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NORMS OF SOME SINGULAR INTEGRAL  
OPERATORS ON WEIGHTED  $L^2$  SPACES

Takahiko Nakazi and Takanori Yamamoto

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NORMS OF SOME SINGULAR INTEGRAL OPERATORS ON  
WEIGHTED  $L^2$  SPACES\*

by

TAKAHIKO NAKAZI  
Department of Mathematics  
Faculty of Science  
Hokkaido University  
Sapporo 060-0810, Japan  
nakazi@math.sci.hokudai.ac.jp

and

TAKANORI YAMAMOTO  
Department of Mathematics  
Hokkai-Gakuen University  
Sapporo 062-8605, Japan  
yamatk@hucc.hokudai.ac.jp

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Ministry of Education.

Abstract. Let  $\alpha$  and  $\beta$  be measurable functions on the unit circle  $T$ , and let  $W$  be a positive function on  $T$  such that the Riesz projection  $P_+$  is bounded on the weighted space  $L^2(W)$  on  $T$ . The singular integral operator  $S_{\alpha,\beta}$  is defined by  $S_{\alpha,\beta}f = \alpha P_+f + \beta P_-f$ , ( $f \in L^2(W)$ ) where  $P_- = I - P_+$ . Let  $h$  be an outer function such that  $W = |h|^2$ , and let  $\phi$  be a unimodular function such that  $\phi = \bar{h}/h$ . In this paper, the norm of  $S_{\alpha,\beta}$  on  $L^2(W)$  is calculated in general, using  $\alpha, \beta$  and  $\phi$ . Moreover, if  $\alpha$  and  $\beta$  are constant functions, then we give the another proof of the Feldman-Krupnik-Marcus theorem. If  $\alpha\bar{\beta}$  belongs to the Hardy space  $H^\infty$ , we give the theorem which is similar to the Feldman-Krupnik-Marcus theorem.

Keywords: Singular integral operator, Norm, Hardy space, Helson-Szegö weight,  $(A_2)$ condition.

AMS Subject Classification: Primary 45E10, 47B35; Secondary 46J15.

## 1. INTRODUCTION

Let  $m$  denote the normalized Lebesgue measure on the unit circle  $T = \{\zeta; |\zeta| = 1\}$ . That is,  $dm(\zeta) = d\theta/2\pi$  for  $\zeta = e^{i\theta}$ . For functions  $f$  and  $g$  satisfying  $fg \in L^1$ , we define the quantity  $(f, g)$  according to

$$(f, g) = \int_T f(\zeta)\overline{g(\zeta)}dm(\zeta).$$

If  $f, g \in L^2$ , then this becomes the inner product. Let  $H^2$  (resp.  $H^\infty$ ) be the Hardy space of functions  $f \in L^2$  (resp.  $f \in L^\infty$ ) whose negative Fourier coefficients are zero. Let  $S$  be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi i} \int_T \frac{f(\eta)}{\eta - \zeta} d\eta \quad (a.e. \zeta \in T),$$

where the integral is understood in the sense of Cauchy's principal value (cf. [6, p.11]). If  $f$  is in  $L^1$ , then  $Sf(\zeta)$  exists for almost everywhere  $\zeta$  on  $T$ , and  $Sf$  becomes a measurable function on  $T$ . For a positive function  $W \in L^1$ , the norm in  $L^2(W)$  is defined by the formula

$$\|f\|_{L^2(W)} = (Wf, f)^{1/2} = \left\{ \int_T |f(\zeta)|^2 W(\zeta) dm(\zeta) \right\}^{1/2}.$$

Let  $A$  (resp.  $\overline{A_0}$ ) be the subspace of continuous functions  $f$  on  $T$  whose negative (resp. positive) Fourier coefficients are zero. Let  $A + \overline{A_0} = \{f_1 + f_2; f_1 \in A, f_2 \in \overline{A_0}\}$ . Then  $A + \overline{A_0}$  is dense in  $L^2(W)$  in norm. Two projections  $P_+$  and  $P_-$  are defined by

$$P_+ = (I + S)/2, \quad \text{and} \quad P_- = (I - S)/2.$$

where  $I$  denotes the identity operator.  $P_+$  is the Riesz projection. For  $\alpha, \beta \in L^\infty$ , let  $S_{\alpha, \beta}$  be the singular integral operator on  $L^2(W)$  defined by

$$S_{\alpha, \beta} f = \alpha P_+ f + \beta P_- f, \quad (f \in L^2(W)).$$

Then,  $S_{1,1} = I$ ,  $S_{1,-1} = S$ ,  $S_{1,0} = P_+$  and  $S_{0,1} = P_-$ . Let  $\|S_{\alpha, \beta}\|_{L^2(W)}$  denote the operator norm of  $S_{\alpha, \beta}$  on  $L^2(W)$ . That is,

$$\|S_{\alpha, \beta}\|_{L^2(W)} = \sup \left\{ \|S_{\alpha, \beta} f\|_{L^2(W)}; f \in L^2(W), \|f\|_{L^2(W)} = 1 \right\}.$$

Let  $W$  be a positive function in  $L^1$  on  $T$  such that  $S$  becomes a bounded operator on  $L^2(W)$ . The relation between the norms of the operators  $S, P_+, P_-$  on the space  $L^2(W)$ ,

$$\|P_+\|_{L^2(W)} = \|P_-\|_{L^2(W)} = \frac{\|S\|_{L^2(W)} + \|S\|_{L^2(W)}^{-1}}{2}$$

was remarked by Spitkovskii ([16]). Let  $h$  be an outer function such that  $W = |h|^2$ , and let  $\phi$  be a unimodular function such that  $\phi = \bar{h}/h$ . Let

$$c = \inf_{k \in H^\infty} \|\phi - k\|_\infty.$$

Then

$$\|S\|_{L^2(W)} = \|S_{1,-1}\|_{L^2(W)} = \sqrt{\frac{1+c}{1-c}},$$

$$\|P_+\|_{L^2(W)} = \|S_{1,0}\|_{L^2(W)} = \frac{1}{\sqrt{1-c^2}}.$$

(cf. [4]). For  $\zeta_0 \in T$ , and  $-1 < \delta < 1$ , let  $W(\zeta) = |\zeta - \zeta_0|^\delta$ . Then the equality  $\|S\|_{L^2(W)} = \cot \frac{\pi(1-|\delta|)}{4}$  was obtained by Krupnik and Verbitskii [12]. Hence  $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi\delta/2)}$ . For continuous functions  $\alpha$  and  $\beta$ , the essential norm of  $S_{\alpha,\beta}$  on  $L^2(W)$  was calculated by Krupnik and Avendanio (cf. [10, p.57, Corollary 6.1], [1]). For constant functions  $\alpha, \beta$  and a positive function  $W \in L^1$  such that  $\|P_+\|_{L^2(W)} < \infty$ , the equality

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2},$$

where

$$\gamma := \left|\frac{\alpha - \beta}{2}\right|^2 (\|P_+\|_{L^2(W)}^2 - 1) = \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{c^2}{1-c^2}\right)$$

was obtained by Feldman, Krupnik and Markus (cf. [3], [11], [6, Section 13.5], [5], [18]). In this paper, for functions  $\alpha, \beta \in L^\infty$ , and a positive function  $W \in L^1$  on the unit circle  $T$ , we will give three formulae of the norm  $\|S_{\alpha,\beta}\|_{L^2(W)}$ . It follows from the Koosis theorem [9] that there exist different functions  $\alpha$  and  $\beta$  such that  $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$  if and only if  $W^{-1} \in L^1$  (cf. [13]). If  $\log W \notin L^1$ , then  $W^{-1} \notin L^1$ . In this case  $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$  implies that  $\alpha \equiv \beta$ . Hence,  $\|S_{\alpha,\beta}\|_{L^2(W)} = \|\alpha I\|_{L^2(W)} = \|\alpha\|_\infty$ . Therefore we assume that  $\log W \in L^1$ . In Section 2, for functions  $\alpha, \beta \in L^\infty$ , and a positive function  $W \in L^1$  on the unit circle  $T$ , we will give the first formula (Theorem 1) of the norm  $\|S_{\alpha,\beta}\|_{L^2(W)}$  using  $\alpha, \beta$  and  $\phi$ . We will also give the another proof of the Feldman-Krupnik-Marcus theorem in Corollary 4. In Section 3, we will give the second formula (Theorem 2) of the norm  $\|S_{\alpha,\beta}\|_{L^2(W)}$ . If  $\alpha\bar{\beta}$  belongs to the Hardy space  $H^\infty$ , we give the theorem which is similar to the Feldman-Krupnik-Marcus theorem. In Section 4, we will give the third formula (Theorem 3) of the norm  $\|S_{\alpha,\beta}\|_{L^2(W)}$ .

## 2. THE FIRST FORMULA OF NORM OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 1 is the first formula of  $\|S_{\alpha,\beta}\|_{L^2(W)}$ . We will use Theorem 1 to prove Theorems 2 and 3 in the following sections. We give some lemmas to prove Theorem 1.

