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On regularizing–decay rate estimates for solutions to the Navier–Stokes initial value problem

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Abstract. It is known that an L^n–valued continuous solution in time interval $(0, T)$ of the Navier–Stokes equations in $\mathbf{R}^n$ is regular for positive time. In this paper regularizing rate estimates similar to a solution of the heat equation are established. The estimates also provide analyticity in space variables as well as decay estimates on derivatives for large time. The solutions need not be small. Our results are obtained by estimating the integral equation with a new version of the Gronwall type inequality originally obtained in [2].

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Key words. Navier–Stokes equations, mild solution, regularizing–decay rate, spatial analyticity, Gronwall inequality.

1. Introduction.

We consider the Navier–Stokes initial value problem in $\mathbf{R}^n$ $(n \geq 2)$:

$$\text{(NS)} \quad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in } \mathbf{R}^n \times (0, T), \\ u|_{t=0} = u_0 \quad \text{in } \mathbf{R}^n, \quad (\text{with } \nabla \cdot u_0 = 0). \end{cases}$$

Here $u = u(x, t) = (u^1(x, t), u^2(x, t), \ldots, u^n(x, t))$ and $p = p(x, t)$ represents, respectively, unknown velocity and pressure at a point $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ and a time $t > 0$; u_0 is a given initial velocity and f is a given vector–valued function which represents the external force. We have used a standard notation about derivatives; $u_t = \partial u/\partial t$, $(u, \nabla) = \sum_{i=1}^n u^i \partial_i$, $\partial_i = \partial/\partial x_i$, $\Delta = \sum_{i=1}^n \partial_i^2$, $\nabla \cdot u = \sum_{i=1}^n \partial_i u^i$ and $\nabla p = (\partial_1 p, \ldots, \partial_n p)$.

It is well known that this problem admits a unique locally–in–time solution at least when $u_0 \in (L^n(\mathbf{R}^n))^n$, see [7], [11]. It is also well known that the solution becomes smooth for $t > 0$ since the problem is parabolic; see e.g. [7], [4], [19]. For the solution v of the heat equation $v_t - \Delta v = 0$ with initial data v_0 such a regularizing effect is quantified in the form

$$\text{(R)} \qquad \|\partial_x^\beta v(t)\|_q \le C t^{-\frac{|\beta|}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|v_0\|_r, \qquad t > 0 \ \text{ and } \ 1 \le r \le q \le \infty$$

with some constant C depending only on multi–index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}_0^n$ and n. Here, by $\|\partial_x^\beta v(t)\|_p$ we mean the spatial L^p–norm of $\partial_x^\beta v(\cdot, t)$. We have used the standard convention $\partial_x^\beta = \partial_1^{\beta_1} \cdot \cdots \cdot \partial_n^{\beta_n}$ and $|\beta| = \beta_1 + \cdots + \beta_n$ for $\beta \in \mathbf{N}_0^n$, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $\mathbf{N}$ is the set of all positive integers. This type of estimates is well known for the heat equation since the solution v is of the form $v = e^{t\Delta} v_0 = G_t * v_0$, where $G_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ is the Gauss kernel; see e.g. Lemma 2.1 for the proof. However, for a solution of (NS) as far as the authors know, the estimate like (R) has not been well studied in the literature except for first derivatives; see e.g. [11]. For vorticity equation [12] derived several derivative estimates when the initial vorticity is a finite Radon measure and $n = 2$. His estimates exclude the case of $q = 1$ and $q = \infty$ and his method appeals to a complicated interpolation theory. In [2] estimates are extended so that they include these extreme cases; moreover, their proof is elementary; see also [5].

Our goal in this paper is to derive several regularizing rate estimates like (R) for the solution u of (NS). To state our results precisely we rather discuss the solution of the integral equation

$$\text{(INT)} \quad u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} \mathbf{P}(u \otimes u)(s) ds + \int_0^t e^{(t-s)\Delta} \mathbf{P} f(s) ds,$$

which is formally equivalent to (NS). Here $\mathbf{P}$ denotes the orthogonal projection onto the solenoidal subspace of $(L^2(\mathbf{R}^n))^n$ and it is written as an $n \times n$ matrix operator $\mathbf{P} = (P_{ij}) = (\delta_{ij} + R_i R_j)$, where δ_{ij} denotes the Kronecker's delta (times the identity operator), and R_i is the Riesz operator formally defined by $R_i = \partial_i(-\Delta)^{-1/2}$. Here, $u \otimes u$ is a tensor whose ij–component is $u^i u^j$. In the formal level once one finds the solution of u of (INT), (u, p) with $p = \sum_{i,j=1}^n R_i R_j u^i u^j$ solves (NS). A solution of (INT) is often called a *mild solution* so we use this terminology here.

