NORMS OF SOME SINGULAR INTEGRAL OPERATORS ON WEIGHTED L^2 SPACES

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by

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Abstract. Let α and β be measurable functions on the unit circle T, and let W be a positive function on T such that the Riesz projection P_+ is bounded on the weighted space $L^2(W)$ on T. The singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f$, $(f \in L^2(W))$ where $P_- = I - P_+$. Let h be an outer function such that $W = |h|^2$, and let ϕ be an unimodular function such that $\phi = \bar{h}/h$. In this paper, the norm of $S_{\alpha,\beta}$ on $L^2(W)$ is calculated in general, using α, β and ϕ. Moreover, if α and β are constant functions, then we give the another proof of the Feldman-Krupnik-Marcus theorem. If $\alpha\bar{\beta}$ belongs to the Hardy space H^∞, we give the theorem which is similar to the Feldman-Krupnik-Marcus theorem.

Keywords: Singular integral operator, Norm, Hardy space, Helson-Szegö weight, (A_2)condition.

AMS Subject Classification: Primary 45E10, 47B35; Secondary 46J15.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $T = \{\zeta; |\zeta| = 1\}$. That is, $dm(\zeta) = d\theta/2\pi$ for $\zeta = e^{i\theta}$. For functions f and g satisfying $fg \in L^1$, we define the quantity (f, g) according to

$$(f, g) = \int_T f(\zeta)\overline{g(\zeta)}dm(\zeta).$$

If $f, g \in L^2$, then this becomes the inner product. Let H^2 (resp. H^∞) be the Hardy space of functions $f \in L^2$ (resp. $f \in L^\infty$) whose negative Fourier coefficients are zero. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi i}\int_T \frac{f(\eta)}{\eta - \zeta}d\eta \quad (a.e. \zeta \in T),$$

where the integral is understood in the sense of Cauchy's principal value (cf. [6, p.11]). If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ on T, and Sf becomes a measurable function on T. For a positive function $W \in L^1$, the norm in $L^2(W)$ is defined by the formula

$$\|f\|_{L^2(W)} = (Wf, f)^{1/2} = \left\{\int_T |f(\zeta)|^2 W(\zeta)dm(\zeta)\right\}^{1/2}.$$

Let A (resp. $\overline{A_0}$) be the subspace of continuous functions f on T whose negative (resp. positive) Fourier coefficients are zero. Let $A + \overline{A_0} = \{f_1 + f_2; f_1 \in A, f_2 \in \overline{A_0}\}$. Then $A + \overline{A_0}$ is dense in $L^2(W)$ in norm. Two projections P_+ and P_- are defined by

$$P_+ = (I + S)/2, \quad \text{and} \quad P_- = (I - S)/2.$$

where I denotes the identity operator. P_+ is the Riesz projection. For $\alpha, \beta \in L^\infty$, let $S_{\alpha,\beta}$ be the singular integral operator on $L^2(W)$ defined by

$$S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f, \quad (f \in L^2(W)).$$

Then, $S_{1,1} = I$, $S_{1,-1} = S$, $S_{1,0} = P_+$ and $S_{0,1} = P_-$. Let $\|S_{\alpha,\beta}\|_{L^2(W)}$ denote the operator norm of $S_{\alpha,\beta}$ on $L^2(W)$. That is,

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \sup\left\{\|S_{\alpha,\beta}f\|_{L^2(W)}; f \in L^2(W), \|f\|_{L^2(W)} = 1\right\}.$$

Let W be a positive function in L^1 on T such that S becomes a bounded operator on $L^2(W)$. The relation between the norms of the operators S, P_+, P_- on the space $L^2(W)$,

$$\|P_+\|_{L^2(W)} = \|P_-\|_{L^2(W)} = \frac{\|S\|_{L^2(W)} + \|S\|^{-1}_{L^2(W)}}{2}$$

was remarked by Spitkovskii ([16]). Let h be an outer function such that $W = |h|^2$, and let ϕ be an unimodular function such that $\phi = \bar{h}/h$. Let

$$c = \inf_{k \in H^\infty} \|\phi - k\|_\infty.$$

Then

$$\|S\|_{L^2(W)} = \|S_{1,-1}\|_{L^2(W)} = \sqrt{\frac{1+c}{1-c}},$$

$$\|P_+\|_{L^2(W)} = \|S_{1,0}\|_{L^2(W)} = \frac{1}{\sqrt{1-c^2}}.$$

(cf. [4]). For $\zeta_0 \in T$, and $-1 < \delta < 1$, let $W(\zeta) = |\zeta - \zeta_0|^\delta$. Then the equality $\|S\|_{L^2(W)} = \cot \frac{\pi(1-|\delta|)}{4}$ was obtained by Krupnik and Verbitskii [12]. Hence $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi\delta/2)}$. For continuous functions α and β, the essential norm of $S_{\alpha,\beta}$ on $L^2(W)$ was calculated by Krupnik and Avendanio (cf. [10, p.57, Corollary 6.1], [1]). For constant functions α, β and a positive function $W \in L^1$ such that $\|P_+\|_{L^2(W)} < \infty$, the equality

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha|-|\beta|}{2}\right)^2},$$

where

$$\gamma := \left|\frac{\alpha-\beta}{2}\right|^2 (\|P_+\|^2_{L^2(W)} - 1) = \left|\frac{\alpha-\beta}{2}\right|^2 \left(\frac{c^2}{1-c^2}\right)$$

was obtained by Feldman, Krupnik and Markus (cf. [3], [11], [6, Section 13.5], [5], [18]). In this paper, for functions $\alpha, \beta \in L^\infty$, and a positive function $W \in L^1$ on the unit circle T, we will give three formulae of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$. It follows from the Koosis theorem [9] that there exist different functions α and β such that $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ if and only if $W^{-1} \in L^1$ (cf. [13]). If $\log W \notin L^1$, then $W^{-1} \notin L^1$. In this case $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ implies that $\alpha \equiv \beta$. Hence, $\|S_{\alpha,\beta}\|_{L^2(W)} = \|\alpha I\|_{L^2(W)} = \|\alpha\|_\infty$. Therefore we assume that $\log W \in L^1$. In Section 2, for functions $\alpha, \beta \in L^\infty$, and a positive function $W \in L^1$ on the unit circle T, we will give the first formula (Theorem 1) of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$ using α, β and ϕ. We will also give the another proof of the Feldman-Krupnik-Marcus theorem in Corollary 4. In Section 3, we will give the second formula (Theorem 2) of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$. If $\alpha\bar{\beta}$ belongs to the Hardy space H^∞, we give the theorem which is similar to the Feldman-Krupnik-Marcus theorem. In Section 4, we will give the third formula (Theorem 3) of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$.

2. THE FIRST FORMULA OF NORM OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 1 is the first formula of $\|S_{\alpha,\beta}\|_{L^2(W)}$. We will use Theorem 1 to prove Theorems 2 and 3 in the following sections. We give some lemmas to prove Theorem 1.

Definition 1. Let $\alpha, \beta \in L^\infty$. For each $\gamma \in L^\infty$, we define the function $G(\gamma) \in L^\infty$ according to

$$(G(\gamma))(\zeta) = \frac{|\alpha(\zeta)|^2 + |\beta(\zeta)|^2}{2} + \sqrt{|\gamma(\zeta)|^2 + \left(\frac{|\alpha(\zeta)|^2 - |\beta(\zeta)|^2}{2}\right)^2}, \quad (\zeta \in T).$$

Definition 2. Let $\alpha, \beta \in L^\infty$, let h be an outer function in H^2, and let $\phi = \bar{h}/h$. We define the function F according to

$$F(x) = \inf_{k \in H^\infty} \left\| G(x - \alpha\bar{\beta} - \bar{\phi}k) \right\|_\infty, \quad (x \geq 0).$$

That is,

$$F(x) = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|x - \alpha\bar{\beta} - \bar{\phi}k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .$$

Lemma 1. For each nonnegative number x, the infimum in the definition of $F(x)$ is attained.

