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EXPOSED POINTS AND EXTREMAL PROBLEMS IN H^1 ON A BIDISC

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Abstract. An essential bounded function ϕ gives a continuous linear functional on the Hardy space H^1 on the bitorus. In this paper, we consider extremal problems on H^1 when ϕ is a rational function, ϕ is a product of one variable functions or $\phi = |f|/f$ for some outer function f in H^1 such that $f(z, w)$ has a good property with respect to w for a.e. z.

1. Introduction

Let m be the normalized Lebesgue measure of the bitorus T^2, the distinguished boundary of the open unit bidisc D^2 in the space $\mathbf{C}^2$ of two complex variables (z, w). Let Z be the set of all integers, Z_+ the set of all nonnegative integers, $Z^2 = Z \times Z$ and $Z_+^2 = Z_+ \times Z_+$. For $1 \le p \le \infty$, $L^p = L^p(T^2, m)$ denotes the Lebesgue space and $H^p = H^p(T^2, m) = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0 \text{ if } (\ell, n) \notin Z_+^2\}$, that is , H^p denotes the usual Hardy space on T^2. Any function f in H^p has an analytic extension in D^2. m_z and m_w denote the normalized Lebesgue measures on the unit circle $T = T_z$ and $T = T_w$, $T^2 = T \times T$ and $m = m_z \times m_w$. For $1 \le p \le \infty$, $H^p(T)$ denotes the usual Hardy space on T. Let $K_0^p = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0 \text{ if } -(\ell, n) \in Z_+^2\}$, then $K_0^p = \{f \in L^p \ ; \ \int_{T^2} fgdm = 0$ if $g \in H^q\}$ where $1/p + 1/q = 1$. Let $C = C(T^2)$ be the set of all continuous functions on T^2, $K_0 = K_0^p \cap C$ and $A = H^p \cap C$. A is usually called the bidisc algebra. When $\alpha = (\alpha_1, \alpha_2) \in Z_+^2$ and $\zeta = (z, w) \in \mathbf{C}^2$, we write $\zeta^\alpha = z^{\alpha_1} w^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. For $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ in Z_+^2, $0 \le \alpha \le \beta$ means that $0 \le \alpha_1 \le \beta_1$ and $0 \le \alpha_2 \le \beta_2$.

If ϕ is a function in L^∞, we denote by B_ϕ the functional defined on H^1 by

$$B_\phi(f) = \int_{T^2} f(\zeta)\phi(\zeta)dm(\zeta).$$

The norm of B_ϕ is $\|B_\phi\| = \sup\{|B_\phi(f)| \ ; \ f \in S\}$ and let S_ϕ denote the set of all f in S for which $B_\phi(f) = \|B_\phi\|$. Here S is the unit sphere of H^1. The functions in S_ϕ are called extremal functions. By the duality relation

$$\|B_\phi\| = \|\phi + K_0^\infty\|.$$

A function $\psi = \phi + k_0$ $(k_0 \in K_0^\infty)$ for which $\|B_\phi\| = \|\psi\|_\infty$ is called an extremal kernel. If S_ϕ is not empty and ψ is an extremal kernel with $\|\psi\|_\infty = 1$, then $\psi = |f|/f$ for some function f in H^1. This can be proved as in one variable case (see [D, §8.1 and §8.2], [Ha]). Let $\langle S_\phi \rangle$ denote the linear span of S_ϕ. Yabuta [Y1], [Y2] and Hasumi [Ha] and Wiegerinck [W] studied S_ϕ when $\langle S_\phi \rangle$ is of dimension one. For example, Yabuta [Y2] showed that if f is not identically zero and $\mathrm{Re} f \ge 0$ *a.e.* on T^2 then $S_\phi = \{\gamma f \ ; \ \gamma > 0, \ \|\gamma f\|_1 = 1\}$. However, S_ϕ has not been studied when $\langle S_\phi \rangle$ is of finite dimension. In this paper we will study such cases. In Section 2, we show that if ϕ is a continuous function on T^2, then S_ϕ is not empty. Moreover three important problems in this area are given. In Section 3, we show that if ϕ is a rational extremal kernel, then $\langle S_\phi \rangle$ is of finite dimension. In Section 4, S_ϕ is studied when ϕ is a product of one variable functions. In Sections 5 and 6, S_ϕ is studied when $\phi = \bar{\zeta}^\alpha |f_0|/f_0$ and $S_{|f_0|/f_0} = \{\gamma f_0 \ ; \ \gamma > 0, \ \|\gamma f_0\|_1 = 1\}$.

