THE HYPERBOLIC GAUSS-BONNET TYPE THEOREM

Shyuichi Izumiya and María Del Carmen Romero Fuster

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THE HYPERBOLIC GAUSS-BONNET TYPE THEOREM

SHYUICHI IZUMIYA AND MARÍA DEL CARMEN ROMERO FUSTER

ABSTRACT. We show that the Gauss-Bonnet type theorem holds for the hyperbolic Gauss-Kronecker curvature of a closed orientable even dimensional hypersurface in hyperbolic space. We also give detailed studies for surfaces.

1. INTRODUCTION

The hyperbolic Gauss map of a surface in hyperbolic space has been indepentetly introduced by Bryant [2] and Epstein [4] in the Poincaré ball model. For fundamental concepts and results in this area, please refer [2, 4, 5, 11]. In [8] we have investigated singularities of hyperbolic Gauss maps of hypersurfaces in hyperbolic n-space $H^n_+(-1)$ by using the model in Minkowski space and defined the notion of hyperbolic Gauss-Kronecker curvature $K_h^{\pm}$. The hyperbolic Gauss-Kronecker curvature is a hyperbolic invariant which describes the contact between hypersurfaces and hyperhorospheres. In this paper we study the global properties of the hyperbolic Gauss-Kronecker curvature. Since the hyperbolic Gauss-Kronecker curvature depends on the choice of the normal direction (i.e., it is not an intrinsic invariant), we need to explicitly use the normal vector of the hypersurface when dealing with global properties. Therefore, in order to define the global hyperbolic Gauss-Kronecker curvature $\mathcal{K}_h$ (cf., §3), we shall need to assume that the hypersurface M is orientable. The main result in this paper is the following hyperbolic Gauss-Bonnet type theorem:

Theorem 1.1. *If M is a closed orientable even-dimensional hypersurface in hyperbolic n-space, then*

$$\int_M \mathcal{K}_h d\mathfrak{v}_M = \frac{1}{2}\gamma_{n-1}\boldsymbol{\chi}(M)$$

where $\boldsymbol{\chi}(M)$ is the Euler characteristic of M, $d\mathfrak{v}_M$ is the volume element of M and the constant γ_{n-1} is the volume of the unit $(n-1)$-sphere S^{n-1}.

We include a quick review of the local properties of the hyperbolic Gauss-Kronecker curvature in §2. We also introduce in §2 the concept of *de Sitter Gauss-Kronecker curvature* of a hypersurface in $H^n_+(-1)$ that will be used in §4. Theorem 1.1 is proven in §3. §4 is devoted to a more detailed study of the case $n = 3$. We show here, as a corollary of Theorem 1.1, that the total mean curvature on a closed orientable surface M in $H^3_+(-1)$ is equal to the area $A(M)$ of M (Theorem 4.1). A further consequence of the main theorem together with the generic classification

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of singularities of hyperbolic Gauss indicatrices is the relation between the Euler characteristics of the image of the hyperbolic Gauss indicatrix and hyperbolic invariants on the closed surface obtained in Theorem 4.2. All arguments in this paper are quite elementary. It might be, however, new results on basic geometric invariants for hypersurfaces in hyperbolic space so far as we know.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2. Local hyperbolic differential geometry

We outline in this section the local differential geometry of hypersurfaces in the hyperbolic n-space developed in a previous paper [8]. We adopt, for this purpose, the model of hyperbolic n-space in Minkowski $(n+1)$-space.

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\ \}$ be an $(n+1)$-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ *Minkowski $(n+1)$-space* and denote it by $\mathbb{R}^{n+1}_1$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define *the hyperplane with pseudo normal $\boldsymbol{v}$* by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *hyperbolic n-space* by $H^n_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1\}$ and *de Sitter n-space* by $S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}$.

Given $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x_0^1 & x_1^1 & \cdots & x_n^1 \\ x_0^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, \dots, x_n^i)$. We can easily show that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, \dots, n)$.

We also define a set $LC^*_+ = \{\boldsymbol{x} = (x_0, \dots x_n) \in LC_0\ |x_0 > 0\ \}$, which is called *future lightcone* at the origin. If $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ is a lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\boldsymbol{x}} = (1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\ \}.$$

We call S^{n-1}_+ *the lightcone $(n-1)$-sphere.*

We now construct the local extrinsic differential geometry on hypersurfaces in $H^n_+(-1)$. Let $\boldsymbol{x} : U \longrightarrow H^n_+(-1)$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We shall identify $M = \boldsymbol{x}(U)$ and U through the embedding $\boldsymbol{x}$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}(u) \rangle \equiv 0 \quad (i = 1, \dots, n-1)$, for any $u = (u_1, \dots u_{n-1}) \in U$. Therefore, if we define