Proof. Let $\{k_n\}$ be a sequence in H^∞ such that

$$F(x) = \lim_{n \to \infty} \left\| G(x - \alpha\bar{\beta} - \bar{\phi}k_n) \right\|_\infty .$$

For any $\varepsilon > 0$, take a positive integer n_0 with

$$\left\| G(x - \alpha\bar{\beta} - \bar{\phi}k_n) \right\|_\infty \leq F(x) + \varepsilon, \quad (n \geq n_0).$$

If $n \geq n_0$, then $\|k_n\|_\infty \leq F(x) + \varepsilon + \|x - \alpha\bar{\beta}\|_\infty < \infty$. Since the closed ball of H^∞ is weak-star compact over L^∞ (cf. [8, p.197]), there exists a subsequence $\{k_{n_j}\}$ and a $k_0 \in H^\infty$ such that

$$\lim_{j \to \infty} (k_{n_j}, g) = (k_0, g), \quad (g \in L^1).$$

Then there exists a sequence $\{h_n\}$ in H^∞ such that each h_n is a finite convex linear combination of the k_{n_j} and

$$\lim_{n \to \infty} \|h_n - k_0\|_{L^2} = 0$$

(cf. [17, p.160, Problem 6]). It follows that there exists a subsequence $\{h_{n_j}\}$ such that

$$\lim_{j\to\infty} |k_0(\zeta) - h_{n_j}(\zeta)| = 0, \quad (a.e. \zeta \in T)$$

(cf. [15, p.68, Theorem 3.12]). Hence there exist nonnegative numbers $\lambda_{j,1}, ..., \lambda_{j,m_j}$ such that $\lambda_{j,1} + ... + \lambda_{j,m_j} = 1$ and

$$h_{n_j} = \lambda_{j,1} k_{n_1} + ... + \lambda_{j,m_j} k_{n_{m_j}}.$$

Since $y = a^2 + \sqrt{t^2 + b^2}$ is a convex function of t, it follows that

$$\begin{aligned}
G(x - \alpha\bar{\beta} - \bar{\phi} h_{n_j}) &= G\left(x - \alpha\bar{\beta} - \bar{\phi}\sum_{i=1}^{m_j} \lambda_{j,i} k_{n_i}\right) \\
&= G\left(\sum_{i=1}^{m_j} \lambda_{j,i}(x - \alpha\bar{\beta} - \bar{\phi} k_{n_i})\right) \\
&\leq G\left(\sum_{i=1}^{m_j} \lambda_{j,i}|x - \alpha\bar{\beta} - \bar{\phi} k_{n_i}|\right) \\
&\leq \sum_{i=1}^{m_j} \lambda_{j,i} G(x - \alpha\bar{\beta} - \bar{\phi} k_{n_i}) \\
&\leq \sum_{i=1}^{m_j} \lambda_{j,i}(F(x) + \varepsilon) \\
&= F(x) + \varepsilon.
\end{aligned}$$

There exists a $g \in L^1$ such that $\|g\|_{L^1} = 1$ and

$$\left\|G(x - \alpha\bar{\beta} - \bar{\phi} k_0)\right\|_\infty \leq \left|\Big(G(x - \alpha\bar{\beta} - \bar{\phi} k_0), g\Big)\right| + \varepsilon.$$

By the Lebesgue theorem,

$$\begin{aligned}
\left|\Big(G(x - \alpha\bar{\beta} - \bar{\phi} k_0), g\Big)\right| &= \lim_{j\to\infty}\left|\Big(G(x - \alpha\bar{\beta} - \bar{\phi} h_{n_j}), g\Big)\right| \\
&\leq \liminf_{j\to\infty}\left\|G(x - \alpha\bar{\beta} - \bar{\phi} h_{n_j})\right\|_\infty \\
&\leq F(x) + \varepsilon.
\end{aligned}$$

Hence,

$$\left\|G(x - \alpha\bar{\beta} - \bar{\phi} k_0)\right\|_\infty \leq F(x) + 2\varepsilon.$$

Let $\varepsilon \to 0$. Then the equality holds, and hence the infimum in the definition of $F(x)$ is attained by $k = k_0$. This completes the proof.

Lemma 2. $F(x)$ is a convex function of x. (Hence it is continuous.)

Proof. Let λ and μ be nonnegative numbers such that $\lambda + \mu = 1$. Since $y = a^2 + \sqrt{t^2 + b^2}$ is a nonnegative, convex, increasing function of $t, t \geq 0$, it follows that

$$\begin{aligned}
\lambda F(x) + \mu F(y) &= \lambda \inf_{k_1 \in H^\infty} \left\| G(x - \alpha\bar{\beta} - \bar{\phi}k_1) \right\|_\infty + \mu \inf_{k_2 \in H^\infty} \left\| G(y - \alpha\bar{\beta} - \bar{\phi}k_2) \right\|_\infty \\
&\geq \inf_{k_1 \in H^\infty} \inf_{k_2 \in H^\infty} \left\| \lambda G(x - \alpha\bar{\beta} - \bar{\phi}k_1) + \mu G(y - \alpha\bar{\beta} - \bar{\phi}k_2) \right\|_\infty \\
&\geq \inf_{k_1 \in H^\infty} \inf_{k_2 \in H^\infty} \left\| G\left(\lambda |x - \alpha\bar{\beta} - \bar{\phi}k_1| + \mu |y - \alpha\bar{\beta} - \bar{\phi}k_2|\right) \right\|_\infty \\
&\geq \inf_{k_1 \in H^\infty} \inf_{k_2 \in H^\infty} \left\| G\left(\lambda x + \mu y - \alpha\bar{\beta} - \bar{\phi}(\lambda k_1 + \mu k_2)\right) \right\|_\infty \\
&\geq \inf_{k \in H^\infty} \left\| G(\lambda x + \mu y - \alpha\bar{\beta} - \bar{\phi}k) \right\|_\infty \\
&= F(\lambda x + \mu y).
\end{aligned}$$

This completes the proof.

Lemma 3. If $x \geq \max\{|\alpha|^2, |\beta|^2\}$, then

$$G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right) = x.$$

Proof. Since $x \geq \max\{|\alpha|^2, |\beta|^2\} \geq (|\alpha|^2 + |\beta|^2)/2$ and

$$\left(x - \frac{|\alpha|^2 + |\beta|^2}{2}\right)^2 - \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2 = (x - |\alpha|^2)(x - |\beta|^2),$$

it follows that

$$\begin{aligned}
x &= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{(x - |\alpha|^2)(x - |\beta|^2) + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \\
&= G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right).
\end{aligned}$$

This completes the proof.

Lemma 4. $F(x) \leq x$ if and only if $x \geq \|S_{\alpha,\beta}\|^2_{L^2(W)}$.

Proof. We prove the "if" part. Suppose $x \geq \|S_{\alpha,\beta}\|^2_{L^2(W)}$. Then,

$$\|S_{\alpha,\beta} f\|^2_{L^2(W)} \leq x \|f\|^2_{L^2(W)}, \quad (f \in A + \overline{A_0}).$$

Hence,

$$\|\alpha f_1 + \beta f_2\|^2_{L^2(W)} \leq x \|f_1 + f_2\|^2_{L^2(W)}, \quad (f_1 \in A, f_2 \in \overline{A_0}).$$

Let $W_1 = (x - |\alpha|^2)W, \quad W_2 = (x - |\beta|^2)W, \quad W_3 = (x - \alpha\bar{\beta})W$, then for any $f_1 \in A$ and $f_2 \in \overline{A_0}$,

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re(W_3 f_1, f_2) \geq 0.$$

By the Cotlar-Sadosky lifting theorem [2], $W_1 \geq 0, W_2 \geq 0$, and there exists a $g \in H^1$ such that

$$|W_3 - g|^2 \leq W_1 W_2.$$

This implies that $x \geq \max\{|\alpha|^2, |\beta|^2\}$, and

$$|(x - \alpha\bar{\beta})W - g|^2 \leq (x - |\alpha|^2)(x - |\beta|^2)W^2.$$

Hence,

$$\left|x - \alpha\bar{\beta} - \frac{g}{W}\right|^2 \leq (x - |\alpha|^2)(x - |\beta|^2).$$

Then,

$$\frac{g}{W} = \frac{g}{|h|^2} = \frac{h}{\bar{h}}\frac{g}{h^2} = \bar{\phi}\frac{g}{h^2}.$$

Let $k = g/h^2$. Then $k \in H^\infty$, and

$$|x - \alpha\bar{\beta} - \bar{\phi}k|^2 \leq (x - |\alpha|^2)(x - |\beta|^2).$$

It follows from Lemma 3 that

$$G(x - \alpha\bar{\beta} - \bar{\phi}k) \leq G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right) = x.$$

