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# A symplectic framework for multiplane gravitational lensing

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## Abstract

We construct a new framework for the study of multiplane gravitational lensing from the view point of Symplectic geometry. Symplectic relations are used to compose the systems and weaker Lagrangian equivalence is applied for classifying the caustics of multiplane gravitational lensing.

## 1 Introduction

Recently there appeared several articles considering gravitational lensing systems as applications of the theory of singularities for smooth mappings. The gravitational lensing is the deflection of light from a distant source (e.g., quasar) by an intervening matter distribution (e.g., a galaxy or a cluster of galaxies). The first gravitational lensed quasar was detected only in 1979. By now gravitational lensing is quite an active field in astrophysics [1, 2, 3].

On the other hand, singularity theory of Lagrangian varieties [4, 5] is the best natural setting for discussing optical systems. In fact, A.O. Petters and his collaborators [6, 1] pointed out that a single gravitational lensing can be described in the framework of symplectic geometry. Especially the caustics in a single gravitational lensing system coincide with caustics in the theory of Lagrangian singularities. Moreover, they also investigate multiplane gravitational lensing as an application of singularity theory [7, 1]. In another paper [8] they pointed out that new, unexpected from the standard singularity theory point of view, caustics might appear

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for multiplane gravitational lensing. However, in their framework for multiplane gravitational lensing generic caustics are the same as those for the single gravitational lensings.

In this paper we propose the symplectic framework for multiplane gravitational lensing based on the notion of symplectic relation, which is a natural generalization of the notion of symplectic transformation (cf. [9]). The original motivation for the paper was an attempt to describe expected nonstandard caustics in gravitational lensing using the weaker, more adequate, versions of Lagrangian equivalence acting on composite symplectic relations. These composites correspond to images of systems of gravitational rays by the subsequent gravitational lens, and their generic forms, described in the paper, reflect the complexity of composition.

In Section 2 we introduce the gravitational lensing problem and in Section 3 the problem is formulated in the language of Lagrangian stability and versality adapted to the product symplectic space of incoming and outgoing rays. The precise meaning of the composition of systems and their actions on the subsequent wave-fronts represented by generating functions, was introduced in Section 4. In Section 5 the local stability of double lensing systems was investigated and it is continued in Section 6 by classification of the normal forms of generating pairs with respect to the natural groups of equivalences.

All manifolds and maps considered here are of class  $C^\infty$  unless stated otherwise.

## 2 Gravitational lensing

In this section we give a quick review of the basic concepts from the theory of gravitational lensing discussed already in [8, 7, 6, 10, 1, 3].

(1) **Single lensing** (cf., [6, 1, 3]) Consider the typical single lens plane gravitational lensing as follows:

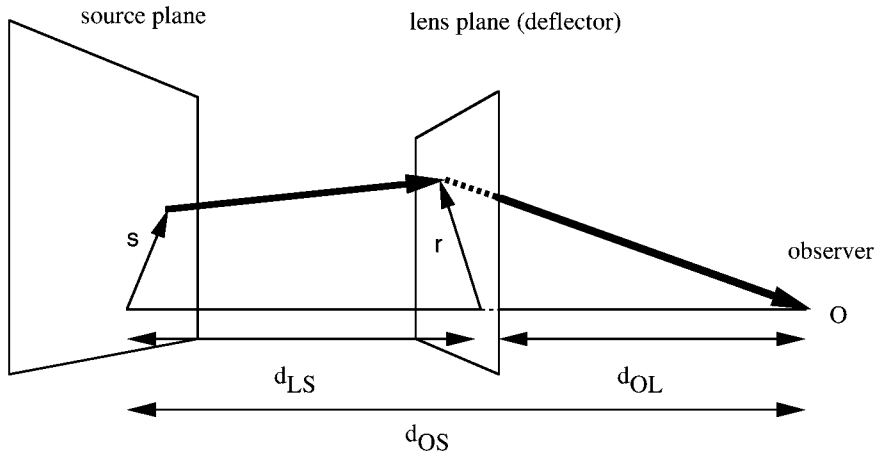


Fig.1

We assume that the deflector is thin and apply the small angle approximation (cf.,[6]). The extra time with respect to the unperturbed ray is giving *the time-delay map*;  $T_{\mathbf{s}} : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}$  defined by

$$T_{\mathbf{s}}(\mathbf{r}) = (1 + z_L) \left[ \frac{d_{OL}d_{OS}}{2d_{LS}} \left\| \frac{\mathbf{s}}{d_{OS}} - \frac{\mathbf{r}}{d_{OL}} \right\|^2 - \Psi(\mathbf{r}) \right].$$

Here,  $z_L$  is the red shift of the lens plane,  $d_{OL}, d_{OS}, d_{LS}$  are angular diameter distances,  $\mathbf{r}$  is the position on the lens plane where the ray hits,  $\mathbf{s}$  is the position of the source, and  $\Psi(\mathbf{r})$

is the two-dimensional potential of the deflector on the lens plane. *The deflector potentials*  $\Psi$  occurring in the time-delay map are given by

$$\Psi(\mathbf{r}) = 4 \int_{\mathbb{R}^n} \sigma(\mathbf{r}') \ln \left\| \frac{\mathbf{r}' - \mathbf{r}}{d_{OL}} \right\|.$$

They are solutions of two-dimensional Poisson equation  $\Delta\Psi(\mathbf{r}) = 8\pi\sigma(\mathbf{r})$ , where  $\sigma(\mathbf{r})$  is the surface mass density. By suitable coordinate transformations, we can express the time-delay map in the convenient form:

$$T_{\mathbf{y}}(\mathbf{x}) = \Theta_L \left[ \frac{\|\mathbf{x} - \mathbf{y}\|^2}{2} - \psi(\mathbf{x}) \right] \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^2).$$

Here  $\mathbf{y}$  corresponds to the point on the source  $\mathbf{s}$ -plane and  $\mathbf{x}$  corresponds to the point on the lens plane  $\mathbf{r}$ .

Fermat's principle yields the critical points of the time delay map  $T_{\mathbf{y}}(\mathbf{x})$  with respect to variations in  $\mathbf{x}$  determining those rays that are real light rays (cf., [6]). For this reason, a critical point of  $T_{\mathbf{y}}(\mathbf{x})$  relative to  $\mathbf{x}$  is called *an image* of the point source at  $\mathbf{y}$ . *The magnification* of an image  $\mathbf{x}$  of a source at  $\mathbf{y}$  is defined by

$$A_{\mathbf{y}}(\mathbf{x}) = \frac{1}{\|\det T_{\mathbf{x}\mathbf{x}}(\mathbf{x}; \mathbf{y})\|},$$

where  $T(\mathbf{x}; \mathbf{y}) = T_{\mathbf{y}}(\mathbf{x})$  and  $T_{\mathbf{x}\mathbf{x}}(\mathbf{x}; \mathbf{y})$  is the Hessian matrix with respect to  $\mathbf{x}$ . A *caustic point* in gravitational lensing is a position  $\mathbf{y} \in \mathbb{R}^2$  for which a source at  $\mathbf{y}$  will have at least one image of infinite magnification. In other words, caustics are source positions  $\mathbf{y} \in \mathbb{R}^2$  for which the time-delay map  $T_{\mathbf{y}}(\mathbf{x})$  has at least one degenerate critical point (i.e.,  $\det T_{\mathbf{x}\mathbf{x}}(\mathbf{x}; \mathbf{y}) = 0$ ). So, we may consider that the time-delay map is *the generating family* of a certain Lagrangian submanifold in  $T^*\mathbb{R}^2$  (cf., §3).

**(2) Multiplane gravitational lensing (cf., [8, 10, 1, 3])** Although we can consider a general  $k$ -planes gravitational lensing, we now only consider the case when  $k = 2$  (i.e, *a double plane gravitational lensing*) for convenience.

