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Singularities of C^∞-lightlike hypersurfaces in Minkowski 4-space

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Abstract

We classify singularities of lightlike hypersurfaces in Minkowski 4-space via the contact invariants for corresponding spacelike surfaces and lightcones.

1 Introduction

The objective of this paper (and [5–9]) is to link the differential geometry of lightlike hypersurfaces in Minkowski 4-space with the modern theory of Legendrian singularities. Lightlike hypersurfaces are ruled 3-manifolds whose induced first fundamental form is positive semi definite. Extending these ruling lines defines a natural completion which contain (nonimmersive) singular points. The generic intersection of such a hypersurface with a spacelike 3-plane is an immersed 2-manifold which encodes the local differential geometry of the lightlike hypersurface [8, 9]. However this approach does not efficiently adapt to more general spacetimes. As an alternative we will use Montaldi's characterization of submanifold contact in terms of $\mathcal{K}$-equivalent submersions, which provides a technical linkage to Legendrian singularity theory. As a consequence we provide a local classification of lightlike hypersurface singularities in terms of algebraic invariants (an $\mathbb{R}$-algebra) and differential geometric invariants (the lightcone indicatrix).

In Section 2 we begin by describing Cartan's frame method adapted to spacelike surfaces and lightlike hypersurfaces (See [6] for a more detailed discussion.) This is used to define the lightcone indicatrix. In Section 3 we describe the (multivalued) Legendrian distance squared function whose discriminant is a given lightlike hypersurface. The given hypersurface is now the wave front set of this function, as described in Legendrian singularity theory [1]. Section 4 applies Montaldi's theorem to describe generic contact between a given lightcone and a spacelike surface. Singularities in the hypersurface are now characterized as points of higher order contact. In Section 5 we present the classification of lightlike hypersurface singularities and tangent lightcone indicatrices, which is based on the theory of Legendrian singularities [1, 18]. (See the appendix for a brief description). As a source of examples and motivation

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Section 6 indicates that generic lightlike hypersurface singularities occurs in the solutions to the eikonal PDE on Minkowski space. Section 7 indicates how these methods can be adopted to general spacetimes.

We assume throughout the paper that all manifolds and maps are C^∞ unless the contrary is explicitly stated.

2 Local differential geometry of spacelike surfaces

In [6] we have introduced the basic geometric tools for the study of spacelike surfaces in Minkowski 4-space. Here we briefly review the part of the theory related to this paper.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *Minkowski 4-space*. We will write $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^4_1 \setminus \{0\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

Let $\mathbb{R}^4_1$ be oriented and timelike oriented (i.e., a 4-volume form dV, and future time-like vector field, have been chosen), and $\boldsymbol{X} : U \to \mathbb{R}^4_1$ a regular surface (i.e., an immersion), where $U \subset \mathbb{R}^2$ is an open subset. We identify M and U through the immersion $\boldsymbol{X}$, $M = \boldsymbol{X}(U)$.

We call M a *spacelike surface* if the tangent plane T_pM of M is a spacelike plane (i.e., consists of spacelike vectors) for any point $p \in M$. In this case, the normal space N_pM is a timelike plane (i.e., Lorentz plane) (cf.,[15]). Let $\{\boldsymbol{e}_3(x, y), \boldsymbol{e}_4(x, y); p = (x, y)\}$ be an orthonormal frame of T_pM and $\{\boldsymbol{e}_1(x, y), \boldsymbol{e}_2(x, y); p = (x, y)\}$ a pseudo-orthonormal frame of N_pM. Here, $\boldsymbol{e}_1(p)$ is a timelike vector and $\boldsymbol{e}_i$; $i = 2, 3, 4$ are spacelike vectors.

In order to establish the fundamental formula for a spacelike surface in $\mathbb{R}^4_1$, we define some notions similar to those of Little [10].

Lets denote $d\boldsymbol{X} = \sum_{i=1}^{4} \omega_i \boldsymbol{e}_i$ and $d\boldsymbol{e}_i = \sum_{j=1}^{4} \omega_{ij} \boldsymbol{e}_j$; $i = 1, 2, 3, 4$. Then we can write $\omega_i = \delta(\boldsymbol{e}_i)\langle d\boldsymbol{X}, \boldsymbol{e}_i \rangle$ and $\omega_{ij} = \delta(\boldsymbol{e}_j)\langle d\boldsymbol{e}_i, \boldsymbol{e}_j \rangle$, where

$$\delta(\boldsymbol{e}_i) = Sign(\boldsymbol{e}_i) = \begin{cases} 1 & i = 2, 3, 4 \\ -1 & i = 1 \end{cases}$$

We have the Codazzi type equations:

$$\begin{cases} d\omega_i = \sum_{j=1}^{4} \delta(\boldsymbol{e}_i)\delta(\boldsymbol{e}_j)\omega_{ij} \wedge \omega_j \\ d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj}, \end{cases} \tag{1}$$

where d is the exterior derivative. And also,

$$\omega_{ij} = -\delta(\boldsymbol{e}_i)\delta(\boldsymbol{e}_j)\omega_{ji}. \tag{2}$$

In particular, $\omega_{ii} = 0$; $i = 1, 2, 3, 4$.

It follows from the fact $\langle d\boldsymbol{X}, \boldsymbol{e}_1 \rangle = \langle d\boldsymbol{X}, \boldsymbol{e}_2 \rangle = 0$ that

$$\omega_1 = \omega_2 = 0. \tag{3}$$

Therefore we have

$$\begin{cases} 0 = d\omega_1 = \sum_{j=1}^{4} \delta(\boldsymbol{e}_1)\delta(\boldsymbol{e}_j)\omega_{1j} \wedge \omega_j = \sum_{j=3}^{4} \delta(\boldsymbol{e}_j)\omega_{1j} \wedge \omega_j = -\omega_{13} \wedge \omega_3 - \omega_{14} \wedge \omega_4 \\ 0 = d\omega_2 = \sum_{j=1}^{4} \delta(\boldsymbol{e}_2)\delta(\boldsymbol{e}_j)\omega_{2j} \wedge \omega_j = \sum_{j=3}^{4} \delta(\boldsymbol{e}_j)\omega_{2j} \wedge \omega_j = \omega_{23} \wedge \omega_3 + \omega_{24} \wedge \omega_4. \end{cases}$$

By Cartan's lemma, we can write

$$\begin{cases} \omega_{13} = a\omega_3 + b\omega_4, \omega_{14} = b\omega_3 + c\omega_4 \\ \omega_{23} = e\omega_3 + f\omega_4, \omega_{24} = f\omega_3 + g\omega_4. \end{cases} \tag{4}$$

for appropriate functions a, b, c, e, f and g. Then we have a vector-valued quadratic form:

$$-\langle d^2\boldsymbol{X}, \boldsymbol{e}_1\rangle\boldsymbol{e}_1 + \langle d^2\boldsymbol{X}, e_2\rangle\boldsymbol{e}_2 = (a\omega_3^2 + 2b\omega_3\omega_4 + c\omega_4^2)\boldsymbol{e}_1 - (e\omega_3^2 + 2f\omega_3\omega_4 + g\omega_4^2)\boldsymbol{e}_2 \tag{5}$$

which is called the *second fundamental form* of the spacelike surface. It follows from equations (2) that

$$d\begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{12} & 0 & \omega_{23} & \omega_{24} \\ \omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ \omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix}. \tag{6}$$