We are now in position to state a typical form of our results.

Theorem 1.1. *Assume that $n \ge 2$. Assume that f is conservative i.e., $\mathbf{P} f = 0$. Assume that $u_0 \in (L^n(\mathbf{R}^n))^n$ satisfying $\nabla \cdot u_0 = 0$. Let u be a mild solution on $[0, T]$ such that $u \in C([0, T); (L^n(\mathbf{R}^n))^n) \cap C((0, T); (L^p(\mathbf{R}^n))^n)$ for some $T > 0$ and some $p \in (n, \infty]$. Then there exist constants K_1 and K_2 depending only on n, p, M_1 and M_2*

$$(1.1) \qquad M_1 \equiv \sup_{0 \le t < T} \|u(t)\|_n,$$

$$M_2 \equiv \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_p \tag{1.2}$$

such that

$$\|\partial_x^\beta u(t)\|_q \le K_1(K_2|\beta|)^{|\beta|} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \tag{1.3}$$

for all $q \in [n,\infty]$, $t \in (0,T)$ *and* $\beta \in \mathbf{N}_0^n$, *provided that* M_2 *is finite.*

Remark 1.1. (i) Note that K_1 and K_2 do not depend on T explicitly, so this estimate (1.3) yields the decay of higher derivatives of solution as $t \to \infty$ once we have a bound for M_1 and M_2 independent of T. (If $\|u_0\|_n$ is small, this situation actually occurs; see [11].) The estimate (1.3) also shows that the solution u is spatial analytic. We also note that (1.3) is still valid even if f is not conservative under suitable assumptions on f, for example,

$$\|\partial_x^\beta \mathbf{P}f(t)\|_n \le M_3|\beta|^{|\beta|}t^{-|\beta|/2}$$

for all $0 < t < T$ and $\beta \in \mathbf{N}_0^n$. The constants K_1 and K_2 in (1.3) now depend also on M_3. If f is analytic in x, then f satisfies above inequality obviously, so we also get the spatial analyticity of u.

(ii) We may remove the condition (1.1) if $p \in (n, 2n]$. Indeed, if $n < p \le 2n$ and assume that (1.2), then one can get (1.1) for some M_1 depending only on n, p and M_2. We are also able to obtain (1.3) if we replace (1.2) by

$$M_4 \equiv \sup_{0<t<T} t^{\frac{\alpha}{2}}\|(-\Delta)^{\frac{\alpha}{2}}u(t)\|_n < \infty$$

for some $\alpha > 0$. Of course, the constants K_1 and K_2 now depend also on M_4 and α. The proof parallels that of Theorem 1.1.

(iii) Even if L^n is replaced by L^r with $r \in (n,p)$ we still have (1.3) type estimates with n replaced by r but K_1 and K_2 also depend on T.

The decay rate estimate (1.3) for $|\beta| \le 1$ is known by T. Kato [11] under smallness assumption on $\|u_0\|_n$. His result is extended by C. He and C. Hsiao [9] for all β but smallness of $\|u_0\|_n$ depends on β. Moreover, β–dependence of (1.3) is not clarified in [9].

Using L^2–theory, M. E. Schonbek [16] obtained the decay rates of m-th order derivatives of the globally–in–time strong solutions to (NS) with $u_0 \in ((L^1 \cap H^m)(\mathbf{R}^2))^2$. (There are large literature on the decay of u itself. The reader is referred to articles cited in [16], [17].) Her proof is often called the Fourier splitting method, that is, she splits the integration in phase–space into two time–dependent domains. Her results are extended by many authors including [17] and [15]. Especially, M. Oliver and E. S. Titi [15] obtained the decay rates of higher derivatives of Leray–Hopf solutions by using the Gevrey norm (see (1.5));

$$\|\partial_x^\beta u(t)\|_2 \le C\Big(\frac{2|\beta|}{e}\Big)^{|\beta|}(1+t)^{-\frac{n}{2}-\frac{|\beta|}{2}} \tag{1.4}$$

for $u_0 \in ((L^1 \cap L^2)(\mathbf{R}^n))^n$ if $\liminf_{t\to\infty} \|u(t)\|_{H^r} < \infty$ (for some $r > n/2$). This estimate yields the spatial analyticity for $L^1 \cap L^2$ data.