The typical double lens plane gravitational lensing situation is given as follows (Fig 2):

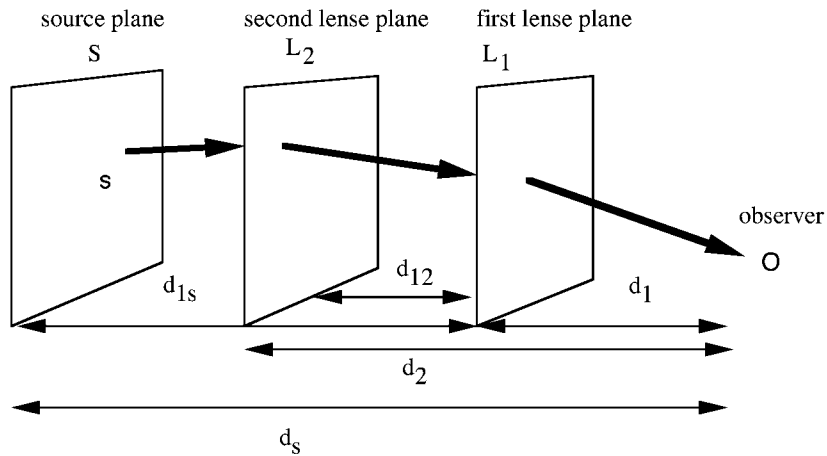


Fig.2

There are two lens planes with “thin” deflectors in each plane. The deflectors are assumed to be independent, that is, the lens planes are sufficiently spaced so that they do not interact. Furthermore, the small angle approximation is assumed. We also parameterize all rays originating from the point source at  $\mathbf{s}$ , deflected by two gravitational lens, using the 4-dimensional vectors  $(\mathbf{r}_1, \mathbf{r}_2)$ . Relative to these approximations the extra time  $T_{\mathbf{s}}$  to reach the indicated observer from  $\mathbf{s}$  is given by the *time-delay map*. It is the function  $T_{\mathbf{s}} : U_1 \times U_2 \subset \mathbb{R}^4 \rightarrow \mathbb{R}$  with each domain  $U_i \subset \mathbb{R}^2$  being an open subset, defined by

$$T_{\mathbf{s}}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{i=1}^2 (1 + z_i) \left[ \frac{d_i d_{i+1}}{2d_{i,i+1}} \left\| \frac{\mathbf{r}_{i+1}}{d_{i+1}} - \frac{\mathbf{r}_i}{d_i} \right\|^2 - \Psi_i(\mathbf{r}_i) \right].$$

Here,  $z_i$  is the redshift of the  $i$ th lens plane,  $d_{ij}$  is the angular diameter distance separating the  $i$ th and  $j$ th lens planes,  $d_i$  is the angular diameter distance from the observer to the  $i$ th lens plane with  $d_{k+1} \equiv d_S$  the distance to the source plane,  $\mathbf{r}_i$  is the position on the  $i$ th plane where the ray hits,  $\mathbf{r}_{k+1} \equiv \mathbf{s}$ , and  $\Psi_i(\mathbf{r}_i)$  is the two-dimensional potential of the deflector on the  $i$ th lens plane.

By suitable coordinate transformations, the double plane time-delay map can be expressed conveniently as follows:

$$T_{\mathbf{y}}(\mathbf{x}_1, \mathbf{x}_2) = \Theta_1 \left[ \frac{\|\mathbf{x}_2 - \mathbf{x}_1\|^2}{2} - \beta_{1,2}\psi_1(\mathbf{x}_1) \right] + \Theta_2 \left[ \frac{\|\mathbf{x}_2 - \mathbf{y}\|^2}{2} - \beta_{2,3}\psi_2(\mathbf{x}_2) \right], \quad (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in \mathbb{R}^2).$$

In [10], Fermat’s principle has been adapted exactly in the same way as it was used already for the single lens plane case, so that *the image* of a gravitational lensed point like light source at position  $\mathbf{y}$  are identified to the critical points of  $T_{\mathbf{y}}$  e.g., the set of images is given as follows:

$$\{(\mathbf{x}_1, \mathbf{x}_2) | \text{grad}_{\mathbf{x}_i} T_{\mathbf{y}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0}, \quad i = 1, 2\}.$$

If we adapt this principle simply, we cannot distinguish the effect of individual lens planes. In fact, in [10] two lens planes are treated like a single lens altogether. It is, however, pointed out in [8] that double-folds or handkerchiefs might appear as the generic caustics for double plane lensings. These singularities do not appear as generic caustics under the above construction. Therefore, our opinion is that we need to have another interpretation of Fermat’s principle.

### 3 A symplectic framework for single gravitational lensing

In [6] Petters pointed out that single gravitational lensing can be described in the framework of symplectic geometry (i.e, Lagrangian singularity theory). In the first place we briefly review the Lagrangian singularity theory [4]. Let  $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$  be the cotangent bundle over  $\mathbb{R}^n$ . We may consider that  $T^*\mathbb{R}^n = \mathbb{R}^{2n}$  and  $\pi(q_1, \dots, q_n; p_1, \dots, p_n) = (q_1, \dots, q_n)$ , where  $(q_1, \dots, q_n; p_1, \dots, p_n)$  are the canonical coordinates on  $T^*\mathbb{R}^n$ . There exists *the Liouville 1-form*  $\alpha = \sum_{i=1}^n p_i dq_i$  on  $T^*\mathbb{R}^n$ . We call the 2-form  $\omega = d\alpha = \sum_{i=1}^n dp_i \wedge dq_i$  *the canonical symplectic structure* on  $T^*\mathbb{R}^n$ . A *Lagrangian submanifold*  $\iota : L \subset T^*\mathbb{R}^n$  is a submanifold with  $\dim L = n$  and  $\iota^*\omega = 0$ . We call a map  $\pi \circ \iota : L \rightarrow \mathbb{R}^n$  a *Lagrangian map*.

There is the notion of generating families for Lagrangian immersion germs as follows: Define an  $n$ -parameter family of function germs  $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  to be a *Morse family* if the

map germ

$$\frac{\partial F}{\partial \lambda} : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k, 0)$$

is non singular, where  $\partial F / \partial \lambda(\lambda, q) = (\partial F / \partial \lambda_1(\lambda, q), \dots, \partial F / \partial \lambda_k(\lambda, q))$ . It follows that  $\Sigma(F) = (\partial F / \partial \lambda)^{-1}(0)$  is a smooth submanifold germ in  $(\mathbb{R}^k \times \mathbb{R}^n, 0)$ . For a Morse family  $F$ , we define a map germ

$$\Phi_F : \Sigma(F) \rightarrow T^*\mathbb{R}^n; \quad \Phi_F(\lambda, q) = \left( q, \frac{\partial F}{\partial q}(\lambda, q) \right).$$

Then it is easy to see that  $\Phi_F$  is a Lagrangian immersion germ. We also have the following well-known result [4]:

**Proposition 3.1** *All Lagrangian immersion germs are constructed by the above method.*

We call  $F$  a *generating family* of the Lagrangian submanifold germ  $\Phi_F(\Sigma(F))$ . By Proposition 3.1, we can interpret the local property of Lagrangian immersions by using the notion of generating family, so that the singularity theory of function germs has been applied [4].

There is a natural equivalence among Lagrangian map germs. Let  $\pi \circ \iota_i : (L_i, z_i) \rightarrow (\mathbb{R}^n, \pi(z_i))$  ( $i = 1, 2$ ) be Lagrangian map germs. We say that  $\pi \circ \iota_1 : (L_1, z_1) \rightarrow (\mathbb{R}^n, \pi(z_1))$  and  $\pi \circ \iota_2 : (L_2, z_2) \rightarrow (\mathbb{R}^n, \pi(z_2))$  are *Lagrangian equivalent* if there exist a symplectic diffeomorphism germ  $\Phi : (T^*\mathbb{R}^n, z_1) \rightarrow (T^*\mathbb{R}^n, z_2)$  with the form  $\Phi(q, p) = (\phi_2(q), \phi_1(q, p))$  and a diffeomorphism germ  $\phi : (L_1, z_1) \rightarrow (L_2, z_2)$  such that  $\Phi \circ \iota_1 = \iota_2 \circ \phi$ .

We also have natural corresponding equivalences among the Morse families. Let  $F_i : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  ( $i = 1, 2$ ) be two Morse families. We say that  $F_1$  and  $F_2$  are  *$R^+$ -equivalent* (respectively,  *$R$ -equivalent*) if there exist a diffeomorphism germ  $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k \times \mathbb{R}^n, 0)$  with the form  $\Psi(\lambda, q) = (\psi_1(\lambda, q), \psi_2(q))$  and a function germ  $\alpha : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$  such that  $F_1(\lambda, q) = F_2 \circ \Psi(\lambda, q) + \alpha(q)$  (respectively,  $F_1(\lambda, q) = F_2 \circ \Psi(\lambda, q)$ , i.e.,  $\alpha$  is constantly equal to 0). We also need the following generalized equivalence relation: For two Morse families  $F_i : (\mathbb{R}^{k_i} \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  ( $i = 1, 2$ ), we say that  $F_1$  and  $F_2$  are *stably  $R^+$ -equivalent* if there exist nondegenerate quadratic forms  $Q_1(\bar{\lambda}), Q_2(\tilde{\lambda})$ , ( $\bar{q} \in \mathbb{R}^{k_3}, \tilde{q} \in \mathbb{R}^{k_4}$ ) with  $k_1 + k_3 = k_2 + k_4$  such that  $F_1 + Q_1$  and  $F_2 + Q_2$  are  $R^+$ -equivalent.