And from these we get the following equations:

$$d\begin{pmatrix} \boldsymbol{e}_1 - \boldsymbol{e}_2 \\ \boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_{12} & \omega_{13} - \omega_{23} & \omega_{14} - \omega_{24} \\ \omega_{12} & 0 & \omega_{13} + \omega_{23} & \omega_{14} + \omega_{24} \\ \dfrac{\omega_{13} - \omega_{23}}{2} & \dfrac{\omega_{13} + \omega_{23}}{2} & 0 & \omega_{34} \\ \dfrac{\omega_{14} - \omega_{24}}{2} & \dfrac{\omega_{14} + \omega_{24}}{2} & -\omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \boldsymbol{e}_1 - \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} \tag{7}$$

On the other hand, we define

$$LC_p = \{\boldsymbol{x} \in \mathbb{R}^4_1 |\ \boldsymbol{x} = -(x_1 - p_1)^2 + \sum_{i=2}^{4}(x_i - p_i)^2 = 0\}$$

and

$$S^2_+ = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in LC_0 \mid x_1 = 1\},$$

where $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_1$. We call S^2_+ the *(future) spacelike unit sphere* and $LC^*_p = LC_p \backslash \{p\}$ the *lightcone with deleted vertex* at p. We also define

$$LC^*_+ = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in LC^*_0 \mid x_1 > 0 \ \}$$

and we call it a *future lightcone at the origin.* For any lightlike vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, we have

$$\widetilde{\boldsymbol{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1},) \in S^2_+.$$

Let $\boldsymbol{e}_1 = (a_1, a_2, a_3, a_4)$, $\boldsymbol{e}_2 = (b_1, b_2, b_3, b_4)$. Clearly, we have

$$d(\boldsymbol{e}_1 \pm \boldsymbol{e}_2) = d(a_1 \pm b_1)(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}) + (a_1 \pm b_1)d(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}).$$

And finally, we get the following fundamental formula:

$$d\begin{pmatrix} \widetilde{\boldsymbol{e}_1 - \boldsymbol{e}_2} \\ \widetilde{\boldsymbol{e}_1 + \boldsymbol{e}_2} \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_{12} - \frac{d(a_1-b_1)}{a_1-b_1} & \frac{\omega_{13}-\omega_{23}}{a_1-b_1} & \frac{\omega_{14}-\omega_{24}}{a_1-b_1} \\ \omega_{12} - \frac{d(a_1+b_1)}{a_1+b_1} & 0 & \frac{\omega_{13}+\omega_{23}}{a_1+b_1} & \frac{\omega_{14}+\omega_{24}}{a_1+b_1} \\ \frac{\omega_{13}-\omega_{23}}{2} & \frac{\omega_{13}+\omega_{23}}{2} & 0 & \omega_{34} \\ \frac{\omega_{14}-\omega_{24}}{2} & \frac{\omega_{14}+\omega_{24}}{2} & -\omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{e}_1 + \boldsymbol{e}_2} \\ \widetilde{\boldsymbol{e}_1 - \boldsymbol{e}_2} \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} \tag{8}$$

For a given normal vector $\boldsymbol{v} = \xi\boldsymbol{e}_1 + \eta\boldsymbol{e}_2 \in N_pM$, we have $d\boldsymbol{v} = d\xi\boldsymbol{e}_1 + \xi d\boldsymbol{e}_1 + d\eta\boldsymbol{e}_2 + \eta d\boldsymbol{e}_2$ and hence

$$\begin{aligned} \langle d\boldsymbol{v}, \boldsymbol{e}_3\rangle \wedge \langle d\boldsymbol{v}, \boldsymbol{e}_4\rangle &= [(a\xi + e\eta)(c\xi + g\eta) - (b\xi + f\eta)^2]\omega_3 \wedge \omega_4 \\ &= [(ac - b^2)\xi^2 + (ec + ag - 2bf)\xi\eta + (eg - f^2)\eta^2]\omega_3 \wedge \omega_4. \end{aligned}$$

We define a function $\mathcal{K}_l$ as follows:

$$\mathcal{K}_l(\boldsymbol{v})(p) = \mathcal{K}_l(\xi, \eta)(p) = (ac - b^2)\xi^2 + (ec + ag - 2bf)\xi\eta + (eg - f^2)\eta^2. \tag{9}$$

We also define the *mean curvature vector* $\mathfrak{H}$ by

$$\mathfrak{H}(p) = \frac{1}{2}(a + c)\boldsymbol{e}_1 - \frac{1}{2}(e + g)\boldsymbol{e}_2. \tag{10}$$

On the other hand, we define a pair of hypersurfaces

$$LH_M^{\pm} : M \times \mathbb{R} \longrightarrow \mathbb{R}^4_1$$

by

$$LH_M^{\pm}(p, u) = LH_M^{\pm}(x, y, u) = \boldsymbol{X}(x, y) + u(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(x, y),$$

where $p = \boldsymbol{X}(x, y)$. We call $LH_M^{\pm}$ the *lightlike hypersurface* along M.

In [6, 8, 9] we defined the lightcone Gauss maps of M, $LG_M^{\pm} : M \longrightarrow S^2_+$, as $LG_M^{\pm}(x, y) = (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(x, y)$. We proved that $p_0 = \boldsymbol{X}(x_0, y_0)$ is a singular point of $LG_M^{\pm}$ if and only if $\mathcal{K}_l(1, \pm 1)(p_0) = 0$. We now have the following proposition:

Proposition 2.1 *Let $p_0 = \boldsymbol{X}(x_0, y_0) \in M$ be a point with $\mathcal{K}_l(1, \pm 1)(p_0) = 0$. Then the lightlike hypersurface $LH_M^{\pm}$ is immersive at (p_0, u_0) for any $u_0 \in \mathbb{R}$.*

Proof. By definition, we have

$$\frac{\partial LH_M^{\pm}}{\partial u}(p_0, u_0) = (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0)$$

$$\frac{\partial LH_M^{\pm}}{\partial x}(p_0, u_0) = \boldsymbol{X}_x(p_0) + u_0(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_x(p_0)$$

$$\frac{\partial LH_M^{\pm}}{\partial y}(p_0, u_0) = \boldsymbol{X}_y(p_0) + u_0(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_y(p_0).$$

It then follows that

$$\left(\frac{\partial LH_M^{\pm}}{\partial u} \wedge \frac{\partial LH_M^{\pm}}{\partial x} \wedge \frac{\partial LH_M^{\pm}}{\partial y}\right)(p_0, u_0) \tag{11}$$

$$= \left((\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}) \wedge \boldsymbol{X}_x \wedge \boldsymbol{X}_y\right)(p_0) + u_0^2\left((\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}) \wedge (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_x \wedge (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_y\right)(p_0). \tag{12}$$

Since $(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}), \boldsymbol{X}_x, \boldsymbol{X}_y$ are linearly independent, we have that

$$\left((\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}) \wedge \boldsymbol{X}_x \wedge \boldsymbol{X}_y\right)(p_0) \neq 0.$$

The assumption $\mathcal{K}_l(1, \pm 1)(p_0) = 0$ means that the lightcone Gauss map $LG_M^{\pm}$ is singular at p_0. Therefore $(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_x(p_0), (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_y(p_0)$ are linearly dependent, so we get

$$\left((\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}) \wedge (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_x \wedge (\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})_y\right)(p_0) = 0.$$

And hence,

$$\left(\frac{\partial LH_M^{\pm}}{\partial u} \wedge \frac{\partial LH_M^{\pm}}{\partial x} \wedge \frac{\partial LH_M^{\pm}}{\partial y}\right)(p_0, u_0) \neq 0.$$

□

Since we are interested in the singular points of the lightlike hypersurface $LH_M^{\pm}$, we shall only consider the cases where $\mathcal{K}_l(1, \pm 1) \neq 0$.