The analyticity of solutions of the Navier–Stokes equations has been investigated by a number of authors previously. In particular, K. Masuda [13] (see also [14]) showed that if Ω is a bounded domain and f is analytic in t and x, then $L^2(\Omega)$ solution u is analytic in Ω. The first author of this paper [3] improved Masuda's result. He treated $L^p(\Omega)$ solutions for any $n/2 < p < \infty$ and showed that it is analytic up to the boundary. C. Kahane [10] proved the interior analyticity that if f is analytic in x, then the solutions are analytic in x. C. Foias and R. Temam [1] considered (NS) with space periodicity boundary condition, and introduced the Gevrey norm:

$$\|A^r e^{\tau A} u\|_2 \qquad \text{for} \quad r > 0, \qquad \text{where} \quad A = \sqrt{-\Delta}. \tag{1.5}$$

They showed L^2 solutions are spatially analytic if f is analytic in x. Z. Grujić and I. Kukavica [8] also constructed the spatially–locally analytic solution for L^p data for $n < p < \infty$. Our result has different features since it also quantifies regularizing rate and does not involve to construct the solutions.

We do not discuss the time analyticity in this paper but it seems to be not difficult. It is not clear whether or not $|\beta|^{|\beta|}$ in (1.3) can be replaced by $|\beta|^{|\beta|/2}$ as for the heat equation (Lemma 2.1).

To prove (1.3) we actually prove an equivalent estimate

$$\|\partial_x^\beta u(t)\|_q \le K_1 (K_2 |\beta|)^{|\beta|-\delta} t^{-\frac{|\beta|}{2} - \frac{n}{2}(\frac{1}{n} - \frac{1}{q})} \tag{1.6}$$

for some $\delta \in (\frac{1}{2}, 1]$. We differentiate the both hand sides of (INT) and take L^q norm. Because of derivative $\nabla\cdot$ in $u \otimes u$ term, the right hand side also contains the highest order derivatives. However, it is expected to be non–integrable near $s = 0$ so the conventional Gronwall inequality does not apply. We divide the integral $\int_0^t$ into two parts as $\int_0^{t(1-\varepsilon)} + \int_{t(1-\varepsilon)}^t$ and apply a new but elementary Gronwall type inequality (Lemma 2.4) in [2], [5] with induction on $|\beta|$ to get (1.6). Here ε is taken small such that $\varepsilon \approx 1/|\beta|$ for large $|\beta|$.

Throughout this paper we denote positive constants by C the value of which may differ from one occasion to another. The variables of constant $C(\cdot, \cdot, \dots, \cdot)$ indicate the dependence of the parameters.

2. Preliminary.

In this section we prepare regularizing rate estimates for heat semigroup, an estimate of multiplication of multi–sequences and a new Gronwall type inequality.

We first give the $L^p - L^q$ estimate for derivatives of a solution to the heat equation.

Lemma 2.1. *Assume that $n \geq 1$, $1 \leq p \leq q \leq \infty$ and $1 < \theta < \infty$. Then*

$$\|\partial_x^\beta e^{t\Delta} f\|_q \leq \Big\{4\pi\Big(1-\frac{1}{\theta}\Big)\Big\}^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Big(\frac{\theta|\beta|}{\pi}\Big)^{\frac{|\beta|}{2}} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\beta|}{2}}\|f\|_p \tag{2.1}$$

for all $t > 0$, $\beta \in \mathbf{N}_0^n$ and $f \in L^p(\mathbf{R}^n)$.

Proof. The proof is standard but we give it for completeness. Since $\|\partial_i G_t\|_1 \leq \pi^{-1/2}t^{-1/2}$, the Young inequality yields

$$\|\partial_i e^{t\Delta} f\|_p \leq \pi^{-\frac{1}{2}} t^{-\frac{1}{2}} \|f\|_p \tag{2.2}$$

for all $i = 1, 2, \ldots, n$, $1 \leq p \leq \infty$, $t > 0$ and $f \in L^p$. Since $\|G_t\|_r \leq (4\pi t)^{-\frac{1}{2}} t^{-\frac{n}{2}(1-\frac{1}{r})}$ for $1 \leq r \leq \infty$, the Young inequality yields

$$\|e^{t\Delta} f\|_q \leq (4\pi t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p \tag{2.3}$$

for all $1 \leq p \leq q \leq \infty$, $t > 0$ and $f \in L^p$. For $1 < \theta < \infty$ let θ' denote the conjugate of θ (i.e., $\frac{1}{\theta} + \frac{1}{\theta'} = 1$). Using the semigroup property of $e^{t\Delta}$ we get

$$\partial_x^\beta e^{t\Delta} f = e^{\frac{t}{\theta'}\Delta} \prod_{i=1}^n (\partial_i e^{\frac{t}{\theta|\beta|}\Delta})^{\beta_i} f.$$

