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Yumiharu Nakano

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Yumiharu Nakano
Division of Mathematics, Graduate School of Science
Hokkaido University, Sapporo 060-0810, Japan
E-mail: nakano_y@math.sci.hokudai.ac.jp

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Abstract

We consider a contingent claim in a jump-diffusion model of complete market. Given initial wealth less than the replicating cost, we explicitly solve the problem of minimizing the expected value of hedging loss weighted by power functions. We show that the optimal portfolio is the difference between the perfect hedging portfolio of the contingent claim and the optimal portfolio of a utility minimization problem. We also give an explicit formula for the value function. These results hold for every European-type contingent claim.

Key words: shortfall risk, jump-diffusion model.

1 Introduction

In this paper, we consider a frictionless, complete financial market consisting of one riskless bond and two risky assets S_i, $i = 1, 2$, that are traded up to a finite time horizon T. We suppose that the dynamics of S_i are described by jump-diffusion processes. Given a contingent claim H and an initial wealth x, we study the following optimization problem:

$$V(x) = \inf_{\pi \in \mathcal{A}} E\left[\ell_p\left((H - X^{x,\pi}(T))_+\right)\right], \tag{1}$$

where $\ell_p(x)$ is the power function x^p/p with $p > 1$, $X^{x,\pi}$ is the wealth process, and $\mathcal{A}$ is a set of admissible portfolios.

To explain the problem (1), we consider an investor with initial wealth x. If x is greater than the replication cost of H, say x_H, then the investor can hedge the contingent claim H without risk, by the completeness of the market. However, if x is strictly less than x_H, he/she faces the possibility of shortfall, i.e., for any portfolio π, the shortfall $(H - X^{x,\pi}(T))_+$ may be positive. In this situation, one method of hedging H is to follow a portfolio that minimizes the shortfall risk $E\left[\ell_p\left((H - X^{x,\pi}(T))_+\right)\right]$.

For work related to the problem (1), see, e.g., Föllmer and Leukert [FL] and Pham [P]. See also Nakano [N1] and [N2]. In [FL], general semimartingale models and general loss functions are considered. They impose the nonnegativity constraint on the wealth processes, and use the arguments involving Neyman-Pearson-type lemmas. This setting is crucial in solving their problem. In this paper, however, instead of the nonnegativity constraint, we impose only an integrability condition on the wealth processes (see Definition 2 below). Thus, our setting is similar to that of [P], except that we work in a jump-diffusion model of continuous-time markets. By requiring only the integrability condition on the wealth processes, we can obtain not only the optimal terminal wealth but also the optimal portfolio explicitly.

In Section 2, we explain the model and state the precise formulation of our problem. We then show that we can separate the problem into two problems, that is, the perfect hedging problem of H and the utility minimization problem

$$J(x_H - x) = \inf_{\pi \in \mathcal{A}_0(x_H - x)} E[\ell_p(X^{x_H - x,\pi}(T))], \tag{2}$$

where $A_0(x_H - x)$ is the set of portfolios. We prove that the optimal portfolio of (1) is represented as the difference between the perfect hedging portfolio of H and the optimal portfolio of the problem (2). As in the standard utility maximization problems (cf. Karatzas and Shreve [KS, Chapter 3] and Jeanblanc-Picqué and Pontier [JP]), we can solve the problem (2) by using the martingale method. In our main theorem, we give closed form expressions for the optimal portfolio and the value function $V(x)$. These results hold for every European-type contingent claim, such as, claims that can take negative values and path-dependent options. All the proofs of the results are given in Appendix A.

2 The model and main results

We consider a frictionless financial market consisting of one riskless bond B and two risky asset S_i, $i = 1, 2$, that are traded up to a finite time horizon T. We suppose that B satisfies the equation

$$dB(t) = r(t)B(t)dt, \quad B(t) = 1.$$

We also suppose that S_i, $i = 1, 2$, satisfy the stochastic differential equations

$$\begin{cases} dS_i(t) = S_i(t-)(\mu_i(t)dt + \sigma_i(t)dW(t) + \gamma_i(t)dN(t)), \\ S_i(0) = s_i \in (0, \infty) \qquad\qquad\qquad\qquad (i = 1, 2), \end{cases} \tag{3}$$

where W is a one-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, and the process N is a Poisson process with intensity $\lambda(\cdot)$, which is independent of W. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is the P-augmentation of the natural filtration generated by W and N. Then W is a $(P, \mathcal{F}_t)$-Brownian motion, N is a $(P, \mathcal{F}_t)$-Poisson process with intensity $\lambda(\cdot)$, and the process $M(t) := N(t) - \int_0^t \lambda(s)ds$ is a P-martingale.