This implies that $F(x) \leq x$. We prove the "only if" part. Suppose $F(x) \leq x$. By Lemma 1, the infimum in the definition of $F(x)$ is attained. Hence, there exists a $k \in H^\infty$ such that

$$G(x - \alpha\bar{\beta} - \bar{\phi}k) \leq x.$$

It follows from Lemma 3 that

$$G(x - \alpha\bar{\beta} - \bar{\phi}k) \leq G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right).$$

Hence,

$$|x - \alpha\bar{\beta} - \bar{\phi}k| \leq \sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}.$$

Since $\phi = \bar{h}/h$ and $W = |h|^2$, it follows that

$$\left|(x - \alpha\bar{\beta})W - h^2k\right| \leq \sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W.$$

Since $h^2k \in H^1$, it follows that

$$\begin{aligned}
& x\|f_1 + f_2\|^2_{L^2(W)} - \|\alpha f_1 + \beta f_2\|^2_{L^2(W)} \\
= \; & x\left(W(f_1 + f_2), f_1 + f_2\right) - \left(W(\alpha f_1 + \beta f_2), \alpha f_1 + \beta f_2\right) \\
= \; & \Big((x - |\alpha|^2)Wf_1, f_1\Big) + \Big((x - |\beta|^2)Wf_2, f_2\Big) + 2Re\left((x - \alpha\bar{\beta})Wf_1, f_2\right) \\
\geq \; & 2\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\right) - 2\left|\left((x - \alpha\bar{\beta})Wf_1, f_2\right)\right| \\
= \; & 2\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\right) - 2\left|\left(\left((x - \alpha\bar{\beta})W - h^2k\right)f_1, f_2\right)\right| \\
\geq \; & 2\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\right) - 2\left(\left|(x - \alpha\bar{\beta})W - h^2k\right| |f_1|, |f_2|\right) \\
\geq \; & 0, \quad (f_1 \in A, f_2 \in \overline{A_0}).
\end{aligned}$$

Hence, $x \geq \|S_{\alpha,\beta}\|^2_{L^2(W)}$. This completes the proof.

Lemma 5. If $x \geq 0$, then

$$F(x) \leq x \inf_{k \in H^\infty} \|\phi - k\|_\infty + 2\max\{\|\alpha\|^2_\infty, \|\beta\|^2_\infty\}.$$

Proof. Since $\sqrt{|a|^2+|b|^2} \leq |a|+|b|$, it follows that

$$\begin{aligned}
F(x) &= \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|x-\alpha\bar{\beta}-\bar{\phi}k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right\|_\infty \\
&\leq \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2+|\beta|^2}{2} + |x-\alpha\bar{\beta}-\bar{\phi}k| + \left| \frac{|\alpha|^2-|\beta|^2}{2} \right| \right\|_\infty \\
&\leq \inf_{k \in H^\infty} \|x-\bar{\phi}k\|_\infty + \left\| \max\{|\alpha|^2, |\beta|^2\} + |\alpha\beta| \right\|_\infty \\
&\leq x \inf_{k \in H^\infty} \|\phi - k\|_\infty + 2\|\max\{|\alpha|, |\beta|\}\|^2_\infty.
\end{aligned}$$

This completes the proof.

Theorem 1. Let $\alpha, \beta \in L^\infty$. Let ϕ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\phi = \bar{h}/h$ and $W = |h|^2$. Then
(1) $F\left(\|S_{\alpha,\beta}\|^2_{L^2(W)}\right) = \|S_{\alpha,\beta}\|^2_{L^2(W)}$.
(2) If $\inf_{k \in H^\infty} \|\phi - k\|_\infty < 1$ then $x = \|S_{\alpha,\beta}\|^2_{L^2(W)}$ is the unique solution of the equation $F(x) = x$.

Proof. We prove (1). Let $s = \|S_{\alpha,\beta}\|^2_{L^2(W)}$. We prove that $x = s$ is the solution of the equation $F(x) = x$. It follows from Lemma 4 that $s \geq F(s)$. It follows from Lemma 2 that $F(x)$ is a continuous function of x. It follows from Lemma 4 that

$$F(x) > x, \quad (x < s).$$

Hence,

$$s \geq F(s) = \lim_{x \to s} F(x) = \lim_{x \to s-0} F(x) \geq \lim_{x \to s-0} x = s.$$

Therefore, $F(s) = s$. We prove (2). Suppose there exists a t such that $t \neq s$ and $F(t) = t$. It follows from Lemma 4 that $t > s$. Let x be any number satisfying $x > t$. Since $s < t < x$, it follows that there exist positive numbers λ, μ such that $\lambda + \mu = 1$ and $t = \lambda s + \mu x$. It follows from Lemma 2 that

$$F(t) = F(\lambda s + \mu x) \leq \lambda F(s) + \mu F(x).$$

Since $F(s) = s$ and $F(t) = t$, it follows that

$$t \leq \lambda s + \mu F(x).$$

Hence,

$$\mu x = t - \lambda s \leq \mu F(x).$$

Since $\mu > 0$, this implies that $x \leq F(x)$. Since $x > s$, it follows from Lemma 4 that $F(x) \leq x$. Therefore,

$$F(x) = x, \quad (x > t).$$

By Lemma 5, this implies that

$$x \leq x \inf_{k \in H^\infty} \|\phi - k\|_\infty + 2 \max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\}, \quad (x > t).$$

Hence,

$$\inf_{k \in H^\infty} \|\phi - k\|_\infty \geq 1.$$

This is a contradiction. Therefore, there does not exist a t such that $t \neq s$ and $F(t) = t$. This completes the proof.

If W is a constant function, then $F(x)$ becomes a constant function. In this case, the formula of $\|S_{\alpha,\beta}\|_{L^2}^2$ follows easily from Theorem 1(1) as Corollary 1. In the preceding paper [14], we gave Corollary 1.

Corollary 1. Let $\alpha, \beta \in L^\infty$. Then

$$\|S_{\alpha,\beta}\|_{L^2}^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .$$

The infimum is attained.

Proof. In the statement of Theorem 1, if $W = 1$, then h and ϕ are constants. If $k \in H^\infty$, then $k - \phi x \in H^\infty$. Hence,

$$\begin{aligned} F(x) &= \inf_{k \in H^\infty} \left\| G(x - \alpha\bar{\beta} - \bar{\phi}k) \right\|_\infty \\ &= \inf_{k \in H^\infty} \left\| G(0 - \alpha\bar{\beta} - \bar{\phi}(k - \phi x)) \right\|_\infty \\ &= \inf_{k \in H^\infty} \left\| G(0 - \alpha\bar{\beta} - \bar{\phi}k) \right\|_\infty = F(0). \end{aligned}$$

Hence $F(x)$ is a constant function of x. It follows from Theorem 1 that

$$\|S_{\alpha,\beta}\|_{L^2}^2 = F\left(\|S_{\alpha,\beta}\|_{L^2}^2\right) = F(0) = \inf_{k \in H^\infty} \left\| G(\alpha\bar{\beta} - k) \right\|_\infty .$$

By Lemma 1, the infimum is attained. This completes the proof.

Corollary 2. Let $\alpha, \beta \in L^\infty$. Then

$$\max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} \leq \|S_{\alpha,\beta}\|_{L^2}^2 \leq \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty.$$

If $\alpha\bar{\beta} \in H^\infty$, then the equality holds.

Proof. It follows from Corollary 1 that

$$\begin{aligned}
\left\|\max\{|\alpha|^2, |\beta|^2\}\right\|_\infty &\leq \|S_{\alpha,\beta}\|_{L^2}^2 \\
&= \inf_{k\in H^\infty} \|G(\alpha\bar{\beta} - k)\|_\infty \\
&\leq \inf_{k\in H^\infty} \left\|\max\{|\alpha|^2, |\beta|^2\} + |\alpha\bar{\beta} - k|\right\|_\infty \\
&\leq \left\|\max\{|\alpha|^2, |\beta|^2\}\right\|_\infty + \inf_{k\in H^\infty} \|\alpha\bar{\beta} - k\|_\infty.
\end{aligned}$$

This completes the proof.

Corollary 3. Let $\alpha, \beta \in L^\infty$. Let ϕ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\phi = \bar{h}/h$ and $W = |h|^2$.