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|},$$

we have $\langle \boldsymbol{e}(u), \boldsymbol{x}_{u_i}(u) \rangle \equiv \langle \boldsymbol{e}(u), \boldsymbol{x}(u) \rangle \equiv 0$, $\langle \boldsymbol{e}(u), \boldsymbol{e}(u) \rangle \equiv 1$. And thus the vector $\boldsymbol{x}(u) \pm \boldsymbol{e}(u)$ is lightlike. Since $\boldsymbol{x}(u) \in H^n_+(-1)$ and $\boldsymbol{e}(u) \in S^n_1$, we have $\boldsymbol{x}(u) \pm \boldsymbol{e}(u) \in$

LC_+^* and hence we can define a map

$$\mathbb{L}^{\pm} : U \longrightarrow LC_+^*$$

by $\mathbb{L}^{\pm}(u) = \boldsymbol{x}(u) \pm \boldsymbol{e}(u)$ which is called *the hyperbolic Gauss indicatrix* (or *the lightcone dual*) of $\boldsymbol{x}$. We also define a map

$$\widetilde{\mathbb{L}^{\pm}} : U \longrightarrow S_+^{n-1}$$

by $\widetilde{\mathbb{L}^{\pm}}(u) = \widetilde{\mathbb{L}^{\pm}(u)}$ and call it *the hyperbolic Gauss map* of $\boldsymbol{x}$. We remark that for $n = 3$, our definition of hyperbolic Gauss map is equivalent to the one introduced in [2, 4]. We also define a further map $\mathbb{E} : U \longrightarrow S_1^n$ by $\mathbb{E}(u) = \boldsymbol{e}(u)$ and call it *the de Sitter Gauss indicatrix* of $\boldsymbol{x}$.

In order to define the hyperbolic Gauss-Kronecker curvature and the hyperbolic mean curvature of the hypersurface $M = \boldsymbol{x}(U)$, we have shown in [8] that $D_v\mathbb{E} \in T_pM$ for any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, so that $D_v\mathbb{L}^{\pm} \in T_pM$. Here, D_v denotes *the covariant derivative* with respect to the tangent vector $\boldsymbol{v}$. Under the identification of U and M by the embedding $\boldsymbol{x}$, the derivative $d\boldsymbol{x}(u_0)$ can be identified to the identity mapping id_{T_pM} on the tangent space T_pM, where $p = \boldsymbol{x}(u_0)$. Therefore, $d\mathbb{E}(u_0)$ can be considered as a linear transformation on the tangent space T_pM. This means that $d\mathbb{L}^{\pm}(u_0) = id_{T_pM} \pm d\mathbb{E}(u_0)$ is also a linear transformation on the tangent space T_pM. We call the linear transformation $S_p^{\pm} = -d\mathbb{L}(u_0) : T_pM \longrightarrow T_pM$ *the hyperbolic shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We also call the linear transformation $A_p = -d\mathbb{E}(u_0) : T_pM \longrightarrow T_pM$ *the de Sitter shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We denote the eigenvalue of $S_p^{\pm}$ by $\bar{\kappa}_p^{\pm}$ and the eigenvalue of A_p by κ_p. The relation $S_p^{\pm} = -id_{T_pM} \pm A_p$ implies that $S_p^{\pm}$ and A_p have same eigenvectors, moreover $\bar{\kappa}_p^{\pm} = -1 \pm \kappa_p$.

We now define the notion of hyperbolic curvatures as follows: *The hyperbolic Gauss-Kronecker curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$K_h^{\pm}(u_0) = \det S_p^{\pm}.$$

The hyperbolic mean curvature of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$H_h^{\pm}(u_0) = \frac{1}{n-1}\mathrm{Trace}S_p^{\pm}.$$

The *de Sitter Gauss-Kronecker curvature* is defined to be

$$K_d(u_0) = \det A_p$$

and the *de Sitter mean curvature* is

$$H_d(u_0) = \frac{1}{n-1}\mathrm{Trace}A_p.$$

We remark that the de Sitter mean curvature is actually the mean curvature of M. We, clearly, have that $H_h^{\pm}(u) = \pm H_d(u) - 1$. Surfaces with $H \equiv \pm 1$ represent the most important class among those with constant mean curvature in hyperbolic space. These surfaces have vanishing hyperbolic mean curvature and might thus be called *hyperbolic minimal surfaces.*

We establish next the hyperbolic (respectively, de Sitter) version of the Weingarten formula. Since $\boldsymbol{x}_{u_i}$ $(i = 1, \ldots n-1)$ are spacelike vectors, we have the Riemannian metric (*hyperbolic first fundamental form*) given by $ds^2 = \sum_{i=1}^{n-1} g_{ij}du_idu_j$ on $M = \boldsymbol{x}(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ and the *hyperbolic (respectively, de Sitter) second fundamental invariant* defined by $\bar{h}_{ij}^{\pm}(u) = \langle -\mathbb{L}_{u_i}^{\pm}(u), \boldsymbol{x}_{u_j}(u)\rangle$

(respectively, $h_{ij}(u)=-\langle \boldsymbol{e}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$) for any $u\in U$. They satisfy the relation $\bar{h}^{\pm}_{ij}(u)=-g_{ij}(u)\pm h_{ij}(u)$.

