L}^\pm(s,\theta) = (\cosh\phi \pm \sinh\phi)\{\gamma(s) \pm (\cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{e}(s))\}.$$

We now define a diffeomorphism

$$\mathcal{M}_c : LC^*_+ \longrightarrow LC^*_+$$

by $\mathcal{M}_c(\boldsymbol{v}) = c\boldsymbol{v}$ for a fixed number $c \in \mathbb{R}$. Then we have the following lemma:

LEMMA 5.1. *Under the above notations, we have*

$$\mathcal{M}_c \circ HS\gamma_\phi(s,\theta) = \mathbb{L}^+(s,\theta),$$

where $c = \cosh\phi + \sinh\phi$.

By Lemma 5.1, the horospherical surface of γ is diffeomorphic to the hyperbolic indicatrix of the hyperbolic canal surface of γ. Therefore we have the following theorem as a corollary of Theorem 2.2:

THEOREM 5.2. *There exists an open and dense subset* $\mathcal{O} \subset \mathrm{Emb}\,(I, H^3_+(-1))$ *such that for any* $\gamma \in \mathcal{O}$, *the hyperbolic canal surface* $HC\gamma_\phi$ *(for sufficiently small*$|\phi|$*) has the excellent hyperbolic Gauss indicatrix.*

By Theorems 2.1, 2.2, 2.4 and 3.4, we have the following corollary:

COROLLARY 5.3. *There exists an open and dense subset* $\mathcal{O} \subset \mathrm{Emb}\,(I, H^3_+(-1))$ *such that for any* $\gamma \in \mathcal{O}$, *the following conditions are equivalent:*
(1) *The horospherical surface* HS_γ *of* γ *is locally diffeomorphic to the swallow tail* SW *at* (s_0, θ_0).
(2) $\cos\theta_0 = 1/\kappa_h(s_0)$, $\sigma_h(s_0) = 0$ *and* $\sigma'_h(s_0) \neq 0$.
(3) *The osculating horosphere and* γ *have 4-point contact at* $s = s_0$.
(4) *The hyperbolic Gauss indicatrix* $\mathbb{L}^+$ *for the hyperbolic canal surface* $HC\gamma_\phi$ *has the swallowtail* SW *at* (s_0, θ_0)
(5) $\text{H-ord}^+(HC\gamma_\phi, (s_0, \theta_0)) = 3$.

Here, $|\phi|$ *is sufficiently small fixed real number,* $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ *and* $\boldsymbol{v}_0^+ = (\cosh\phi + \sinh\phi)\boldsymbol{v}_0$.

We remark that we also have other conditions (in Theorem 3.4) which characterize the swallowtail point of the hyperbolic indicatrix for the canal surface $HC\gamma_\phi$ of γ. We do not, however, mention here to avoid the complicated description. The above corollary asserts that the contact between curves and horospheres generically corresponds to the contact between canal surfaces of curves and horospheres. We can asserts that such a correspondence holds in general as an application of the theory of Legendrian singularities.

We now define a contact diffeomorphism

$$\widetilde{\mathcal{M}}_c : PT^*(LC^*_+) \longrightarrow PT^*(LC^*_+)$$

by $\widetilde{\mathcal{M}}_c(\boldsymbol{v}, [\xi]) = (c\boldsymbol{v}, [\xi])$ for a fixed number $c \in \mathbb{R}$, which is the unique contact lift of the diffeomorphism $\mathcal{M}_c : LC^*_+ \longrightarrow LC^*_+$. Then we have the following proposition:

PROPOSITION 5.4. *Let* $\gamma : I \longrightarrow H^3_+(-1)$ *be a unit speed hyperbolic space curve. Then we have*

$$\widetilde{\mathcal{M}}_c \circ \mathcal{L}_\gamma(s, \theta) = \mathcal{L}^+(s, \theta),$$

where $c = \cosh\phi + \sinh\phi$ *and* $\mathcal{L}^+$ *is the lift of the hyperbolic Gauss indicatrix of* $HC\gamma_\phi$.

Therefore, the Legendrian lift $\mathcal{L}^+$ *of the hyperbolic Gauss indicatrix of* $HC\gamma_\phi$ *is Legendrian equivalent to* $\mathcal{L}_\gamma$.

We now consider the contact between curves (or, surfaces) and horospheres. The main tools belong to the theory of contact due to Montaldi [8]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [8], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory. He has shown the following theorem:

THEOREM 5.5. *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of* $\mathbb{R}^n$ *with* $\dim X_1 = \dim X_2$ *and* $\dim Y_1 = \dim Y_2$. *Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

In §2 we have defined the osculating horosphere of a hyperbolic space curve γ with $\kappa_h(s) \neq 0$. We have also defined the tangent horosphere of a surface $\boldsymbol{x}$ in Hyperbolic space. Here we consider the relation between the osculating horosphere of a hyperbolic space curve and the tangent horosphere of the canal surface of the curve. By definition $HS^2(\boldsymbol{v}_0, -1)$ is the osculating horosphere when $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ and $\cos\theta_0 = 1/\kappa_h(s_0)$. In this case $HS^2(\boldsymbol{v}_0^+, -1)$ are respectively tangent horospheres of $HC\gamma_\phi$ at (s_0, θ_0) where $\boldsymbol{v}_0^+ = (\cosh\phi + \sinh\phi)\boldsymbol{v}_0$. Then we have the following theorem.