We now apply (2.2) and (2.3) to get

$$\begin{aligned}\|\partial_x^\beta e^{t\Delta} f\|_q &\leq \|e^{\frac{t}{\theta'}\Delta}\|_{L^p\to L^q}\Big\{\prod_{i=1}^n \|\partial_i e^{\frac{t}{\theta|\beta|}\Delta}\|_{L^p\to L^p}^{\beta_i}\Big\}\|f\|_p \\ &\leq \left(4\pi\frac{t}{\theta'}\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\left\{\pi^{-\frac{1}{2}}\left(\frac{t}{\theta|\beta|}\right)^{-\frac{1}{2}}\right\}^{|\beta|}\|f\|_p.\end{aligned}$$

Here $\|\cdot\|_{L^p\to L^q}$ stands for the operator–norm from L^p to L^q. □

We next recall the boundedness of $t^{\frac{1}{2}}(e^{t\Delta}\mathbf{P}\partial_i)$ in every $(L^p(\mathbf{R}^n))^n$ for $1 \leq p \leq \infty$. Of course, if $1 < p < \infty$, then $\mathbf{P}$ is a bounded operator in L^p, so (2.2) yields this assertion. However, since $\mathbf{P}$ is not a bounded operator in L^1 or L^∞ we need some extra argument. Fortunately, this difficulty is overcome by [6] and we have:

Lemma 2.2. *Let $n \geq 1$. Then there exists a positive constant $C = C(n)$ such that*

$$\|e^{t\Delta}\mathbf{P}\partial_i f\|_p \leq Ct^{-\frac{1}{2}}\|f\|_p \tag{2.4}$$

for all $1 \leq p \leq \infty$, $1 \leq i \leq n$, $t > 0$ and $f \in (L^p(\mathbf{R}^n))^n$.

The proof in [6] appeals to the Hardy space estimate of $\partial_i G_t$. It yields $\|P_{kl}\partial_i G_t\|_1 \leq Ct^{-\frac{1}{2}}$, which yields (2.4) with C independent of p. We note that there is an elementally proof by Y. Shibata and S. Shimizu [18].

We shall recall the estimate for multiplication of multi–sequences, which has been proved by C. Kahane [10, Lemma 2.1].

Lemma 2.3. *Let $\delta > \frac{1}{2}$. Then there exists a positive constant λ depending only on δ such that*

$$\sum_{\gamma \le \beta} \binom{\beta}{\gamma} |\gamma|^{|\gamma|-\delta} |\beta - \gamma|^{|\beta-\gamma|-\delta} \le \lambda |\beta|^{|\beta|-\delta} \qquad \text{for all } \beta \in \mathbf{N}_0^n.$$

Here, for multi–indices β and γ, $\gamma \le \beta$ means $\gamma_i \le \beta_i$ for all i, and $\binom{\beta}{\gamma} = \prod_{i=1}^n \frac{\beta_i!}{\gamma_i!(\beta_i-\gamma_i)!}$. The proof of this lemma is based on Stirling's formula. We do not present the proof here.

We shall recall a variant of the Gronwall type inequality [5, Lemma 6.1], which has been proved in [2]. Although the proof is essentially the same as in [2] and [5], we give a proof for completeness.