The following theorem is the principal result of the Lagrangian singularity theory [4, 5]:

**Theorem 3.2** *Let  $F_i : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  ( $i = 1, 2$ ) be two Morse families.*

- (1) *If  $F_1$  and  $F_2$  induce the same Lagrangian submanifold germ, then  $F_1$  and  $F_2$  are  $R$ -equivalent.*
- (2) *Lagrangian manifold germs  $\Phi_{F_1}(\Sigma(F_1))$  and  $\Phi_{F_2}(\Sigma(F_2))$  are Lagrangian equivalent if and only if  $F_1$  and  $F_2$  are stably  $R^+$ -equivalent.*

We define the notion of stability of Lagrangian map germs as follows: A Lagrangian map germ is said to be *Lagrangian stable* if for every map representing the given map-germ there is a neighborhood  $\mathcal{V}$  in the space of Lagrangian maps (in the  $C^\infty$ -topology) and a neighbourhood  $\mathcal{U}$  of the source point of the germ, such that for each Lagrangian map belonging to  $\mathcal{V}$  there is a point in  $\mathcal{U}$  at which the germ of Lagrangian map-germ is Lagrangian equivalent to the original germ. The corresponding infinitesimal notion for generating family is given as follows: Let  $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  be a Morse family. We say that  $F$  is *infinitesimally  $R^+$ -versal* if

$$\mathcal{E}_\lambda = \left\langle \frac{\partial f}{\partial \lambda_1}, \dots, \frac{\partial f}{\partial \lambda_k} \right\rangle_{\mathcal{E}_\lambda} + \left\langle \frac{\partial F}{\partial q_1} \Big|_{\mathbb{R}^k \times \{0\}}, \dots, \frac{\partial F}{\partial q_n} \Big|_{\mathbb{R}^k \times \{0\}}, 1 \right\rangle_{\mathbb{R}},$$

where  $f(q) = F(q, 0)$  and  $\mathcal{E}_\lambda$  is the local ring of function germs  $(\mathbb{R}^k, 0) \rightarrow \mathbb{R}$ . Then we have the following theorem (cf. [4]).



**Theorem 3.3** *Let  $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  be a Morse family. Then the Lagrangian map germ  $\pi \circ \Phi_F$  is Lagrangian stable if and only if  $F$  is infinitesimally  $R^+$ -versal.*

Now let us recall the time-delay map  $T_{\mathbf{y}}(\mathbf{x})$ . If we consider the family of functions  $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$F(\lambda_1, \lambda_2, q_1, q_2) = \Theta \left[ \frac{\|(\lambda_1, \lambda_2) - (q_1, q_2)\|^2}{2} - \psi(\lambda_1, \lambda_2) \right] = T_q(\lambda),$$

we can easily verify that  $F$  is a Morse family, so  $\Phi_F : \Sigma(F) \rightarrow T^*\mathbb{R}^2$  is a Lagrangian immersion. Here, we have

$$\Sigma(F) = \{(\mathbf{x}, \mathbf{y}) \mid \text{grad}_{\mathbf{x}} T = \mathbf{0}\},$$

so that the Lagrangian immersion is corresponding to those rays that are actual light rays. The set of critical values of the Lagrangian map  $\pi \circ \Phi_F$  is the caustic.

On the other hand, we present another symplectic framework for single gravitational lensing, which is essentially the same as the above framework. Our framework will be, however, much useful when we try to generalize this framework to the case of multiple planes gravitational lensing (cf., §4).

We consider the *product symplectic space*

$$\mathcal{M}_{(x,y)} = (T^*M_x \times T^*M_y, \omega_{M_y} \ominus \omega_{M_x})$$

where  $\omega_{M_y}$  and  $\omega_{M_x}$  are the corresponding canonical symplectic forms, and  $\Omega_{(x,y)} = \omega_{M_y} \ominus \omega_{M_x} = \pi_{M_y}^* \omega_{M_y} - \pi_{M_x}^* \omega_{M_x}$ , where  $\pi_{M_x}, \pi_{M_y}$  are the canonical projections of the product  $T^*M_x \times T^*M_y$ . The corresponding phase spaces  $(T^*M_x, \omega_{M_x})$  and  $(T^*M_y, \omega_{M_y})$  are called the *observer space* and the *source space* respectively. In our 2-dimensional case  $M_x = \mathbb{R}^2$  and  $M_y = \mathbb{R}^2$ . The concrete realized single lensing system is represented (following [6]) by the Lagrangian submanifold

$$L_\psi = \left\{ \left( (\mathbf{x}, \text{grad}_{\mathbf{x}} T), (\mathbf{y}, \text{grad}_{\mathbf{y}} T) \right) \mid (\mathbf{x}, \mathbf{y}) \in M_x \times M_y \right\}.$$

This means that the generating function of  $L_\psi$  is the time-delay map

$$T(\mathbf{x}; \mathbf{y}) = \Theta_L \left[ \frac{\|\mathbf{x} - \mathbf{y}\|^2}{2} - \psi(\mathbf{x}) \right] \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^2).$$

By the previous arguments, light rays are given by  $\{(\mathbf{x}, \mathbf{y}) \mid \text{grad}_{\mathbf{x}} T = \mathbf{0}\}$  and the set of point sources for light rays is the Lagrangian submanifold

$$L_S = \{(\mathbf{y}, \text{grad}_{\mathbf{y}} T) \in T^*M_y \mid (\mathbf{x}, \mathbf{y}) \in M_x \times M_y, \text{grad}_{\mathbf{x}} T = \mathbf{0}\}$$

of  $T^*M_y$ . Then we have

$$\pi_{M_y}(L_\psi \cap ((M_x \times \{\mathbf{0}\}) \times T^*M_y)) = L_S.$$

Let us recall the basic notions of the theory of symplectic relations ([11]). Let  $X_1, X_2$  be smooth manifolds with the same dimension. We consider the product symplectic manifold

$$(T^*X_1 \times T^*X_2, \omega_{X_2} \ominus \omega_{X_1}),$$

where  $\omega_{X_2} \ominus \omega_{X_1} = \pi_2^* \omega_{X_2} - \pi_1^* \omega_{X_1}$ . We define a symplectic relation from  $T^*X_1$  to  $T^*X_2$  as a Lagrangian submanifold  $R$  of  $(T^*X_1 \times T^*X_2, \omega_{X_2} \ominus \omega_{X_1})$ . If the restriction of the projection

$$\pi_{X_1} \times \pi_{X_2} : T^*X_1 \times T^*X_2 \longrightarrow X_1 \times X_2$$

to  $R$  is always non-singular, we call  $R$  the *elementary symplectic relation*. Let  $R$  be a symplectic relation in  $(T^*X_1 \times T^*X_2, \omega_{X_2} \ominus \omega_{X_1})$  and  $S$  be a subset of  $T^*X_1$ , then the symplectic image of  $S$  by  $R$  is defined as

$$R(S) = \{p_2 \in T^*X_2 : \exists p_1 \in S (p_1, p_2) \in R\}.$$

If  $S$  is Lagrangian submanifold in  $(T^*X_1, \omega_{X_1})$ , then  $R(S)$  is a Lagrangian subset in  $(T^*X_2, \omega_{X_2})$ .

Since both of  $S$  and  $R$  are Lagrangian submanifolds, we have their generating families at least locally. We only consider the local situation here, so that we assume that  $X_1 = X_2 = \mathbb{R}^n$ . Let  $F_1 : (\mathbb{R}^{k_1} \times X_1, 0) \longrightarrow \mathbb{R}$  be a generating family of a Lagrangian submanifold germ  $S \subset T^*X_1$  and  $F_2 : (\mathbb{R}^{k_2} \times (X_1 \times X_2), 0) \longrightarrow \mathbb{R}$  be a generating family of a symplectic relation  $R \subset T^*X_1 \times T^*X_2$ . Then we have a function germ

$$F : ((\mathbb{R}^{k_1} \times X_1 \times \mathbb{R}^{k_2}) \times X_2, 0) \longrightarrow \mathbb{R}$$

defined by

$$F((\lambda_1, q_1, \lambda_2), q_2) = F_1(\lambda_1, q_1) + F_2(\lambda_2, q_1, q_2).$$

Hence we have the following lemma:

**Lemma 3.4** *If  $F$  is a Morse family, then  $F$  is a generating family of the Lagrangian submanifold  $R(S) \subset T^*X_2$ .*

*Proof.* By definition, we have

$$S = \left\{ \left( q_1, \frac{\partial F_1}{\partial q_1}(\lambda_1, q_1) \right) \mid \frac{\partial F_1}{\partial \lambda_1} = 0 \right\},$$

$$R = \left\{ \left( q_1, q_2, -\frac{\partial F_2}{\partial q_1}(\lambda_2, q_1), \frac{\partial F_2}{\partial q_2}(\lambda_2, q_2) \right) \mid \frac{\partial F_2}{\partial \lambda_2} = 0 \right\}.$$