**Definition 1.** Let  $\alpha, \beta \in L^\infty$ . For each  $\gamma \in L^\infty$ , we define the function  $G(\gamma) \in L^\infty$  according to

$$(G(\gamma))(\zeta) = \frac{|\alpha(\zeta)|^2 + |\beta(\zeta)|^2}{2} + \sqrt{|\gamma(\zeta)|^2 + \left(\frac{|\alpha(\zeta)|^2 - |\beta(\zeta)|^2}{2}\right)^2}, \quad (\zeta \in T).$$

**Definition 2.** Let  $\alpha, \beta \in L^\infty$ , let  $h$  be an outer function in  $H^2$ , and let  $\phi = \bar{h}/h$ . We define the function  $F$  according to

$$F(x) = \inf_{k \in H^\infty} \|G(x - \alpha\bar{\beta} - \bar{\phi}k)\|_\infty, \quad (x \geq 0).$$

That is,

$$F(x) = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|x - \alpha\bar{\beta} - \bar{\phi}k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty.$$

**Lemma 1.** For each nonnegative number  $x$ , the infimum in the definition of  $F(x)$  is attained.

*Proof.* Let  $\{k_n\}$  be a sequence in  $H^\infty$  such that

$$F(x) = \lim_{n \rightarrow \infty} \|G(x - \alpha\bar{\beta} - \bar{\phi}k_n)\|_\infty.$$

For any  $\varepsilon > 0$ , take a positive integer  $n_0$  with

$$\|G(x - \alpha\bar{\beta} - \bar{\phi}k_n)\|_\infty \leq F(x) + \varepsilon, \quad (n \geq n_0).$$

If  $n \geq n_0$ , then  $\|k_n\|_\infty \leq F(x) + \varepsilon + \|x - \alpha\bar{\beta}\|_\infty < \infty$ . Since the closed ball of  $H^\infty$  is weak-star compact over  $L^\infty$  (cf. [8, p.197]), there exists a subsequence  $\{k_{n_j}\}$  and a  $k_0 \in H^\infty$  such that

$$\lim_{j \rightarrow \infty} (k_{n_j}, g) = (k_0, g), \quad (g \in L^1).$$

Then there exists a sequence  $\{h_n\}$  in  $H^\infty$  such that each  $h_n$  is a finite convex linear combination of the  $k_{n_j}$  and

$$\lim_{n \rightarrow \infty} \|h_n - k_0\|_{L^2} = 0$$



(cf. [17, p.160, Problem 6]). It follows that there exists a subsequence  $\{h_{n_j}\}$  such that

$$\lim_{j \rightarrow \infty} |k_0(\zeta) - h_{n_j}(\zeta)| = 0, \quad (a.e. \zeta \in T)$$

(cf. [15, p.68, Theorem 3.12]). Hence there exist nonnegative numbers  $\lambda_{j,1}, \dots, \lambda_{j,m_j}$  such that  $\lambda_{j,1} + \dots + \lambda_{j,m_j} = 1$  and

$$h_{n_j} = \lambda_{j,1}k_{n_{j,1}} + \dots + \lambda_{j,m_j}k_{n_{j,m_j}}.$$

Since  $y = a^2 + \sqrt{t^2 + b^2}$  is a convex function of  $t$ , it follows that

$$\begin{aligned} G(x - \alpha\bar{\beta} - \bar{\phi}h_{n_j}) &= G\left(x - \alpha\bar{\beta} - \bar{\phi} \sum_{i=1}^{m_j} \lambda_{j,i}k_{n_i}\right) \\ &= G\left(\sum_{i=1}^{m_j} \lambda_{j,i}(x - \alpha\bar{\beta} - \bar{\phi}k_{n_i})\right) \\ &\leq G\left(\sum_{i=1}^{m_j} \lambda_{j,i}|x - \alpha\bar{\beta} - \bar{\phi}k_{n_i}|\right) \\ &\leq \sum_{i=1}^{m_j} \lambda_{j,i}G(x - \alpha\bar{\beta} - \bar{\phi}k_{n_i}) \\ &\leq \sum_{i=1}^{m_j} \lambda_{j,i}(F(x) + \varepsilon) \\ &= F(x) + \varepsilon. \end{aligned}$$

There exists a  $g \in L^1$  such that  $\|g\|_{L^1} = 1$  and

$$\|G(x - \alpha\bar{\beta} - \bar{\phi}k_0)\|_{\infty} \leq |(G(x - \alpha\bar{\beta} - \bar{\phi}k_0), g)| + \varepsilon.$$

By the Lebesgue theorem,

$$\begin{aligned} |(G(x - \alpha\bar{\beta} - \bar{\phi}k_0), g)| &= \lim_{j \rightarrow \infty} |(G(x - \alpha\bar{\beta} - \bar{\phi}h_{n_j}), g)| \\ &\leq \liminf_{j \rightarrow \infty} \|G(x - \alpha\bar{\beta} - \bar{\phi}h_{n_j})\|_{\infty} \\ &\leq F(x) + \varepsilon. \end{aligned}$$

Hence,

$$\|G(x - \alpha\bar{\beta} - \bar{\phi}k_0)\|_{\infty} \leq F(x) + 2\varepsilon.$$

Let  $\varepsilon \rightarrow 0$ . Then the equality holds, and hence the infimum in the definition of  $F(x)$  is attained by  $k = k_0$ . This completes the proof.

**Lemma 2.**  $F(x)$  is a convex function of  $x$ . (Hence it is continuous.)

*Proof.* Let  $\lambda$  and  $\mu$  be nonnegative numbers such that  $\lambda + \mu = 1$ . Since  $y = a^2 + \sqrt{t^2 + b^2}$  is a nonnegative, convex, increasing function of  $t, t \geq 0$ , it follows that

$$\begin{aligned}
\lambda F(x) + \mu F(y) &= \lambda \inf_{k_1 \in H^\infty} \|G(x - \alpha\bar{\beta} - \bar{\phi}k_1)\|_\infty + \mu \inf_{k_2 \in H^\infty} \|G(y - \alpha\bar{\beta} - \bar{\phi}k_2)\|_\infty \\
&\geq \inf_{k_1 \in H^\infty} \inf_{k_2 \in H^\infty} \|\lambda G(x - \alpha\bar{\beta} - \bar{\phi}k_1) + \mu G(y - \alpha\bar{\beta} - \bar{\phi}k_2)\|_\infty \\
&\geq \inf_{k_1 \in H^\infty} \inf_{k_2 \in H^\infty} \|G(\lambda|x - \alpha\bar{\beta} - \bar{\phi}k_1| + \mu|y - \alpha\bar{\beta} - \bar{\phi}k_2|)\|_\infty \\
&\geq \inf_{k_1 \in H^\infty} \inf_{k_2 \in H^\infty} \|G(\lambda x + \mu y - \alpha\bar{\beta} - \bar{\phi}(\lambda k_1 + \mu k_2))\|_\infty \\
&\geq \inf_{k \in H^\infty} \|G(\lambda x + \mu y - \alpha\bar{\beta} - \bar{\phi}k)\|_\infty \\
&= F(\lambda x + \mu y).
\end{aligned}$$

This completes the proof.

**Lemma 3.** If  $x \geq \max\{|\alpha|^2, |\beta|^2\}$ , then

$$G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right) = x.$$

*Proof.* Since  $x \geq \max\{|\alpha|^2, |\beta|^2\} \geq (|\alpha|^2 + |\beta|^2)/2$  and

$$\left(x - \frac{|\alpha|^2 + |\beta|^2}{2}\right)^2 - \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2 = (x - |\alpha|^2)(x - |\beta|^2),$$

it follows that

$$\begin{aligned}
x &= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{(x - |\alpha|^2)(x - |\beta|^2) + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \\
&= G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right).
\end{aligned}$$

This completes the proof.