Lemma 2.4. *Let ψ_0 be a measurable and locally bounded function in $(0,T)$. Let $\{\psi_j\}_{j=1}^\infty$ be a sequence of measurable functions in $(0,T)$. Assume that $\alpha \in \mathbf{R}$ and that μ, $\nu > 0$ satisfying $\mu + \nu = 1$. Let $b_\varepsilon > 0$ be a number depending on $\varepsilon \in (0,1)$, and assume that b_ε is nonincreasing with respect to ε. Assume that there is a positive constant σ such that*

$$0 \le \psi_0(t) \le b_\varepsilon t^{-\alpha} + \sigma \int_{t(1-\varepsilon)}^t (t-s)^{-\mu} s^{-\nu} \psi_0(s)ds \tag{2.5}$$

and

$$0 \le \psi_{j+1}(t) \le b_\varepsilon t^{-\alpha} + \sigma \int_{t(1-\varepsilon)}^t (t-s)^{-\mu} s^{-\nu} \psi_j(s)ds \tag{2.6}$$

for all $j \ge 0$, $t \in (0,T)$ and $\varepsilon \in (0,1)$. Let ε_0 be a unique positive number such that $I(\varepsilon_0) = \min\{\frac{1}{2\sigma}, I(1)\}$ with $I(\varepsilon) = \int_{1-\varepsilon}^1 (1-\tau)^{-\mu}\tau^{-\alpha-\nu}d\tau$. Then

$$\psi_j(t) \le 2b_{\varepsilon_0} t^{-\alpha} \tag{2.7}$$

for all $j \ge 0$ and $t \in (0,T)$.

Proof. We first show the case of $j = 0$. By (2.5) we have

$$t^\alpha \psi_0(t) \le b_\varepsilon + \sigma t^\alpha \int_{t(1-\varepsilon)}^t (t-s)^{-\mu} s^{-\alpha-\nu} s^\alpha \psi_0(s)ds.$$

For small $\eta > 0$ (say, $\eta < t$) we set $\phi_0(t) = \sup_{\eta \le s \le t} s^\alpha \psi_0(s)$ for $t > 0$. This quantity is always finite. For $t > \frac{\eta}{1-\varepsilon}$ we have

$$\begin{aligned}\phi_0(t) &\le b_\varepsilon + \sigma t^\alpha \phi_0(t) \int_{t(1-\varepsilon)}^t (t-s)^{-\mu} s^{-\alpha-\nu} ds \\ &\le b_\varepsilon + \sigma \phi_0(t) \int_{1-\varepsilon}^1 (1-\tau)^{-\mu} \tau^{-\alpha-\nu} d\tau.\end{aligned}$$

We set $I(\varepsilon) = \int_{1-\varepsilon}^{1} (1-\tau)^{-\mu}\tau^{-\alpha-\nu}d\tau$ and observe that $I(\varepsilon)$ is continuous in $[0,1)$ and $I(0)=0$. Moreover, $I(\varepsilon)$ is strictly monotone decreasing in ε. Thus, for $\sigma > 0$ there exists a unique $\varepsilon_0 > 0$ such that $I(\varepsilon_0) = \min\{1/2\sigma, I(1)\}$. The value $I(1)$ may be infinity. For such an ε_0 we have

$$\phi_0(t) \le b_{\varepsilon_0} + \frac{1}{2}\phi_0(t), \qquad t \in \left(\frac{\eta}{1-\varepsilon_0}, T\right).$$

Hence, $t^\alpha \psi_0(t) \le 2b_{\varepsilon_0}$ for $t \in (0,T)$.

The estimates for ψ_j is similar. Indeed, for small $\eta > 0$ we set $\phi_j(t) = \sup_{\eta \le s \le t} s^\alpha \psi_j(s)$ for $j \ge 1$ and $t > 0$. Similarly as above, by (2.6) we have

$$\phi_{j+1}(t) \le b_\varepsilon + \sigma\phi_j(t)\int_{1-\varepsilon}^{1}(1-\tau)^{-\mu}\tau^{\alpha-\nu}d\tau$$

for $t > \frac{\eta}{1-\varepsilon}$. We set ε_0 the same as above to get

$$\phi_{j+1}(t) \le b_{\varepsilon_0} + \frac{1}{2}\phi_j(t), \qquad j \ge 0 \quad \text{and} \quad t \in \left(\frac{\eta}{1-\varepsilon_0}, T\right).$$

We have already obtained $\phi_0(t) \le 2b_{\varepsilon_0}$, the last successive estimates yield $\phi_j(t) \le 2b_\varepsilon$ for all $j \ge 1$ and $t \in (0,T)$. We have thus obtained (2.7). □

3. Proof of Theorem.

We first prove a variant of Theorem 1.1 under extra regularity assumptions.