3 Lorentzian distance-squared functions on spacelike surfaces

In this section we introduce the notion of Lorentzian distance-squared functions on spacelike surfaces which is useful for the study of singularities of lightlike hypersurfaces.

First we define a family of functions on a spacelike surface $M = \boldsymbol{X}(U)$

$$G : M \times \mathbb{R}_1^4 \longrightarrow \mathbb{R}$$

by

$$G(p, \boldsymbol{\lambda}) = G(x, y, \boldsymbol{\lambda}) = \langle \boldsymbol{X}(x, y) - \boldsymbol{\lambda}, \boldsymbol{X}(x, y) - \boldsymbol{\lambda} \rangle,$$

where $p = \boldsymbol{X}(x, y)$ We call G the *Lorentzian distance-squared function* on the spacelike surface M. For any fixed $\lambda_0 \in \mathbb{R}_1^4$, we write $g(p) = G_{\lambda_0}(p) = G(p, \boldsymbol{\lambda}_0)$ and we have the following proposition:

Proposition 3.1 *Let M be a spacelike surface such that $\mathcal{K}_l(1,\pm 1)(p_0) \neq 0$ at a point $p_0 \in M$ and $G : M \times \mathbb{R}^4_1 \to \mathbb{R}$ the Lorentzian distance-squared function on M. Suppose that $p_0 \neq \boldsymbol{\lambda}_0$ then we have the following:*
(1) $g(p_0) = \dfrac{\partial g}{\partial x}(p_0) = \dfrac{\partial g}{\partial y}(p_0) = 0$ *if and only if* $p_0 - \boldsymbol{\lambda}_0 = \mu(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0)$ *for some* $\mu \in \mathbb{R} \setminus \{0\}$.
(2) $g(p_0) = \dfrac{\partial g}{\partial x}(p_0) = \dfrac{\partial g}{\partial y}(p_0) = det\mathcal{H}(g)(p_0) = 0$ *($det\mathcal{H}(g)(p_0)$ is the determinant of the Hessian matrix) if and only if*

$$p_0 - \boldsymbol{\lambda}_0 = \mu(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0),\ \mathcal{K}_l(1,\pm 1)(p_0) < 0,\ \mu = \pm\sqrt{\frac{-1}{\mathcal{K}_l(1,\pm 1)(p_0)}}$$

and the mean curvature vector $\mathfrak{H}(p_0)$ is lightlike.

Proof. (1) The condition $g(p) = \langle \boldsymbol{X}(x,y) - \boldsymbol{\lambda}_0, \boldsymbol{X}(x,y) - \boldsymbol{\lambda}_0 \rangle = 0$ means that $\boldsymbol{X}(x,y) - \boldsymbol{\lambda}_0 \in LC_0$. We can observe that $dg(p) = \langle d\boldsymbol{X}(x,y), \boldsymbol{X}(x,y) - \boldsymbol{\lambda}_0 \rangle = 0$ if and only if $\boldsymbol{X}(x,y) - \lambda_0 \in N_pM$. Hence $g(p_0) = dg(p_0) = 0$ if and only if $p_0 - \boldsymbol{\lambda}_0 \in N_pM \cap LC_0$. This is equivalent to the condition that $p_0 - \lambda_0 = \mu(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0)$ for some $\mu \in \mathbb{R} \setminus \{0\}$.
(2) By a Lorentzian motion, we may assume that p_0 is the origin of $\mathbb{R}^4_1$. We can choose local coordinates such that $\boldsymbol{X}$ is given by the Monge form

$$\boldsymbol{X}(x,y) = (f_1(x,y), f_2(x,y), x, y),$$

so that we have $\boldsymbol{e}_1(p_0) = (1,0,0,0)$, $\boldsymbol{e}_2(p_0) = (0,1,0,0)$.

Under the condition (1), we have the following calculations:

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2} &= g_{xx} = 2(\langle \boldsymbol{X}_{xx}, \boldsymbol{X} - \boldsymbol{\lambda}_0 \rangle + \langle \boldsymbol{X}_x, \boldsymbol{X}_x \rangle) \\
&= 2(\langle (f_{1_{xx}}, f_{2_{xx}}, 0, 0), \mu(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0) \rangle + 2\langle (f_{1_x}, f_{2_x}, 1, 0), (f_{1_x}, f_{2_x}, 1, 0) \rangle) \\
&= -2\mu(a_1 \pm b_1)a + 2\mu(a_2 \pm b_2)e + 2,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 g}{\partial x \partial y} &= g_{xy} = 2(\langle \boldsymbol{X}_{xy}, \boldsymbol{X} - \boldsymbol{\lambda}_0 \rangle + \langle \boldsymbol{X}_x, \boldsymbol{X}_y \rangle) \\
&= 2\langle (f_{1_{xy}}, f_{2_{xy}}, 0, 0), \mu(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0) \rangle + 2\langle (f_{1_x}, f_{2_x}, 1, 0), (f_{1_y}, f_{2_y}, 0, 1) \rangle \\
&= -2\mu(a_1 \pm b_1)b + 2\mu(a_2 \pm b_2)f,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 g}{\partial y^2} &= g_{yy} = 2(\langle \boldsymbol{X}_{yy}, \boldsymbol{X} - \boldsymbol{\lambda}_0 \rangle + \langle \boldsymbol{X}_y, \boldsymbol{X}_y \rangle) \\
&= 2\langle (f_{1_{yy}}, f_{2_{yy}}, 0, 0), \mu(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0) \rangle + 2\langle (f_{1_y}, f_{2_y}, 0, 1), (f_{1_y}, f_{2_y}, 0, 1) \rangle \\
&= -2\mu(a_1 \pm b_1)c + 2\mu(a_2 \pm b_2)g + 2,
\end{aligned}$$

where $\boldsymbol{e}_1(p) = (a_1, a_2, a_3, a_4)$ and $\boldsymbol{e}_2(p) = (b_1, b_2, b_3, b_4)$. Since $(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0) = (1, \pm 1, 0, 0)$, we have

$$\begin{vmatrix} (g_\lambda)_{xx} & (g_\lambda)_{xy} \\ (g_\lambda)_{xy} & (g_\lambda)_{yy} \end{vmatrix} = \begin{vmatrix} -\mu(a_1 \pm b_1)a + \mu(a_2 \pm b_2)e + 1 & -\mu(a_1 \pm b_1)b + \mu(a_2 \pm b_2)f \\ -\mu(a_1 \pm b_1)b + \mu(a_2 \pm b_2)f & -\mu(a_1 \pm b_1)c + \mu(a_2 \pm b_2)g + 1 \end{vmatrix} = 0,$$

namely

$$\det\mathcal{H}(g_\lambda)(p_0) = \begin{vmatrix} -\mu a \pm \mu e + 1 & -\mu b \pm \mu f \\ -\mu b \pm \mu f & -\mu c \pm \mu g + 1 \end{vmatrix} = 0.$$