Assumption 1. For $i = 1, 2$, λ, r, μ_i, σ_i, and γ_i are bounded, measurable, deterministic functions on $[0, T]$ that satisfy the following conditions:

(i) $\lambda(t) > 0$, $r(t) \geq 0$, $\sigma_i(t) > 0$, $\gamma_i(t) > -1$, and $\gamma_i(t) \neq 0$ for $t \in [0, T]$ and $i = 1, 2$;

(ii) there exists $c_1 \in (0, \infty)$ such that, for $t \in [0, T]$,

$$|\sigma_1(t)\gamma_2(t) - \sigma_2(t)\gamma_1(t)| \geq c_1;$$

(iii) there exists $c_2 \in (0, \infty)$ such that, for $t \in [0, T]$,

$$\frac{(\mu_2(t) - r(t))\sigma_1(t) - (\mu_1(t) - r(t))\sigma_2(t)}{\lambda(t)(\sigma_2(t)\gamma_1(t) - \sigma_1(t)\gamma_2(t))} \geq c_2.$$

In this paper, we define the investor's wealth process $(X(t))_{0 \leq t \leq T}$ in the standard self-financing way. Thus we assume that $X(\cdot) \equiv X^{x,\pi}(\cdot)$ satisfies

$$\begin{cases} dX^{x,\pi}(t) = r(t)X^{x,\pi}(t)dt \\ \qquad\qquad + \displaystyle\sum_{i=1}^{2} \pi_i(t) \left\{ (\mu_i(t) - r(t))dt + \sigma_i(t)dW(t) + \gamma_i(t)dN(t) \right\}, \\ X^{x,\pi}(0) = x, \end{cases} \tag{4}$$

where $x \in \mathbf{R}$ is an initial wealth and the portfolio process $\pi(t) = (\pi_1(t), \pi_2(t))$ is an $\mathbf{R}^2$-valued $\mathcal{F}_t$-predictable process such that all the integrals in (4) are well-defined. The process $(\pi_1(t), \pi_2(t))$ represents the actual amounts of money invested in the risky assets $(S_1(t), S_2(t))$.

Throughout this paper, we fix $p \in (1, \infty)$, and consider the loss function ℓ_p defined by

$$\ell_p(x) = \frac{x^p}{p} \quad (x \geq 0).$$

We are concerned with the minimization of $E[\ell_p((H - X^{x,\pi}(T))_+)]$ over some suitable class of portfolios. To this end, we define the class of admissible portfolios as follows.

Definition 2. A portfolio process $(\pi(t))_{0 \leq t \leq T}$ is said to be *admissible* if

$$E\left[\sup_{0 \leq t \leq T} |X^{0,\pi}(t)|^p\right] < \infty.$$

We write $\mathcal{A}$ for the class of all such π.

Put, for $t \in [0,T]$,

$$\theta(t) := \frac{(\mu_2(t) - r(t))\gamma_1(t) - (\mu_1(t) - r(t))\gamma_2(t)}{\sigma_2(t)\gamma_1(t) - \sigma_1(t)\gamma_2(t)};$$
$$\beta(t) := \frac{(\mu_2(t) - r(t))\sigma_1(t) - (\mu_1(t) - r(t))\sigma_2(t)}{\lambda(t)(\sigma_2(t)\gamma_1(t) - \sigma_1(t)\gamma_2(t))}.$$

Then, by Assumption 1, the functions θ and β are bounded, and β is positive. Moreover we have

$$\mu_i(t) - r(t) - \sigma_i(t)\theta(t) + \lambda\gamma_i(t)\beta(t) = 0, \quad i = 1,2. \tag{5}$$

We consider the exponential local martingale

$$L(t) := \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds\right)$$
$$\times \exp\left(\int_0^t \log\beta(s)dN(s) + \int_0^t \lambda(s)(1-\beta(s))ds\right).$$