**Lemma 4.**  $F(x) \leq x$  if and only if  $x \geq \|S_{\alpha,\beta}\|_{L^2(W)}^2$ .

*Proof.* We prove the "if" part. Suppose  $x \geq \|S_{\alpha,\beta}\|_{L^2(W)}^2$ . Then,

$$\|S_{\alpha,\beta}f\|_{L^2(W)}^2 \leq x\|f\|_{L^2(W)}^2, \quad (f \in A + \overline{A_0}).$$

Hence,

$$\|\alpha f_1 + \beta f_2\|_{L^2(W)}^2 \leq x\|f_1 + f_2\|_{L^2(W)}^2, \quad (f_1 \in A, f_2 \in \overline{A_0}).$$

Let  $W_1 = (x - |\alpha|^2)W$ ,  $W_2 = (x - |\beta|^2)W$ ,  $W_3 = (x - \alpha\bar{\beta})W$ , then for any  $f_1 \in A$  and  $f_2 \in \overline{A_0}$ ,

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2\operatorname{Re}(W_3 f_1, f_2) \geq 0.$$

By the Cotlar-Sadosky lifting theorem [2],  $W_1 \geq 0, W_2 \geq 0$ , and there exists a  $g \in H^1$  such that

$$|W_3 - g|^2 \leq W_1 W_2.$$

This implies that  $x \geq \max\{|\alpha|^2, |\beta|^2\}$ , and

$$|(x - \alpha\bar{\beta})W - g|^2 \leq (x - |\alpha|^2)(x - |\beta|^2)W^2.$$

Hence,

$$\left| x - \alpha\bar{\beta} - \frac{g}{W} \right|^2 \leq (x - |\alpha|^2)(x - |\beta|^2).$$

Then,

$$\frac{g}{W} = \frac{g}{|h|^2} = \frac{h}{\bar{h}} \frac{g}{h^2} = \bar{\phi} \frac{g}{h^2}.$$

Let  $k = g/h^2$ . Then  $k \in H^\infty$ , and

$$|x - \alpha\bar{\beta} - \bar{\phi}k|^2 \leq (x - |\alpha|^2)(x - |\beta|^2).$$

It follows from Lemma 3 that

$$G(x - \alpha\bar{\beta} - \bar{\phi}k) \leq G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right) = x.$$

This implies that  $F(x) \leq x$ . We prove the "only if" part. Suppose  $F(x) \leq x$ . By Lemma 1, the infimum in the definition of  $F(x)$  is attained. Hence, there exists a  $k \in H^\infty$  such that

$$G(x - \alpha\bar{\beta} - \bar{\phi}k) \leq x.$$

It follows from Lemma 3 that

$$G(x - \alpha\bar{\beta} - \bar{\phi}k) \leq G\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\right).$$

Hence,

$$|x - \alpha\bar{\beta} - \bar{\phi}k| \leq \sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}.$$

Since  $\phi = \bar{h}/h$  and  $W = |h|^2$ , it follows that

$$|(x - \alpha\bar{\beta})W - h^2k| \leq \sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W.$$

Since  $h^2k \in H^1$ , it follows that

$$\begin{aligned} & x \|f_1 + f_2\|_{L^2(W)}^2 - \|\alpha f_1 + \beta f_2\|_{L^2(W)}^2 \\ &= x (W(f_1 + f_2), f_1 + f_2) - (W(\alpha f_1 + \beta f_2), \alpha f_1 + \beta f_2) \\ &= ((x - |\alpha|^2)W f_1, f_1) + ((x - |\beta|^2)W f_2, f_2) + 2\operatorname{Re}((x - \alpha\bar{\beta})W f_1, f_2) \\ &\geq 2\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\right) - 2\left|((x - \alpha\bar{\beta})W f_1, f_2)\right| \\ &= 2\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\right) - 2\left|((x - \alpha\bar{\beta})W - h^2k) f_1, f_2\right| \\ &\geq 2\left(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\right) - 2\left|((x - \alpha\bar{\beta})W - h^2k) f_1, f_2\right| \\ &\geq 0, \quad (f_1 \in A, f_2 \in \overline{A_0}). \end{aligned}$$

Hence,  $x \geq \|S_{\alpha,\beta}\|_{L^2(W)}^2$ . This completes the proof.

**Lemma 5.** If  $x \geq 0$ , then

$$F(x) \leq x \inf_{k \in H^\infty} \|\phi - k\|_\infty + 2 \max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\}.$$

*Proof.* Since  $\sqrt{|a|^2 + |b|^2} \leq |a| + |b|$ , it follows that

$$\begin{aligned} F(x) &= \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|x - \alpha\bar{\beta} - \bar{\phi}k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty \\ &\leq \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + |x - \alpha\bar{\beta} - \bar{\phi}k| + \left\| \frac{|\alpha|^2 - |\beta|^2}{2} \right\|_\infty \right\|_\infty \\ &\leq \inf_{k \in H^\infty} \|x - \bar{\phi}k\|_\infty + \left\| \max\{|\alpha|^2, |\beta|^2\} + |\alpha\beta| \right\|_\infty \\ &\leq x \inf_{k \in H^\infty} \|\phi - k\|_\infty + 2 \left\| \max\{|\alpha|, |\beta|\} \right\|_\infty^2. \end{aligned}$$

This completes the proof.

**Theorem 1.** Let  $\alpha, \beta \in L^\infty$ . Let  $\phi$  and  $W$  be functions such that there exists an outer function  $h \in H^2$  satisfying  $\phi = \bar{h}/h$  and  $W = |h|^2$ . Then

- (1)  $F(\|S_{\alpha,\beta}\|_{L^2(W)}^2) = \|S_{\alpha,\beta}\|_{L^2(W)}^2$ .
- (2) If  $\inf_{k \in H^\infty} \|\phi - k\|_\infty < 1$  then  $x = \|S_{\alpha,\beta}\|_{L^2(W)}^2$  is the unique solution of the equation  $F(x) = x$ .

*Proof.* We prove (1). Let  $s = \|S_{\alpha,\beta}\|_{L^2(W)}^2$ . We prove that  $x = s$  is the solution of the equation  $F(x) = x$ . It follows from Lemma 4 that  $s \geq F(s)$ . It follows from Lemma 2 that  $F(x)$  is a continuous function of  $x$ . It follows from Lemma 4 that

$$F(x) > x, \quad (x < s).$$

Hence,

$$s \geq F(s) = \lim_{x \rightarrow s} F(x) = \lim_{x \rightarrow s-0} F(x) \geq \lim_{x \rightarrow s-0} x = s.$$

Therefore,  $F(s) = s$ . We prove (2). Suppose there exists a  $t$  such that  $t \neq s$  and  $F(t) = t$ . It follows from Lemma 4 that  $t > s$ . Let  $x$  be any number satisfying  $x > t$ . Since  $s < t < x$ , it follows that there exist positive numbers  $\lambda, \mu$  such that  $\lambda + \mu = 1$  and  $t = \lambda s + \mu x$ . It follows from Lemma 2 that

$$F(t) = F(\lambda s + \mu x) \leq \lambda F(s) + \mu F(x).$$

Since  $F(s) = s$  and  $F(t) = t$ , it follows that

$$t \leq \lambda s + \mu F(x).$$

Hence,

$$\mu x = t - \lambda s \leq \mu F(x).$$

Since  $\mu > 0$ , this implies that  $x \leq F(x)$ . Since  $x > s$ , it follows from Lemma 4 that  $F(x) \leq x$ . Therefore,

$$F(x) = x, \quad (x > t).$$

By Lemma 5, this implies that

$$x \leq x \inf_{k \in H^\infty} \|\phi - k\|_\infty + 2 \max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\}, \quad (x > t).$$

Hence,

$$\inf_{k \in H^\infty} \|\phi - k\|_\infty \geq 1.$$

This is a contradiction. Therefore, there does not exist a  $t$  such that  $t \neq s$  and  $F(t) = t$ . This completes the proof.

If  $W$  is a constant function, then  $F(x)$  becomes a constant function. In this case, the formula of  $\|S_{\alpha,\beta}\|_{L^2}^2$  follows easily from Theorem 1(1) as Corollary 1. In the preceding paper [14], we gave Corollary 1.

**Corollary 1.** Let  $\alpha, \beta \in L^\infty$ . Then

$$\|S_{\alpha,\beta}\|_{L^2}^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty.$$

The infimum is attained.

*Proof.* In the statement of Theorem 1, if  $W = 1$ , then  $h$  and  $\phi$  are constants. If  $k \in H^\infty$ , then  $k - \phi x \in H^\infty$ . Hence,

$$\begin{aligned} F(x) &= \inf_{k \in H^\infty} \|G(x - \alpha\bar{\beta} - \bar{\phi}k)\|_\infty \\ &= \inf_{k \in H^\infty} \|G(0 - \alpha\bar{\beta} - \bar{\phi}(k - \phi x))\|_\infty \\ &= \inf_{k \in H^\infty} \|G(0 - \alpha\bar{\beta} - \bar{\phi}k)\|_\infty = F(0). \end{aligned}$$

Hence  $F(x)$  is a constant function of  $x$ . It follows from Theorem 1 that

$$\|S_{\alpha,\beta}\|_{L^2}^2 = F(\|S_{\alpha,\beta}\|_{L^2}^2) = F(0) = \inf_{k \in H^\infty} \|G(\alpha\bar{\beta} - k)\|_\infty.$$

By Lemma 1, the infimum is attained. This completes the proof.

**Corollary 2.** Let  $\alpha, \beta \in L^\infty$ . Then

$$\max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\} \leq \|S_{\alpha,\beta}\|_{L^2}^2 \leq \max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\} + \inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty.$$

If  $\alpha\bar{\beta} \in H^\infty$ , then the equality holds.

*Proof.* It follows from Corollary 1 that

$$\begin{aligned} \left\| \max\{|\alpha|^2, |\beta|^2\} \right\|_\infty &\leq \|S_{\alpha,\beta}\|_{L^2}^2 \\ &= \inf_{k \in H^\infty} \|G(\alpha\bar{\beta} - k)\|_\infty \\ &\leq \inf_{k \in H^\infty} \left\| \max\{|\alpha|^2, |\beta|^2\} + |\alpha\bar{\beta} - k| \right\|_\infty \\ &\leq \left\| \max\{|\alpha|^2, |\beta|^2\} \right\|_\infty + \inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty. \end{aligned}$$

This completes the proof.