Proposition 2.1. *Under the above notations, we have the following formulae:*

(1) $\mathbb{L}^{\pm}_{u_i}=-\sum_{j=1}^{n-1}\left(\bar{h}^{\pm}\right)_i^j \boldsymbol{x}_{u_j}$ (*The hyperbolic Weingarten formula*),

(2) $\mathbb{E}_{u_i}=-\sum_{j=1}^{n-1}\left(h_i^j\right)\boldsymbol{x}_{u_j}$ (*The de Sitter Weingarten formula*),

where $\left(\left(\bar{h}^{\pm}\right)_i^j\right)=\left(\bar{h}^{\pm}_{ik}\right)\left(g^{kj}\right)$, $\left(h_i^j\right)=\left(\bar{h}^{\pm}_{ik}\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right)=\left(g_{kj}\right)^{-1}$.

Proof. Since the hyperbolic Weingarten formula has been shown in [8], we only give here the proof of the de Sitter Weingarten formula.

There exist real numbers λ,μ,Γ_i^j such that $\mathbb{E}_{u_i}=\lambda\boldsymbol{e}+\mu\boldsymbol{x}+\sum_{j=1}^{n-1}\Gamma_i^j\boldsymbol{x}_{u_j}$. Since $\langle\mathbb{E},\mathbb{E}\rangle=1$, we have $0=\langle\mathbb{E}_{u_i},\mathbb{E}\rangle=\langle\lambda\boldsymbol{e}+\mu\boldsymbol{x},\boldsymbol{e}\rangle=\lambda$. Therefore, $\mathbb{E}_{u_i}=\mu\boldsymbol{x}+\sum_{j=1}^{n-1}\Gamma_i^j\boldsymbol{x}_{u_j}$. On the other hand, it follows from the definition $-h_{i\beta}=\sum_{\alpha=1}^{n-1}\Gamma_i^\alpha\langle\boldsymbol{x}_{u_\alpha},\boldsymbol{x}_{u_\beta}\rangle=\sum_{\alpha=1}^{n-1}\Gamma_i^\alpha g_{\alpha\beta}$. Hence, we have $-h_i^j=-\sum_{\beta=1}^{n-1}\bar{h}^{\pm}_{i\beta}g^{\beta j}=\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n-1}\Gamma_i^\alpha g_{\alpha\beta}g^{\beta j}=\Gamma_i^j$.

Moreover, $\langle\mathbb{E},\boldsymbol{x}\rangle=0$ and $\langle\mathbb{E},\boldsymbol{x}_{u_i}\rangle=0$, and thus $\mu=-\mu\langle\boldsymbol{x},\boldsymbol{x}\rangle=\langle\mathbb{E}_{u_i},\boldsymbol{x}\rangle=0$. This completes the proof of the de Sitter Weingarten formula. □

As a corollary of the above proposition, we have an explicit expression of the hyperbolic (respectively, de Sitter) Gauss-Kronecker curvature in terms of the Riemannian metric and the hyperbolic (respectively, de Sitter) second fundamental invariant.

Corollary 2.2. *Under the same notations as in the above proposition, we have the following formulae:*

$$K_h^{\pm}=\frac{\det\left(\bar{h}^{\pm}_{ij}\right)}{\det\left(g_{\alpha\beta}\right)},\quad K_d=\frac{\det\left(h_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Proof. By the hyperbolic Weingarten formula, the representation matrix of the hyperbolic shape operator $S_p^{\pm}$ with respect to the basis $\{\boldsymbol{x}_{u_1},\dots,\boldsymbol{x}_{u_{n-1}}\}$ is

$$\left(\left(\bar{h}^{\pm}\right)_i^j\right)=\left(\bar{h}^{\pm}_{i\beta}\right)\left(g^{\beta j}\right).$$

It follows from this fact that

$$K_h^{\pm}=\det S_p^{\pm}=\det\left(\left(\bar{h}^{\pm}\right)_i^j\right)=\det\left(\bar{h}^{\pm}_{i\beta}\right)\left(g^{\beta j}\right)=\frac{\det\left(\bar{h}^{\pm}_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

By Proposition 2.1, the representaion matrix of de Sitter shape operator A_p is also given by (h_i^j), the formula for the de Sitter Gauss-Kronecker curvature follows. □

We say that a point $p=\boldsymbol{x}(u_0)$ is *a (positive or negative) horo-parabolic point* of $\boldsymbol{x}:U\longrightarrow H^n_+(-1)$ if $K_h^{\pm}(u_0)=0$.