From the boundedness of θ and β, the process $(L(t))_{0\le t\le T}$ is a strictly positive P-martingale, and satisfies, for every $a \in \mathbf{R}$,

$$E[(L(t))^a] < \infty. \tag{6}$$

We consider the probability measure P_0 on $(\Omega, \mathcal{F}_T)$ defined by

$$\frac{dP_0}{dP} = L(T).$$

Then, the process

$$W_0(t) := W(t) + \int_0^t \theta(s)ds \quad (0 \le t \le T)$$

is a $(P_0, \mathcal{F}_t)$-Brownian motion, $(N(t))_{0\le t\le T}$ is a $(P_0, \mathcal{F}_t)$-Poisson process with intensity $\lambda(t)\beta(t)$, and the process

$$M_0(t) := N(t) - \int_0^t \lambda(s)\beta(s)ds \quad (0 \le t \le T)$$

is a P_0-martingale (cf. Brémaud [B]). Using (5), we have, for $i = 1,2$,

$$d\tilde{S}_i(t) = \tilde{S}_i(t-)(\sigma_i(t)dW_0(t) + \gamma_i(t)dM_0(t)),$$

where $\tilde{S}_i(t) = S_i(t)/B(t)$. Thus, $(\tilde{S}_i(t))_{0\le t\le T}$, $i = 1,2$, are also P_0-martingales.

In what follows, we use the following notation:

Notation. *For a process $(Y(t))_{0\le t\le T}$, we denote by $\tilde{Y}(t)$ the discounted value of $Y(t)$, i.e., $\tilde{Y}(t) := Y(t)/B(t)$.*

For $x \in \mathbf{R}$ and $\pi \in \mathcal{A}$, the discounted wealth process $\tilde{X}^{x,\pi}$ satisfies

$$\tilde{X}^{x,\pi}(t) = x + \sum_{i=1}^{2} \int_0^t \tilde{\pi}_i(s) \left\{ \sigma_i(s) dW_0(s) + \gamma_i(s) dM_0(s) \right\}. \tag{7}$$

Thus, $\tilde{X}^{x,\pi}$ is a local P_0-martingale. However, since $\pi \in \mathcal{A}$, we have from Hölder's inequality and (6) that $E_0[\sup_{0\le t\le T} |\tilde{X}^{x,\pi}(t)|] < \infty$, where $E_0[\cdot]$ denotes the expectation with respect to P_0. This implies that $\tilde{X}^{x,\pi}$ is a P_0-martingale.

Now, the market above is complete in the following sense:

Proposition 3. *Let $H \in L^{p+\varepsilon}(\Omega, \mathcal{F}_T, P)$ for some $\varepsilon > 0$, and put*

$$x_H := E_0 \left[\frac{H}{B(T)} \right]. \tag{8}$$

Then there exists unique $\pi_H \in \mathcal{A}$ such that, for $t \in [0, T]$,

$$\tilde{X}_t^{x_H, \pi_H} = E_0 \left[\frac{H}{B(T)} \middle| \mathcal{F}_t \right] \quad a.s. \tag{9}$$

Let $H \in L^{p+\varepsilon}(\Omega, \mathcal{F}_T, P)$. We interpret H as the investor's liability. By Proposition 3, starting with initial wealth $x_H := E_0[H/B(T)]$, the investor can find the replicating portfolio $\pi \in \mathcal{A}$ for H. However, if the initial wealth x is less than x_H, he/she faces the possibility of shortfall. In such a situation, one is naturally led to the minimization of shortfall in an adequate sense. Thus we consider the following stochastic control problem:

$$V(x) := \inf_{\pi \in \mathcal{A}} E\left[\ell_p((H - X^{x,\pi}(T))_+)\right], \quad x < x_H.$$

By definitions of H and $\mathcal{A}$, we easily find that $V(x) < \infty$.

As we stated in Section 1, we can separate the problem above into two problems, that is, the perfect hedging problem of H and a utility minimization problem. For $z > 0$, we denote by $\mathcal{A}_0(z)$ the set of portfolio processes $(\pi(t))_{0\le t\le T}$ satisfying

$$X^{z,\pi}(t) \ge 0 \quad t \in [0, T] \quad \text{a.s.},$$

and

$$E\left[\sup_{0\le t\le T} |X^{0,\pi}(t)|^p\right] < \infty.$$

We consider another optimization problem, that is,

$$J(z) := \inf_{\pi \in \mathcal{A}_0(z)} E\left[\ell_p(X^{z,\pi}(T))\right], \quad z > 0. \tag{10}$$

Proposition 4. *For every $z \in (0,\infty)$, there exists $\pi_0 \in \mathcal{A}_0(z)$ that is optimal for the problem* (10).