This is equivalent to the condition that

$$(ac + eg \mp ag \mp ce - b^2 - f^2 \pm 2bf)\mu^2 + (\pm e + \pm g - a - c)\mu + 1 = 0.$$

Therefore we have $\mathcal{K}_l(1, \pm 1)\mu^2 + \widetilde{(\pm e} + \pm g - a - c)\mu + \widetilde{1 = 0}$. If we proceed the calculations above but using the vector $-(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)(p_0)$ instead of $(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)(p_0)$, then we can see that $-\mu$ is also a solution of the above quadratic equation. Hence we have $\mathcal{K}_l(1, \pm 1)\mu^2 + 1 = 0$. This means that $\mathcal{K}_l(1, \pm 1) < 0$, $\mu = \pm\sqrt{-1/\mathcal{K}_l(1, \pm 1)}$ and $a + c \pm (e + g) = 0$. The last condition is equivalent to the condition that the mean curvature $\mathfrak{H}(p_0)$ is lightlike. $\square$

Thus Proposition 3.1 means that the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G = \left\{ \boldsymbol{\lambda} \mid \boldsymbol{\lambda} = \boldsymbol{X}(p) + u(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p),\ p \in M, u \in \mathbb{R} \right\},$$

which is the image of the lightlike hypersurface along M. Therefore a singular point of the lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(p_0) + u_0(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(p_0)$ at which $K_l(1, \pm 1)(p_0) < 0$, $u_0 = \pm\sqrt{-1/\mathcal{K}_l(1, \pm 1)(p_0)}$ and the mean curvature vector $\mathfrak{H}(p_0)$ is lightlike.

We now explain the reason why such a correspondence exists from the view point of contact geometry. Given a point $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4_1$. Let's take the projective cotangent bundle $\pi : PT^*(\mathbb{R}^4_1) \longrightarrow \mathbb{R}^4_1$ with its canonical contact structure. We review next the geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^4_1) \to PT^*(\mathbb{R}^4_1)$ and the differential map $d\pi : TPT^*(\mathbb{R}^4_1) \to \mathbb{R}^4_1$ of π. For any $X \in TPT^*(\mathbb{R}^4_1)$, there exists an element $\alpha \in T^*(\mathbb{R}^4_1)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^4_1)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^4_1)$ by

$$K = \{X \in TPT^*(\mathbb{R}^4_1) | \tau(X)(d\pi(X)) = 0\}.$$

Via the coordinates (v_1, v_2, v_3, v_4), we have the trivialization $PT^*(\mathbb{R}^4_1) \cong \mathbb{R}^4_1 \times P(\mathbb{R}^3)^*$, and we call

$$((v_1, v_2, v_3, v_4), [\xi_1 : \xi_2 : \xi_3 : \xi_4])$$

homogeneous coordinates, where $[\xi_1 : \xi_2 : \xi_3 : \xi_4]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^3)^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=2}^4 \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=2}^4 \mu_i \frac{\partial}{\partial v_i}$. An immersion $i : L \to PT^*(\mathbb{R}^4_1)$ is said to be *a Legendrian immersion* if $\dim L = 3$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called *the Legendrian map* and the set $W(i) = \mathrm{image}\, \pi \circ i$, the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

In the appendix, we give a quick survey of the theory of Legendrian singularities . For additional definitions and basic results on generating families, we refer to [1]. By the previous arguments, the lightlike hypersurface $LH^{\pm}_M$ is the discriminant set of the Lorentzian distance-squared function G. We have the following proposition.

Proposition 3.2 *Let G be the Lorentzian distance-squared function on M. For any point $((x,y),\boldsymbol{\lambda}) \in G^{-1}(0)$, G is a Morse family around $((x,y),\boldsymbol{\lambda})$. See the appendix for the definition of a Morse family.*

Proof. Denote

$$\boldsymbol{X}(x,y) = (X_1(x,y), X_2(x,y), X_3(x,y), X_4(x,y)) \text{ and } \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

By definition, we have

$$G(x,y,\boldsymbol{\lambda}) = -(X_1(x,y)-\lambda_2)^2 + (X_2(x,y)-\lambda_2)^2 + (X_3(x,y)-\lambda_3)^2 + (X_4(x,y)-\lambda_4)^2.$$

We now prove that the mapping

$$\Delta^* G = \left(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

is non-singular at $((x,y),\boldsymbol{\lambda}) \in G^{-1}(0)$. In fact, the Jacobian matrix of $\Delta^* G$ is given by:

$$\left(\begin{array}{c|cccc} \mathrm{A} & \begin{matrix} 2(X_1-\lambda_1) & -2(X_2-\lambda_2) & -2(X_3-\lambda_3) & -2(X_4-\lambda_4) \\ 2X_{1x} & -2X_{2x} & -2X_{3x} & -2X_{4x} \\ 2X_{1y} & -2X_{2y} & -2X_{3y} & -2X_{4y} \end{matrix} \end{array} \right),$$

where

$$\mathrm{A} = \begin{pmatrix} 2\langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_x \rangle & 2\langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_y \rangle \\ 2(\langle \boldsymbol{X}_x, \boldsymbol{X}_x \rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{xx} \rangle) & 2(\langle \boldsymbol{X}_x, \boldsymbol{X}_y \rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{xy} \rangle) \\ 2(\langle \boldsymbol{X}_y, \boldsymbol{X}_x \rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{yx} \rangle) & 2(\langle \boldsymbol{X}_y, \boldsymbol{X}_y \rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{yy} \rangle) \end{pmatrix}.$$

Since $\boldsymbol{X}$ is an immersion, the rank of the matrix

$$\begin{pmatrix} 2X_{1x} & -2X_{2x} & -2X_{3x} & -2X_{4x} \\ 2X_{1y} & -2X_{2y} & -2X_{3y} & -2X_{4y} \end{pmatrix}.$$

is equal to two. Moreover, $\boldsymbol{X}-\boldsymbol{\lambda}$ is lightlike, so that it is linearly independent on tangent vectors $\boldsymbol{X}_x, \boldsymbol{X}_y$. This means that the rank of the matrix

$$\begin{pmatrix} 2(X_1-\lambda_1) & -2(X_2-\lambda_2) & -2(X_3-\lambda_3) & -2(X_4-\lambda_4) \\ 2X_{1x} & -2X_{2x} & -2X_{3x} & -2X_{4x} \\ 2X_{1y} & -2X_{2y} & -2X_{3y} & -2X_{4y} \end{pmatrix}.$$

is equal to three. Therefore the Jacobi matrix of $\Delta^* G$ is non-singular at $((x,y),\boldsymbol{\lambda}) \in G^{-1}(0)$. □

We observe that these considerations allow us to assert that the lightlike hypersurface $LH^{\pm}_M$ is a wave front and the Lorentzian distance-squared function G on M gives a Minkowski-canonical generating family for the Legendrian lift of $LH^{\pm}_M$.

4 Contact with lightcones

In this section we describe Montaldi's characterization of submanifolds contact in terms of $\mathcal{K}$-equivalence. It is then adapted to lightlike hypersurfaces and their indicatrices. We begin with the following basic observations.