We also get in this context the *hyperbolic Gauss equations* as we shall see next and it will be used in §4. Since $\boldsymbol{x}(U)=M$ is a Riemannian manifold, it makes sense to consider the *Christoffel symbols*:

$$\left\{\begin{matrix} k \\ i\ j\end{matrix}\right\}=\frac{1}{2}\sum_m g^{km}\left\{\frac{\partial g_{jm}}{\partial u_i}+\frac{\partial g_{im}}{\partial u_j}-\frac{\partial g_{ij}}{\partial u_m}\right\}.$$

Proposition 2.3. *Let $\boldsymbol{x} : U \longrightarrow H^n_+(-1)$ be a hypersurface. Then we have the following hyperbolic Gauss equations:*

$$\boldsymbol{x}_{u_i u_j} = \sum_k \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \boldsymbol{x}_{u_k} + h_{ij}\boldsymbol{e} + g_{ij}\boldsymbol{x}.$$

Proof. Since $\{\boldsymbol{e}, \boldsymbol{x}, \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is a pseudo-orthonormal frame of $\mathbb{R}^{n+1}_1$, we can write $\boldsymbol{x}_{u_i u_j} = \sum_k \Gamma^k_{ij}\boldsymbol{x}_{u_k} + \Gamma_{ij}\boldsymbol{e} + \Gamma^{ij}\boldsymbol{x}$. We now have

$$\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}_{u_\ell} \rangle = \sum_k \Gamma^k_{ij} \langle \boldsymbol{x}_{u_k}, \boldsymbol{x}_{u_\ell} \rangle = \sum_k \Gamma^k_{ij} g_{k\ell}.$$

Therefore, $\frac{\partial g_{i\ell}}{\partial u_j} = \langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}_{u_\ell} \rangle + \langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_\ell u_j} \rangle$. And since $\boldsymbol{x}_{u_i u_j} = \boldsymbol{x}_{u_j u_i}$, we get $\Gamma^k_{ij} = \Gamma^k_{ji}$, $\Gamma_{ij} = \Gamma_{ji}$, $\Gamma^{ij} = \Gamma^{ji}$. Then by exactly the same calculation as those applied in the case of hypersurfaces in Euclidean space, it follows $\Gamma^k_{ij} = \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}$.

On the other hand, $\Gamma_{ij} = \langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{e} \rangle = h_{ij}$. Moreover $\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x} \rangle = -\Gamma^{ij}$. and since $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle = 0$, we have $\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x} \rangle = -\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_j} \rangle = -g_{ij}$. which implies that $\Gamma^{ij} = g_{ij}$. □

3. Proof of the main result

In this section we give a proof for the hyperbolic Gauss-Bonnet type theorem. Let M be a closed orientable $(n-1)$-dimensional manifold and $f : M \longrightarrow H^n_+(-1)$ an embedding. We consider the canonical projection $\pi : \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (0, x_1, \dots, x_n)$. Then we have orientation preserving diffeomorphisms $\pi | H^n_+(-1) : H^n_+(-1) \longrightarrow \mathbb{R}^n$ and $\pi | S^{n-1}_+ : S^{n-1}_+ \longrightarrow S^{n-1}$.

Consider the unit normal $\mathbb{E}$ of $f(M)$ and write $f(p) = (f_0(p), f_1(p), \dots, f_n(p))$ and $\mathbb{E}(p) = (e_0(p), e_1(p), \dots, e_n(p))$ in local coordinates. Since $\langle f(p), \mathbb{E}(p) \rangle = 0$, we get $f_1(p)e_1(p) + \cdots + f_n(p)e_n(p) = f_0(p)e_0(p)$. But $\mathbb{E}(p) \in T_{f(p)}H^n_+(-1)$, and hence $e_0(p) = 0$ if and only if $f(p) = \boldsymbol{e}_0$. Now, taking into account that $f_0(p) \geq 1$, we deduce that $\langle \pi \circ f(p), \pi \circ \mathbb{E}(p) \rangle$ may vanish at at most one point. So we conclude that either $\langle \pi \circ f(p), \pi \circ \mathbb{E}(p) \rangle \geq 0$ at any $p \in M$, or $\langle \pi \circ f(p), \pi \circ \mathbb{E}(p) \rangle \leq 0$ at any point $p \in M$. We can thus define the hyperbolic Gauss indicatrix in the global

$$\mathbb{L} : M \longrightarrow LC^*_+$$

either by

$$\mathbb{L}(p) = f(p) + \mathbb{E}(p)$$

provided $\langle \pi \circ f, \pi \circ \mathbb{E} \rangle \geq 0$, or

$$\mathbb{L}(p) = f(p) - \mathbb{E}(p)$$

in case $\langle \pi \circ f, \pi \circ \mathbb{E} \rangle \leq 0$.

The *global hyperbolic Gauss-Kronecker curvature function* $\mathcal{K}_h : M \longrightarrow \mathbb{R}$ is then defined in the usual way in terms of the global hyperbolic Gauss indicatrix $\mathbb{L}$.

Consider now the hyperbolic Gauss map

$$\widetilde{\mathbb{L}} : M \longrightarrow S^{n-1}_+$$

on $f(M)$ and the (Euclidean) Gauss map

$$\mathbb{N} : M \longrightarrow S^{n-1}$$

on $\pi \circ f(M)$.