**Corollary 3.** Let  $\alpha, \beta \in L^\infty$ . Let  $\phi$  and  $W$  be functions such that there exists an outer function  $h \in H^2$  satisfying  $\phi = \bar{h}/h$  and  $W = |h|^2$ .

(1) If  $|\alpha|, |\beta|$  are constants, then

$$\inf_{k \in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2} \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}.$$

(2) If  $|\alpha|, |\beta|$  are constants and  $\alpha\bar{\beta} \in H^\infty$ , then

$$\inf_{k \in H^\infty} \left\| \left( \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right) (\phi - k) \right\|_\infty = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2} \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}.$$

*Proof.* We prove (1). Since  $|\alpha|, |\beta|$  are constants, it follows from Theorem 1 that

$$\begin{aligned} \|S_{\alpha,\beta}\|_{L^2(W)}^2 &= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\inf_{k \in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty^2 + \left( \frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \\ &= G \left( \inf_{k \in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty \right). \end{aligned}$$

Since  $\|S_{\alpha,\beta}\|_{L^2(W)}^2 \geq \max\{|\alpha|^2, |\beta|^2\}$ , it follows from Lemma 3 that

$$\|S_{\alpha,\beta}\|_{L^2(W)}^2 = G \left( \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2} \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2} \right).$$

Therefore,

$$G \left( \inf_{k \in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty \right) = G \left( \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2} \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2} \right).$$

This proves (1). We prove (2). Since  $\alpha\bar{\beta} \in H^\infty$  and  $\operatorname{Re} \left( \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right) \geq 0$ , it follows that  $\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}$  is an outer function or a zero function. Hence,

$$\begin{aligned} &\inf_{k \in H^\infty} \left\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} - \bar{\phi}k \right\|_\infty \\ &= \inf_{k \in H^\infty} \left\| \left( \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right) \left( \phi - \frac{k}{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}} \right) \right\|_\infty \\ &= \inf_{k \in H^\infty} \left\| \left( \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right) (\phi - k) \right\|_\infty. \end{aligned}$$

By (1), this implies (2). This completes the proof.

The statement (3) in the following corollary was given by Feldman, Krupnik and Marcus (cf. [3]). By the Helson-Szegö theorem [7],  $\|P_+\|_{L^2(W)} < \infty$  if and only if  $\inf_{k \in H^\infty} \|\phi - k\|_\infty < 1$ .

**Corollary 4.** Let  $\alpha$  and  $\beta$  be complex numbers satisfying  $\alpha \neq \beta$ . Let  $\phi$  and  $W$  be functions such that there exists an outer function  $h \in H^2$  satisfying  $\phi = \bar{h}/h$  and  $W = |h|^2$ . Let  $c = \inf_{k \in H^\infty} \|\phi - k\|_\infty$ . If  $c < 1$ , then the following equalities hold.

$$\begin{aligned} (1) \quad & c \left| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right| = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2} \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}. \\ (2) \quad & \left| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right| = |\alpha - \beta| \|S_{\alpha,\beta}\|_{L^2(W)} \|P_+\|_{L^2(W)} = \frac{|\alpha - \beta| \|S_{\alpha,\beta}\|_{L^2(W)}}{\sqrt{1 - c^2}}. \\ (3) \quad & \|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}, \end{aligned}$$

where

$$\gamma := \left| \frac{\alpha - \beta}{2} \right|^2 \left( \frac{c^2}{1 - c^2} \right) = \left| \frac{\alpha - \beta}{2} \right|^2 (\|P_+\|_{L^2(W)}^2 - 1).$$

*Proof.* Since  $\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta}$  is a constant, it follows from Corollary 3(2) that (1) holds. We prove (2). Since  $P_+ = S_{1,0}$ , it follows from (1) that

$$c \|P_+\|_{L^2(W)}^2 = \|P_+\|_{L^2(W)} \sqrt{\|P_+\|_{L^2(W)}^2 - 1}.$$

Hence,  $(1 - c^2) \|P_+\|_{L^2(W)}^2 = 1$ . It follows from (1) that

$$\begin{aligned} c^2 \left| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right|^2 &= \left( \|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2 \right) \left( \|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2 \right) \\ &= \left| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right|^2 - |\alpha - \beta|^2 \|S_{\alpha,\beta}\|_{L^2(W)}^2. \end{aligned}$$

Hence,

$$(1 - c^2) \left| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\bar{\beta} \right|^2 = |\alpha - \beta|^2 \|S_{\alpha,\beta}\|_{L^2(W)}^2.$$

Since  $(1 - c^2) \|P_+\|_{L^2(W)}^2 = 1$ , this implies (2). We prove (3). Let  $s = \|S_{\alpha,\beta}\|_{L^2(W)}$ . It follows from (1) that

$$c^2 |s - \alpha\bar{\beta}|^2 = (s - |\alpha|^2)(s - |\beta|^2).$$

Then

$$(1 - c^2)s^2 - \left\{ |\alpha - \beta|^2 + 2(1 - c^2) \operatorname{Re}(\alpha\bar{\beta}) \right\} s + (1 - c^2)|\alpha\beta|^2 = 0.$$

Since  $c < 1$ , it follows that

$$s^2 - \left\{ \frac{|\alpha - \beta|^2}{1 - c^2} + 2 \operatorname{Re}(\alpha\bar{\beta}) \right\} s + |\alpha\beta|^2 = 0.$$

Since  $s \geq \max\{|\alpha|^2, |\beta|^2\} \geq |\alpha\beta|$ , it follows that

$$s = t + \sqrt{t^2 - |\alpha\beta|^2},$$

where

$$t := \frac{|\alpha - \beta|^2}{2(1 - c^2)} + \operatorname{Re}(\alpha\bar{\beta}).$$

Hence,

$$\begin{aligned} \|S_{\alpha,\beta}\|_{L^2(W)} &= \sqrt{s} \\ &= \sqrt{t + \sqrt{t^2 - |\alpha\beta|^2}} \\ &= \sqrt{\frac{t + |\alpha\beta|}{2}} + \sqrt{\frac{t - |\alpha\beta|}{2}} \\ &= \sqrt{\frac{|\alpha - \beta|^2}{4(1 - c^2)} + \frac{\operatorname{Re}(\alpha\bar{\beta}) + |\alpha\beta|}{2}} + \sqrt{\frac{|\alpha - \beta|^2}{4(1 - c^2)} + \frac{\operatorname{Re}(\alpha\bar{\beta}) - |\alpha\beta|}{2}} \\ &= \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}, \end{aligned}$$

where

$$\gamma := \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{c^2}{1 - c^2}\right).$$

Then,

$$\|P_+\|_{L^2(W)} = \|S_{1,0}\|_{L^2(W)} = 2\sqrt{\frac{c^2}{4(1 - c^2)} + \frac{1}{4}} = \frac{1}{\sqrt{1 - c^2}}.$$

Hence,

$$\gamma = \left|\frac{\alpha - \beta}{2}\right|^2 (\|P_+\|_{L^2(W)}^2 - 1).$$

This completes the proof.

### 3. THE SECOND FORMULA OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 2 is the second formula of  $\|S_{\alpha,\beta}\|_{L^2(W)}$ . It is the generalization of the Feldman-Krupnik-Markus theorem. Since the formula is symmetric with respect to  $\alpha$  and  $\beta$ , it follows that if  $\alpha\bar{\beta}$  is a constant then

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \|S_{\beta,\alpha}\|_{L^2(W)}.$$

By Theorem 2, if  $W(\zeta) = 1$  and  $\alpha\bar{\beta} \in H^\infty$ , then

$$\|S_{\alpha,\beta}\|_{L^2} = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$



If  $\alpha(\zeta) = \zeta, \beta(\zeta) = 1, W(\zeta) = 1$ , then  $\|S_{\zeta,1}\|_{L^2} = 1$ . It follows from Corollary 1 that

$$\|S_{1,\zeta}\|_{L^2} = \inf_{k \in H^\infty} \left\| 1 + |\bar{\zeta} - k| \right\|_\infty^{1/2} = \sqrt{1 + \inf_{k \in H^\infty} \|\bar{\zeta} - k\|_\infty} = \sqrt{2} \neq \|S_{\zeta,1}\|_{L^2}.$$

**Theorem 2.** Let  $\alpha, \beta \in L^\infty$ . Let  $\phi$  and  $W$  be functions such that there exists an outer function  $h \in H^2$  satisfying  $\phi = \bar{h}/h$  and  $W = |h|^2$ . If  $\alpha\bar{\beta}$  belongs to  $H^\infty$  and  $|\alpha - \beta| > 0$ , then

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty,$$

where

$$\gamma_k := \left| \frac{\alpha - \beta}{2} \right|^2 \left( \frac{|\phi - k|^2}{1 - |\phi - k|^2} \right).$$