Define $q \in (0,\infty)$ by $(1/p) + (1/q) = 1$. Here is the main theorem of this paper.

Theorem 5. (i) *Let π_0 be as in Proposition 4 with $z = x_H - x$ and let π_H be as in Proposition 3. Then $\pi_H - \pi_0$ is optimal for the problem* (1).

(ii) *For $(t,u) \in [0,T] \times (0,\infty)$, let (Π_1, Π_2) be the unique solution to the linear system*

$$\begin{cases} \sigma_1(t)\Pi_1(t,u) + \sigma_2(t)\Pi_2(t,u) = -\dfrac{\theta(t)}{p-1}u, \\ \gamma_1(t)\Pi_1(t,u) + \gamma_2(t)\Pi_2(t,u) = (\beta(t))^{q-1}u. \end{cases}$$

Then the optimal portfolio $\pi_0 \in \mathcal{A}_0(z)$ of the problem (10) *is given by $(\Pi_1(t, X^{z,\pi_0}(t-)), \Pi_2(t, X^{z,\pi_0}(t-)))$.*

(iii) *The value function $V(x)$ in* (1) *is given by*

$$V(x) = \ell_p(x_H - x)\exp\left(-(p-1)\int_0^T a(s)ds\right) \quad (x < x_H),$$

where $a(\cdot)$ is defined by

$$a(s) = -qr(s) + \frac{1}{2}q(q-1)\theta^2(s) - \lambda(s)\left((q-1) - q\beta(s) + (\beta(s))^q\right). \quad (11)$$

(iv) *The optimal terminal wealth in* (1) *is given by*

$$\begin{aligned} &X^{x,\pi_H-\pi_0}(T) \\ &= H - (x_H - x)(L(T))^{q-1}\exp\left(-\int_0^T \left(a(s) + \frac{r(s)}{p-1}\right)ds\right). \end{aligned}$$

By Theorem 5, the problem (1) is reduced to the perfect hedging problem of H. If we can obtain the hedging portfolio π_H, then Theorem 5 implies that we can minimize the shortfall risk by following the portfolio $\pi_H - \pi_0$.

Remark 6. As in the most utility maximization problems, we can associate the problem (10) with a HJB equation. We define the wealth process $X^{t,z,\pi}$ with initial condition $(t,z) \in [0,T] \times (0,\infty)$ as in (4). We also define the class $\mathcal{A}_0(t,z)$ of portfolio processes in a way similar to the definition of $\mathcal{A}_0(z)$, and put $J(t,z) := \inf_{\pi \in \mathcal{A}_0(t,z)} E[\ell_p(X^{t,z,\pi}(T))]$. Then, as in [KS, Chapter 3] and [JP, Proposition 5.1], we can prove that the function $J(t,z)$ satisfies the following HJB equation:

$$\begin{cases} \dfrac{\partial J(t,z)}{\partial t} + \inf\limits_{\pi \in \mathbf{R}^2} \mathcal{L}J(t,z) = 0, \\ J(T,z) = \ell_p(z), \qquad (t,z) \in [0,T) \times (0,\infty), \end{cases}$$

where

$$\mathcal{L}J(t,z) = zr(t)\frac{\partial J}{\partial z}(t,z) + \sum_{i=1}^{2}(\mu_i(t) - r(t))\frac{\partial J}{\partial z}(t,z)$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{2}\pi_i\sigma_i(t)\right)^2 \frac{\partial^2 J}{\partial z^2}(t,z) + \lambda\left\{J\left(t, z + \sum_{i=1}^{2}\pi_i\gamma_i(t)\right) - J(t,z)\right\}.$$

A Proofs

A.1 Proof of Proposition 3

Proof of Proposition 4. Let $H \in L^{p+\varepsilon}(\Omega, \mathcal{F}_T, P_0)$, and put $x_H := E_0[H/B(T)]$. Then, as in [JP, Proposition 2.1], there exists a portfolio process π_H such that (9) holds. We can easily show the admissibility of π_H using the P_0-martingale property of $\tilde{X}$, Hölder's inequality, (6), and Doob's maximal inequality. $\square$

A.2 Proof of Proposition 4

As in the references on the expected utility maximization such as [KS, Chapter 3], we use the martingale method.