The proof of Theorem 1.1 is based in the following key lemma:

Lemma 3.1. *Under the choice of a suitable direction of* $\mathbb{N}$, $\pi \circ \widetilde{\mathbb{L}}$ *and* $\mathbb{N}$ *are homotopic.*

Proof. Since $\mathbb{E}(p)$ is normal to $f(M)$ in $H^n_+(-1)$, $\pi \circ \mathbb{E}(p)$ is transverse to $\pi \circ f(M)$ in $\mathbb{R}^n$. It follows that $\langle \pi \circ \mathbb{E}(p), \mathbb{N}(p)\rangle \neq 0$ at any $p \in M$.

We now construct a homotopy between $\pi \circ \widetilde{\mathbb{L}}$ and $\mathbb{N}$. Let

$$F : M \times [0,1] \longrightarrow S^{n-1}$$

be defined by

$$F(p,t) = \frac{t\mathbb{N}(p) + (1-t)\pi \circ \widetilde{\mathbb{L}}(p)}{\|t\mathbb{N}(p) + (1-t)\pi \circ \widetilde{\mathbb{L}}(p)\|},$$

where $\|\cdot\|$ is the Euclidean norm.

If there exists $t' \in [0,1]$ and $p' \in M$ such that $t'\mathbb{N}(p') + (1-t')\pi \circ \widetilde{\mathbb{L}}(p') = \mathbf{0}$, then we have $\mathbb{N}(p') = -\pi \circ \widetilde{\mathbb{L}}(p')$. Since $T_{\pi\circ f(p')}\pi \circ f(M) = d(\pi \circ f)_{p'}(T_{p'}M)$, we get

$$\begin{cases} -v_0(f_0(p') + e_0(p')) + v_1(f_1(p') + e_1(p')) + \cdots + v_n(f_n(p') + e_n(p')) = 0 \\ v_1 N_1(p') + \cdots + v_n N_n(p') = 0 \end{cases}$$

for any $\boldsymbol{v} = (v_0, v_1, \ldots, v_n) \in df_{p'}(T_{p'}M)$, where $\mathbb{N}(p) = (0, N_1(p), \ldots, N_n(p))$. It follows that $v_0(f_0(p') + e_0(p')) = 0$. But $f_0(p') + e_0(p') \neq 0$ and hence $v_0 = 0$. This means that $df_{p'}(T_{p'}M) \subset T_{f(p')}\{0\} \times \mathbb{R}^n \subset T_{f(p')}\mathbb{R}^{n+1}_1$. Moreover, $\langle \mathbb{E}(p'), \boldsymbol{v}\rangle = 0$ for any $\boldsymbol{v} \in df_{p'}(T_{p'}M)$. Therefore $\pi \circ \mathbb{E}(p')$ is a normal vector of $\pi \circ f(M)$ at $p' \in M$, so there exists a non-zero real number λ such that $\pi \circ \mathbb{E}(p') = \lambda \mathbb{N}(p')$.

We now distinguish two cases. Suppose first that $\langle \pi \circ f(p), \pi \circ \mathbb{E}(p)\rangle \geq 0$ at any $p \in M$. Then we choose the direction of $\mathbb{N}$ that makes $\langle \pi \circ \mathbb{E}(p), \mathbb{N}(p)\rangle \geq 0$. It then follows that $\lambda = \langle \lambda\mathbb{N}(p'), \mathbb{N}(p')\rangle = \langle \pi \circ \mathbb{E}(p'), \mathbb{N}(p')\rangle > 0$. In this case, we have

$$\pi \circ \mathbb{L}(p') = \pi \circ f(p') + \pi \circ \mathbb{E}(p'),$$

so

$$\pi \circ f(p') = -\pi \circ \mathbb{E}(p') - (f_0(p') + e_0(p'))\mathbb{N}(p').$$

Therefore we get

$$\begin{aligned} \langle \pi \circ f(p'), \pi \circ \mathbb{E}(p')\rangle &= -\langle \pi \circ \mathbb{E}(p'), \pi \circ \mathbb{E}(p')\rangle - (f_0(p') + e_0(p'))\langle \mathbb{N}(p'), \pi \circ \mathbb{E}(p')\rangle \\ &= -\lambda^2 - (f_0(p') + e_0(p'))\lambda < 0 \end{aligned}$$

which contradicts to the choice of the direction of $\mathbb{N}$.