*Proof.* Let  $s = \|S_{\alpha,\beta}\|_{L^2(W)}^2$ . It is well known that  $\max\{|\alpha|^2, |\beta|^2\} \leq s$ . For any  $k \in H^\infty$  satisfying  $|\phi - k| < 1$ , we define the quantity  $N_k$  according to

$$N_k = \left\| \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty.$$

We prove that  $\|S_{\alpha,\beta}\|_{L^2(W)} \geq \inf\{N_k; k \in H^\infty, |\phi - k| < 1\}$ . It follows from Theorem 1 that  $F(s) = s$ . Hence there exists a function  $g \in H^\infty$  such that

$$G(s - \alpha\bar{\beta} - \bar{\phi}g) \leq s.$$

It follows from Lemma 3 that  $G(\sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2}) = s$ . Hence,

$$G(s - \alpha\bar{\beta} - \bar{\phi}g) \leq G(\sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2}).$$

Hence,

$$|s - \alpha\bar{\beta} - \bar{\phi}g|^2 \leq (s - |\alpha|^2)(s - |\beta|^2).$$

Suppose  $g = 0$ . Then,

$$|s - \alpha\bar{\beta}|^2 \leq (s - |\alpha|^2)(s - |\beta|^2).$$

Hence  $|\alpha - \beta|^2 s \leq 0$ . Since  $\max\{|\alpha|^2, |\beta|^2\} \leq s$  and  $|\alpha - \beta| > 0$ , it follows that  $s > 0$ . Hence  $\alpha \equiv \beta$ . This contradiction implies  $g \neq 0$ . Since

$$|g| - |s - \alpha\bar{\beta}| \leq |s - \alpha\bar{\beta} - \bar{\phi}g| \leq \sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2} \leq |s - \alpha\bar{\beta}|,$$

it follows that

$$0 < |g| < 2|s - \alpha\bar{\beta}|.$$

Since

$$\operatorname{Re}(s - \alpha\bar{\beta}) \geq s - \max\{|\alpha|^2, |\beta|^2\} \geq 0,$$

and since  $\alpha\bar{\beta} \in H^\infty$ , it follows that  $s - \alpha\bar{\beta}$  is an outer function. We define the function  $k$  according to

$$k = \frac{g}{s - \alpha\bar{\beta}}.$$

Since  $|\alpha - \beta| > 0$ , it follows that

$$|\phi - k|^2 = \left| \phi - \frac{g}{s - \alpha\bar{\beta}} \right|^2 \leq \frac{(s - |\alpha|^2)(s - |\beta|^2)}{|s - \alpha\bar{\beta}|^2} = 1 - \frac{|\alpha - \beta|^2 s}{|s - \alpha\bar{\beta}|^2} < 1.$$

Hence,  $k \in H^\infty$ ,  $|\phi - k| < 1$ , and

$$|s - \alpha\bar{\beta}|^2 - \frac{|\alpha - \beta|^2}{1 - |\phi - k|^2} s \geq 0.$$

Therefore

$$s^2 - \left( \frac{|\alpha - \beta|^2}{1 - |\phi - k|^2} + 2\operatorname{Re}(\alpha\bar{\beta}) \right) s + |\alpha\beta|^2 \geq 0.$$

Hence,

$$0 \leq s \leq t - \sqrt{t^2 - |\alpha\beta|^2},$$

or

$$s \geq t + \sqrt{t^2 - |\alpha\beta|^2},$$

where

$$t := \left( \frac{|\alpha - \beta|^2}{2} \frac{1}{1 - |\phi - k|^2} + \operatorname{Re}(\alpha\bar{\beta}) \right).$$

Hence,

$$\sqrt{s} \leq \sqrt{t - \sqrt{t^2 - |\alpha\beta|^2}} = \sqrt{\frac{t + |\alpha\beta|}{2}} - \sqrt{\frac{t - |\alpha\beta|}{2}},$$

or

$$\sqrt{s} \geq \sqrt{t + \sqrt{t^2 - |\alpha\beta|^2}} = \sqrt{\frac{t + |\alpha\beta|}{2}} + \sqrt{\frac{t - |\alpha\beta|}{2}}.$$

Suppose

$$\sqrt{s} \leq \sqrt{\frac{t + |\alpha\beta|}{2}} - \sqrt{\frac{t - |\alpha\beta|}{2}}$$

on some measurable subset  $E$  of  $T$ . Then

$$\sqrt{s} \leq \sqrt{\gamma_k + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} - \sqrt{\gamma_k + \left( \frac{|\alpha| - |\beta|}{2} \right)^2}$$

on  $E$ . Since  $\max\{|\alpha|, |\beta|\} \leq \sqrt{s}$ , it follows that

$$\begin{aligned}
& \frac{|\alpha| + |\beta|}{2} + \left| \frac{|\alpha| - |\beta|}{2} \right| + \sqrt{\gamma_k + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} \\
& \leq \sqrt{s} + \sqrt{\gamma_k + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} \\
& \leq \sqrt{\gamma_k + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} \\
& \leq \sqrt{\gamma_k + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} + \frac{|\alpha| + |\beta|}{2}.
\end{aligned}$$

on  $E$ . Hence  $|\alpha| = |\beta|$  on  $E$ . Hence  $|\alpha| + \sqrt{\gamma_k} = \sqrt{\gamma_k + |\alpha|^2}$  on  $E$ . Hence  $\alpha\gamma_k = 0$  on  $E$ . Hence  $\gamma_k = 0$  on  $E$ . Therefore,

$$\sqrt{s} \geq \sqrt{\gamma_k + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} + \sqrt{\gamma_k + \left( \frac{|\alpha| - |\beta|}{2} \right)^2}$$

on  $T$ . Hence,

$$\|S_{\alpha, \beta}\|_{L^2(W)} \geq \inf_{k \in H^\infty, |\phi - k| < 1} N_k.$$

Next we prove the reverse inequality. This is the easy direction of the theorem. Since

$$N_k \geq \sqrt{\gamma_k + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} + \sqrt{\gamma_k + \left( \frac{|\alpha| - |\beta|}{2} \right)^2},$$

squaring both sides, it follows that

$$N_k^2 \geq 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left( 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} \right)^2 - |\alpha\beta|^2}.$$

Hence,

$$N_k^4 - (4\gamma_k + |\alpha|^2 + |\beta|^2)N_k^2 + |\alpha\beta|^2 \geq 0.$$

Hence,

$$4\gamma_k N_k^2 \leq (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Hence,

$$N_k^2 |\alpha - \beta|^2 \left( \frac{|\phi - k|^2}{1 - |\phi - k|^2} \right) \leq (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Since

$$N_k^2 |\alpha - \beta|^2 = |N_k^2 - \alpha\bar{\beta}|^2 - (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2),$$

it follows that

$$\left(|N_k^2 - \alpha\bar{\beta}|^2 - (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2)\right) \left(\frac{|\phi - k|^2}{1 - |\phi - k|^2}\right) \leq (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Hence,

$$|N_k^2 - \alpha\bar{\beta}|^2 |\phi - k|^2 \leq (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

It follows from Lemma 3 that

$$G\left(|N_k^2 - \alpha\bar{\beta}| |\phi - k|\right) \leq G\left(\sqrt{N_k^2 - |\alpha|^2} \sqrt{N_k^2 - |\beta|^2}\right) = N_k^2.$$

Since

$$|N_k^2 - \alpha\bar{\beta}| |\phi - k| = |(N_k^2 - \alpha\bar{\beta})(1 - \bar{\phi}k)| = |N_k^2 - \alpha\bar{\beta} - \bar{\phi}(N_k^2 - \alpha\bar{\beta})k|$$

and  $(N_k^2 - \alpha\bar{\beta})k \in H^\infty$ , it follows that  $F(N_k^2) \leq N_k^2$ . It follows from Lemma 4 that  $N_k^2 \geq \|S_{\alpha,\beta}\|_{L^2(W)}^2$ . This completes the proof.