Therefore we have

$$R(S) = \left\{ \left( q_2, \frac{\partial F_2}{\partial q_2}(\lambda_2, q_1, q_2) \right) \mid \frac{\partial F_1}{\partial \lambda_1} = \frac{\partial F_2}{\partial \lambda_2} = 0, \frac{\partial F_1}{\partial q_1} = -\frac{\partial F_2}{\partial q_1} \right\}.$$

Since  $\partial F / \partial \lambda_1 = \partial F_1 / \partial \lambda_1$ ,  $\partial F / \partial \lambda_2 = \partial F_2 / \partial \lambda_2$ ,  $\partial F / \partial q_1 = \partial F_1 / \partial q_1 + \partial F_2 / \partial q_1$  and  $\partial F / \partial q_2 = \partial F_2 / \partial q_2$ , we have

$$R(S) = \left\{ \left( q_2, \frac{\partial F}{\partial q_2}((\lambda_1, q_1, \lambda_2), q_2) \right) \mid \frac{\partial F}{\partial \lambda_1} = \frac{\partial F}{\partial \lambda_2} = \frac{\partial F}{\partial q_1} = 0 \right\},$$

so that  $F$  is a generating family of  $R(S)$ . Q.E.D.

In the case of single gravitational lensing, if  $S_0$  denotes the observer Lagrangian submanifold of system of gravitational rays then the source Lagrangian submanifold of rays is an image

$$L_\psi(S_0) \subset T^*M_y.$$

In the standard setting (cf. [6] and the previous arguments)  $S_0$  is the zero section of the cotangent bundle  $T^*M_x$ . Therefore we have

$$L_\psi(S_0) = \{(\mathbf{y}, \text{grad}_{\mathbf{y}}T) \mid \text{grad}_{\mathbf{x}}T = \mathbf{0}\},$$

so that the generating family for  $L_\psi(S_0)$  is given by

$$F(\lambda_1, \lambda_2, q_1, q_2) = \Theta \left[ \frac{\|(\lambda_1, \lambda_2) - (q_1, q_2)\|^2}{2} - \psi(\lambda_1, \lambda_2) \right],$$

which is the same generating family as that of the source Lagrangian submanifold in the previous framework in [6]. We call the pair  $(S, R)$  a (*single*) *lensing system* if  $S$  is a Lagrangian submanifold of  $T^*X_1$  and  $R$  is a symplectic relation from  $T^*X_1$  to  $T^*X_2$ . If the projection  $\pi_1|_R : R \rightarrow T^*X_1$  is non-singular,  $R$  is the graph of a symplectomorphism  $H : T^*M_x \rightarrow T^*M_y$ . In this case we say that  $(S, R)$  is *regular*. Moreover, if  $S$  is the zero section of  $T^*X_1$ , we call  $(S, R)$  a *special lensing system*. Therefore, the single gravitational lensing is a regular special lensing system.

## 4 A symplectic framework for multiplane gravitational lensing

In this section we will construct the intrinsic framework for the study of gravitational lensing.

We can summarize the main problem in this paper as follows:

**Problem.** How can we construct the intrinsic framework for the study of double planes gravitational lensing?

In order to tackle this problem, we now interpret the Fermat's principle from another view point. We define

$$\begin{aligned} T_{\beta_{23}}(\mathbf{x}_2; \mathbf{y}) &= \Theta_2 \left[ \frac{\|\mathbf{y} - \mathbf{x}_2\|^2}{2} - \beta_{23}(\mathbf{x}_2) \right] \\ T_{\beta_{12}}(\mathbf{x}_1; \mathbf{x}_2) &= \Theta_1 \left[ \frac{\|\mathbf{x}_2 - \mathbf{x}_1\|^2}{2} - \beta_{12}(\mathbf{x}_1) \right]. \end{aligned}$$

Then  $T_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = T_{\beta_{12}}(\mathbf{x}_1, \mathbf{x}_2) + T_{\beta_{23}}(\mathbf{x}_2, \mathbf{y})$ . We may consider that  $T_{\beta_{23}}(\mathbf{x}_1, \mathbf{x}_2)$  and  $T_{\beta_{12}}(\mathbf{x}_1; \mathbf{y})$  are respectively single time-delay maps. It is clear that

$$\begin{aligned} \text{grad}_{\mathbf{x}_1} T_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) &= \text{grad}_{\mathbf{x}_1} T_{\beta_{23}}(\mathbf{x}_1; \mathbf{x}_2) \\ \text{grad}_{\mathbf{x}_2} T_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) &= \text{grad}_{\mathbf{x}_2} T_{\beta_{12}}(\mathbf{x}_1; \mathbf{x}_2) + \text{grad}_{\mathbf{x}_2} T_{\beta_{23}}(\mathbf{x}_2; \mathbf{y}). \end{aligned}$$

These formulae suggest us that the Fermat's principle can be interpreted as

$$\begin{cases} \text{grad}_{\mathbf{x}_1} T_{\beta_{23}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0} \\ \text{grad}_{\mathbf{x}_2} T_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \mathbf{0}. \end{cases}$$

It follows that we consider the following families of functions:  $F_1(\mathbf{x}_1, \mathbf{x}_2) = T_{\beta_{23}}(\mathbf{x}_1, \mathbf{x}_2)$ ,  $F_2(\mathbf{x}_2, \mathbf{y}) = T_{\beta_{12}}(\mathbf{x}_2, \mathbf{y})$  and  $F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = F_1(\mathbf{x}_1, \mathbf{x}_2) + F_2(\mathbf{x}_2, \mathbf{y})$ . By Fermat's principle, we have two submanifolds:

$$\begin{aligned}\Sigma(F_1) &= \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^4 \mid \text{grad}_{\mathbf{x}_1} F_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0}\} \\ \Sigma(F) &= \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \in \mathbb{R}^6 \mid \text{grad}_{\mathbf{x}_1} F(\mathbf{x}_1; \mathbf{x}_2) = \text{grad}_{\mathbf{x}_2} F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \mathbf{0}\}.\end{aligned}$$

Dimensions of both submanifolds are two. Moreover, we define the following mappings

$$\Phi_{F_1} : \Sigma(F_1) \longrightarrow T^*\mathbb{R}^2$$

by  $\Phi_{F_1}(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_2, \text{grad}_{\mathbf{x}_2} F_1(\mathbf{x}_1, \mathbf{x}_2))$  and

$$\Phi_F : \Sigma(F) \longrightarrow T^*\mathbb{R}^2$$

by  $\Phi_F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = (\mathbf{y}, \text{grad}_{\mathbf{y}} F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}))$ . By the previous arguments, images of both mappings are Lagrangian submanifolds. The later Lagrangian submanifold corresponds to the light source. The first Lagrangian submanifold corresponds to the light ray image on the second lens. Since the distance between the first lens and the second lens is so long that there might be no interactions between these lenses. So we have to consider the stability of light ray under the independent perturbations of each lens planes.

On the other hand, we might consider that  $F_2(\mathbf{x}_2, \mathbf{y})$  is a generating function of the graph of a certain symplectomorphism  $H : T^*\mathbb{R}^2 \longrightarrow T^*\mathbb{R}^2$  (cf., [11]). In this case, the Lagrangian submanifold  $\Phi_F(\Sigma(F))$  can be considered as the image  $H(\Phi_{F_1}(\Sigma(F_1)))$ .

Since the single lensing system can be described under the framework of symplectic relations, we might construct the framework for the double gravitation lensing by using the composition of two symplectic relations.