First we write $I(\cdot)$ for the inverse function of $\ell_p'(\cdot)$, that is, $I(y) = y^{q-1}$ for $y > 0$, where $1/p + 1/q = 1$. Let U_p be the minus of the Legendre transform of ℓ_p:

$$U_p(y) = -\sup_{x>0}(xy - \ell_p(x)) = \inf_{x>0}(\ell_p(x) - xy) \tag{12}$$
$$= \ell_p(I(y)) - yI(y), \quad y > 0.$$

The infimum in (12) is attained by $x = I(y)$.

For $(t,y) \in [0,T] \times (0,\infty)$, we define the process

$$Y^{t,y}(s) = y\exp\left(-\int_t^s r(u)du - \int_t^s \theta(s)dW(u) - \frac{1}{2}\int_t^s |\theta(s)|^2 du\right)$$
$$\times \exp\left(\int_t^s \log\beta(u)dN(u) + \int_t^s (1-\beta(u))\lambda du\right), \quad t \le s \le T.$$

Then $Y^{t,y}$ satisfies

$$\begin{cases} dY^{t,y}(s) = Y^{t,y}(s-)\left(-r(s)ds - \theta(s)dW(s) + (\beta(s)-1)dM(s)\right), \\ Y^{t,y}(t) = y. \end{cases}$$

We put, for $(t,y) \in [0,T] \times (0,\infty)$,

$$\mathcal{X}(t,y) := E_0\left[e^{-\int_t^T r(s)ds} I(Y^{t,y}(T))\right].$$

Then, we easily see that

$$\mathcal{X}(t,y) = y^{1/(p-1)} \exp\left(\int_t^T a(s)ds\right), \tag{13}$$

where $a(\cdot)$ is given by (11). We write $\mathcal{Y}(t,\cdot)$ for the inverse function of $\mathcal{X}(t,\cdot)$, that is,

$$\mathcal{Y}(t,z) = z^{p-1} \exp\left(-(p-1)\int_t^T a(s)ds\right). \tag{14}$$

Proof of Proposition 4. By (12), we find that, for $y > 0$ and $\pi \in \mathcal{A}_0(z)$,

$$\begin{aligned} &E[\ell_p(X^{z,\pi}(T))] \\ &= E[\ell_p(X^{z,\pi}(T)) - Y^{0,y}(T)X^{z,\pi}(T)] + E[Y^{0,y}(T)X^{z,\pi}(T)] \\ &= E[\ell_p(X^{z,\pi}(T)) - Y^{0,y}(T)X^{z,\pi}(T)] + yz \\ &\geq E[U_p(Y^{0,y}(T))] + yz. \end{aligned} \tag{15}$$

The equality in (15) holds if and only if

$$X^{z,\pi}(T) = I(Y^{0,y}(T)).$$

However, from Proposition 3, there exists $\pi_0 \in \mathcal{A}_0(z)$ such that

$$\tilde{X}^{z,\pi_0}(t) = E_0\left[e^{-\int_0^T r(s)ds} I(Y^{0,\mathcal{Y}(0,z)}(T)) \middle| \mathcal{F}_t\right]. \tag{16}$$

Then (15) implies that $E[\ell_p(X^{z,\pi_0}(T))] = J(z)$. □

A.3 Proof of Theorem 5

Proof of Theorem 5. We consider the stochastic Legendre transform $U(y,\omega)$ of $-\ell_p(H(\omega)-z)$: for $y > 0$,

$$U(y,\omega) = \sup_{-\infty < z \leq H(\omega)} \{-\ell_p(H(\omega)-z) - yz\} = \ell_q(y) - yH(\omega). \tag{17}$$

The supremum in (17) is attained by $H(\omega) - I(y)$.