**Corollary 5.** If  $\inf_{k \in H^\infty} \|\phi - k\|_\infty = c < 1$ ,  $\alpha\bar{\beta} \in H^\infty$  and  $|\alpha - \beta| > 0$ , then

$$\begin{aligned} (1) \quad & \|S_{\alpha,\beta}\|_{L^2(W)} \geq \inf_{k \in H^\infty} \left\| \sqrt{\max\{|\alpha|^2, |\beta|^2\} + \frac{|\alpha - \beta|^2 |\phi - k|^2}{1 - |\phi - k|^2}} \right\|_\infty, \\ (2) \quad & \|S_{\alpha,\beta}\|_{L^2(W)} \leq \left\| \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty, \\ (3) \quad & \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \|\max\{|\alpha|, |\beta|\} + 2\sqrt{\gamma}\|_\infty, \end{aligned}$$

where

$$\gamma := \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{c^2}{1 - c^2}\right) = \left|\frac{\alpha - \beta}{2}\right|^2 (\|P_+\|_{L^2(W)}^2 - 1).$$

*Proof.* We prove (1). Let  $\gamma_k = \left|\frac{\alpha - \beta}{2}\right|^2 \left(\frac{|\phi - k|^2}{1 - |\phi - k|^2}\right)$ . Then

$$\begin{aligned} & \left(\sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}\right)^2 \\ &= 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} + 2\sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \\ &\geq 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} + 2\gamma_k + \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| \\ &= \max\{|\alpha|^2, |\beta|^2\} + 4\gamma_k. \end{aligned}$$

By Theorem 2, this implies (1). We prove (2). Since

$$\gamma_k = \left| \frac{\alpha - \beta}{2} \right|^2 \left( \frac{|\phi - k|^2}{1 - |\phi - k|^2} \right) \leq \left| \frac{\alpha - \beta}{2} \right|^2 \left( \frac{\|\phi - k\|_\infty^2}{1 - \|\phi - k\|_\infty^2} \right) \leq \left| \frac{\alpha - \beta}{2} \right|^2 \left( \frac{c^2}{1 - c^2} \right) = \gamma,$$

(2) follows from Theorem 2. Since

$$\begin{aligned} & \sqrt{\gamma + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} + \sqrt{\gamma + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} \\ & \leq \sqrt{\gamma} + \frac{|\alpha| + |\beta|}{2} + \sqrt{\gamma} + \left| \frac{|\alpha| - |\beta|}{2} \right| \\ & = \max\{|\alpha|, |\beta|\} + 2\sqrt{\gamma}, \end{aligned}$$

(3) follows from (2). This completes the proof.

**Example 1.** Let  $\zeta = e^{i\theta}$ ,  $\alpha(\zeta) = \zeta + 1$ ,  $\beta(\zeta) = 1$  and  $W(\zeta) = |\zeta + 1|^{1/2}$ . Then  $\alpha\bar{\beta} \in H^\infty$ ,  $\phi(e^{i\theta}) = e^{-i\theta/4}$ ,  $\zeta = e^{i\theta}$ ,  $-\pi \leq \theta < \pi$ . Hence,  $|\phi - \frac{1}{\sqrt{2}}| \leq \frac{1}{\sqrt{2}} < 1$ . By Theorem 2,

$$\begin{aligned} 2 \leq \|S_{\alpha,\beta}\|_{L^2(W)} & \leq \left\| \sqrt{\gamma_{1/\sqrt{2}} + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} + \sqrt{\gamma_{1/\sqrt{2}} + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} \right\|_\infty, \\ \gamma_{1/\sqrt{2}} & = \frac{1}{4} \left( \frac{1}{1 - |\phi - \frac{1}{\sqrt{2}}|^2} - 1 \right). \end{aligned}$$

Hence,

$$\begin{aligned} 2 \leq \|S_{\alpha,\beta}\|_{L^2(W)} & \leq \sup_{-\pi \leq \theta < \pi} \left\{ \sqrt{\frac{3 - 2\sqrt{2} \cos(\theta/4)}{4(2\sqrt{2} \cos(\theta/4) - 1)} + \left( \frac{\sqrt{2}(1 + \cos \theta) + 1}{2} \right)^2} \right. \\ & \quad \left. + \sqrt{\frac{3 - 2\sqrt{2} \cos(\theta/4)}{4(2\sqrt{2} \cos(\theta/4) - 1)} + \left( \frac{\sqrt{2}(1 + \cos \theta) - 1}{2} \right)^2} \right\} \\ & = \sqrt{\frac{29 + 2\sqrt{2}}{14}} + \sqrt{\frac{1 + 2\sqrt{2}}{14}} < 2.04. \end{aligned}$$

Since  $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi/4)} = \sqrt{2}$  (cf. [12]),

$$\gamma = \left| \frac{\alpha - \beta}{2} \right|^2 (\|P_+\|_{L^2(W)}^2 - 1) = \frac{1}{4},$$

and hence

$$\begin{aligned} & \left\| \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_{\infty} \\ &= \sqrt{\frac{1}{4} + \left(\frac{3}{2}\right)^2} + \sqrt{\frac{1}{4} + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{10} + \sqrt{2}}{2} > 2.28. \end{aligned}$$

This example shows that it is not able to change the function  $\gamma_k$  by the function  $\gamma$  in Theorem 2, and that

$$2 \leq \|S_{\alpha, \beta}\|_{L^2(W)} < 2.04.$$

**Corollary 6.** If  $|\alpha(\zeta)| > 0$ , then

$$\|\alpha P_+\|_{L^2(W)} = \|\alpha P_-\|_{L^2(W)} = \inf_{k \in H^\infty} \left\| \frac{\alpha}{\sqrt{1 - |\phi - k|^2}} \right\|_{\infty}.$$

**Corollary 7.** Let  $\alpha \in H^\infty$ . Then

(1) If  $W^{-1} \in L^1$ , then there exists  $\varepsilon_n \in L^\infty, \varepsilon_n > 0, \|\varepsilon_n\|_{\infty} \rightarrow 0, (n \rightarrow \infty)$  such that

$$\|\alpha P_+\|_{L^2(W)} = \lim_{n \rightarrow \infty} \|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} = \lim_{n \rightarrow \infty} \inf_{k \in H^\infty} \left\| \frac{|\alpha| + \varepsilon_n}{\sqrt{1 - |\phi - k|^2}} \right\|_{\infty}.$$

(2) If  $\inf_{k \in H^\infty} \|\phi - k\|_{\infty} < 1$ , then we can take  $\varepsilon_n = \frac{1}{n}$ .

*Proof.* We prove (1). Since  $|\alpha| + \varepsilon_n > 0$ , it follows from Corollary 6 that

$$\|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} = \left\| \frac{|\alpha| + \varepsilon_n}{\sqrt{1 - |\phi - k|^2}} \right\|_{\infty}.$$

By the Koosis theorem [9], if  $W^{-1} \in L^1$ , then there exists  $U \in L^1, U > 0$  such that

$$\|P_+ f\|_{L^2(U)} \leq \|f\|_{L^2(W)}, \quad (f \in L^2(W)).$$

Let  $\varepsilon_n = \frac{1}{n} \sqrt{\frac{U}{W}}$ . Since  $U \leq W$ , it follows that  $\varepsilon_n \leq \frac{1}{n}$  and

$$\|\varepsilon_n P_+ f\|_{L^2(W)} \leq \frac{1}{n} \|f\|_{L^2(W)}, \quad (f \in L^2(W)).$$

Hence,

$$\lim_{n \rightarrow \infty} \left| \|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} - \|\alpha P_+\|_{L^2(W)} \right| \leq \lim_{n \rightarrow \infty} \|\varepsilon_n P_+\|_{L^2(W)} \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

We prove (2). By the Helson-Szegö theorem [7], if  $\inf_{k \in H^\infty} \|\phi - k\|_\infty < 1$ , then  $\|P_+\|_{L^2(W)} < \infty$ . Hence,

$$\lim_{n \rightarrow \infty} \left\| \left( |\alpha| + \frac{1}{n} \right) P_+ \right\|_{L^2(W)} - \|\alpha P_+\|_{L^2(W)} \leq \lim_{n \rightarrow \infty} \left( \frac{1}{n} \|P_+\|_{L^2(W)} \right) = 0.$$

This completes the proof.

**Example 2.** Suppose  $W(\zeta) = |\zeta + 1|^{1/2}$ ,  $h(\zeta) = (\zeta + 1)^{1/2}$ ,  $\zeta = e^{i\theta}$ ,  $\phi(\zeta) = \overline{h(\zeta)}/h(\zeta) = e^{-i\theta/4}$  and  $E \subset T$ . By Corollaries 8 and 9,

$$\begin{aligned} \|\chi_E P_+\|_{L^2(W)} &= \lim_{n \rightarrow \infty} \left\| \left( \chi_E + \frac{1}{n} \right) P_+ \right\|_{L^2(W)} = \lim_{n \rightarrow \infty} \inf_{k \in H^\infty} \left\| \frac{\chi_E + \frac{1}{n}}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty \\ &\leq \left\| \frac{\chi_E + 0.1}{\sqrt{1 - |e^{-i\theta/4} - \frac{1}{\sqrt{2}}|^2}} \right\|_\infty = \left\| \frac{\chi_E + 0.1}{\sqrt{\sqrt{2} \cos(\frac{\theta}{4}) - \frac{1}{2}}} \right\|_\infty. \end{aligned}$$

If  $-\pi \leq \theta < \pi$ , then  $\frac{1}{\sqrt{2}} \leq \cos(\frac{\theta}{4}) \leq 1$ . Hence,

$$\sqrt{\frac{2}{2\sqrt{2} - 1}} \leq \frac{1}{\sqrt{\sqrt{2} \cos(\frac{\theta}{4}) - \frac{1}{2}}} \leq \sqrt{2}.$$

Hence, on  $E^c$ ,

$$\frac{\chi_E + 0.1}{\sqrt{\sqrt{2} \cos(\frac{\theta}{4}) - \frac{1}{2}}} < 0.15.$$

If  $\theta = 0$ , then

$$\frac{1}{\sqrt{\sqrt{2} \cos(\frac{\theta}{4}) - \frac{1}{2}}} = \sqrt{\frac{2}{2\sqrt{2} - 1}} = 1.04\dots < 1.05.$$

Hence, for sufficiently small  $\varepsilon > 0$ ,  $E = (-\varepsilon, \varepsilon)$  satisfies

$$\|\chi_E P_+\|_{L^2(W)} < 1.05 \cdot 1.1 = 1.155.$$

Since  $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi/4)} = \sqrt{2}$  (cf. [12]), it follows that

$$\|\chi_E P_+\|_{L^2(W)} < \|\chi_E\|_\infty \|P_+\|_{L^2(W)}.$$

Hence,  $\|\alpha P_+\|_{L^2(W)} = \|\alpha\|_\infty \|P_+\|_{L^2(W)}$  does not hold in general.