Proposition 4.1 *There exists a point* $\boldsymbol{\lambda}_0 \in \mathbb{R}^4_1$ *such that* $M \subset LC_{\lambda_0}$ *if and only if* $\boldsymbol{\lambda}_0$ *is an isolated singular point of the lightlike hypersurface* $LH^{\pm}_M$ *and* $LH^{\pm}_M(U) \subset LC_{\lambda_0}$.

Proof. By definition, there exists a point $\boldsymbol{\lambda}_0 \in \mathbb{R}^4_1$ such that $M \subset LC_{\lambda_0}$ if and only if $g_{\lambda_0}(x,y) \equiv 0$ for any $(x,y) \in U$, where $g_{\lambda_0}(x,y) = G(x,y,\boldsymbol{\lambda}_0)$ is the Lorentzian distance-squared function on M. It follows from Proposition 3.1 that there exists a smooth function $\mu : U \longrightarrow \mathbb{R}$ such that

$$\boldsymbol{X}(x,y) = \boldsymbol{\lambda}_0 + \mu(x,y)\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}(x,y).$$

Therefore we have

$$LH^{\pm}_M(x,y) = \boldsymbol{\lambda}_0 + (u + \mu(x,y))\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}(x,y).$$

Hence we have $LH^{\pm}_M(U) \subset LC_{\lambda_0}$. Moreover,

$$\frac{\partial LH^{\pm}_M}{\partial u} = \widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}(x,y),$$

$$\frac{\partial LH^{\pm}_M}{\partial x} = \mu_x(x,y)\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}(x,y) + (u+\mu(x,y))\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}_x(x,y)$$

and

$$\frac{\partial LH^{\pm}_M}{\partial y} = \mu_y(x,y)\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}(x,y) + (u+\mu(x,y))\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}_y(x,y).$$

It follows that

$$\left(\frac{\partial LH^{\pm}_M}{\partial u} \wedge \frac{\partial LH^{\pm}_M}{\partial x} \wedge \frac{\partial LH^{\pm}_M}{\partial y}\right) = (u+\mu(x,y))^2\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)} \wedge \widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}_x \wedge \widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}_y.$$

By the similar arguments in the proof of Proposition 2.1, the above vector vanishes if and only if $u + \mu(x,y) = 0$ under the assumption that $\mathcal{K}_l \neq 0$. This means that $\boldsymbol{\lambda}_0$ is an isolated singularity of $LH^{\pm}_M$. The converse assertion is trivial. □

Motivated by the above proposition, we now consider the contact between spacelike surfaces and lightcones by way of Montaldi's theorem [13]. Let X_i, Y_i $(i = 1,2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of* X_1 and Y_1 at y_1 is same type as *the contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Since this definition of the contact is local, we can replace $\mathbb{R}^n$ by any n-manifold. Montaldi gives in [13] the following characterization of contact by using $\mathcal{K}$-equivalence.

Theorem 4.2 *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

Turning to lightlike hypersurfaces we now consider the function $\mathcal{G} : \mathbb{R}^4_1 \times \mathbb{R}^4_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{G}(\boldsymbol{x}, \boldsymbol{\lambda}) = \langle \boldsymbol{x} - \boldsymbol{\lambda}, \boldsymbol{x} - \boldsymbol{\lambda} \rangle$. Given $\boldsymbol{\lambda}_0 \in \mathbb{R}^4_1$, we denote $\mathfrak{g}_{\lambda_0}(\boldsymbol{x}) = \mathcal{G}(\boldsymbol{x}, \boldsymbol{\lambda}_0)$, so we have that $\mathfrak{g}^{-1}_{\lambda_0}(0) = LC_{\lambda_0}$. For any $(x_0, y_0) \in U$, we take the point $\boldsymbol{\lambda}_0^{\pm} = \boldsymbol{X}(x_0, y_0) + u_0(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(x_0, y_0)$ and we have

$$\mathfrak{g}_{\lambda_0^{\pm}} \circ \boldsymbol{X}(x_0, y_0)) = \mathcal{G} \circ (\boldsymbol{X} \times id_{\mathbb{R}^4_1})((x_0, y_0), \boldsymbol{\lambda}_0^{\pm}) = G(x_0, y_0, \boldsymbol{\lambda}_0^{\pm}) = 0,$$

where $u_0 = \pm\sqrt{-1/\mathcal{K}_l(1, \pm 1)(x_0, y_0)}$. We also have relations

$$\frac{\partial \mathfrak{g}_{\lambda_0^{\pm}} \circ \boldsymbol{X}}{\partial x}(p_0) = \frac{\partial G}{\partial x}(p_0, \boldsymbol{\lambda}_0^{\pm}) = 0$$

and

$$\frac{\partial \mathfrak{g}_{\lambda_0^{\pm}} \circ \boldsymbol{X}}{\partial y}(p_0) = \frac{\partial G}{\partial y}(p_0, \boldsymbol{\lambda}_0^{\pm}) = 0.$$

They imply that the lightcone $\mathfrak{g}^{-1}_{\lambda_0^{\pm}}(0) = LC_{\lambda_0^{\pm}}$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(x_0, y_0)$. In this case, we call each $LC_{\lambda_0^{\pm}}$ the *tangent lightcone* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(x_0, y_0)$.

We now describe the contacts between spacelike surfaces and lightcones. Let $LH^{\sigma}_{M,i} : (U, (x_i, y_i)) \longrightarrow (LC^*_+, \boldsymbol{v}^{\sigma}_i)$ $(i = 1, 2)$ be two lightlike hypersurface germs of spacelike surface germs $\boldsymbol{X}_i : (U, (x_i, y_i)) \longrightarrow (\mathbb{R}^4_1, p_i)$, where $\sigma = \pm$. We say that $LH^{\sigma}_{M,1}$ and $LH^{\sigma}_{M,2}$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (U, (x_1, y_1)) \longrightarrow (U, x_2, y_2))$ and $\Phi : (\mathbb{R}^4_1, \boldsymbol{\lambda}^{\sigma}_1) \longrightarrow (\mathbb{R}^4_1, \boldsymbol{\lambda}^{\sigma}_2)$ such that $\Phi \circ LH^{\sigma}_{M,1} = LM^{\sigma}_{M,2} \circ \phi$. If both of the regular sets of $LM^{\sigma}_{M,i}$ are dense in $(U, (x_i, y_i))$, it follows from Proposition A.2 of the appendix that $LH^{\sigma}_{M,1}$ and $LH^{\sigma}_{M,2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian lift germs are Legendrian equivalent. This condition is also equivalent to the condition that two generating families G_1 and G_2 are P-$\mathcal{K}$-equivalent by Theorem A.3. Here, $G_i : (U \times \mathbb{R}^4_1, ((x_i, y_i), \boldsymbol{\lambda}^{\sigma}_i)) \longrightarrow \mathbb{R}$ is the Lorentzian distance-squared function germ of $\boldsymbol{X}_i$.