(1) If $|\alpha|, |\beta|$ are constants, then

$$\inf_{k\in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}.$$

(2) If $|\alpha|, |\beta|$ are constants and $\alpha\bar{\beta} \in H^\infty$, then

$$\inf_{k\in H^\infty} \left\| \left(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}\right)(\phi - k) \right\|_\infty = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}.$$

Proof. We prove (1). Since $|\alpha|, |\beta|$ are constants, it follows from Theorem 1 that

$$\begin{aligned}
\|S_{\alpha,\beta}\|_{L^2(W)}^2 &= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\inf_{k\in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \\
&= G\left(\inf_{k\in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty\right).
\end{aligned}$$

Since $\|S_{\alpha,\beta}\|_{L^2(W)}^2 \geq \max\{|\alpha|^2, |\beta|^2\}$, it follows from Lemma 3 that

$$\|S_{\alpha,\beta}\|_{L^2(W)}^2 = G\left(\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}\right).$$

Therefore,

$$G\left(\inf_{k\in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty\right) = G\left(\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}\right).$$

This proves (1). We prove (2). Since $\alpha\bar{\beta} \in H^\infty$ and $Re\left(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}\right) \geq 0$, it follows that $\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}$ is an outer function or a zero function. Hence,

$$\begin{aligned}
&\inf_{k\in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty \\
= &\inf_{k\in H^\infty} \left\| \left(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}\right)\left(\phi - \frac{k}{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}}\right) \right\|_\infty \\
= &\inf_{k\in H^\infty} \left\| \left(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}\right)(\phi - k) \right\|_\infty.
\end{aligned}$$

By (1), this implies (2). This completes the proof.

The statement (3) in the following corollary was given by Feldman, Krupnik and Marcus (cf. [3]). By the Helson-Szegö theorem [7], $\|P_+\|_{L^2(W)} < \infty$ if and only if $\inf_{k\in H^\infty} \|\phi - k\|_\infty < 1$.

Corollary 4. Let α and β be complex numbers satisfying $\alpha \neq \beta$. Let ϕ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\phi = \bar{h}/h$ and $W = |h|^2$. Let $c = \inf_{k\in H^\infty} \|\phi - k\|_\infty$. If $c < 1$, then the following equalities hold.

(1) $c\left|\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\bar{\beta}\right| = \sqrt{\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\beta|^2}.$

(2) $\left|\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\bar{\beta}\right| = |\alpha - \beta|\|S_{\alpha,\beta}\|_{L^2(W)}\|P_+\|_{L^2(W)} = \dfrac{|\alpha - \beta|\|S_{\alpha,\beta}\|_{L^2(W)}}{\sqrt{1-c^2}}.$

(3) $\|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma + \left(\dfrac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\dfrac{|\alpha| - |\beta|}{2}\right)^2},$

where

$$\gamma := \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{c^2}{1-c^2}\right) = \left|\frac{\alpha - \beta}{2}\right|^2 (\|P_+\|^2_{L^2(W)} - 1).$$

Proof. Since $\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\bar{\beta}$ is a constant, it follows from Corollary 3(2) that (1) holds. We prove (2). Since $P_+ = S_{1,0}$, it follows from (1) that

$$c\|P_+\|^2_{L^2(W)} = \|P_+\|_{L^2(W)}\sqrt{\|P_+\|^2_{L^2(W)} - 1}.$$

Hence, $(1 - c^2)\|P_+\|^2_{L^2(W)} = 1$. It follows from (1) that

$$\begin{aligned} c^2\left|\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\bar{\beta}\right|^2 &= \left(\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\alpha|^2\right)\left(\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\beta|^2\right) \\ &= \left|\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\bar{\beta}\right|^2 - |\alpha - \beta|^2\|S_{\alpha,\beta}\|^2_{L^2(W)}. \end{aligned}$$

Hence,

$$(1 - c^2)\left|\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\bar{\beta}\right|^2 = |\alpha - \beta|^2\|S_{\alpha,\beta}\|^2_{L^2(W)}.$$

Since $(1 - c^2)\|P_+\|^2_{L^2(W)} = 1$, this implies (2). We prove (3). Let $s = \|S_{\alpha,\beta}\|_{L^2(W)}$. It follows from (1) that

$$c^2|s - \alpha\bar{\beta}|^2 = (s - |\alpha|^2)(s - |\beta|^2).$$

Then

$$(1 - c^2)s^2 - \left\{|\alpha - \beta|^2 + 2(1 - c^2)Re(\alpha\bar{\beta})\right\}s + (1 - c^2)|\alpha\beta|^2 = 0.$$

Since $c < 1$, it follows that

$$s^2 - \left\{\frac{|\alpha - \beta|^2}{1 - c^2} + 2Re(\alpha\bar{\beta})\right\}s + |\alpha\beta|^2 = 0.$$

Since $s \geq \max\{|\alpha|^2, |\beta|^2\} \geq |\alpha\beta|$, it follows that

$$s = t + \sqrt{t^2 - |\alpha\beta|^2},$$

where

$$t := \frac{|\alpha - \beta|^2}{2(1-c^2)} + Re(\alpha\bar{\beta}).$$

Hence,

$$\begin{aligned}
\|S_{\alpha,\beta}\|_{L^2(W)} &= \sqrt{s} \\
&= \sqrt{t + \sqrt{t^2 - |\alpha\beta|^2}} \\
&= \sqrt{\frac{t + |\alpha\beta|}{2}} + \sqrt{\frac{t - |\alpha\beta|}{2}} \\
&= \sqrt{\frac{|\alpha-\beta|^2}{4(1-c^2)} + \frac{Re(\alpha\bar{\beta}) + |\alpha\beta|}{2}} + \sqrt{\frac{|\alpha-\beta|^2}{4(1-c^2)} + \frac{Re(\alpha\bar{\beta}) - |\alpha\beta|}{2}} \\
&= \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2},
\end{aligned}$$

where

$$\gamma := \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{c^2}{1-c^2}\right).$$

Then,

$$\|P_+\|_{L^2(W)} = \|S_{1,0}\|_{L^2(W)} = 2\sqrt{\frac{c^2}{4(1-c^2)} + \frac{1}{4}} = \frac{1}{\sqrt{1-c^2}}.$$

Hence,

$$\gamma = \left|\frac{\alpha - \beta}{2}\right|^2 (\|P_+\|^2_{L^2(W)} - 1).$$

This completes the proof.

3. THE SECOND FORMULA OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 2 is the second formula of $\|S_{\alpha,\beta}\|_{L^2(W)}$. It is the generalization of the Feldman-Krupnik-Markus theorem. Since the formula is symmetric with respect to α and β, it follows that if $\alpha\bar{\beta}$ is a constant then

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \|S_{\beta,\alpha}\|_{L^2(W)}.$$

By Theorem 2, if $W(\zeta) = 1$ and $\alpha\bar{\beta} \in H^\infty$, then

$$\|S_{\alpha,\beta}\|_{L^2} = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

If $\alpha(\zeta) = \zeta, \beta(\zeta) = 1, W(\zeta) = 1$, then $\|S_{\zeta,1}\|_{L^2} = 1$. It follows from Corollary 1 that

$$\|S_{1,\zeta}\|_{L^2} = \inf_{k \in H^\infty} \left\| 1 + |\bar{\zeta} - k| \right\|_\infty^{1/2} = \sqrt{1 + \inf_{k \in H^\infty} \|\bar{\zeta} - k\|_\infty} = \sqrt{2} \neq \|S_{\zeta,1}\|_{L^2}.$$

Theorem 2. Let $\alpha, \beta \in L^\infty$. Let ϕ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\phi = \bar{h}/h$ and $W = |h|^2$. If $\alpha\bar{\beta}$ belongs to H^∞ and $|\alpha - \beta| > 0$, then

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty,$$

where

$$\gamma_k := \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{|\phi - k|^2}{1 - |\phi - k|^2}\right).$$

Proof. Let $s = \|S_{\alpha,\beta}\|_{L^2(W)}^2$. It is well known that $\max\{|\alpha|^2, |\beta|^2\} \leq s$. For any $k \in H^\infty$ satisfying $|\phi - k| < 1$, we define the quantity N_k according to

$$N_k = \left\| \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty.$$

We prove that $\|S_{\alpha,\beta}\|_{L^2(W)} \geq \inf\{N_k; k \in H^\infty, |\phi - k| < 1\}$. It follows from Theorem 1 that $F(s) = s$. Hence there exists a function $g \in H^\infty$ such that

$$G\left(s - \alpha\bar{\beta} - \bar{\phi}g\right) \leq s.$$

It follows from Lemma 3 that $G\left(\sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2}\right) = s$. Hence,

$$G\left(s - \alpha\bar{\beta} - \bar{\phi}g\right) \leq G\left(\sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2}\right).$$