THEOREM 5.6. *Let $\gamma_i : I \longrightarrow H^3_+(-1)$ $(i = 1, 2)$ be unit speed curves in $H^3_+(-1)$. Then*

$$K(\gamma_1, HS^2(\boldsymbol{v}_1, -1); \gamma_1(s_0)) = K(\gamma_2, HS^2(\boldsymbol{v}_2, -1); \gamma_2(s_0))$$

if and only if

$$K(HC\gamma_{1,\phi}, HS^2(\boldsymbol{v}_1^+, -1); HC\gamma_{1,\phi}(s_0, \theta_0)) = K(HC\gamma_{2,\phi}, HS^2(\boldsymbol{v}_2^+, -1); HC\gamma_{2,\phi}(s_0, \theta_0))$$

Here, $|\phi|$ is sufficiently small fixed real number, $\boldsymbol{v}_i = \gamma_i(s_0) + \cos\theta_0 \boldsymbol{n}_i(s_0) + \sin\theta_0 \boldsymbol{e}_i(s_0)$ and $\boldsymbol{v}_i^+ = (\cosh\phi + \sinh\phi)\boldsymbol{v}_i$.

Proof. We consider the function $\mathfrak{H} : H^3_+(-1) \times LC^*_+ \longrightarrow \mathbb{R}$ defined by $\mathfrak{H}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle + 1$. This function has been used to define the tangent horosphere of a surface in §3.

On the other hand, consider a unit speed curve $\gamma : I \longrightarrow H^3_+(-1)$, then we have $\mathfrak{h}_{v_0} \circ \gamma(s) = H(s, \boldsymbol{v}_0) = h(s)$, where H is the horospherical height function on γ. Therefore, $HS^2(\boldsymbol{v}_0, -1) = h_{v_0}^{-1}(0)$ is an osculating horosphere of γ at $\gamma(s_0)$ if and only if $h(s_0) = h'(s_0) = h''(s_0) = 0$. By Proposition 5.1, we have $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$.

Let $H_i : I \times LC^*_+ \longrightarrow \mathbb{R}$ be the horospherical height function of γ_i, where $i = 1, 2$. By Theorem 5.5, $K(\gamma_1, HS^2(\boldsymbol{v}_1, -1); \gamma_1(s_0)) = K(\gamma_2, HS^2(\boldsymbol{v}_2, -1); \gamma_2(s_0))$ if and only if h_{v_1} and h_{v_2} are $\mathcal{K}$-equivalent, where $h_{v_i}(s) = H_i(s, \boldsymbol{v}_i)$ $(i = 1, 2)$.

It also follows from Theorem 5.5 that

$$K(HC\gamma_{1,\phi}, HS^2(\boldsymbol{v}_1^+, -1); HC\gamma_{1,\phi}(s_0, \theta_0)) = K(HC\gamma_{2,\phi}, HS^2(\boldsymbol{v}_2^+, -1); HC\gamma_{2,\phi}(s_0, \theta_0))$$

if and only if $\widetilde{h}_{v_1^+}$ and $\widetilde{h}_{v_2^+}$ are $\mathcal{K}$-equivalent, where $\widetilde{h}_{v_i^+}(s, \theta) = \mathfrak{H}(HC\gamma_{i,\phi}(s, \theta), \boldsymbol{v}_i^+)$ $(i = 1, 2)$.

On the other hand, the horospherical height function $\mathcal{H} : I \times J \times LC^*_+ \longrightarrow \mathbb{R}$ on the canal surface $HC\gamma_\phi$ is a generating family of the Legendrian lift $\mathcal{L}^+$ of $\mathbb{L}^+$. Moreover, the horospherical height function $H : I \times LC^*_+ \longrightarrow \mathbb{R}$ on γ is a generating family of $\mathcal{L}_gamma$. By Proposition 5.4 and Theorem 6.3, $\mathcal{H}$ and H are stably P-$\mathcal{K}$-equivalent. It follows that h_{v_1} and h_{v_2} are $\mathcal{K}$-equivalent if and only if $\widetilde{h}_{v_1^+}$ and $\widetilde{h}_{v_2^+}$ are $\mathcal{K}$-equivalent. This completes the proof. □

We also have the following theorem:

THEOREM 5.7. *Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed curve in $H^3_+(-1)$. The following conditions are equivalent:*

(1) *The osculating horosphere and* γ *have* $k+1$*-point contact for* $s = s_0$.
(2) $\text{H-ord}^+(HC\gamma_\phi, (s_0, \theta_0)) = k$.
(3) $\cos\theta_0 = 1/\kappa_h(s_0)$, $\sigma_h(s_0)^{(\ell)} = 0$ *for* $0 \le \ell \le k-3$ *and* $\sigma_h^{(k-2)}(s_0) \ne 0$.
Here, $|\phi|$ *is sufficiently small fixed real number,* $\boldsymbol{v}_0 = \gamma(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ *and* $\boldsymbol{v}_0^+ = (\cosh\phi + \sinh\phi)\boldsymbol{v}_0$.