On the other hand, if we suppose that $\langle \pi \circ f(p), \pi \circ \mathbb{E}(p)\rangle \leq 0$ at any point $p \in M$, we also have $\lambda = \langle \lambda\mathbb{N}(p'), \mathbb{N}(p')\rangle = \langle \pi \circ \mathbb{E}(p'), \mathbb{N}(p')\rangle < 0$. Then

$$\pi \circ \mathbb{L}(p') = \pi \circ f(p') - \pi \circ \mathbb{E}(p'),$$

and thus

$$\pi \circ f(p') = \pi \circ \mathbb{E}(p') - (f_0(p') + e_0(p'))\mathbb{N}(p').$$

If we choose now the direction of $\mathbb{N}$ such that $\langle \pi \circ \mathbb{E}(p), \mathbb{N}(p)\rangle < 0$, we get

$$\langle \pi \circ f(p'), \pi \circ \mathbb{E}(p')\rangle = \lambda^2 - (f_0(p') - e_0(p'))\lambda > 0.$$

But this contradicts the fact that $\langle \pi \circ \mathbb{E}(p), \mathbb{N}(p)\rangle < 0$.

It follows from the above arguments that F is a continuous mapping satisfying that $F(p,0)=\pi\circ\widetilde{\mathbb{L}^{\pm}}(p)$ and $F(p,1)=\mathbb{N}(p)$ for any $p\in M$. $\square$

Since the mapping degree is a homotopy invariant, we have the following corollary (cf., [6], Chapter 4, §9).

Corollary 3.2. *If M is a closed orientable, even-dimensional hypersurface in $H^n_+(-1)$, then we have*

$$\deg\widetilde{\mathbb{L}}=\frac{1}{2}\chi(M),$$

where $\deg\widetilde{\mathbb{L}}$ is the mapping degree of $\widetilde{\mathbb{L}}$.

It remains to show that the hyperbolic Gauss-Kronecker curvature function $\mathcal{K}_h$ is the Jacobian of the hyperbolic Gauss map $\widetilde{\mathbb{L}}:M\longrightarrow S^{n-1}_+$.

Proposition 3.3. *There exist local coordinates $(U,(u_1,\dots,u_{n-1}))$ of M and $(V,(v_1,\dots,v_{n-1}))$ of S^{n-1}_+ such that the corresponding matrix $\left(\bar{h}^j_i\right)$ is the Jacobi matrix of $-\widetilde{\mathbb{L}}$.*

Here $\left(\bar{h}^j_i\right)=\left((\bar{h}^{\pm})^j_i\right)$ is the matrix given in Proposition 2.1.

Proof. We define a projection $\Pi:LC^*_+\longrightarrow S^{n-1}_+$ by $\Pi(\boldsymbol{v})=\widetilde{\boldsymbol{v}}$. We use the local notation in §2 here. Therefore, on a local coordinates $(U,(u_1,\dots,u_{n-1})$ of M we denote that $f|U=\boldsymbol{x}:U\longrightarrow H^n_+(-1)$ and assume that $\mathbb{L}(u)=\boldsymbol{x}(u)+\boldsymbol{e}(u)$. We observe that the tangent space of LC^*_+ at $\boldsymbol{v}\in LC^*_+$ is the hyperplane $HP(\boldsymbol{v},0)$. Since $\langle\mathbb{L},\boldsymbol{x}_{u_i}\rangle=0$, $\boldsymbol{x}_{u_i}(u)$ is a tangent vector of LC^*_+ at $\mathbb{L}(u)$, it follows that $\{\mathbb{L}(u),\boldsymbol{x}_{u_1}(u),\dots,\boldsymbol{x}_{u_{n-1}}(u)\}$ is a basis of the tangent space of LC^*_+ at $\mathbb{L}(u)$. The tangent direction of the fibre of Π is given by the lightlike vector $\mathbb{L}(u)$, and hence $\{d\Pi(\boldsymbol{x}_{u_1}),\dots,d\Pi(\boldsymbol{x}_{u_{n-1}})\}$ is a basis of the tangent space of S^{n-1}_+ at $d\Pi(\mathbb{L}(u))$. On the other hand, we have the hyperbolic Weingarten formula (cf., Proposition 2.3): $\mathbb{L}^{\pm}_{u_i}=-\sum_{j=1}^{n-1}\bar{h}^j_i\boldsymbol{x}_{u_j}$. Therefore,

$$d\Pi(\mathbb{L}_{u_i}(u))=-\sum_{j=1}^{n-1}\bar{h}^j_i d\Pi(\boldsymbol{x}_{u_j}(u)).$$

We can choose local coordinate $(V,(v_1,\dots v_{n-1}))$ of S^{n-1}_+ around $d\Pi\mathbb{L}(u)$ such that $(\partial/\partial v_i)=d\Pi(\boldsymbol{x}_{u_j}(u))$. This means that the Jacobi matrix of $-\widetilde{\mathbb{L}}$ at $u\in U$ in the local coordinates $(U,(u_1,\dots,u_{n-1}))$ of M and $(V,(v_1,\dots,v_{n-1}))$ of S^{n-1}_+ is $(\bar{h}^j_i)$. $\square$

If we denote by $J_{\widetilde{L}}$ the Jacobian of $\widetilde{\mathbb{L}}$, we obtain:

$$\int_M\mathcal{K}_h d\mathfrak{v}_M=\int_M J_{\widetilde{L}}d\mathfrak{v}_M=\int_M\widetilde{\mathbb{L}}^*d\mathfrak{v}_{S^{n-1}_+}=\deg(\widetilde{\mathbb{L}})\int_{S^{n-1}_+}d\mathfrak{v}_{S^{n-1}_+}=\deg(\widetilde{\mathbb{L}})\gamma_{n-1}.$$

And the proof of Theorem 1.1 is now completed as a consequence of Corollary 3.2.