Next, we compare  $\|S_{\alpha,\beta}\|_{L^2(W)}$  and  $\|S\|_{L^2(W)}$ . Let  $\alpha, \beta \in L^\infty$ . Since  $S_{\alpha,\beta} = \frac{\alpha+\beta}{2} I + \frac{\alpha-\beta}{2} S$ , it follows that

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leq \left\| \frac{\alpha + \beta}{2} \right\|_\infty + \left\| \frac{\alpha - \beta}{2} \right\|_\infty \|S\|_{L^2(W)}.$$

By Theorem 2, we have

**Corollary 8.** If  $\alpha, \beta \in H^\infty$  and  $|\alpha - \beta| > 0$ , then

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|S_{\alpha, \beta}\|_{L^2(W)} \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \|S\|_{L^2(W)}.$$

*Proof.* Without loss of generality, we assume that  $\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} = 1$  and  $\|S\|_{L^2(W)} < \infty$ . Let  $c = \inf_{k \in H^\infty} \|\phi - k\|_\infty$ . By the Helson-Szegö theorem [7],  $c < 1$ . It follows from Theorem 2 that

$$\|S_{\alpha, \beta}\|_{L^2(W)} = \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \sqrt{\gamma_k + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma_k + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty,$$

where

$$\gamma_k := \left| \frac{\alpha - \beta}{2} \right|^2 \left( \frac{|\phi - k|^2}{1 - |\phi - k|^2} \right).$$

Since  $\max\{|\alpha|, |\beta|\} \leq 1$ , it follows that

$$\gamma_k \leq \frac{|\phi - k|^2}{1 - |\phi - k|^2} \leq \frac{c^2}{1 - c^2}.$$

Hence,

$$\|S_{\alpha, \beta}\|_{L^2(W)} \leq \left\| \sqrt{\frac{c^2}{1 - c^2} + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\frac{c^2}{1 - c^2} + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \right\|_\infty.$$

It follows from Theorem 2 that

$$\begin{aligned} \|S\|_{L^2(W)} &= \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \sqrt{\frac{|\phi - k|^2}{1 - |\phi - k|^2} + 1} + \sqrt{\frac{|\phi - k|^2}{1 - |\phi - k|^2}} \right\|_\infty \\ &= \sqrt{\frac{c^2}{1 - c^2} + 1} + \sqrt{\frac{c^2}{1 - c^2}}. \end{aligned}$$

Let  $x = \frac{c^2}{1 - c^2}$ . It is sufficient to show that

$$\sqrt{x + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{x + \left(\frac{|\alpha| - |\beta|}{2}\right)^2} \leq \sqrt{x + 1} + \sqrt{x}.$$

Since  $(1 - |\alpha|^2)(1 - |\beta|^2) \geq 0$ , it follows that

$$\left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2 \leq \left(1 - \frac{|\alpha|^2 + |\beta|^2}{2}\right)^2.$$



Hence,

$$\left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2 + 2(|\alpha|^2 + |\beta|^2)x \leq \left(1 - \frac{|\alpha|^2 + |\beta|^2}{2}\right)^2 + 4x.$$

Hence,

$$\begin{aligned} & 4 \left( x + \left( \frac{|\alpha| + |\beta|}{2} \right)^2 \right) \left( x + \left( \frac{|\alpha| - |\beta|}{2} \right)^2 \right) \\ & \leq \left( 1 - \frac{|\alpha|^2 + |\beta|^2}{2} \right)^2 + 4x(x+1) \\ & \leq \left( 1 - \frac{|\alpha|^2 + |\beta|^2}{2} \right)^2 + 4 \left( 1 - \frac{|\alpha|^2 + |\beta|^2}{2} \right) \sqrt{x(x+1)} + 4x(x+1). \end{aligned}$$

Hence,

$$\begin{aligned} & 2x + \frac{|\alpha|^2 + |\beta|^2}{2} + 2\sqrt{x + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} \sqrt{x + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} \\ & \leq 2x + 1 + 2\sqrt{x(x+1)}. \end{aligned}$$

Hence,

$$\left( \sqrt{x + \left( \frac{|\alpha| + |\beta|}{2} \right)^2} + \sqrt{x + \left( \frac{|\alpha| - |\beta|}{2} \right)^2} \right)^2 \leq (\sqrt{x+1} + \sqrt{x})^2.$$

This completes the proof.

#### 4. THE THIRD FORMULA OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 3 is the third formula of  $\|S_{\alpha,\beta}\|_{L^2(W)}$ . Suppose  $\alpha(\zeta) = 1, \beta(\zeta) = \zeta$ . Then  $\alpha\bar{\beta} \notin H^\infty$ . By Corollary 1,  $\|S_{\alpha,\beta}\|_{L^2} = \sqrt{2}$ . Since  $\|S\|_{L^2} = 1$ , it follows that  $\|S_{\alpha,\beta}\|_{L^2} > \|S\|_{L^2}$ . Hence Corollary 8 does not hold in general. Hence Theorem 2 does not hold in general. But the following Theorem 3 holds even when  $\alpha\bar{\beta} \notin H^\infty$ . We give Lemma 6 to prove Theorem 3.

**Lemma 6.** If  $|g| < 1$ , then

$$(1) \quad \frac{|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g)}{1 - |g|^2} = |\alpha|^2 + \frac{|\bar{\beta} - \bar{\alpha}g|^2}{1 - |g|^2} = |\beta|^2 + \frac{|\alpha - \beta g|^2}{1 - |g|^2}.$$

$$(2) \quad \max\{|\alpha|^2, |\beta|^2\} \leq \frac{|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g)}{1 - |g|^2}.$$

$$(3) \quad \frac{(|\alpha| - |\beta|)^2}{1 - |g|^2} \leq \frac{\max\{|\bar{\beta} - \bar{\alpha}g|^2, |\alpha - \beta g|^2\}}{1 - |g|^2} \leq \frac{|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g)}{1 - |g|^2} \leq \frac{(|\alpha| + |\beta|)^2}{1 - |g|^2}.$$

*Proof.* Since

$$|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g) - |\alpha|^2(1 - |g|^2) = |\bar{\beta} - \bar{\alpha}g|^2$$

and

$$|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g) - |\beta|^2(1 - |g|^2) = |\alpha - \beta g|^2,$$

we have (1). (1) implies (2). We prove (3). Since  $|g| < 1$ , it follows that

$$|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g) \leq (|\alpha| + |\beta|)^2.$$

If  $|\alpha| \leq |\beta|$ , then  $|\bar{\beta} - \bar{\alpha}g|^2 \geq (|\beta| - |\alpha g|)^2 \geq (|\beta| - |\alpha|)^2$ .

If  $|\alpha| \geq |\beta|$ , then  $|\alpha - \beta g|^2 \geq (|\alpha| - |\beta g|)^2 \geq (|\alpha| - |\beta|)^2$ .

This completes the proof.

**Theorem 3.** Let  $\alpha, \beta \in L^\infty$ . Let  $\phi$  and  $W$  be functions such that there exists an outer function  $h \in H^2$  satisfying  $\phi = \bar{h}/h$  and  $W = |h|^2$ .

(1) If  $\|\alpha\| - \|\beta\| > 0$ , then

$$\begin{aligned} \|S_{\alpha, \beta}\|_{L^2(W)}^2 &= \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \frac{|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta(1 - \bar{\phi}k))}{1 - |\phi - k|^2} \right\|_\infty \\ &= \inf_{k \in H^\infty, |\phi - k| < 1} \left\| |\alpha|^2 + \frac{|\bar{\beta} - \bar{\alpha}(1 - \bar{\phi}k)|^2}{1 - |\phi - k|^2} \right\|_\infty \\ &= \inf_{k \in H^\infty, |\phi - k| < 1} \left\| |\beta|^2 + \frac{|\alpha - \beta(1 - \bar{\phi}k)|^2}{1 - |\phi - k|^2} \right\|_\infty. \end{aligned}$$

The infimum is attained.