Hence,

$$|s - \alpha\bar{\beta} - \bar{\phi}g|^2 \leq (s - |\alpha|^2)(s - |\beta|^2).$$

Suppose $g = 0$. Then,

$$|s - \alpha\bar{\beta}|^2 \leq (s - |\alpha|^2)(s - |\beta|^2).$$

Hence $|\alpha - \beta|^2 s \leq 0$. Since $\max\{|\alpha|^2, |\beta|^2\} \leq s$ and $|\alpha - \beta| > 0$, it follows that $s > 0$. Hence $\alpha \equiv \beta$. This contradiction implies $g \neq 0$. Since

$$|g| - |s - \alpha\bar{\beta}| \leq |s - \alpha\bar{\beta} - \bar{\phi}g| \leq \sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2} \leq |s - \alpha\bar{\beta}|,$$

it follows that

$$0 < |g| < 2|s - \alpha\bar{\beta}|.$$

Since

$$Re(s - \alpha\bar{\beta}) \geq s - \max\{|\alpha|^2, |\beta|^2\} \geq 0,$$

and since $\alpha\bar{\beta} \in H^\infty$, it follows that $s - \alpha\bar{\beta}$ is an outer function. We define the function k according to

$$k = \frac{g}{s - \alpha\bar{\beta}}.$$

Since $|\alpha - \beta| > 0$, it follows that

$$|\phi - k|^2 = \left|\phi - \frac{g}{s - \alpha\bar{\beta}}\right|^2 \leq \frac{(s - |\alpha|^2)(s - |\beta|^2)}{|s - \alpha\bar{\beta}|^2} = 1 - \frac{|\alpha - \beta|^2 s}{|s - \alpha\bar{\beta}|^2} < 1.$$

Hence, $k \in H^\infty$, $\quad |\phi - k| < 1$, and

$$|s - \alpha\bar{\beta}|^2 - \frac{|\alpha - \beta|^2}{1 - |\phi - k|^2} s \geq 0.$$

Therefore

$$s^2 - \left(\frac{|\alpha - \beta|^2}{1 - |\phi - k|^2} + 2Re(\alpha\bar{\beta})\right) s + |\alpha\beta|^2 \geq 0.$$

Hence,

$$0 \leq s \leq t - \sqrt{t^2 - |\alpha\beta|^2},$$

or

$$s \geq t + \sqrt{t^2 - |\alpha\beta|^2},$$

where

$$t := \left(\frac{|\alpha - \beta|^2}{2} \frac{1}{1 - |\phi - k|^2} + Re(\alpha\bar{\beta})\right).$$

Hence,

$$\sqrt{s} \leq \sqrt{t - \sqrt{t^2 - |\alpha\beta|^2}} = \sqrt{\frac{t + |\alpha\beta|}{2}} - \sqrt{\frac{t - |\alpha\beta|}{2}},$$

or

$$\sqrt{s} \geq \sqrt{t + \sqrt{t^2 - |\alpha\beta|^2}} = \sqrt{\frac{t + |\alpha\beta|}{2}} + \sqrt{\frac{t - |\alpha\beta|}{2}}.$$

Suppose

$$\sqrt{s} \leq \sqrt{\frac{t + |\alpha\beta|}{2}} - \sqrt{\frac{t - |\alpha\beta|}{2}}$$

on some measurable subset E of T. Then

$$\sqrt{s} \leq \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} - \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}$$

on E. Since $\max\{|\alpha|, |\beta|\} \leq \sqrt{s}$, it follows that

$$\begin{aligned}
&\frac{|\alpha|+|\beta|}{2} + \left|\frac{|\alpha|-|\beta|}{2}\right| + \sqrt{\gamma_k + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} \\
\leq\ & \sqrt{s} + \sqrt{\gamma_k + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} \\
\leq\ & \sqrt{\gamma_k + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} \\
\leq\ & \sqrt{\gamma_k + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} + \frac{|\alpha|+|\beta|}{2}.
\end{aligned}$$

on E. Hence $|\alpha| = |\beta|$ on E. Hence $|\alpha| + \sqrt{\gamma_k} = \sqrt{\gamma_k + |\alpha|^2}$ on E. Hence $\alpha\gamma_k = 0$ on E. Hence $\gamma_k = 0$ on E. Therefore,

$$\sqrt{s} \geq \sqrt{\gamma_k + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha|-|\beta|}{2}\right)^2}$$

on T. Hence,

$$\|S_{\alpha,\beta}\|_{L^2(W)} \geq \inf_{k \in H^\infty, |\phi - k| < 1} N_k.$$

Next we prove the reverse inequality. This is the easy direction of the theorem. Since

$$N_k \geq \sqrt{\gamma_k + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha|-|\beta|}{2}\right)^2},$$

squaring both sides, it follows that

$$N_k^2 \geq 2\gamma_k + \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{\left(2\gamma_k + \frac{|\alpha|^2+|\beta|^2}{2}\right)^2 - |\alpha\beta|^2}.$$

Hence,

$$N_k^4 - (4\gamma_k + |\alpha|^2 + |\beta|^2)N_k^2 + |\alpha\beta|^2 \geq 0.$$

Hence,

$$4\gamma_k N_k^2 \leq (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Hence,

$$N_k^2 |\alpha - \beta|^2 \left(\frac{|\phi - k|^2}{1 - |\phi - k|^2}\right) \leq (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Since

$$N_k^2 |\alpha - \beta|^2 = |N_k^2 - \alpha\bar{\beta}|^2 - (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2),$$

it follows that

$$\left(|N_k^2-\alpha\bar{\beta}|^2-(N_k^2-|\alpha|^2)(N_k^2-|\beta|^2)\right)\left(\frac{|\phi-k|^2}{1-|\phi-k|^2}\right)\le (N_k^2-|\alpha|^2)(N_k^2-|\beta|^2).$$

Hence,

$$|N_k^2-\alpha\bar{\beta}|^2|\phi-k|^2\le (N_k^2-|\alpha|^2)(N_k^2-|\beta|^2).$$

It follows from Lemma 3 that

$$G\left(|N_k^2-\alpha\bar{\beta}||\phi-k|\right)\le G\left(\sqrt{N_k^2-|\alpha|^2}\sqrt{N_k^2-|\beta|^2}\right)=N_k^2.$$

Since

$$\left|N_k^2-\alpha\bar{\beta}\right||\phi-k|=\left|(N_k^2-\alpha\bar{\beta})(1-\bar{\phi}k)\right|=\left|N_k^2-\alpha\bar{\beta}-\bar{\phi}(N_k^2-\alpha\bar{\beta})k\right|$$

and $(N_k^2-\alpha\bar{\beta})k\in H^\infty$, it follows that $F(N_k^2)\le N_k^2$. It follows from Lemma 4 that $N_k^2\ge \|S_{\alpha,\beta}\|^2_{L^2(W)}$. This completes the proof.

Corollary 5. If $\inf_{k\in H^\infty}\|\phi-k\|_\infty=c<1$, $\alpha\bar{\beta}\in H^\infty$ and $|\alpha-\beta|>0$, then

$$(1)\quad \|S_{\alpha,\beta}\|_{L^2(W)}\ge \inf_{k\in H^\infty}\left\|\sqrt{\max\{|\alpha|^2,|\beta|^2\}+\frac{|\alpha-\beta|^2|\phi-k|^2}{1-|\phi-k|^2}}\right\|_\infty,$$

$$(2)\quad \|S_{\alpha,\beta}\|_{L^2(W)}\le \left\|\sqrt{\gamma+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\right\|_\infty,$$

$$(3)\quad \max\{\|\alpha\|_\infty,\|\beta\|_\infty\}\le \|S_{\alpha,\beta}\|_{L^2(W)}\le \|\max\{|\alpha|,|\beta|\}+2\sqrt{\gamma}\|_\infty,$$

where

$$\gamma:=\left|\frac{\alpha-\beta}{2}\right|^2\left(\frac{c^2}{1-c^2}\right)=\left|\frac{\alpha-\beta}{2}\right|^2\left(\|P_+\|^2_{L^2(W)}-1\right).$$