The composition  $R_2 \circ R_1 \subset (T^*X_1 \times T^*X_3, \omega_{X_3} \ominus \omega_{X_1})$  of two symplectic relations  $R_1 \subset (T^*X_1 \times T^*X_2, \omega_{X_2} \ominus \omega_{X_1})$ ,  $R_2 \subset (T^*X_2 \times T^*X_3, \omega_{X_3} \ominus \omega_{X_2})$  is defined as follows:

$$R_2 \circ R_1 = \{(p_1, p_3) \in T^*X_1 \times T^*X_3; \exists_{p_2 \in T^*X_2} (p_1, p_2) \in R_1 \text{ and } (p_2, p_3) \in R_2\}.$$

If  $S$  is Lagrangian submanifold in  $T^*X_1$ , then we have two symplectic images

$$R_1(S) \text{ and } R_2 \circ R_1(S).$$

By the previous arguments, a double gravitational lensing is represented by the pair of symplectic relations  $(L_{\beta_{12}}, L_{\beta_{23}})$ ,

$$L_{\beta_{12}} \subset (T^*M_{x_1} \times T^*M_{x_2}, \omega_{M_{x_2}} \ominus \omega_{M_{x_1}}),$$

$$L_{\beta_{23}} \subset (T^*M_{x_2} \times T^*M_y, \omega_{M_y} \ominus \omega_{M_{x_2}}).$$

Now the source Lagrangian subspace of the system is the image by the composition;  $L_{\beta_{23}} \circ L_{\beta_{12}}(S_0)$ , where  $S_0$  is the zero section of  $T^*M_{x_1}$ . Since  $T_{\beta_{12}}(\mathbf{x}_1, \mathbf{x}_2)$  and  $T_{\beta_{23}}(\mathbf{x}_2, \mathbf{y})$  are the corresponding generating functions for the symplectic relations (the graphs of symplectomorphism)  $L_{\beta_{12}}$  and  $L_{\beta_{23}}$ , then the configurational positions of rays emitted from the source at a point  $\mathbf{y}$  are the points  $((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_2, \mathbf{y})) \in L_{\beta_{12}} \times L_{\beta_{23}}$  given by the solutions  $(\mathbf{x}_1, \mathbf{x}_2)$  of the system of equations

$$\text{grad}_{\mathbf{x}_1} T_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \mathbf{0} \quad (1)$$

$$\text{grad}_{\mathbf{x}_2} T_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \mathbf{0}. \quad (2)$$

If we now consider a family of functions  $F : \mathbb{R}^2 \times \mathbb{R}^2 \times M_y \longrightarrow \mathbb{R}$  defined by

$$F(q, \lambda, \mu) = T_2(q, \lambda, \mu) = T_{\beta_{12}}(q, \lambda) + T_{\beta_{23}}(\lambda, \mu),$$

then  $F$  is a generating family for the image Lagrangian subspace

$$L_{\beta_{23}} \circ L_{\beta_{12}}(S_0) \subset T^*M_y.$$

According to the above arguments, we say that  $(S, R_1, R_2)$  is a *double lensing system* if  $S$  is a Lagrangian submanifold of  $T^*X_1$  and  $R_i$  is a symplectic relation from  $T^*X_i$  to  $T^*X_{i+1}$ , where  $i = 1, 2$ . We also say that the double lensing system  $(S, R_1, R_2)$  is *regular* if both of  $R_i$  ( $i = 1, 2$ ) are graphs of symplectomorphisms. Moreover, we say that a double lensing system  $(S, R_1, R_2)$  is *special* if  $S$  is the zero section of  $T^*X_1$ . Therefore, a double gravitation lensing is a regular special double lensing system.

We now define a natural equivalence relation among double lensing system germs. Let  $(S, R_1, R_2), (S', R'_1, R'_2)$  be double lensing system germs. We say that  $(S, R_1, R_2), (S', R'_1, R'_2)$  are *Lagrangian equivalent* if there exist a symplectic diffeomorphism germ

$$\Phi_1 : (T^*X_1, z_1) \longrightarrow (T^*X_1, z'_1),$$

Lagrangian equivalences (symplectic diffeomorphisms preserving the cotangent bundle fibration)

$$\Phi_2 : (T^*X_2, z_2) \longrightarrow (T^*X_2, z'_2) \text{ and } \Phi_3 : (T^*X_3, z_3) \longrightarrow (T^*X_3, z'_3)$$

such that  $\Phi_1(S) = S', (\Phi_1 \times \Phi_2)(R_1) = R'_1$  and  $(\Phi_2 \times \Phi_3)(R_2) = R'_2$ .

Since there always exists a symplectic diffeomorphism germ  $\Phi_1 : (T^*X_1, z_1) \longrightarrow (T^*X_1, z'_1)$  such that  $\Phi_1(S) = S'$ , we may assume that  $S = S'$  are equal to the zero section of  $T^*X_1$ . Therefore, without the loss of generality, we stick to special double lensings. In this case we say that  $(S, R_1, R_2), (S, R'_1, R'_2)$  are *strictly Lagrangian equivalent* if there exists a Lagrangian equivalences

$$\Phi_2 : (T^*X_2, z_2) \longrightarrow (T^*X_2, z'_2) \text{ and } \Phi_3 : (T^*X_3, z_3) \longrightarrow (T^*X_3, z'_3)$$

such that  $(\text{id}_{T^*X_1} \times \Phi_2)(R_1) = R'_1$  and  $(\Phi_2 \times \Phi_3)(R_2) = R'_2$ . This equivalence relation might be, however, too strong to give a classification of double lensing system germs. Therefore, we need an appropriate equivalence relation among special double lensing system germs.

Now we give a candidate of such a natural equivalence. We say that the special double lensing system germs  $(S, R_1, R_2), (S, R'_1, R'_2)$  are *weakly Lagrangian equivalent* if there exist a Lagrangian equivalence germs

$$\tilde{\Phi}_3 : (T^*X_3, z_3) \longrightarrow (T^*X_3, z'_3) \text{ and } \tilde{\Phi} : ((T^*X_2 \times T^*X_3, (z_2, z_3)) \longrightarrow (T^*X_2 \times T^*X_3, (z'_2, z'_3)))$$

such that  $\pi_{X_3} \circ \tilde{\Phi} = \tilde{\Phi}_3 \circ \pi_{X_3}$  and  $\tilde{\Phi}(R_1(S) \times R_2 \circ R_1(S)) = R'_1(S) \times R'_2 \circ R'_1(S)$ .

We have the following proposition.

**Proposition 4.1** *Suppose that  $(S, R_1, R_2), (S, R'_1, R'_2)$  are weakly Lagrangian equivalent, then both of the Lagrangian submanifolds  $R_1(S), R'_1(S)$  and  $R_2 \circ R_1(S), R'_2 \circ R'_1(S)$  are Lagrangian equivalent.*

*Proof.* By definition we have a diffeomorphism germ  $\Phi : (X_2 \times X_3, (\pi_2(z_2), \pi_3(z_3))) \longrightarrow (X_2 \times X_3, (\pi_2(z'_2), \pi_3(z'_3)))$  of the form  $\Phi(x_2, x_3) = (\phi_2(x_2, x_3), \phi_3(x_3))$  and a symplectic diffeomorphism  $\tilde{\Phi} : ((T^*X_2 \times T^*X_3, (z_2, z_3)) \longrightarrow (T^*X_2 \times T^*X_3, (z'_2, z'_3)))$  of the form

$$\tilde{\Phi}((x_2, p_2), (x_3, p_3)) = ((\phi_2(x_2, x_3), \psi_2(x_2, x_3, p_2, p_3)), \phi_3(x_3), \psi_3(x_3, p_3))$$

such that  $\tilde{\Phi}(R_1(S) \times R_2 \circ R_3(S)) = R'_1(S) \times R'_2 \circ R'_1(S)$ . Therefore we have  $\tilde{\Phi}(R_1(S) \times \{z_3\}) = (R'_1(S) \times \{z'_3\})$ . We identify symplectic manifolds:  $T^*X_2 = T^*X_2 \times \{z_3\} = T^*X_2 \times \{z'_3\}$ . Under this identification, the above equality means that  $R_1(S)$  and  $R'_1(S)$  are Lagrangian equivalent.

By definition, we have  $\tilde{\Phi}_3(R_2 \circ R_1(S)) = R'_2 \circ R'_1(S)$ . This fact means that  $R_2 \circ R_1(S), R'_2 \circ R'_1(S)$  are Lagrangian equivalent. Q.E.D.

By the above proposition, the weak Lagrangian equivalence among double lensing system germs preserve both of caustics of the first and the second deflectors. It is the caustic equivalence already introduced in the classification of coisotropic varieties in [12].

## 5 Generating pairs for double lensing systems

In this section we consider the problem how to construct a kind of the notion of generating families for double lensing systems. We already have a solution because a double plane gravitational lensing is described by the pair of time-delay maps. In which we only consider local properties, so that we assume that  $X_1 = X_2 = X_3 = \mathbb{R}^n$ .

For any double lensing system germ  $(S, R_1, R_2)$ , we have generating families  $F_0 : (\mathbb{R}^{k_0} \times X_1, 0) \longrightarrow \mathbb{R}$  of  $S$ ,  $F_1 : (\mathbb{R}^{k_1} \times (X_1 \times X_2), 0) \longrightarrow \mathbb{R}$  of  $R_1$  and  $F_2 : (\mathbb{R}^{k_2} \times (X_2 \times X_3), 0) \longrightarrow \mathbb{R}$  of  $R_2$ .