From (9) and (16), we have

$$X^{x_H,\pi_H}(T) = H, \quad X^{x_H - x,\pi_0}(T) = I\left(Y^{0,y}(T)\right)$$

and

$$E_0\left[e^{-\int_0^T r(u)du} I\left(Y^{0,y}(T)\right)\right] = x_H - x,$$

where $y := \mathcal{Y}(0, x_H - x)$. It follows that

$$X^{x,\pi_H-\pi_0}(T) = X^{x_H,\pi_H}(T) - X^{x_H-x,\pi_0}(T) = H - I\left(Y^{0,y}(T)\right). \tag{18}$$

Now, by (17), we see that, for every $\pi \in \mathcal{A}$,

$$\begin{aligned}
&(x_H - x)\mathcal{Y}(0, x_H - x) - E\left[\ell_q\left(\mathcal{Y}(0, x_H - x)e^{-\int_0^T r(u)du}L(T)\right)\right] \\
&= E\left[Y^{0,y}(T)H - \ell_q\left(Y^{0,y}(T)\right)\right] - x\mathcal{Y}(0, x_H - x) \\
&\le E\left[Y^{0,y}(T)H - \ell_q\left(Y^{0,y}(T)\right) - Y^{0,y}(T)\left(H \wedge X^{x,\pi}(T)\right)\right] \\
&\le E\left[\ell_p(H - X^{x,\pi}(T) \wedge H)\right] = E\left[\ell_p((H - X^{x,\pi}(T))_+)\right].
\end{aligned}$$

Both equalities hold in the above inequalities if and only if

$$H \wedge X^{x,\pi}(T) = H - I\left(Y^{0,y}(T)\right). \tag{19}$$

However, (18) implies that the portfolio $\pi_H - \pi_0$ satisfies (19). Therefore,

$$\begin{aligned}
V(x) &= E\left[\ell_p((H - X^{x,\pi_H - \pi_0}(T))_+)\right] \\
&= (x_H - x)\mathcal{Y}(0, x_H - x) - E\left[\ell_q\left(\mathcal{Y}(0, x_H - x)e^{-\int_0^T r(u)du}L(T)\right)\right].
\end{aligned} \tag{20}$$

Thus Theorem 5 (i) follows.

From (16) and the Markov property of the process $Y^{t,y}$, we have, for $z > 0$,

$$\begin{aligned}
X^{z,\pi_0}(t) &= E_0\left[e^{-\int_t^T r(s)ds} I\left(Y^{0,\mathcal{Y}(0,z)}(T)\right)\middle| \mathcal{F}_t\right] \\
&= \mathcal{X}\left(t, Y^{0,\mathcal{Y}(0,z)}(t)\right),
\end{aligned}$$

where π_0 is as in Proposition 4. Thus

$$Y^{0,\mathcal{Y}(0,z)}(t) = \mathcal{Y}(t, X^{z,\pi_0}(t)). \tag{21}$$

Itô formula and (13) imply that

$$\begin{aligned}
d\left(e^{-\int_0^t r(s)ds}\mathcal{X}(t, Y^{0,y}(t))\right) &= -\theta(t)e^{-\int_0^t r(s)ds}Y^{0,y}(t)\mathcal{X}_y(t, Y^{0,y}(t))dW_0(t) \\
&\quad + \left\{\mathcal{X}(t, \beta(t)Y^{0,y}(t)) - \mathcal{X}(t, Y^{0,y}(t))\right\} dM_0(t).
\end{aligned}$$

From this and (21), we see that

$$\begin{aligned}
&\tilde{X}^{z,\pi_0}(t) - z \\
&= -\int_0^t \theta(s)e^{-\int_0^s r(u)du}\frac{\mathcal{Y}(s, X^{z,\pi_0}(s))}{\mathcal{Y}_z(s, X^{z,\pi_0}(s))}dW_0(s) \\
&\quad + \int_0^t e^{-\int_0^s r(u)du}\left\{\mathcal{X}(s, \beta(s)\mathcal{Y}(s, X^{z,\pi_0}(s-))) - X^{z,\pi_0}(s-)\right\} dM_0(s).
\end{aligned} \tag{22}$$

Therefore, by (14) we have

$$-\theta(t)\frac{\mathcal{Y}(t,u)}{\mathcal{Y}_z(t,u)} = -\frac{\theta(t)}{p-1}u = \sum_{i=1}^{2}\sigma_i(t)\Pi_i(t,u),$$

$$\mathcal{X}(t, \beta(t)\mathcal{Y}(t,u)) - u = \beta(t)^{1/(p-1)}u = \sum_{i=1}^{2}\gamma_i(t)\Pi_i(t,u).$$

Thus, by (22), Theorem 5 (ii) follows.

By (14), (20), and easy computation similar to that of (13), we have

$$V(x) = \ell_p(x_H - x) \exp\left(-(p-1)\int_0^T a(s)ds\right),$$

which proves Theorem 5 (iii). Finally, Theorem (5) (iv) follows immediately from (18). □

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