Proof. We prove (1). Let $\gamma_k=\left|\frac{\alpha-\beta}{2}\right|^2\left(\frac{|\phi-k|^2}{1-|\phi-k|^2}\right)$. Then

$$\begin{aligned}
&\left(\sqrt{\gamma_k+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma_k+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\right)^2\\
=\;& 2\gamma_k+\frac{|\alpha|^2+|\beta|^2}{2}+2\sqrt{\gamma_k+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}\sqrt{\gamma_k+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\\
\ge\;& 2\gamma_k+\frac{|\alpha|^2+|\beta|^2}{2}+2\gamma_k+\left|\frac{|\alpha|^2-|\beta|^2}{2}\right|\\
=\;& \max\{|\alpha|^2,|\beta|^2\}+4\gamma_k.
\end{aligned}$$

By Theorem 2, this implies (1). We prove (2). Since

$$\gamma_k = \left|\frac{\alpha-\beta}{2}\right|^2 \left(\frac{|\phi-k|^2}{1-|\phi-k|^2}\right) \le \left|\frac{\alpha-\beta}{2}\right|^2 \left(\frac{\|\phi-k\|_\infty^2}{1-\|\phi-k\|_\infty^2}\right) \le \left|\frac{\alpha-\beta}{2}\right|^2 \left(\frac{c^2}{1-c^2}\right) = \gamma,$$

(2) follows from Theorem 2. Since

$$\begin{aligned} & \sqrt{\gamma + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} \\ \le \; & \sqrt{\gamma} + \frac{|\alpha|+|\beta|}{2} + \sqrt{\gamma} + \left|\frac{|\alpha|-|\beta|}{2}\right| \\ = \; & \max\{|\alpha|, |\beta|\} + 2\sqrt{\gamma}, \end{aligned}$$

(3) follows from (2). This completes the proof.

Example 1. Let $\zeta = e^{i\theta}$, $\alpha(\zeta) = \zeta + 1$, $\beta(\zeta) = 1$ and $W(\zeta) = |\zeta+1|^{1/2}$. Then $\alpha\bar{\beta} \in H^\infty, \phi(e^{i\theta}) = e^{-i\theta/4}, \zeta = e^{i\theta}, -\pi \le \theta < \pi$. Hence, $|\phi - \frac{1}{\sqrt{2}}| \le \frac{1}{\sqrt{2}} < 1$. By Theorem 2,

$$2 \le \|S_{\alpha,\beta}\|_{L^2(W)} \le \left\| \sqrt{\gamma_{1/\sqrt{2}} + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma_{1/\sqrt{2}} + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} \right\|_\infty,$$

$$\gamma_{1/\sqrt{2}} = \frac{1}{4}\left(\frac{1}{1-|\phi-\frac{1}{\sqrt{2}}|^2} - 1\right).$$

Hence,

$$\begin{aligned} 2 \le \|S_{\alpha,\beta}\|_{L^2(W)} \le \sup_{-\pi\le\theta<\pi} & \left\{ \sqrt{\frac{3-2\sqrt{2}\cos(\theta/4)}{4(2\sqrt{2}\cos(\theta/4)-1)} + \left(\frac{\sqrt{2(1+\cos\theta)}+1}{2}\right)^2} \right. \\ & \left. + \sqrt{\frac{3-2\sqrt{2}\cos(\theta/4)}{4(2\sqrt{2}\cos(\theta/4)-1)} + \left(\frac{\sqrt{2(1+\cos\theta)}-1}{2}\right)^2} \right\} \\ = & \sqrt{\frac{29+2\sqrt{2}}{14}} + \sqrt{\frac{1+2\sqrt{2}}{14}} < 2.04. \end{aligned}$$

Since $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi/4)} = \sqrt{2}$ (cf. [12]),

$$\gamma = \left|\frac{\alpha-\beta}{2}\right|^2 (\|P_+\|_{L^2(W)}^2 - 1) = \frac{1}{4},$$

and hence

$$\left\| \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty$$

$$= \sqrt{\frac{1}{4} + \left(\frac{3}{2}\right)^2} + \sqrt{\frac{1}{4} + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{10} + \sqrt{2}}{2} > 2.28.$$

This example shows that it is not able to change the function γ_k by the function γ in Theorem 2, and that

$$2 \leq \|S_{\alpha,\beta}\|_{L^2(W)} < 2.04.$$

Corollary 6. If $|\alpha(\zeta)| > 0$, then

$$\|\alpha P_+\|_{L^2(W)} = \|\alpha P_-\|_{L^2(W)} = \inf_{k \in H^\infty} \left\| \frac{\alpha}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty .$$

Corollary 7. Let $\alpha \in H^\infty$. Then

(1) If $W^{-1} \in L^1$, then there exists $\varepsilon_n \in L^\infty, \varepsilon_n > 0, \|\varepsilon_n\|_\infty \to 0, (n \to \infty)$ such that

$$\|\alpha P_+\|_{L^2(W)} = \lim_{n\to\infty} \|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} = \lim_{n\to\infty} \inf_{k\in H^\infty} \left\| \frac{|\alpha| + \varepsilon_n}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty .$$

(2) If $\inf_{k\in H^\infty} \|\phi - k\|_\infty < 1$, then we can take $\varepsilon_n = \frac{1}{n}$.

Proof. We prove (1). Since $|\alpha| + \varepsilon_n > 0$, it follows from Corollary 6 that

$$\|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} = \left\| \frac{|\alpha| + \varepsilon_n}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty .$$

By the Koosis theorem [9], if $W^{-1} \in L^1$, then there exists $U \in L^1, U > 0$ such that

$$\|P_+ f\|_{L^2(U)} \leq \|f\|_{L^2(W)}, \quad (f \in L^2(W)).$$

Let $\varepsilon_n = \frac{1}{n}\sqrt{\frac{U}{W}}$. Since $U \leq W$, it follows that $\varepsilon_n \leq \frac{1}{n}$ and

$$\|\varepsilon_n P_+ f\|_{L^2(W)} \leq \frac{1}{n}\|f\|_{L^2(W)}, \quad (f \in L^2(W)).$$

Hence,

$$\lim_{n\to\infty} \left| \|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} - \|\alpha P_+\|_{L^2(W)} \right| \leq \lim_{n\to\infty} \|\varepsilon_n P_+\|_{L^2(W)} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

We prove (2). By the Helson-Szegö theorem [7], if $\inf_{k\in H^\infty}\|\phi-k\|_\infty<1$, then $\|P_+\|_{L^2(W)}<\infty$. Hence,

$$\lim_{n\to\infty}\left|\left\|\left(|\alpha|+\frac{1}{n}\right)P_+\right\|_{L^2(W)}-\|\alpha P_+\|_{L^2(W)}\right|\le\lim_{n\to\infty}\left(\frac{1}{n}\|P_+\|_{L^2(W)}\right)=0.$$

This completes the proof.

Example 2. Suppose $W(\zeta)=|\zeta+1|^{1/2}, h(\zeta)=(\zeta+1)^{1/2}, \zeta=e^{i\theta}, \phi(\zeta)=\overline{h(\zeta)}/h(\zeta)=e^{-i\theta/4}$ and $E\subset T$. By Corollaries 8 and 9,

$$\begin{aligned}\|\chi_E P_+\|_{L^2(W)} &= \lim_{n\to\infty}\|(\chi_E+\frac{1}{n})P_+\|_{L^2(W)}=\lim_{n\to\infty}\inf_{k\in H^\infty}\left\|\frac{\chi_E+\frac{1}{n}}{\sqrt{1-|\phi-k|^2}}\right\|_\infty\\ &\le \left\|\frac{\chi_E+0.1}{\sqrt{1-|e^{-i\theta/4}-\frac{1}{\sqrt{2}}|^2}}\right\|_\infty=\left\|\frac{\chi_E+0.1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4})-\frac{1}{2}}}\right\|_\infty.\end{aligned}$$

If $-\pi\le\theta<\pi$, then $\frac{1}{\sqrt{2}}\le\cos(\frac{\theta}{4})\le 1$. Hence,

$$\sqrt{\frac{2}{2\sqrt{2}-1}}\le\frac{1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4})-\frac{1}{2}}}\le\sqrt{2}.$$

Hence, on E^c,

$$\frac{\chi_E+0.1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4})-\frac{1}{2}}}<0.15.$$

If $\theta=0$, then

$$\frac{1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4})-\frac{1}{2}}}=\sqrt{\frac{2}{2\sqrt{2}-1}}=1.04...<1.05.$$

Hence, for sufficiently small $\varepsilon>0$, $E=(-\varepsilon,\varepsilon)$ satisfies

$$\|\chi_E P_+\|_{L^2(W)}<1.05\cdot 1.1=1.155.$$

Since $\|P_+\|_{L^2(W)}=\frac{1}{\cos(\pi/4)}=\sqrt{2}$ (cf. [12]), it follows that

$$\|\chi_E P_+\|_{L^2(W)}<\|\chi_E\|_\infty\|P_+\|_{L^2(W)}.$$

Hence, $\|\alpha P_+\|_{L^2(W)}=\|\alpha\|_\infty\|P_+\|_{L^2(W)}$ does not hold in general.