Proposition 3.1. *Assume that same hypotheses of Theorem 1.1. Assume furthermore that*

$$\partial_x^\beta u \in C((0,T);(L^q(\mathbf{R}^n))^n) \tag{3.1}$$

for all $q \in [n,\infty]$ and $\beta \in \mathbf{N}_0^n$. For $\delta \in (\frac{1}{2},1]$ there exist positive constants K_1 and K_2 depending only on n, p, M_1, M_2 and δ such that

$$\|\partial_x^\beta u(t)\|_q \le K_1(K_2|\beta|)^{|\beta|-\delta} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \tag{3.2}$$

for all $t \in (0,T]$, $q \in [n,\infty]$ and $\beta \in \mathbf{N}_0^n$.

Since the solution u in Theorem 1.1 always satisfies the regularity property (3.1) as shown in Proposition 3.2 and its Remark 3.1, Theorem 1.1 evidently follows from Proposition 3.1.

Proof. We argue by an induction on $m = |\beta|$.

(Step 1). We shall prove (3.2) for $m = 0$. It suffices to prove

$$\|u(t)\|_\infty \le C_1 t^{-\frac{1}{2}} \tag{3.3}$$

with some constant $C_1 = C_1(n, p, M_1, M_2)$. Indeed, once (3.3) is proved, the estimate (3.2) for $m = 0$ and $n < q < \infty$ is obtained by interpolating (3.3) and L^n–estimate $\|u(t)\|_n \leq M_1$.

We may assume that $n < p < \infty$ in the assumption (1.2) since (3.3) directly follows from (1.2) when $p = \infty$. We take L^∞–norm of the both sides of (INT) and for $0 < \varepsilon < 1$ we divide the integral into two parts:

$$\begin{aligned} \|u(t)\|_\infty &\leq \|e^{t\Delta}u_0\|_\infty + \left(\int_0^{t(1-\varepsilon)} + \int_{t(1-\varepsilon)}^t\right) \|e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)(s)\|_\infty ds \\ &\equiv A_1 + A_2 + A_3. \end{aligned}$$

We shall estimate each term. Here and hereafter C_j $(j = 1, 2, \ldots)$ denotes a positive constant depending only on n, p, M_1 and M_2 while C depends only on n and p.

We first estimate A_1. By (2.3) we have

$$A_1 \leq (4\pi t)^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{\infty})}\|u_0\|_n \leq (4\pi)^{-\frac{1}{2}}M_1 t^{-\frac{1}{2}}.$$

We next estimate A_2. By Lemma 2.1 with $\theta = \pi$ (this choice of θ is kept throughout this Section) and the boundedness of $\mathbf{P}$ in L^q for $q \in (1, \infty)$, we observe that

$$\begin{aligned} A_2 &\leq \int_0^{t(1-\varepsilon)} \|e^{(t-s)\Delta}\nabla\|_{L^{\frac{p}{2}} \to L^\infty} \|\mathbf{P}\|_{L^{\frac{p}{2}} \to L^{\frac{p}{2}}} \|u \otimes u(s)\|_{\frac{p}{2}} ds \\ &\leq C \int_0^{t(1-\varepsilon)} (t-s)^{-\frac{n}{p}-\frac{1}{2}} \|u(s)\|_p^2 ds \\ &\leq C \int_0^{t(1-\varepsilon)} (t-s)^{-\frac{n}{p}-\frac{1}{2}} s^{-n(\frac{1}{n}-\frac{1}{p})} M_2^2 ds \leq C_2 \varepsilon^{-\frac{3}{2}} t^{-\frac{1}{2}}. \end{aligned}$$

Similarly, A_3 is estimated as

$$\begin{aligned} A_3 &\leq \int_{t(1-\varepsilon)}^t \|e^{(t-s)\Delta}\nabla\|_{L^p \to L^\infty} \|\mathbf{P}\|_{L^p \to L^p} \|u \otimes u(s)\|_p ds \\ &\leq C \int_{t(1-\varepsilon)}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \|u(s)\|_p \|u(s)\|_\infty ds \\ &\leq C_3 \int_{t(1-\varepsilon)}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} s^{-\frac{1}{2}+\frac{n}{2p}} \|u(s)\|_\infty ds. \end{aligned}$$

Combining these three estimates and setting $b_\varepsilon = (4\pi)^{-\frac{1}{2}}M_1 + C_2\varepsilon^{-\frac{3}{2}}$, we obtain

$$\|u(t)\|_\infty \leq b_\varepsilon t^{-\frac{1}{2}} + C_3 \int_{t(1-\varepsilon)}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} s^{-\frac{1}{2}+\frac{n}{2p}} \|u(s)\|_\infty ds.$$

We now apply Lemma 2.4 to the above inequality to get (3.3) with $C_1 = 2b_{\varepsilon_0}$, where $\varepsilon_0 = \varepsilon_0(n, p, M_2) \in (0,1)$.