(2) If  $m\{\zeta \in T; |\alpha(\zeta)| = |\beta(\zeta)|\} > 0$  and  $W^{-1} \in L^1$ , then there exists  $\varepsilon_n \in L^\infty, \varepsilon_n > 0, \|\varepsilon_n\|_\infty \rightarrow 0, (n \rightarrow \infty)$  such that  $\|\alpha_n\| - \|\beta\| > 0$  and

$$\|S_{\alpha, \beta}\|_{L^2(W)} = \lim_{n \rightarrow \infty} \inf_{k \in H^\infty, |\phi - k| < 1} \|S_{\alpha_n, \beta}\|_{L^2(W)},$$

where

$$\begin{aligned} \alpha_n &:= \alpha + (\alpha\chi_{E_1} + \chi_{E_0})\varepsilon_n \\ E_0 &:= \{\zeta \in T; \alpha(\zeta) = \beta(\zeta) = 0\}, \\ E_1 &:= \{\zeta \in T; |\alpha(\zeta)| = |\beta(\zeta)| > 0\}. \end{aligned}$$

*Proof.* We prove (1). Let  $s = \|S_{\alpha, \beta}\|_{L^2(W)}^2$ . It is well known that  $\max\{|\alpha|^2, |\beta|^2\} \leq s$ . Since  $\|\alpha\| - \|\beta\| > 0$ , it follows that  $s > 0$ . For any  $k \in H^\infty$  satisfying  $|\phi - k| < 1$ , we define the quantity  $M_k$  according to

$$M_k = \left\| \frac{|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta(1 - \bar{\phi}k))}{1 - |\phi - k|^2} \right\|_\infty.$$

We prove that  $s \geq \inf\{M_k; k \in H^\infty, |\phi - k| < 1\}$ . It follows from Theorem 1 that  $F(s) = s$ . Hence there exists a function  $k \in H^\infty$  such that

$$G(s - \alpha\bar{\beta} - \bar{\phi}k) \leq s.$$

It follows from Lemma 3 that

$$G\left(\sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2}\right) = s.$$

Since

$$\left|\alpha\bar{\beta} - s\left(1 - \bar{\phi}\frac{k}{s}\right)\right|^2 = |s - \alpha\bar{\beta} - \bar{\phi}k|^2 \leq (s - |\alpha|^2)(s - |\beta|^2),$$

it follows that there exists a  $k_0 \in H^\infty$  such that

$$|\alpha\bar{\beta} - s(1 - \bar{\phi}k_0)|^2 \leq (s - |\alpha|^2)(s - |\beta|^2).$$

Hence,

$$|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta(1 - \bar{\phi}k_0)) \leq s(1 - |\phi - k_0|^2).$$

Since

$$s|\phi - k_0| \leq \sqrt{s - |\alpha|^2}\sqrt{s - |\beta|^2} + |\alpha\beta| \leq s,$$

it follows that  $|\phi - k_0| \leq 1$ . Since  $||\alpha| - |\beta|| > 0$ , it follows that  $|\phi - k_0| < 1$ . Hence,

$$\inf_{k \in H^\infty, |\phi - k| < 1} M_k \leq M_{k_0} \leq s.$$

We prove that  $s \leq \inf\{M_k; k \in H^\infty, |\phi - k| < 1\}$ . This is the easy direction of the theorem. Since

$$|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta(1 - \bar{\phi}k)) \leq M_k(1 - |\phi - k|^2),$$

it follows that

$$\begin{aligned} |M_k - \alpha\bar{\beta} - \bar{\phi}M_kk|^2 &= |M_k(1 - \bar{\phi}k) - \alpha\bar{\beta}|^2 \\ &= |M_k(1 - \bar{\phi}k)|^2 - 2M_k\operatorname{Re}(\bar{\alpha}\beta(1 - \bar{\phi}k)) + |\alpha\beta|^2 \\ &\leq M_k^2 - (|\alpha|^2 + |\beta|^2)M_k + |\alpha\beta|^2 \\ &= (M_k - |\alpha|^2)(M_k - |\beta|^2). \end{aligned}$$

It follows from Lemma 6 that  $\max\{|\alpha|^2, |\beta|^2\} \leq M_k$ . It follows from Lemma 3 that

$$G\left(\sqrt{M_k - |\alpha|^2}\sqrt{M_k - |\beta|^2}\right) = M_k.$$

Hence,

$$G(M_k - \alpha\bar{\beta} - \bar{\phi}M_kk) \leq M_k.$$

Hence  $F(M_k) \leq M_k$ . It follows from Lemma 4 that  $s \leq M_k$ . Therefore,

$$\inf_{k \in H^\infty, |\phi - k| < 1} M_k \leq M_{k_0} \leq s \leq \inf_{k \in H^\infty, |\phi - k| < 1} M_k.$$

Hence the equalities hold, and the infimum is attained by  $k = k_0$ . We prove (2). By Corollary 7, there exists  $\varepsilon_n \in L^\infty, \varepsilon_n > 0, \|\varepsilon_n\|_\infty \rightarrow 0, (n \rightarrow \infty)$  such that  $\|\alpha_n\| - \|\beta\| > 0$ , and

$$\|\varepsilon_n P_+\|_{L^2(W)} \rightarrow 0, (n \rightarrow \infty).$$

Since

$$\|(\alpha_n - \alpha)P_+\|_{L^2(W)} = \|(\alpha\chi_{E_1} + \chi_{E_0})\varepsilon_n P_+\|_{L^2(W)} \leq \|(\alpha\chi_{E_1} + \chi_{E_0})\|_\infty \|\varepsilon_n P_+\|_{L^2(W)},$$

it follows that

$$\|(\alpha_n - \alpha)P_+\|_{L^2(W)} \rightarrow 0, (n \rightarrow \infty).$$

Since

$$\left| \|S_{\alpha_n, \beta}\|_{L^2(W)} - \|S_{\alpha, \beta}\|_{L^2(W)} \right| \leq \|S_{\alpha_n, \beta} - S_{\alpha, \beta}\|_{L^2(W)} = \|(\alpha_n - \alpha)P_+\|_{L^2(W)},$$

it follows that

$$\left| \|S_{\alpha_n, \beta}\|_{L^2(W)} - \|S_{\alpha, \beta}\|_{L^2(W)} \right| \rightarrow 0, (n \rightarrow \infty).$$

This completes the proof.

If  $W$  is a constant function, then  $\phi$  becomes a constant function. In this case, the formula of  $\|S_{\alpha, \beta}\|_{L^2}$  was given by Corollary 1. The another formula of  $\|S_{\alpha, \beta}\|_{L^2}$  is given by Corollary 9.

**Corollary 9.** Let  $\alpha, \beta \in L^\infty$ . If  $\|\alpha\| - \|\beta\| > 0$ , then

$$\begin{aligned} \|S_{\alpha, \beta}\|_{L^2}^2 &= \inf_{g \in H^\infty, |g| < 1} \left\| \frac{|\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\bar{\alpha}\beta g)}{1 - |g|^2} \right\|_\infty \\ &= \inf_{g \in H^\infty, |g| < 1} \left\| |\alpha|^2 + \frac{|\bar{\beta} - \bar{\alpha}g|^2}{1 - |g|^2} \right\|_\infty \\ &= \inf_{g \in H^\infty, |g| < 1} \left\| |\beta|^2 + \frac{|\alpha - \beta g|^2}{1 - |g|^2} \right\|_\infty. \end{aligned}$$

**Corollary 10.** Let  $\alpha, \beta \in L^\infty$ . Let  $\phi$  and  $W$  be functions such that there exists an outer function  $h \in H^2$  satisfying  $\phi = \bar{h}/h$  and  $W = |h|^2$ . If  $\|\alpha\| - \|\beta\| > 0$ , then

$$(1) \quad \inf_{k \in H^\infty} \left\| \frac{|\alpha| - |\beta|}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty \leq \inf_{k \in H^\infty} \left\| \frac{\max\{|\bar{\beta} - \bar{\alpha}(1 - \bar{\phi}k)|, |\alpha - \beta(1 - \bar{\phi}k)|\}}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty$$

$$\begin{aligned}
&\leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \inf_{k \in H^\infty} \left\| \frac{|\alpha| + |\beta|}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty. \\
(2) \quad &\max \left\{ \|\alpha\|_\infty, \|\beta\|_\infty, \inf_{k \in H^\infty} \left\| \frac{|\alpha| - |\beta|}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty \right\} \\
&\leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \inf_{k \in H^\infty} \left\| \frac{\alpha - \beta}{\sqrt{1 - |\phi - k|^2}} \right\|_\infty + \min\{\|\alpha\|_\infty, \|\beta\|_\infty\}.
\end{aligned}$$

*Proof.* (1) follows from Theorem 3(1) and Lemma 6. (2) follows from Corollary 6 and

$$\begin{aligned}
\|S_{\alpha,\beta}\|_{L^2(W)} &\leq \|(\alpha - \beta)P_+\|_{L^2(W)} + \|\beta\|_\infty, \\
\|S_{\alpha,\beta}\|_{L^2(W)} &\leq \|(\beta - \alpha)P_-\|_{L^2(W)} + \|\alpha\|_\infty.
\end{aligned}$$

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