Next, we compare $\|S_{\alpha,\beta}\|_{L^2(W)}$ and $\|S\|_{L^2(W)}$. Let $\alpha,\beta\in L^\infty$. Since $S_{\alpha,\beta}=\frac{\alpha+\beta}{2}I+\frac{\alpha-\beta}{2}S$, it follows that

$$\|S_{\alpha,\beta}\|_{L^2(W)}\le\left\|\frac{\alpha+\beta}{2}\right\|_\infty+\left\|\frac{\alpha-\beta}{2}\right\|_\infty\|S\|_{L^2(W)}.$$

By Theorem 2, we have

Corollary 8. If $\alpha\bar{\beta} \in H^\infty$ and $|\alpha - \beta| > 0$, then

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}\|S\|_{L^2(W)}.$$

Proof. Without loss of generality, we assume that $\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} = 1$ and $\|S\|_{L^2(W)} < \infty$. Let $c = \inf_{k \in H^\infty} \|\phi - k\|_\infty$. By the Helson-Szegö theorem [7], $c < 1$. It follows from Theorem 2 that

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty,$$

where

$$\gamma_k := \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{|\phi - k|^2}{1 - |\phi - k|^2}\right).$$

Since $\max\{|\alpha|, |\beta|\} \leq 1$, it follows that

$$\gamma_k \leq \frac{|\phi - k|^2}{1 - |\phi - k|^2} \leq \frac{c^2}{1 - c^2}.$$

Hence,

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leq \left\| \sqrt{\frac{c^2}{1 - c^2} + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\frac{c^2}{1 - c^2} + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty.$$

It follows from Theorem 2 that

$$\begin{aligned} \|S\|_{L^2(W)} &= \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \sqrt{\frac{|\phi - k|^2}{1 - |\phi - k|^2} + 1} + \sqrt{\frac{|\phi - k|^2}{1 - |\phi - k|^2}} \right\|_\infty \\ &= \sqrt{\frac{c^2}{1 - c^2} + 1} + \sqrt{\frac{c^2}{1 - c^2}}. \end{aligned}$$

Let $x = \frac{c^2}{1 - c^2}$. It is sufficient to show that

$$\sqrt{x + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{x + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \leq \sqrt{x + 1} + \sqrt{x}.$$

Since $(1 - |\alpha|^2)(1 - |\beta|^2) \geq 0$, it follows that

$$\left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2 \leq \left(1 - \frac{|\alpha|^2 + |\beta|^2}{2}\right)^2.$$

Hence,

$$\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2+2(|\alpha|^2+|\beta|^2)x\le\left(1-\frac{|\alpha|^2+|\beta|^2}{2}\right)^2+4x.$$

Hence,

$$\begin{aligned}
& 4\left(x+\left(\frac{|\alpha|+|\beta|}{2}\right)^2\right)\left(x+\left(\frac{|\alpha|-|\beta|}{2}\right)^2\right)\\
\le\ & \left(1-\frac{|\alpha|^2+|\beta|^2}{2}\right)^2+4x(x+1)\\
\le\ & \left(1-\frac{|\alpha|^2+|\beta|^2}{2}\right)^2+4\left(1-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sqrt{x(x+1)}+4x(x+1).
\end{aligned}$$

Hence,

$$\begin{aligned}
& 2x+\frac{|\alpha|^2+|\beta|^2}{2}+2\sqrt{x+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}\sqrt{x+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\\
\le\ & 2x+1+2\sqrt{x(x+1)}.
\end{aligned}$$

Hence,

$$\left(\sqrt{x+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{x+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\right)^2\le\left(\sqrt{x+1}+\sqrt{x}\right)^2.$$

This completes the proof.

4. THE THIRD FORMULA OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 3 is the third formula of $\|S_{\alpha,\beta}\|_{L^2(W)}$. Suppose $\alpha(\zeta)=1,\beta(\zeta)=\zeta$. Then $\alpha\bar{\beta}\notin H^\infty$. By Corollary 1, $\|S_{\alpha,\beta}\|_{L^2}=\sqrt{2}$. Since $\|S\|_{L^2}=1$, it follows that $\|S_{\alpha,\beta}\|_{L^2}>\|S\|_{L^2}$. Hence Corollary 8 does not hold in general. Hence Theorem 2 does not hold in general. But the following Theorem 3 holds even when $\alpha\bar{\beta}\notin H^\infty$. We give Lemma 6 to prove Theorem 3.

Lemma 6. If $|g|<1$, then

$$\text{(1)}\qquad \frac{|\alpha|^2+|\beta|^2-2Re(\bar{\alpha}\beta g)}{1-|g|^2}=|\alpha|^2+\frac{|\bar{\beta}-\bar{\alpha}g|^2}{1-|g|^2}=|\beta|^2+\frac{|\alpha-\beta g|^2}{1-|g|^2}.$$

$$\text{(2)}\qquad \max\{|\alpha|^2,|\beta|^2\}\le\frac{|\alpha|^2+|\beta|^2-2Re(\bar{\alpha}\beta g)}{1-|g|^2}.$$

(3) $\dfrac{(|\alpha|-|\beta|)^2}{1-|g|^2} \le \dfrac{\max\{|\bar\beta-\bar\alpha g|^2, |\alpha-\beta g|^2\}}{1-|g|^2} \le \dfrac{|\alpha|^2+|\beta|^2-2Re(\bar\alpha\beta g)}{1-|g|^2} \le \dfrac{(|\alpha|+|\beta|)^2}{1-|g|^2}.$

Proof. Since

$$|\alpha|^2+|\beta|^2-2Re(\bar\alpha\beta g)-|\alpha|^2(1-|g|^2)=|\bar\beta-\bar\alpha g|^2$$

and

$$|\alpha|^2+|\beta|^2-2Re(\bar\alpha\beta g)-|\beta|^2(1-|g|^2)=|\alpha-\beta g|^2,$$

we have (1). (1) implies (2). We prove (3). Since $|g|<1$, it follows that

$$|\alpha|^2+|\beta|^2-2Re(\bar\alpha\beta g)\le(|\alpha|+|\beta|)^2.$$

If $|\alpha|\le|\beta|$, then $|\bar\beta-\bar\alpha g|^2\ge(|\beta|-|\alpha g|)^2\ge(|\beta|-|\alpha|)^2$.
If $|\alpha|\ge|\beta|$, then $|\alpha-\beta g|^2\ge(|\alpha|-|\beta g|)^2\ge(|\alpha|-|\beta|)^2$.
This completes the proof.

Theorem 3. Let $\alpha,\beta\in L^\infty$. Let ϕ and W be functions such that there exists an outer function $h\in H^2$ satisfying $\phi=\bar h/h$ and $W=|h|^2$.
(1) If $||\alpha|-|\beta||>0$, then

$$\begin{aligned}\|S_{\alpha,\beta}\|^2_{L^2(W)} &= \inf_{k\in H^\infty,|\phi-k|<1}\left\|\frac{|\alpha|^2+|\beta|^2-2Re(\bar\alpha\beta(1-\bar\phi k))}{1-|\phi-k|^2}\right\|_\infty\\ &= \inf_{k\in H^\infty,|\phi-k|<1}\left\||\alpha|^2+\frac{|\bar\beta-\bar\alpha(1-\bar\phi k)|^2}{1-|\phi-k|^2}\right\|_\infty\\ &= \inf_{k\in H^\infty,|\phi-k|<1}\left\||\beta|^2+\frac{|\alpha-\beta(1-\bar\phi k)|^2}{1-|\phi-k|^2}\right\|_\infty.\end{aligned}$$