On the other hand, if we denote by $g_{i,\lambda^{\sigma}_i}(x, y) = G_i(x, y, \boldsymbol{\lambda}^{\sigma}_i)$, then we have $g_{i,\lambda^{\pm}_i}(x, y) = \mathfrak{g}_{\lambda^{\pm}_i} \circ \boldsymbol{x}_i(x, y)$. By Theorem 4.1, $K(\boldsymbol{X}_1(U), LC_{\lambda^{\sigma}_1}, \boldsymbol{\lambda}^{\sigma}_1) = K(\boldsymbol{x}_2(U), LC\lambda^{\sigma}_2, \boldsymbol{\lambda}^{\sigma}_2)$ if and only if $\widetilde{g}_{1,\lambda_1}$ and $\widetilde{g}_{2,\lambda_2}$ are $\mathcal{K}$-equivalent. Therefore, we can apply Proposition A.4 to our situation. We denote $Q^{\sigma}(\boldsymbol{X}, (x_0, y_0))$ the local ring of the function germ $\widetilde{g}_{\lambda^{\sigma}_0} : (U, (x_0, y_0)) \longrightarrow \mathbb{R}$, where $\boldsymbol{\lambda}^{\sigma}_0 = LC^{\sigma}_M((x_0, y_0), u_0)$. We remark that we can explicitly write the local ring as follows:

$$Q^{\pm}(\boldsymbol{X}, (x_0, y_0)) = \frac{C^{\infty}_{(x_0, y_0)}(U)}{\langle \langle \boldsymbol{X}(x, y), \widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(x_0, y_0) \rangle - 1 \rangle_{C^{\infty}_{(x_0, y_0)}(U)}},$$

where $C^{\infty}_{(x_0, y_0)}(U)$ is the local ring of function germs at (x_0, y_0).

Theorem 4.3 *Let $\boldsymbol{X}_i : (U, (x_i, y_i)) \longrightarrow (\mathbb{R}^4_1, \boldsymbol{X}_i((x_i, y_i)))$ $(i = 1, 2)$ be spacelike surface germs such that the corresponding Legendrian lift germs are Legendrian stable. For $\sigma = +$ or $-$, the following conditions are equivalent:*

(1) *The lightlike hypersurface germs* $LH^{\sigma}_{M_1}$ *and* $LH^{\sigma}_{M_2}$ *are* $\mathcal{A}$*-equivalent.*
(2) G_1 *and* G_2 *are* P*-*$\mathcal{K}$*-equivalent.*
(3) g_{1,λ_1} *and* g_{2,λ_2} *are* $\mathcal{K}$*-equivalent.*
(4) $K(\boldsymbol{X}_1(U), LC_{\lambda_1^\sigma}, \boldsymbol{\lambda}_1^\sigma) = K(\boldsymbol{X}_2(U), LC\lambda_2^\sigma, \boldsymbol{\lambda}_2^\sigma)$
(5) $Q^\sigma(\boldsymbol{X}_1, (x_1, y_1))$ *and* $Q^\sigma(\boldsymbol{X}_2, (x_2, y_2))$ *are isomorphic as* $\mathbb{R}$*-algebras.*

Proof. The previous arguments has been shown that conditions (3) and (4) are equivalent. The other assertions follow from Proposition A.4. □

Given a spacelike surface germ $\boldsymbol{X} : (U, (x_0, y_0)) \longrightarrow (\mathbb{R}^4_1, \boldsymbol{X}(x_0, y_0))$, we call

$$(\boldsymbol{X}^{-1}(LC_{\lambda^\pm}), (x_0, y_0))$$

the *tangent lightcone indicatrix germ* of $\boldsymbol{X}$, where $\boldsymbol{\lambda}^\pm = \boldsymbol{X}(x_0, y_0) + u_0(\widetilde{\boldsymbol{e}_1 \pm \boldsymbol{e}_2})(x_0, y_0)$ and $u_0 = \pm\sqrt{-1/\mathcal{K}_l(x_0, y_0)}$. We have the following corollary of Theorem 4.3 :

Corollary 4.4 *Under the assumptions of Theorem 4.3, if the lightlike hypersurface germs* $LH^{\sigma}_{M_1}$ *and* $LH^{\sigma}_{M_2}$ *are* $\mathcal{A}$*-equivalent, then tangent lightcone indicatrix germs*

$$(\boldsymbol{X}_1^{-1}(LC_{\lambda_1^\pm}), (x_1, y_1)) \quad \text{and} \quad (\boldsymbol{X}_2^{-1}(LC_{\lambda_2^\pm}), (x_2, y_2))$$

are diffeomorphic as set germs.

Proof. Notice that the tangent lightcone indicatrix germ of $\boldsymbol{X}_i$ is the zero level set of g_{i,λ_i}. Since $\mathcal{K}$-equivalence among function germs preserves the zero-level sets of function germs, the assertion follows from Theorem 4.3. □

5 Classification of singularities of lightlike hypersurfaces

In this section we provide a generic classification of the singularities of lightlike hypersurfaces in $\mathbb{R}^4_1$. We consider the space of spacelike embeddings $\mathrm{Emb}^s(U, \mathbb{R}^4_1)$ with the Whitney C^∞-topology. We also consider a function $\mathcal{G} : \mathbb{R}^4_1 \times \mathbb{R}^4_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{G}(\boldsymbol{v}, \boldsymbol{\lambda}) = \langle \boldsymbol{v} - \boldsymbol{\lambda}, \boldsymbol{v} - \boldsymbol{\lambda} \rangle$, and claim that $\mathcal{G}_\lambda$ is a submersion at $\boldsymbol{v} \neq \boldsymbol{\lambda}$ for any $\boldsymbol{\lambda} \in \mathbb{R}^4_1$, where $\mathcal{G}_\lambda(\boldsymbol{v}) = \mathcal{G}(\boldsymbol{v}, \boldsymbol{\lambda})$. Given $\boldsymbol{X} \in \mathrm{Emb}^s(U, \mathbb{R}^4_1)$, we have $G = \mathcal{G} \circ (\boldsymbol{X} \times id_{\mathbb{R}^4_1})$. We also have the ℓ-jet extension

$$j^\ell_1 G : U \times \mathbb{R}^4_1 \longrightarrow J^\ell(U, \mathbb{R})$$

defined by $j^\ell_1 G(u, \boldsymbol{\lambda}) = j^\ell g_\lambda(u)$, where we write $G(u, \boldsymbol{\lambda}) = g_\lambda(u)$. Consider the trivialization $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(2, 1)$. For any submanifold $Q \subset J^\ell(2, 1)$, we denote $\widetilde{Q} = U \times \{0\} \times Q$. Since $\mathrm{Emb}^s(U, \mathbb{R}^4_1)$ is an open subset of the space of all embeddings $\mathrm{Emb}(U, \mathbb{R}^4_1)$, we have the following proposition as a corollary of Lemma 6 in Wassermann [17]. (See also Montaldi [14]).

Proposition 5.1 *Let* Q *be a submanifold of* $J^\ell(n-1, 1)$*. Then the set*

$$T_Q = \{\boldsymbol{X} \in \mathrm{Emb}^s(U, \mathbb{R}^4_1) \mid j^\ell_1 G \text{ is transversal to } \widetilde{Q}\ \}$$

is a residual subset of $\mathrm{Emb}^s(U, \mathbb{R}^4_1)$*. If* Q *is a closed subset, then* T_Q *is open.*

On the other hand, we have the stratification given by the set of $\mathcal{K}$-orbits in $J^\ell(2,1)\backslash W^\ell(2,1)$ (For the definition of $W^\ell(2,1)$ and additional properties, refer to [1, 4]). As a consequence of the above proposition, we have the following theorem.