(Step 2). Our next task is to prove (3.2) for $m = 1$. However, the proof is essentially contained in Step 3 (without checking the dependence of m and δ) so we safely omit the proof.

(Step 3). Assume that $m \geq 2$. Assume that we have already proved (3.2) for all $q \in [n, \infty]$ and β with $|\beta| \leq m-1$. We shall prove (3.2) for $|\beta| = m$. For $q \in [n, \infty]$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned}\|\partial_x^\beta u(t)\|_q &\leq \|\partial_x^\beta e^{t\Delta}u_0\|_q + \left(\int_0^{t(1-\varepsilon)} + \int_{t(1-\varepsilon)}^t\right)\|\partial_x^\beta e^{(t-s)\Delta}\mathbf{P}\nabla\cdot(u\otimes u)(s)\|_q ds \\ &\equiv B_1 + B_2 + B_3.\end{aligned}$$

We shall estimate each term.

The term B_1 is easily estimated. By Lemma 2.1 we have

$$\begin{aligned}B_1 &\leq C_4 m^{m/2} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}, \qquad C_4 = CM_1, \\ &\leq C_4 m^{m-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}\end{aligned}$$

since $m/2 \leq m - \delta$ for $m \geq 2$ and $\delta \leq 1$. To estimate B_2 we apply Lemma 2.1 and Lemma 2.2 to get

$$\begin{aligned}B_2 &\leq \int_0^{t(1-\varepsilon)} \|\mathbf{P}e^{\frac{t-s}{2}\Delta}\nabla\|_{L^q\to L^q}\|\partial_x^\beta e^{\frac{t-s}{2}\Delta}\|_{L^{\frac{n}{2}}\to L^q}\|u\otimes u(s)\|_{\frac{n}{2}} ds \\ &\leq Cm^{m/2}\int_0^{t(1-\varepsilon)}\left(\frac{t-s}{2}\right)^{-\frac{n}{2}(\frac{2}{n}-\frac{1}{q})-\frac{m}{2}}\|u(s)\|_n^2 ds \\ &\leq Cm^{m/2}M_1^2\left(\frac{\varepsilon}{2}\right)^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-1} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}} \\ &\leq C_5(2m)^{\frac{m}{2}}\varepsilon^{-\frac{m}{2}-\frac{3}{2}} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}.\end{aligned}$$

We apply Lemma 2.2 to get

$$\begin{aligned}B_3 &\leq \int_{t(1-\varepsilon)}^t \|e^{(t-s)\Delta}\mathbf{P}\nabla\|_{L^q\to L^q}\|\partial_x^\beta(u\otimes u)(s)\|_q ds \\ &\leq C\int_{t(1-\varepsilon)}^t (t-s)^{-\frac{1}{2}}\|\partial_x^\beta(u\otimes u)(s)\|_q ds.\end{aligned}$$

We now calculate $\partial_x^\beta(u\otimes u)$ by the Leibniz rule. We divide the sum into two parts:

$$B_3 \leq 2C\int_{t(1-\varepsilon)}^t (t-s)^{-\frac{1}{2}}\|\partial_x^\beta u(s)\|_q\|u(s)\|_\infty ds$$

$$+C\int_{t(1-\varepsilon)}^{t}(t-s)^{-\frac{1}{2}}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u(s)\|_q\|\partial_x^{\beta-\gamma}u(s)\|_\infty ds$$
$$\equiv B_4+B_5.$$

Here, for multi–indices β and γ, $\gamma<\beta$ means $\gamma\le\beta$ and $|\gamma|<|\beta|$. We shall estimate B_4. Applying L^∞–estimate (3.3) to B_4 yields

$$B_4\le C_6\int_{t(1-\varepsilon)}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\|\partial_x^\beta u(s)\|_q ds,\qquad C_6=2CC_1.$$