4. Surfaces in hyperbolic 3-space

In this section we stick to the case $n=3$. First of all we need to make some local calculations. Let $\boldsymbol{x}:U\longrightarrow H^3_+(-1)$ be a (local) surface, where $U\subset\mathbb{R}^2$ is an open region, and consider the Riemannian curvature tensor

$$R^{\ell}_{ijk} = \frac{\partial}{\partial u_k}\left\{ \begin{matrix} \ell \\ i\ j \end{matrix} \right\} - \frac{\partial}{\partial u_j}\left\{ \begin{matrix} \ell \\ i\ k \end{matrix} \right\} + \sum_m \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\}\left\{ \begin{matrix} \ell \\ m\ k \end{matrix} \right\} - \sum_m \left\{ \begin{matrix} m \\ i\ k \end{matrix} \right\}\left\{ \begin{matrix} \ell \\ m\ j \end{matrix} \right\}.$$

We also consider the tensor $R_{ijk\ell} = \sum_m g_{im} R^m_{jk\ell}$. Standard calculations, analogous to those used in the study of the classical differential geometry on surfaces in Euclidean space (cf., [13]), lead to the following formula.

Proposition 4.1. *Under the above notations, we have*

$$R_{ijk\ell} = h_{jk}h_{i\ell} - h_{j\ell}h_{ik} - g_{jk}g_{i\ell} + g_{j\ell}g_{ik}.$$

From Corollary 2.2 we have

$$K_d = \frac{h_{11}h_{22} - h_{21}h_{12}}{g_{11}g_{22} - g_{12}g_{21}}.$$

And thus we obtain the analogous result of the *Theorema Egregium* of Gauss for the hyperbolic case:

Proposition 4.2. *Under the above notations, we have*

$$K_d = -\frac{R_{1212}}{g} + 1,$$

where $g = g_{11}g_{22} - g_{12}g_{21}$.

We remark that $-R_{1212}/g$ is the *sectional curvature* of the surface, so we denote it by K_s.

On the other hand, let k_i $(i = 1, 2)$ be eigenvalues of (h^i_j) (i.e., de Sitter principal curvatures of the surface). We remind that $\bar{k}^{\pm}_i = -1 \pm k_i$, from which we deduce:

Proposition 4.3. *The following relation holds:*

$$K^{\pm}_h = 1 \mp 2H_d + K_d = 2 \mp 2H_d + K_s.$$

We return to the global situation. Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow H^3_+(-1)$ an embedding. Under the same notations as in §4, we define a global mean curvature function $\mathcal{H}_d : M \longrightarrow \mathbb{R}$ by using the de Sitter Gauss map $\mathbb{E}$ when $\langle \pi \circ f, \pi \circ \mathbb{E} \rangle \geq 0$. Therefore we have the relation

$$\mathcal{K}_h = 1 - 2\mathcal{H}_d + \mathcal{K}_d = 2 - 2\mathcal{H}_d + \mathcal{K}_s,$$

where $\mathcal{K}_d$ is the global de Sitter Gauss-Kronecker curvature function and $\mathcal{K}_s$ is the global sectional curvature function. Then, as a corollary of Theorem 1.1, we obtain:

Theorem 4.4. *Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow H^3_+(-1)$ an embedding. Then we have*

$$\int_M \mathcal{K}_d d\mathfrak{a}_M = 2\pi\chi(M) + A(M) \quad \textit{and} \quad \int_M \mathcal{H}_d d\mathfrak{a}_M = A(M),$$

where $d\mathfrak{a}_M$ is the area element and $A(M)$ is the area of M.

Proof. By the Gauss-Bonnet theorem on M, considered as a Riemannian manifold, we have $\frac{1}{2\pi}\int_M \mathcal{K}_s d\mathfrak{a}_M = \chi(M)$. Since $\mathcal{K}_d = \mathcal{K}_s + 1$, we have the first formula. But then, Theorem 1.1 together with the above relation imply

$$2\left(A(M) - \int_M \mathcal{H}_d d\mathfrak{a}_M\right) = 0$$

and the proof is completed. $\square$

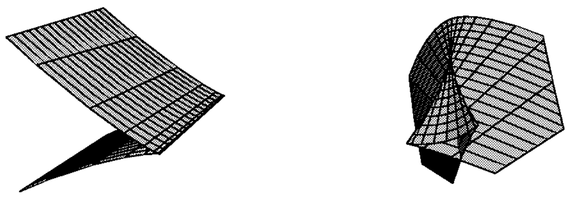
We study in the remaining of the paper some generic properties of surfaces embedded in $H^3_+(-1)$. We gave in [8] the following local classification of singularities for the hyperbolic Gauss indicatrix of a generic local surface in $H^3_+(-1)$:

Theorem 4.5. *Let* $\mathrm{Emb}\,(U, H^3_+(-1))$ *be the space of embeddings from an open region* $U \subset \mathbb{R}^2$ *into* $H^3_+(-1)$ *equipped with the Whitney* C^∞*-topology. There exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}\,(U, H^3_+(-1))$ *such that for any* $\boldsymbol{x} \in \mathcal{O}$, *the following conditions hold:*

(1) *The* $H^\pm$*-parabolic set* $K_h^{-1}(0)$ *is a regular curve. We call such a curve the* $H^\pm$*-parabolic curve.*

(2) *The hyperbolic Gauss indicatrix* $\mathbb{L}^\pm$ *along the* $H^\pm$*-parabolic curve is locally diffeomorphic to the cuspidaledge except at isolated points. At such isolated points,* $\mathbb{L}^\pm$ *is locally diffeomorphic to the swallowtail.*

Here, the cuspidal edge is $C = \{(x_1, x_2, x_3) | {x_1}^2 = {x_2}^3\}$ *and the swallowtail is* $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *(cf., Fig.1).*



cuspidaledge swallowtail

Fig. 1.

The proof of the theorem was given by using an appropriate jet-transversality theorem [8]. When considering a global embedding $f : M \longrightarrow H^3_+(-1)$, one must also pay attention to the multilocal phenomena. So we must add the double point locus, the intersection of a regular surface and the cuspidaledge and the triple point to the list of local normal forms of the singular image of hyperbolic Gauss indicatrices of generic embeddings. These follow from the multi-jet version of the above mentioned jet-transversality theorem. We also studied in [8] the geometric meaning of the singularities of the hyperbolic Gauss indicatrices: Given a point $p_0 \in M$ and the lightlike vector $\boldsymbol{v}_0 = \mathbb{L}(p_0)$, we have seen that the horosphere $HS(\boldsymbol{v}_0, -1) = HP(\boldsymbol{v}_0, -1) \cap H^3_+(-1)$ is tangent to $f(M)$ at $f(p_0)$. We called $HS(\boldsymbol{v}_0, -1)$ the *tangent horosphere* of $f(M)$ at $f(p_0)$. By definition, $\mathbb{L}(p_1) = \mathbb{L}(p_2)$ if and only if $HS(\boldsymbol{v}_1, -1) = HS(\boldsymbol{v}_2, -1)$ where $\boldsymbol{v}_i = \mathbb{L}(p_i)$. Analogously, a triple point of the hyperbolic Gauss indicatrix of $f : M \longrightarrow H^3_+(-1)$ corresponds to a tritangent horosphere. On the other hand one of the characterizations of the swallowtail point $p_0 \in M$ of $\mathbb{L}$ was the following (cf., [8]): For any open neighbourhood U of p_0 in M, there exist two distinct points $p_1, p_2 \in U \subset M$ such that both of p_1, p_2 are not H-parabolic points and the tangent horospheres to $f(M)$ at $f(p_1), f(p_2)$ are equal.

Denote by $T(f)$ the number of tritangent horospheres and by $SW(f)$ the number of swallowtail points of a generic embedding $f : M \longrightarrow H^3_+(-1)$. We saw in [8] that the image of the hyperbolic Gauss indicatrix of a hypersurface can be interpreted as a wave front set in the theory of Legendrian singularities (cf., [1]). Therefore, we have the following formula as a particular case of the relation obtained in [7] for wave fronts:

$$\chi(\mathbb{L}(M)) = \chi(M) + \frac{1}{2}SW(f) + T(f).$$

This together with Theorem 1.1 lead to the following:

Theorem 4.6. *Given a generic embedding $f : M \longrightarrow H^3_+(-1)$, the following relation holds:*

$$\chi(\mathbb{L}(M)) = \frac{1}{2\pi}\int_M \mathcal{K}_h d\mathfrak{a}_M + \frac{1}{2}SW(f) + T(f).$$

This theorem tells us that the Euler number of the image of the hyperbolic Gauss indicatrix of a generic embedding can be obtained in terms of the invariants of the hyperbolic differential geometry.

Finally, we remark that we can also apply other formulae involving the number of swallowtails and triple points on singular surfaces in a 3-manifolds (cf., [9, 10, 12]) to our situation in order to get further relations among invariants of the hyperbolic differential geometry.

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Department of Mathematics, Hokkaido University, Sapporo 060-0810,Japan
e-mail:`izumiya@math.sci.hokudai.ac.jp`

Departament de Geometría i Topología, Universitat de València, 46100 Burjassot (València), Espanya
e-mail:`carmen.romero@post.uv.es`