Theorem 5.2 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}^s(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the Legendrian lift of the corresponding lightlike hypersurface $LH^\pm_M$ at each point is Legendrian stable.*

By the classification results on stable Legendrian mappings, we have the following corollary:

Corollary 5.3 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}^s(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding lightlike hypersurfaces $LH^\pm_M$ at any point $(x, y, u) \in U \times \mathbb{R}$ is $\mathcal{A}$-equivalent to one of the map germs A_k $(1 \le k \le 4)$ or $D_4^\pm$: where, A_k, $D_4^\pm$-map germ $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ are given by*
(A_1) $f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0)$,
(A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$,
(A_3) $f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$,
(A_4) $f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_1, u_2)$,
(D_4^+) $f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$,
(D_4^-) $f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3)$.

Proof. By Theorems 5.2 and 7.3, the Lorentzian distance squared function G is a $\mathcal{K}$-versal deformation of g_{λ_0} at each point $(x_0, y_0, \boldsymbol{\lambda}_0) \in U \times \mathbb{R}$. Therefore we can apply the generic classification of $\mathcal{K}$-versal deformations of function germs up to 4-parameters [1]. The normal forms are given by

$$\begin{aligned} F(x, y, \boldsymbol{\lambda}) &= x^{k+1} \pm y^2 + \lambda_1 + \lambda_2 x + \cdots + \lambda_k x^{k-1} \ (1 \le k \le 4) \\ F(x, y, \boldsymbol{\lambda}) &= x^3 + y^3 + \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 xy \\ F(x, y, \boldsymbol{\lambda}) &= x^3 - xy^2 + \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4(x^2 + y^2). \end{aligned}$$

For example, if we consider the germ given by

$$F(x, y, \boldsymbol{\lambda}) = x^3 + y^3 + \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 xy.$$

Then, we get

$$\Sigma_*(F) = \{(x, y, 2(x^3 + y^3) + \lambda_4 xy, -3x^2 - \lambda_4 y, -3y^2 - \lambda_4 x, \lambda_4) \mid (x, y, \lambda_4) \in \mathbb{R}^4\}.$$

Therefore the corresponding Legendrian map germ is

$$f(u_1, u_2, u_3) = (2(u_1^2 + u_2^2) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3) \quad (D_4^+).$$

The other cases follow from similar arguments, so that we may leave the details to the readers. $\square$

By using the generic normal forms of Lorentzian distance squared functions and Corollary 4.4, we have the following:

Corollary 5.4 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}\,(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding tangent lightcone indicatrix at any point $(x_0, y_0, \boldsymbol{\lambda}_0) \in U \times \mathbb{R}$ is diffeomorphic to one of the germs in the following list:*
(1) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^2 = 0\ \}$ *(ordinary cusp)*
(2) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^4 \pm y^2 = 0\ \}$ *(tachnode or point)*
(3) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^5 + y^2 = 0\ \}$ *(rhamphoid cusp)*
(4) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 - xy^2 = 0\ \}$ *(three lines)*
(5) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^3 = 0\ \}$ *(line)*

Proof. We have the same generic normal forms of Lorentzian distance squared function germs at each point as in the above corollary. By Corollary 4.4, the corresponding lightcone tangent indicatrix germs are diffeomorphic to the zero-level set of the function germ $F|\mathbb{R}^2 \times \{0\}$ of the list. For example if the normal form is given by

$$F(x, y, \boldsymbol{\lambda}) = x^3 + y^3 + \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 xy,$$

then we have $F|\mathbb{R}^2 \times \{0\} = x^3 - xy^2$, so that the corresponding lightcone tangent indicatrix germ is diffeomorphic to the set germ (4) in the above list. $\square$

6 The eikonal equation

As indirect motivation we will show how the constructions above are naturally encountered in solutions to the Minkowski eikonal equation:

$$-\left(\frac{\partial S}{\partial x_1}\right)^2 + \left(\frac{\partial S}{\partial x_2}\right)^2 + \left(\frac{\partial S}{\partial x_3}\right)^2 + \left(\frac{\partial S}{\partial x_4}\right)^2 = 0.$$

Let $\pi : T^*(\mathbb{R}^4_1) \longrightarrow \mathbb{R}^4_1$ the cotangent bundle over $\mathbb{R}^4_1$ and $((x_1, x_2, x_3, x_4), (p_1, p_2, p_3, p_4))$ be the canonical coordinate system, for a single valued solution S we have that $p_i = \partial S/\partial x_i$. Therefore the above eikonal equation can be viewed as a family of cones in $T^*(\mathbb{R}^4_1)$ given by the following equation:

$$H(x_1, \boldsymbol{x}, p_1, \boldsymbol{p}) = \frac{1}{2}(-p_1^2 + \boldsymbol{p} \cdot \boldsymbol{p}) = \frac{1}{2}(-p_1^2 + p_2^2 + p_3^2 + p_4^2) = 0,$$

where $\boldsymbol{x} = (x_2, x_3, x_4)$ and $\boldsymbol{p} = (p_2, p_3, p_4)$. The singularities of the hypersurface $H^{-1}(0)$ correspond to the zero section $\mathbb{R}^4_1 \times \{\boldsymbol{0}\}$ of the cotangent bundle. Consider the one-form on $T^*(\mathbb{R}^4_1)$ given by

$$\theta = -p_1 dx_1 + \boldsymbol{p} \cdot d\boldsymbol{x},$$

where $\boldsymbol{p} \cdot d\boldsymbol{x} = \sum_{i=2}^4 p_i dx_i$. We can show that $\theta|H^{-1}(0)$ is a contact form on the nonsingular part of $H^{-1}(0)$. If we consider a surface $\boldsymbol{X}(U) = M$ in Euclidean space $\mathbb{R}^3 = \{\boldsymbol{x} = (0, x_2, x_3, x_4) \mid \boldsymbol{x} \in \mathbb{R}^4\ \}$ and the unit normal vector $\boldsymbol{n}(x, y)$, then the surface $\boldsymbol{\ell}(x, y) = (0, \boldsymbol{X}(x, y), 1, \boldsymbol{n}(x, y))$ in $T^*\mathbb{R}^4_1$ lies in the hypersurface $H^{-1}(0)$. Since $\boldsymbol{n}(x, y)$ is the normal vector of M, we have $\boldsymbol{\ell}^*\theta = \boldsymbol{n}(x, y) \cdot d\boldsymbol{X}(x, y) = 0$. This means that the surface $\boldsymbol{\ell}(x, y)$ is an integral submanifold of $\theta|H^{-1}(0)$. Moreover, the Hamiltonian vector field along the surface $\boldsymbol{\ell}(x, y)$ is given by

$$X_H = -\frac{\partial}{\partial x_1} + \boldsymbol{n}(x, y) \cdot \frac{\partial}{\partial \boldsymbol{x}}.$$