On the other hand, there always exists a symplectomorphism germ  $\Phi_1 : (T^*X_1, z_1) \longrightarrow (T^*X_1, 0)$  such that  $\Phi_1(S)$  is a zero section germ of  $T^*X_1$ , so that we might assume that  $S$  is a zero section germ of  $T^*X_1$  under the Lagrangian equivalence among double lensing system germs. In other words, it is enough to investigate special double lensing system germs. In this case  $F_0$  can be chosen as a constant function. We call  $(F_1, F_2)$  a *generating pair* of the special lensing system germ  $(S, R_1, R_2)$  if  $F_i$  is a generating family of  $R_i$  ( $i = 1, 2$ ). Then  $F_1$  can be regarded as a generating family of  $R_1(S) \subset T^*X_2$  and  $F = F_1 + F_2$  is a generating family of  $R_2 \circ R_1(S) \subset T^*X_3$  (cf., [11]).

Since a double gravitation lensing is a regular double lensing system, we now pay attention to regular special double lensing systems here. In this case  $F_1 : (X_1 \times X_2, 0) \longrightarrow \mathbb{R}$  is a generating function of  $R_1$  and  $F_2 : (X_2 \times X_3, 0) \longrightarrow \mathbb{R}$  is a generating function of  $R_2$ . By the arguments in the previous paragraph,  $F_1$  is a generating family of the Lagrangian submanifold germ  $R_1(S) \subset T^*X_2$  and a map germ  $F : (X_1 \times X_2) \times X_3, 0 \longrightarrow \mathbb{R}$  defined by

$$F((x_1, x_2), x_3) = F_1(x_1, x_2) + F_2(x_2, x_3)$$

is a generating family of the Lagrangian submanifold germ  $R_2 \circ R_1(S) \subset T^*X_3$ . In other words,  $(F_1, F_2)$  is a *generating pair* of a regular special double lensing system germ  $(S, R_1, R_2)$  if  $(dF_1(X_1 \times X_2), (z_1, z_2)) = (R_1, (z_1, z_2))$  and  $(dF_2(X_2 \times X_3), (z_2, z_3)) = (R_2, (z_2, z_3))$ .

Since any elementary symplectic relation has a generating function at least locally, we have the following fundamental proposition:

**Proposition 5.1** *All regular special double lensing system germs are constructed by the above method*

We can translate equivalence relations among double lensing systems into those of corresponding generating pairs. We consider the ambiguity of the choice for generating pairs of a double lensing system.

**Proposition 5.2** *Let  $(F_1, F_2)$  and  $(G_1, G_2)$  be generating pairs of a common regular special double lensing system germ. Then  $F_1 = G_1 + \text{constant}$  and  $F_2 = G_2 + \text{constant}$ .*

For our purpose, we introduce equivalence relations among generating pairs for double lensing system germs. Let  $F_1, G_1, (X_1 \times X_2, 0) \rightarrow \mathbb{R}$  and  $F_2, G_2, (X_2 \times X_3, 0) \rightarrow \mathbb{R}$  be function germs. We say that  $(F_1, F_2)$  and  $(G_1, G_2)$  are  $(R, L)^+$ -equivalent if there exist diffeomorphism germs  $\Phi_1 : (X_1 \times X_2, 0) \rightarrow (X_1 \times X_2, 0)$  of the form  $\Phi_1(x_1, x_2) = (\phi_1(x_1, x_2), \phi_2(x_2))$ , and  $\Phi_2 : (X_2 \times X_3, 0) \rightarrow (X_2 \times X_3, 0)$  of the form  $\Phi_2(x_2, x_3) = (\phi_2(x_2), \phi_3(x_3))$  and function germs  $\alpha : (X_2, 0) \rightarrow \mathbb{R}, \beta : (X_3, 0) \rightarrow \mathbb{R}$  such that

$$\begin{cases} F_1(x_1, x_2) = G_1 \circ \Phi_1(x_1, x_2) + \alpha(x_2) \\ F_2(x_2, x_3) = G_2 \circ \Phi_2(x_2, x_3) + \beta(x_3), \end{cases}$$

**Proposition 5.3** *Let  $(F_1, F_2)$  and  $(G_1, G_2)$  be respectively generating pairs of special regular double lensing system germs  $(S_0, R_1, R_2)$  and  $(S_0, R'_1, R'_2)$ . Then  $(S_0, R_1, R_2)$  and  $(S_0, R'_1, R'_2)$  are strictly Lagrangian equivalent if and only if  $(F_1, F_2)$  and  $(G_1, G_2)$  are  $(R, L)^+$ -equivalent.*

We also say that  $(F_1, F_2)$  and  $(G_1, G_2)$  are  $R \times L^+$ -equivalent if there exist a diffeomorphism germ

$$\Phi : (X_1 \times X_2 \times X_3, 0) \rightarrow (X_1 \times X_2 \times X_3, 0)$$

of the form  $\Phi(x_1, x_2, x_3) = (\phi_1(x_1, x_2, x_3), \phi_2(x_2, x_3), \phi_3(x_3))$  and a function germ  $\alpha : (X_3, 0) \rightarrow \mathbb{R}$  such that

$$F_1(x_1, x_2) + F_2(x_2, x_3) = G_1(\phi_1(x_1, x_2, x_3), \phi_2(x_2, x_3)) + G_2(\phi_2(x_2, x_3), \phi_3(x_3)) + \alpha(x_3).$$

Suppose that  $(F_1, F_2)$  and  $(G_1, G_2)$  are  $R \times L^+$ -equivalent. If we substitute  $x_3 = 0$  into the both sides of the above equality, then we have

$$F_1(x_1, x_2) + F_2(x_2, 0) = G_1(\phi_1(x_1, x_2, 0), \phi_2(x_2, 0)) + G_2(\phi_2(x_2, 0), 0) + \alpha(0).$$

Therefore,  $F_1$  and  $G_1$  are  $R^+$ -equivalent.

By the general theory for Lagrangian singularities (cf., [4]), we can show the following proposition:

**Proposition 5.4** *Let  $(F_1, F_2)$  and  $(G_1, G_2)$  be respectively generating pairs of special regular double lensing system germs  $(S_0, R_1, R_2)$  and  $(S_0, R'_1, R'_2)$ . Then  $(S_0, R_1, R_2)$  and  $(S_0, R'_1, R'_2)$  are weakly Lagrangian equivalent if and only if  $(F_1, F_2)$  and  $(G_1, G_2)$  are  $R \times L^+$ -equivalent.*

We say that  $(F_1, F_2)$  is *infinitesimally  $R \times L^+$ -stable* if

$$\mathcal{E}_{(x_1, x_2)} + \mathcal{E}_{(x_2, x_3)} \subset \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2, x_3)}} + \left\langle \frac{\partial(F_1 + F_2)}{\partial x_2} \right\rangle_{\mathcal{E}_{(x_2, x_3)}} + \left\langle \frac{\partial F_2}{\partial x_3}, 1 \right\rangle_{\mathcal{E}_{x_3}}.$$

In this case, for  $F_1$  we have

$$\mathcal{E}_{x_1} = \left\langle \frac{\partial f_1}{\partial x_1} \right\rangle_{\mathcal{E}_{x_1}} + \left\langle \frac{\partial F_1}{\partial x_2} \Big|_{x_2=0}, 1 \right\rangle_{\mathbb{R}},$$

where  $f_1(x_1) = F_1(x_1, 0)$ .

Now we have the following proposition:

**Proposition 5.5** *Let  $(F_1, F_2)$  be a generating pair of a special regular double lensing system germ  $(S_0, R_1, R_2)$ . Then the following are equivalent:*

- (1)  $(F_1, F_2)$  is infinitesimally  $R \times L^+$ -stable.
- (2)  $(F_1, F_2)$  satisfies the following condition:

$$\mathcal{E}_{(x_1, x_2)} = \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2)}} + \left\langle \frac{\partial(F_1 + f_2)}{\partial x_2} \right\rangle_{\mathcal{E}_{x_2}} + \left\langle \frac{\partial F_2}{\partial x_3} \Big|_{x_3=0}, 1 \right\rangle_{\mathbb{R}},$$

where  $f_2(x_2) = F_2(x_2, 0)$ .