The infimum is attained.
(2) If $m\{\zeta\in T;|\alpha(\zeta)|=|\beta(\zeta)|\}>0$ and $W^{-1}\in L^1$, then there exists $\varepsilon_n\in L^\infty,\varepsilon_n>0,\|\varepsilon_n\|_\infty\to0,(n\to\infty)$ such that $||\alpha_n|-|\beta||>0$ and

$$\|S_{\alpha,\beta}\|_{L^2(W)}=\lim_{n\to0}\inf_{k\in H^\infty,|\phi-k|<1}\|S_{\alpha_n,\beta}\|_{L^2(W)},$$

where

$$\begin{aligned}&\alpha_n:=\alpha+(\alpha\chi_{E_1}+\chi_{E_0})\varepsilon_n\\ &E_0:=\{\zeta\in T;\alpha(\zeta)=\beta(\zeta)=0\},\\ &E_1:=\{\zeta\in T;|\alpha(\zeta)|=|\beta(\zeta)|>0\}.\end{aligned}$$

Proof. We prove (1). Let $s=\|S_{\alpha,\beta}\|^2_{L^2(W)}$. It is well known that $\max\{|\alpha|^2,|\beta|^2\}\le s$. Since $||\alpha|-|\beta||>0$, it follows that $s>0$. For any $k\in H^\infty$ satisfying $|\phi-k|<1$, we define the quantity M_k according to

$$M_k=\left\|\frac{|\alpha|^2+|\beta|^2-2Re(\bar\alpha\beta(1-\bar\phi k))}{1-|\phi-k|^2}\right\|_\infty.$$

We prove that $s \geq \inf\{M_k; k \in H^\infty, |\phi - k| < 1\}$. It follows from Theorem 1 that $F(s) = s$. Hence there exists a function $k \in H^\infty$ such that

$$G\left(s - \alpha\bar{\beta} - \bar{\phi}k\right) \leq s.$$

It follows from Lemma 3 that

$$G\left(\sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2}\right) = s.$$

Since

$$\left|\alpha\bar{\beta} - s\left(1 - \bar{\phi}\frac{k}{s}\right)\right|^2 = |s - \alpha\bar{\beta} - \bar{\phi}k|^2 \leq (s - |\alpha|^2)(s - |\beta|^2),$$

it follows that there exists a $k_0 \in H^\infty$ such that

$$|\alpha\bar{\beta} - s(1 - \bar{\phi}k_0)|^2 \leq (s - |\alpha|^2)(s - |\beta|^2).$$

Hence,

$$|\alpha|^2 + |\beta|^2 - 2Re(\bar{\alpha}\beta(1 - \bar{\phi}k_0)) \leq s(1 - |\phi - k_0|^2).$$

Since

$$s|\phi - k_0| \leq \sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2} + |\alpha\beta| \leq s,$$

it follows that $|\phi - k_0| \leq 1$. Since $||\alpha| - |\beta|| > 0$, it follows that $|\phi - k_0| < 1$. Hence,

$$\inf_{k \in H^\infty, |\phi - k| < 1} M_k \leq M_{k_0} \leq s.$$

We prove that $s \leq \inf\{M_k; k \in H^\infty, |\phi - k| < 1\}$. This is the easy direction of the theorem. Since

$$|\alpha|^2 + |\beta|^2 - 2Re(\bar{\alpha}\beta(1 - \bar{\phi}k)) \leq M_k(1 - |\phi - k|^2),$$

it follows that

$$\begin{aligned}
|M_k - \alpha\bar{\beta} - \bar{\phi}M_k k|^2 &= |M_k(1 - \bar{\phi}k) - \alpha\bar{\beta}|^2 \\
&= |M_k(1 - \bar{\phi}k)|^2 - 2M_k Re(\bar{\alpha}\beta(1 - \bar{\phi}k)) + |\alpha\beta|^2 \\
&\leq M_k^2 - (|\alpha|^2 + |\beta|^2)M_k + |\alpha\beta|^2 \\
&= (M_k - |\alpha|^2)(M_k - |\beta|^2).
\end{aligned}$$

It follows from Lemma 6 that $\max\{|\alpha|^2, |\beta|^2\} \leq M_k$. It follows from Lemma 3 that

$$G\left(\sqrt{M_k - |\alpha|^2}\sqrt{M_k - |\beta|^2}\right) = M_k.$$

Hence,

$$G\left(M_k - \alpha\bar{\beta} - \bar{\phi}M_k k\right) \leq M_k.$$

Hence $F(M_k) \leq M_k$. It follows from Lemma 4 that $s \leq M_k$. Therefore,

$$\inf_{k\in H^\infty,|\phi-k|<1} M_k \leq M_{k_0} \leq s \leq \inf_{k\in H^\infty,|\phi-k|<1} M_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. We prove (2). By Corollary 7, there exists $\varepsilon_n \in L^\infty, \varepsilon_n > 0, ||\varepsilon_n||_\infty \to 0, (n \to \infty)$ such that $||\alpha_n| - |\beta|| > 0$, and

$$||\varepsilon_n P_+||_{L^2(W)} \to 0, (n \to \infty).$$

Since

$$||(\alpha_n - \alpha)P_+||_{L^2(W)} = ||(\alpha\chi_{E_1} + \chi_{E_0})\varepsilon_n P_+||_{L^2(W)} \leq ||(\alpha\chi_{E_1} + \chi_{E_0})||_\infty ||\varepsilon_n P_+||_{L^2(W)},$$

it follows that

$$||(\alpha_n - \alpha)P_+||_{L^2(W)} \to 0, (n \to \infty).$$

Since

$$\Big| ||S_{\alpha_n,\beta}||_{L^2(W)} - ||S_{\alpha,\beta}||_{L^2(W)} \Big| \leq ||S_{\alpha_n,\beta} - S_{\alpha,\beta}||_{L^2(W)} = ||(\alpha_n - \alpha)P_+||_{L^2(W)},$$

it follows that

$$\Big| ||S_{\alpha_n,\beta}||_{L^2(W)} - ||S_{\alpha,\beta}||_{L^2(W)} \Big| \to 0, (n \to \infty).$$

This completes the proof.

If W is a constant function, then ϕ becomes a constant function. In this case, the formula of $||S_{\alpha,\beta}||_{L^2}$ was given by Corollary 1. The another formula of $||S_{\alpha,\beta}||_{L^2}$ is given by Corollary 9.

Corollary 9. Let $\alpha, \beta \in L^\infty$. If $||\alpha| - |\beta|| > 0$, then

$$\begin{aligned}
||S_{\alpha,\beta}||_{L^2}^2 &= \inf_{g\in H^\infty,|g|<1} \left\| \frac{|\alpha|^2 + |\beta|^2 - 2Re(\bar{\alpha}\beta g))}{1 - |g|^2} \right\|_\infty \\
&= \inf_{g\in H^\infty,|g|<1} \left\| |\alpha|^2 + \frac{|\bar{\beta} - \bar{\alpha}g|^2}{1 - |g|^2} \right\|_\infty \\
&= \inf_{g\in H^\infty,|g|<1} \left\| |\beta|^2 + \frac{|\alpha - \beta g|^2}{1 - |g|^2} \right\|_\infty .
\end{aligned}$$

Corollary 10. Let $\alpha, \beta \in L^\infty$. Let ϕ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\phi = \bar{h}/h$ and $W = |h|^2$. If $||\alpha| - |\beta|| > 0$, then

$$(1) \qquad \inf_{k\in H^\infty} \left\| \frac{|\alpha| - |\beta|}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty \leq \inf_{k\in H^\infty} \left\| \frac{\max\{|\bar{\beta} - \bar{\alpha}(1 - \bar{\phi}k)|, |\alpha - \beta(1 - \bar{\phi}k)|\}}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty$$

$$\leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \inf_{k\in H^\infty} \left\| \frac{|\alpha|+|\beta|}{\sqrt{1-|\phi-k|^2}} \right\|_\infty .$$

(2)
$$\max\left\{ \|\alpha\|_\infty, \|\beta\|_\infty, \inf_{k\in H^\infty} \left\| \frac{|\alpha|-|\beta|}{\sqrt{1-|\phi-k|^2}} \right\|_\infty \right\}$$
$$\leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \inf_{k\in H^\infty} \left\| \frac{\alpha-\beta}{\sqrt{1-|\phi-k|^2}} \right\|_\infty + \min\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Proof. (1) follows from Theorem 3(1) and Lemma 6. (2) follows from Corollary 6 and

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leq \|(\alpha-\beta)P_+\|_{L^2(W)} + \|\beta\|_\infty,$$
$$\|S_{\alpha,\beta}\|_{L^2(W)} \leq \|(\beta-\alpha)P_-\|_{L^2(W)} + \|\alpha\|_\infty.$$

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