It follows that we have a Cauchy problem for the PDE $H(x_1, \boldsymbol{x}, p_1, \boldsymbol{p}) = 0$ with the initial submanifold $\boldsymbol{\ell}(x, y)$. We can apply the characteristic method to obtain a multi-valued solution which is a Legendrian submanifold of $H^{-1}(0)$. In general the solution to this Cauchy problem is an exact multi-valued solution. To see this consider the 3-dimensional submanifold

$$L(x, y, u) = (u, \boldsymbol{X}(x, y) + u\boldsymbol{n}(x, y), 1, \boldsymbol{n}(x, y))$$

in $T^*\mathbb{R}^4_1$. Since $\boldsymbol{n}(x, y)$ is a unit vector, we have $\boldsymbol{n}(x, y) \cdot d\boldsymbol{n}(x, y) = 0$, so that

$$L^*\theta = -du + \boldsymbol{n}(x, y) \cdot d\boldsymbol{X}(x, y) + du + \boldsymbol{n}(x, y) \cdot d\boldsymbol{n}(x, y) = 0.$$

Therefore L is a Legendrian embedding. It is clear that $\operatorname{Image} L \subset H^{-1}(0)$. Moreover if we set $\boldsymbol{e}_1(x, y) = (1, 0, 0, 0)$ and $\boldsymbol{e}_2(x, y) = \boldsymbol{n}(x, y)$, then we have the lightlike hypersurface

$$LH^{\pm}_M(x, y, u) = \boldsymbol{X}(x, y) + u(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)(x, y).$$

We remark that $\widetilde{(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}(x, y) = (\boldsymbol{e}_1 \pm \boldsymbol{e}_2)(x, y)$ in this case. Therefore, the above Legendrian embedding L is the Legendrian lift of the lightlike hypersurface $LH^{\pm}_M$. However, we have examples of lightlike hypersurface which cannot be constructed from a regular surface in $\mathbb{R}^3$ ([8, 9]).

Observe that there is a natural spherical blow up the 7-dimensional cone bundle $\{\ H = 0\ \}$ in $T^*\mathbb{R}^4_1$,

$$\mathbb{R}^4_1 \times \mathbb{R} \times S^2 \longrightarrow T^*\mathbb{R}^4_1$$

where $(x_1, \boldsymbol{x}, t, \theta) \longrightarrow (x_1, \boldsymbol{x}, t(1, \theta))$, $t \in \mathbb{R}$, $\theta \in S^2$. The characteristic line field and the canonical 1-form θ pullback to the cylinder bundle with removable zero points. It follows that the Cauchy problem can be extended to initial which intersect the zero section in $\{\ H = 0\ \} \in T^*\mathbb{R}^4_1$. Moreover there exist C^∞-foliations of $\mathbb{R}^3$ with mild singularities which generate well posed initial data. For example consider a foliation by level surfaces $f(\boldsymbol{x}) = c$ possibly with critical points then the initial data

$$\boldsymbol{x} \longrightarrow (0, \boldsymbol{x}, \sqrt{1 - \|df_x\|}, df_x),$$

will generate a 4-dimension submanifold in $\{\ H = 0\ \}$ which is a family of multivalued 3-dimension Legendrian submanifolds in $\{\ H = 0\ \}$ (i.e., a multivalued solution) on the complement of the critical points. For special cases of $f(\boldsymbol{x}) = c$ this 4-manifold has a C^∞-immersive extension to the missing points. In any case each nonsingular level surface $f(\boldsymbol{x}) = c$ generates a lightlike hypersurface as in the above paragraph. These hypersurfaces are the "level 3-manifolds " of the multivalued solution.

7 Lightlike hypersurface singularities in curved spacetimes

Let g denote a C^∞-Lorentzian (pseudo) Riemannian metric on a neighbourhood of the origin in $\mathbb{R}^4$. We may choose local Gauss coordinates [16] so that the components g_{ij} of g satisfy

$$g_{ij} \equiv \delta_{ij} \bmod \mathcal{M}^4,$$

i.e., agree with the Minkowski metric up to fourth order. Recall that the conformal metric cg, $0 < c \in \mathbb{R}$ has the same unparametrized null geodesics as the original metric g. As in Section 2 the lightlike hypersurfaces of g consist of two parameter families of null geodesics. It follows that a lightlike hypersurface for cg is also a lightlike for g. Thus in Gauss metric cg over the dilation $d_c : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$; $\boldsymbol{x} \longrightarrow 1/\sqrt{c}\boldsymbol{x}$ then for all $c > 0$, g has the same lightlike hypersurface singularities (near the origin in $\mathbb{R}^4$) as the metric

$$d_c^*(cg_{ij}) = \delta_{ij} + \frac{1}{c}(\text{4th order terms}).$$

Thus for sufficiently large c we may use the generic nature of the results in Sections 4 and 5 to conclude that Corollary 5.3 is also valid for an open dense set of C^∞ embeddings $U \longrightarrow (\mathbb{R}^4, g)$.

Appendix Generating families

Here we give a quick survey on the theory of Legendrian singularities mainly developed by Arnol'd-Zakalyukin [1, 18]. Let $F : (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is *a Morse family* if the map germ

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is submersive, where $(q, x) = (q_1, \ldots, q_k, x_1, x_2, x_3, x_4) \in (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0})$. In this case we have a smooth 3-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^4$ defined by

$$\Phi_F(q, x) = \left(x, [\frac{\partial F}{\partial x_1}(q, x) : \frac{\partial F}{\partial x_2}(q, x) : \frac{\partial F}{\partial x_3}(q, x) : \frac{\partial F}{\partial x_4}(q, x)] \right)$$

is a Legendrian immersion. Then we have the following fundamental theorem in the theory of Legendrian singularities [1, 18].

Proposition A.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^4$ are constructed by the above method.*

We call F *a generating family* of Φ_F, and the corresponding wave front is

$$W(\Phi_F) = \left\{ x \in \mathbb{R}^4 \mid \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^4, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^4, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^4, p) \longrightarrow (PT^*\mathbb{R}^4, p')$ such that H preserves fibers of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^4$ at a point is said to be Legendrian stable if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the

Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^4, p)$ is uniquely determined by the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition A.2 *Let $i : (L, p) \subset (PT^*\mathbb{R}^4, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^4, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i, \pi \circ i'$ are dense respectively. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [19]. The assumption in the above proposition is a generic condition for i, i'. Especially, if i, i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\ \}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \frac{\partial F}{\partial x_2}|\mathbb{R}^k \times \{\mathbf{0}\}, \frac{\partial F}{\partial x_3}|, \frac{\partial F}{\partial x_4}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [11].)

The main result in the theory [1, 18] is the following:

Theorem A.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then*
(1) Φ_F and Φ_G are Legendrian equivalent if and only if $F,\ G$ are P-$\mathcal{K}$-equivalent.
(2) Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{\mathbf{0}\}$. Since

F, G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0})$, we do no need the notion of stably P-$\mathcal{K}$-equivalences under this situation (cf., [1]). By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, we have the following classification result of Legendrian stable germs(cf., [5]). For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define *the local ring of f* by $Q(f) = \mathcal{E}_n/f^*(\mathfrak{M}_p)\mathcal{E}_n$.

Proposition A.4 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that Φ_F, Φ_G are Legendrian stable. The the following conditions are equivalent.*

(1) *$(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs.*

(2) *Φ_F and Φ_G are Legendrian equivalent.*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|\mathbb{R}^k \times \{\mathbf{0}\}, g = G|\mathbb{R}^k \times \{\mathbf{0}\}$.*

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