By our induction assumption we get

$$\begin{aligned}B_5&\le C\int_{t(1-\varepsilon)}^{t}(t-s)^{-\frac{1}{2}}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}K_1(K_2|\gamma|)^{|\gamma|-\delta}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\gamma|}{2}}\\&\qquad\times K_1(K_2|\beta-\gamma|)^{|\beta-\gamma|-\delta}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{\infty})-\frac{|\beta-\gamma|}{2}}ds\\&\le CK_1^2K_2^{m-2\delta}\left(\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta}\right)\\&\qquad\times\int_{t(1-\varepsilon)}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}-\frac{1}{2}}ds.\end{aligned}$$

We now apply Lemma 2.3 to get

$$B_5\le C\lambda K_1^2K_2^{m-2\delta}m^{m-\delta}t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}I(\varepsilon)$$

with $I(\varepsilon)=\int_{1-\varepsilon}^{1}(1-\tau)^{-\frac{1}{2}}\tau^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}-\frac{1}{2}}d\tau$.

Combining estimates for B_1–B_5 and setting $\tilde{b}_\varepsilon$ by

$$\tilde{b}_\varepsilon=C_4m^{m-\delta}+C_5(2m)^{\frac{m}{2}}\varepsilon^{-\frac{m}{2}-\frac{3}{2}}+C\lambda I(\varepsilon)K_1^2K_2^{m-2\delta}m^{m-\delta},$$

we obtain

$$\|\partial_x^\beta u(t)\|_q\le\tilde{b}_\varepsilon t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}+C_6\int_{t(1-\varepsilon)}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\|\partial_x^\beta u(s)\|_q ds.$$

We apply Lemma 2.4 to get

$$\|\partial_x^\beta u(t)\|_q\le 2\tilde{b}_{\varepsilon_0}t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}},\qquad t\in(0,T)\tag{3.4}$$

for $\varepsilon\le\varepsilon_0=\varepsilon_0(m)$ for sufficiently large m, say $m\ge m_0=m_0(n,p,M_1,M_2)$. By (3.4) we may assume that (3.2) holds for β, $|\beta|<m_0$ by taking K_1, K_2 large, so we may assume that $m\ge m_0$.

We shall prove that it is possible to choose K_1 and K_2 such that

$$(3.5) \qquad 2\tilde{b}_{1/m} \le K_1(K_2 m)^{m-\delta} \qquad \text{for } m \ge m_0.$$

Since $I(1/m) \le \sqrt{e/m} \le 2$ and $m^{\frac{3}{2}+\delta} \le 8 \cdot 2^{m-\delta}$, we observe that

$$2\tilde{b}_{1/m} \le (C_7 4^{m-\delta} + C' K_1^2 K_2^{m-2\delta}) m^{m-\delta}, \qquad C' = 4C\lambda, \quad C_7 = 2(C_4 + 8C_5).$$

We take $K_1 = 8C_7$ and $K_2 \ge 4$ large enough such that $C' K_1 K_2^{-\delta} < 1/2$ and obtain (3.5). Since K_1 and K_2 are independent of m the proof is now complete. □

We next verify that the locally–in–time solution satisfies (3.1).

Proposition 3.2. *Let $u_0 \in (L^n(\mathbf{R}^n))^n$ satisfying $\nabla \cdot u_0 = 0$. Then there exist $T_0 > 0$ and a unique mild solution u satisfying* (3.1) *and*

$$(3.6) \qquad t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{|\beta|}{2}} \|\partial_x^\beta u(t)\|_q \le K_1'(K_2'|\beta|)^{|\beta|}$$

for all $0 < t < T_0$, $n \le q \le \infty$ and $\beta \in \mathbf{N}_0^n$. Here, K_1' and K_2' are constants depending only on $\|u_0\|_n$, n and T_0.

Remark 3.1. By Proposition 3.2 and the uniqueness of the mild solution (see [11]), we conclude that the mild solution in Theorem 1.1 fulfills (3.1).

Proof. We recall that the solution u is constructed as a limit of u_j by a successive iteration: $u_1(t) = e^{t\Delta} u_0$ and

$$u_{j+1}(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u_j \otimes u_j)(s) ds.$$

In fact, as shown in [11] there exists $T_0 > 0$ satisfying

$$(3.7) \qquad \sup_j \|u_j(t)\|_q \le C_1' \|u_0\|_n t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}, \qquad t \in (0, T_0).$$

Here, C_1' is a positive constant depending only on n and T. Moreover, u_j converges to u in $(0, T_0]$. We set $\psi_j(t) = \|\partial_x^\beta u_{j+1}(t)\|_q$ and argue in the similar way in the proof of Proposition 3.1 to get (3.6) by applying Lemma 2.4. □

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