- (3)  $(F_1, F_2)$  satisfies the following condition:

$$\mathcal{E}_{(x_1, x_2)} + \mathcal{E}_{(x_2, x_3)} = \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2, x_3)}} + \left\langle \frac{\partial(F_1 + F_2)}{\partial x_2} \right\rangle_{\mathcal{E}_{(x_2, x_3)}} + \left\langle \frac{\partial F_2}{\partial x_3}, 1 \right\rangle_{\mathcal{E}_{x_3}}.$$

- (4)  $F_1$  is infinitesimally  $R^+$ -versal and

$$\mathcal{E}_{x_2} \subset \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2)}} + \left\langle \frac{\partial(F_1 + f_2)}{\partial x_2} \right\rangle_{\mathcal{E}_{x_2}} + \left\langle \frac{\partial F_2}{\partial x_3} \Big|_{x_3=0}, 1 \right\rangle_{\mathbb{R}}.$$

*Proof.* We assume that the condition (4) holds. Since  $F_1$  is infinitesimally  $R^+$ -versal deformation of  $f_1$ , we can show that

$$\mathcal{E}_{(x_1, x_2)} = \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2)}} + \left\langle \frac{\partial F_1}{\partial x_2}, 1 \right\rangle_{\mathcal{E}_{x_2}}.$$

We remark that

$$\frac{\partial F_1}{\partial x_2} = \frac{\partial(F_1 + f_2)}{\partial x_2} - \frac{\partial f_2}{\partial x_2},$$

so that we have

$$\mathcal{E}_{(x_1, x_2)} = \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2)}} + \left\langle \frac{\partial(F_1 + f_2)}{\partial x_2} \right\rangle_{\mathcal{E}_{x_2}} + \left\langle \frac{\partial F_2}{\partial x_3} \Big|_{x_3=0}, 1 \right\rangle_{\mathbb{R}} + \mathcal{E}_{x_2}.$$

It follows from the assumption that

$$\mathcal{E}_{(x_1, x_2)} = \left\langle \frac{\partial F_1}{\partial x_1} \right\rangle_{\mathcal{E}_{(x_1, x_2)}} + \left\langle \frac{\partial(F_1 + f_2)}{\partial x_2} \right\rangle_{\mathcal{E}_{x_2}} + \left\langle \frac{\partial F_2}{\partial x_3} \Big|_{x_3=0}, 1 \right\rangle_{\mathbb{R}}.$$

This means that the condition (2) holds. The converse assertion is trivial by definition.

By the Malgrange preparation theorem (cf., [13]), we can easily show that the condition (3) is equivalent to the condition (2). Since  $\mathcal{E}_{(x_1, x_2)} + \mathcal{E}_{x_2} = \mathcal{E}_{(x_1, x_2)}$ , the condition (1) implies the condition (2). It follows from the inclusion  $\mathcal{E}_{(x_1, x_2)} + \mathcal{E}_{(x_2, x_3)} \subset \mathcal{E}_{(x_1, x_2, x_3)}$  that the condition (3) implies the condition (1). This completes the proof.

## 6 Classification of lensing systems and caustics

We recall that the family of functions

$$F(x_1, x_2, x_3) = F_1(x_1, x_2) + F_2(x_2, x_3)$$



is a generating family for the Lagrangian submanifold-germ  $R_2 \circ R_1(S) \subset T^*X_3$  with  $(x_1, x_2)$  being the Morse parameters auxiliary in the reduction process (cf. [15]). A *caustic* of the time-delay map in the case of the double lensing system is defined to be the set of source positions  $x_3 \in \mathbb{R}^2$ , which are critical values of the projection

$$T^*X_3 \ni R_2 \circ R_1(S) \longrightarrow X_3,$$

or the function  $(x_1, x_2) \longrightarrow F(x_1, x_2, x_3)$  has at least one degenerate critical point.

Using the  $R \times L^+$  equivalency group and infinitesimal stability conditions obtained in Propositions 5.4, 5.5 we can construct normal and prenormal forms of infinitesimally stable generating pairs  $(F_1, F_2)$ , or equivalently the functions  $F(x_1, x_2, x_3) = F_1(x_1, x_2) + F_2(x_2, x_3)$ , which belongs to the space of additively composed functions on  $\Delta = \{(x_1, x_2, \tilde{x}_2, x_3) \in X_1 \times X_2 \times X_2 \times X_3 : x_2 = \tilde{x}_2\}$ , i.e.

$$F(x_1, x_2, x_3) = \pi_{12}^* F_1(x_1, x_2) + \pi_{23}^* F_2(\tilde{x}_2, x_3) |_{\Delta},$$

where  $\pi_{12}, \pi_{23}$  are the canonical projections

$$\pi_{ij} : X_1 \times X_2 \times X_2 \times X_3 \rightarrow X_i \times X_j, \quad (ij) = (12), (23).$$

If  $(F_1, F_2)$  is infinitesimally  $R \times L^+$ -stable generating pair, then by Proposition 5.5 (4)  $F_1$  is infinitesimally versal. Now we define the subgroup of  $R \times L^+$ -equivalency group prescribed to  $F_1$  and acting on  $F_2$ . We say that  $F_2$  and  $G_2$  are  $(R \times L^+)_{F_1}$  equivalent if there exists  $\Phi(x_1, x_2, x_3) = (\phi_1(x_1, x_2, x_3), \phi_2(x_2, x_3), \phi(x_3))$  and a function-germ  $\alpha : (X_3, 0) \rightarrow R$  such that  $F_1$  is preserved,  $F_1 \circ (\phi_1, \phi_2) = F_1$  and  $(F_1, F_2)$  and  $(F_1, G_2)$  are  $R \times L^+$ -equivalent by  $\Phi$ . Now we can formulate the following result.

**Proposition 6.1** *Any infinitesimally  $R \times L^+$ -stable pair is  $R \times L^+$ -equivalent to the pair  $(F_1, F_2)$ , where  $F_1$  is a versal unfolding of  $F_1(x_1, 0)$  and  $F_2$  belongs to an open orbit of  $(R \times L^+)_{F_1}$ -action.*

In two dimensional case we simplify the notation and write

$$F(x, u, v) = F_1(x, u) + F_2(u, v), \quad x, u, v \in \mathbb{R}^2.$$

**Proposition 6.2** *Let us assume that  $F_1$  is an infinitesimally  $R^+$ -versal Morse family. Then the generic generating pair-germs  $(F_1, F_2)$  are  $R \times L^+$ -equivalent to one of the following normal forms:*

$$\begin{aligned} (A_1 A_1) : & (F_1(x, u), F_2(u, v)) = (\pm x_1^2 \pm x_2^2, \pm u_1^2 \pm u_2^2) \\ (A_1 A_2) : & (F_1(x, u), F_2(u, v)) = (\pm x_1^2 \pm x_2^2, u_1^3 \pm u_2^2 + v_1 u_1) \\ (A_1 A_3) : & (F_1(x, u), F_2(u, v)) = (\pm x_1^2 \pm x_2^2, \pm u_1^4 \pm u_2^2 + v_1 u_1^2 + v_2 u_1) \\ (A_2 B_2) : & (F_1(x, u), F_2(u, v)) = (x_1^3 \pm x_2^2 + x_1 u_1, \pm u_1^2 \pm u_2^2 + v_1 u_1) \\ (A_2 B_3) : & (F_1(x, u), F_2(u, v)) = (x_1^3 \pm x_2^2 + x_1 u_1, u_1^3 \pm u_2^2 + v_1 u_1 + v_2 u_1^2) \\ (A_2 C_2) : & (F_1(x, u), F_2(u, v)) = (x_1^3 \pm x_2^2 + x_1 u_1, u_1 u_2 \pm u_2^2 + v_1 u_2) \\ (A_2 C_3) : & (F_1(x, u), F_2(u, v)) = (x_1^3 \pm x_2^2 + x_1 u_1, u_1 u_2 + u_2^3 + v_1 u_2 + v_2 u_2^2) \\ (A_3 X) : & (F_1(x, u), F_2(u, v)) = (\pm x_1^4 \pm x_2^2 + x_1^2 u_1 + x_1 u_2, \xi(u, v)), \end{aligned}$$

where  $\xi$  is a smooth function-germ.



Fig. 3.

The first three cases ( $AA$ ) give the standard plane caustics (i.e., non-singular, folds and cusps).

The four cases ( $AB$ ), ( $AC$ ) are the composed  $A$ ,  $B$  and  $C$  boundary type caustics. However, only the case ( $A_2C_3$ ) has the caustics at the origin as a composition. In this case we can calculate that

$$R_2 \circ R_1(s) = \{(-2xy, y, x, x^2) \mid (x, y) \in (\mathbb{R}^2, 0)\} \subset T^*X_3.$$

Therefore the projection onto  $X_3$  is locally represented by  $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ ;  $f(x, y) = (-2xy, y)$ . This map-germ is called the *pinch map* (cf., Fig. 4). This is a famous example which does not admit a Thom stratification ([16], page 24).