Proof. By the proof of Theorem 5.6, $\mathcal{H}$ and H are stably P-$\mathcal{K}$-equivalent. Therefore the condition (1) is equivalent to the condition (2). If we continue the calculation in Proposition 4.1, we can show that $h^{(\ell)}(s_0) = 0$ for $0 \le \ell \le k$ and $h^{(k+1)}(s_0) \ne 0$ if and only if the condition (3) holds. It follows that the condition (1) is equivalent to the condition (3). □

We emphasize that the above two theorems hold not necessary under the generic condition.

6. Appendix: Generating families In which we give a quick survey on the theory of Legendrian singularities mainly due to Arnol'd-Zakalyukin [1, 10].

Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, x_2, x_3) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0})$. In this case we have a smooth surface

$$\Sigma_*(F) = \left\{(q, x) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0\right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^3$ defined by

$$\Phi_F(q, x) = \left(x, [\frac{\partial F}{\partial x_1}(q, x) : \frac{\partial F}{\partial x_2}(q, x) : \frac{\partial F}{\partial x_3}(q, x)]\right)$$

is a Legendrian immersion. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 10].

PROPOSITION 6.1. *All Legendrian submanifold germs in* $PT^*\mathbb{R}^3$ *are constructed by the above method.*

We call F a *generating family* of Φ_F. Therefore the wave front is

$$W(\Phi_F) = \left\{x \in \mathbb{R}^3 \mid \exists q \in \mathbb{R}^k \;;\; F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0\right\}.$$

We sometime denote $\mathcal{D}_F = W(\Phi_F)$ and call it the *discriminant set* of F.

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^3, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^3, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^3, p) \longrightarrow (PT^*\mathbb{R}^3, p')$ such that H preserves fibres of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^3$ at a point is said to be Legendrian stable if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^3, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

PROPOSITION 6.2. *Let $i : (L,p) \subset (PT^*\mathbb{R}^3, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^3, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i, \pi \circ i'$ are dense respectively. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [11]. The assumption in the above proposition is a generic condition for i, i'. Specially, if i, i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_m$ the local ring of function germs $(\mathbb{R}^m, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_m = \{h \in \mathcal{E}_m \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+3}}) = \langle G\rangle_{\mathcal{E}_{k+3}}$. Here $\Psi^* : \mathcal{E}_{k+3} \longrightarrow \mathcal{E}_{k+3}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. For any $F_1 \in \mathfrak{M}_{k+3}$, $F_2 \in \mathfrak{M}_{k'+3}$ we also say that F_1, F_2 are stably *stably P-$\mathcal{K}$-equivalent* if they become P-$\mathcal{K}$-equivalent after the addition to the arguments to q_i of new arguments p_i and to the functions F_i of nondegenerate quadratic forms Q_i in the new arguments (i.e., $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{K}$-equivalent).

Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ a function germ. We say that F is a *$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \frac{\partial F}{\partial x_2}|\mathbb{R}^k \times \{\mathbf{0}\}, \frac{\partial F}{\partial x_3}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [7].)

The main result in Arnol'd-Zakalyukin's theory [1, 10] is the following:

THEOREM 6.3. *Let $F_1 \in \mathfrak{M}_{k+3}$ and $F_2 \in \mathfrak{M}_{k'+3}$ be Morse families. Then*
(1) *Φ_{F_1} and Φ_{F_2} are Legendrian equivalent if and only if F_1, F_2 are stably P-$\mathcal{K}$-equivalent.*
(2) *Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{\mathbf{0}\}$.*

By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition 5.2 and Theorem 5.3, we have the following classification result of Legendrian stable germs. For any function germ $f : (\mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$, we define the *local ring of f* by $Q(f) = \mathcal{E}_k / \langle f\rangle_{\mathcal{E}_n}$.

PROPOSITION 6.4. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that Φ_F, Φ_G are Legendrian stable. The the following conditions are equivalent.*

(1) *$(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs.*

(2) *Φ_F and Φ_G are Legendrian equivalent.*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras.*
Here $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$, $g = G|\mathbb{R}^k \times \{\mathbf{0}\}$.

Proof. Since Φ_F, Φ_G are Legendrian stable, these satisfy the generic condition of Proposition 5.2, so that the conditions (1) and (2) are equivalent. The condition (3) implies that f, g are $\mathcal{K}$-equivalent [7]. By the uniqueness of the $\mathcal{K}$-versal deformation of a function germ, F, G are P-$\mathcal{K}$-equivalent. This means that the condition (2) holds. By Theorem 5.3, the condition (2) implies the condition (3). □

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