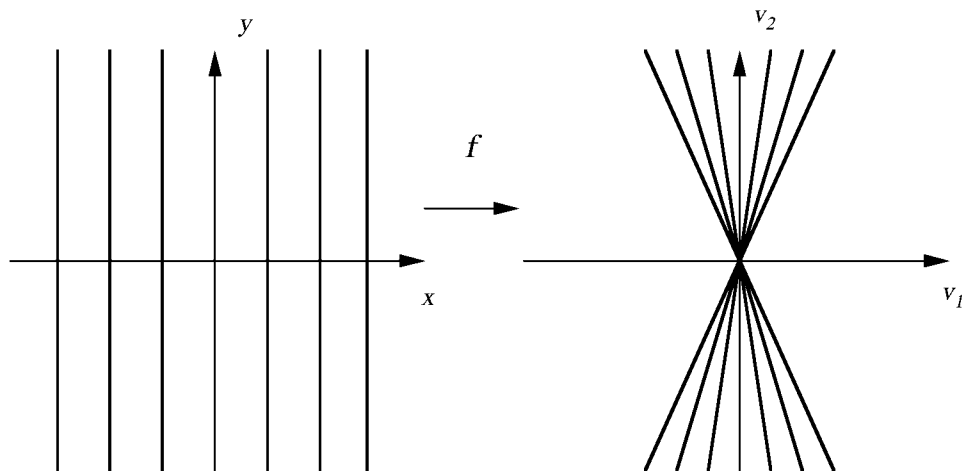


Fig.4

The last case gives other possibilities of compositions with  $A_3$ -caustic. There might be appeared several complicated singularities.

An equivalence  $R \times L^+$  - group acting on  $X_1 \times X_2 \times X_3$  is a subgroup of the  $(r, s)$ -equivalences introduced in [17], where  $r = \dim X_2$  and  $s = \dim X_3$ . We recall that  $(r, s)$ -infinitesimal stability condition

$$\mathcal{E}_{(x,u)} = \left\langle \frac{\partial F_0}{\partial x} \right\rangle_{\mathcal{E}_{(x,u)}} + \left\langle \frac{\partial(F_0)}{\partial u} \right\rangle_{\mathcal{E}_{(u)}} + \left\langle \frac{\partial F}{\partial v} \Big|_{v=0}, 1 \right\rangle_{\mathbb{R}} + \tilde{F}_0^* \mathcal{E}_{(\lambda,\mu)},$$

where  $\tilde{F}_0 = (F_0, u)$ ,  $F_0(x, u) = F(x, u, 0)$ ,  $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^2$ , is weaker than the  $R \times L^+$ -infinitesimal stability condition.

If  $h(x, u, v)$  is an  $(r, s)$ -infinitesimally stable unfolding of  $f_0(x) = F(x, 0, 0)$  then the prenormal form for infinitesimally  $R \times L^+$ -stable unfolding of  $f_0$  is given in the form

$$F(x, u, v) = h(x, u, v) + \sum_{i=1}^2 u_i g_i(u, v),$$

where  $g_i \in \mathcal{E}_{(u,v)}$ .

By  $(r, s)$ -stability theory (cf. [17]), if  $g \in \mathcal{E}_{(x,u,v)}$  is an  $(r, s)$ -stable unfolding of  $\eta(x) = g(x, 0, 0)$ . Then  $g$  is  $(r, s)$ -equivalent to the function-germ

$$F(x, T_\sigma(u, v)),$$

where  $F$  is an  $(r + s)$ -stable unfolding of  $\eta$  and  $T_\sigma(u, v) = T \circ W_\sigma$  is a composition of the polynomial mapping  $T : (\mathbb{R}^2 \times \mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2 \times \mathbb{R}^2, 0)$  and permutation of the variables  $W_\sigma$ . Here we have

$$T(u, v) = (u, v + p(u) + \sum_{i=1}^s v_i \xi_i(u)),$$

where  $p(u)$  is a polynomial mapping  $\mathbb{R}^r \rightarrow \mathbb{R}^s$  with zero constant term and degree at most  $s + 1$  and  $\xi_i(u)$  are polynomial mappings  $\mathbb{R}^r \rightarrow \mathbb{R}^s$  with zero constant terms and degree at most  $s - 1$ . The permutation  $W_\sigma$  acting on  $u, v$ -variables,  $W_\sigma(w_1, \dots, w_{r+s}) = (w_{\sigma(1)}, \dots, w_{\sigma(r+s)})$  is defined as one of the following permutations; Taking  $k \leq \min\{r, s\}$  and integers  $1 \leq i_1 < \dots < i_k \leq r$ ,  $1 \leq j_1 < \dots < j_k \leq s$  we define  $\sigma$  as the product of the following transpositions:  $\sigma = (i_1, r + j_1)(i_2, r + j_2) \dots (i_k, r + j_k)$ .

In our  $(2, 2)$ -case all stable unfoldings are related to the corresponding strata of the family of mappings:

$$T(u, v) = \left( u_1, u_2, v_1 + \sum_{1 \leq i+j \leq 3} a_{ij} u_1^i u_2^j + \sum_{i,j=1}^2 b_{ij} v_i u_j, \right. \\ \left. v_2 + \sum_{1 \leq i+j \leq 3} c_{ij} u_1^i u_2^j + \sum_{i,j=1}^2 d_{ij} v_i u_j \right)$$

These unfoldings were classified in [18, 19] and we may use them in our classification of gravitational caustics.

**Remark 6.3** By the straightforward application of the classification theorem from [18] we find that the generic perturbations of the composed function-germs  $F(x, u, v) = F_1(x, u) + F_2(u, v)$ , are  $(2, 2)$ -equivalent to the following normal forms:

$$\begin{aligned}
F(x, u, v) &= x_1^3 \pm x_2^2 + x_1 u_1, \\
F(x, u, v) &= x_1^3 \pm x_2^2 + x_1(v_1 + u_1^2 + \pm u_2^2), \\
F(x, u, v) &= x_1^3 \pm x_2^2 + x_1(v_1 - u_1^2 - u_2^2), \\
F(x, u, v) &= x_1^3 \pm x_2^2 + x_1(v_1 + u_1^3 + \pm u_2^2 + u_1 v_2), \\
F(x, u, v) &= \pm x_1^4 \pm x_2^2 + x_1^2 u_1 + u_2 x_1, \\
F(x, u, v) &= \pm x_1^4 \pm x_2^2 + x_1^2(\pm u_1^2 + u_2) + x_1(u_2 + v_1), \\
F(x, u, v) &= \pm x_1^4 \pm x_2^2 + x_1^2(u_1^3 + u_1 v_2 + u_2) + x_1(u_2 + v_1), \\
F(x, u, v) &= \pm x_1^4 \pm x_2^2 + x_1^2(\pm u_1^2 + u_2 v_2 + v_1) + x_1 u_2, \\
F(x, u, v) &= \pm x_1^4 \pm x_2^2 + x_1^2 u_1 + x_1(u_1^2 \pm u_2^2 + u_1 v_1 + v_2), \\
F(x, u, v) &= x_1^5 \pm x_2^2 + x_1^3 u_1 + x_1^2(u_2 v_2 + c u_2 + u_1 + v_1) + x_1 u_2, (c \neq -\frac{2}{3}), \\
F(x, u, v) &= x_1^3 + x_2^3 + x_1 x_2 u_1 + x_1 u_2 + x_2(u_2 v_2 + u_1 + c u_2 + v_1), (c \neq 0), \\
F(x, u, v) &= x_1^3 - x_1 x_2^2 + x_1^2(u_1 v_2 + c u_1 + u_2 + v_1) + x_1 u_2 + x_2 u_1, (c \in \mathbb{R}), \\
F(x, u, v) &= x_1^3 - x_1 x_2^2 + x_1^2(u_2 v_2 + u_1 + v_1) + x_1 u_2 + x_2 u_1,
\end{aligned}$$

By definition,  $(2, 2)$ -equivalence destroy the exact structure of the composition of caustics. However the structure of generic perturbation of the caustics are still remained. By the above list, we can calculate the discriminant set

$$\mathcal{D}_F = \{(v_1, v_2) \in X_3 \mid \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial u_1} = \frac{\partial F}{\partial u_2} = 0\}.$$

Such sets are the perturbed caustics of the original double lensing systems. For the function germ  $F(x, u, v) = \pm x_1^4 \pm x_2^2 + x_1^2 u_1 + x_1(u_1^2 \pm u_2^2 + u_1 v_1 + v_2)$  we have  $\mathcal{D}_F = \{(t, 0)\} \cup \{(6s^2 \pm 2s, \pm 20s^3)\}$  which is depicted in Fig. 5. We can observe that two regular curves (ie., folds) have order 3

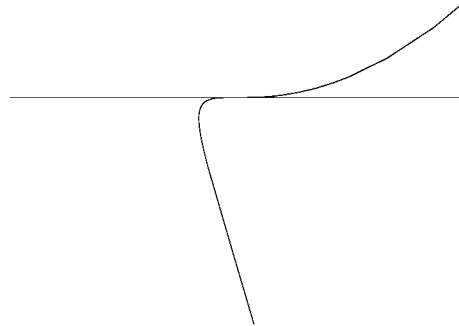


Fig.5

contact at the origin.

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