The function h in H^p is called an outer function if

$$\int_{T^2} \log |h| dm = \log \left| \int_{T^2} h dm \right| > -\infty$$

and a weakly outer function if

$$\int_{T^2} \log |h| dm = \int_T \log \left| \int_T h dm_z \right| dm_w = \int_T \log \left| \int_T h dm_w \right| dm_z > -\infty.$$

When $\phi = |f|/f$ and f is a function in H^1, f is called a strongly outer function if

$$S_\phi = \{\gamma f \ ; \ \gamma > 0, \ \|\gamma f\|_1 = 1\}$$

and f is called a w-strongly outer function if for a.e. $z \in T$

$$S_{\phi_z} = \{\gamma f_z \ ; \ \gamma > 0, \ \|\gamma f_z\|_1 = 1\}$$

where $\phi_z(w) = \phi(z, w)$ and $f_z(w) = f(z, w)$. The following are known from [Ha]. If f is w-strongly outer and z-strongly outer, then f is strongly outer. However there exists a weakly outer function which is not outer. For example, $f(z, w) = z - w$. In the one variable case, a strongly outer function is outer but the converse is not true. In the two variable case, a strongly outer function may not be outer. In fact, it is easy to see that $f(z, w) = z - w$ is strongly outer but not outer.

2. Extremal kernel and problems

For a unimodular function ϕ in L^∞, put $\mathcal{P}_\phi = \{f \in H^1 \ ; \ \phi f$ is nonnegative on $T^2\}$ and $\mathcal{R}_\phi = \{f \in H^1 \ ; \ \phi f$ is real on $T^2\}$. Then $\mathcal{P}_\phi \subseteq \mathcal{R}_\phi$. When ϕ is an extremal kernel and $S_\phi \neq \emptyset$, $\phi = \|B_\phi\| |f|/f$ for some function f in H^1 (see Introduction) and so $S_\phi \subset \mathcal{P}_\phi$.

Proposition 1. *If ϕ is a continuous function on T^2, then S_ϕ is not empty and S_ϕ is weak $*$ compact.*

Proof : By a theorem of Bochner [B, p717], $(C/K_0)^* = H^1$ and so if ϕ is in C, then $S_\phi \neq \emptyset$ and S_ϕ is weak $*$ compact because the unit ball of H^1 is weak $*$ compact. □

Lemma 1. *Let ϕ be a unimodular function in L^∞. Then*

$$\mathcal{R}_\phi + i\mathcal{R}_\phi = H^1 \cap \bar{\phi}^2 \bar{H}^1 \text{ and } \mathcal{R}_\phi = \left\{ \frac{f + \bar{\phi}^2 \bar{f}}{2} \ ; \ f \in H^1 \cap \bar{\phi}^2 \bar{H}^1 \right\}.$$

Proof : If $f \in \mathcal{R}_\phi$, then $\phi^2 f^2$ is non-negative and so $\phi^2 f = \bar{f}$. Hence $f = \bar{\phi}^2 \bar{f}$ and $if = \bar{\phi}^2 \overline{(-if)}$. This implies that $\mathcal{R}_\phi + i\mathcal{R}_\phi \subseteq H^1 \cap \bar{\phi}^2 \bar{H}^1$. Conversely if $f \in H^1 \cap \bar{\phi}^2 \bar{H}^1$, then $f = \bar{\phi}^2 \bar{g}$ for some $g \in H^1$ and so $g = \bar{\phi}^2 \bar{f}$. Put $F = (f + \bar{\phi}^2 \bar{f})/2$, then $\phi^2 F = \bar{F}$ and so ϕF is real-valued. Hence $F \in \mathcal{R}_\phi$. Put $G = -i(f - \bar{\phi}^2 \bar{f})/2$, $\phi^2 G = \bar{G}$ and so ϕG is real-valued. Hence $G \in \mathcal{R}_\phi$. Since

$$F + iG = \frac{f + \bar{\phi}^2 \bar{f}}{2} + \frac{f - \bar{\phi}^2 \bar{f}}{2} = f,$$

$\mathcal{R}_\phi + i\mathcal{R}_\phi \supseteq H^1 \cap \bar{\phi}^2 \bar{H}^1$. This implies the first equality. The second equality follows from the proof of the first one. In fact, if $f \in \mathcal{R}_\phi$ then $f = (f + \bar{\phi}^2 \bar{f})/2$. Comversely if $F = (f + \bar{\phi}^2 \bar{f})/2$ and $f \in H^1 \cap \bar{\phi}^2 \bar{H}^1$, then ϕF is real-valued and $F \in \mathcal{R}_\phi$. □

Proposition 2. *If ϕ is a unimodular function in L^∞ and ϕ is an extremal kernel, then $\{f^2 \ ; \ f \in \mathcal{R}_{\sqrt{\phi}} \cap H^2\} \subset S_\phi \subset \mathcal{R}_\phi$ and so $\langle S_\phi \rangle \subseteq H^1 \cap \bar{\phi}^2 \bar{H}^1$.*

Proof : Lemma 1 implies the proposition. □

Now we give three important problems in this area. These have been solved affirmatively in one variable case by the author [N]. In the one variable case, $S_{\bar{\zeta}^\alpha} = \left\{ \lambda \prod_{j=1}^{\alpha} (\zeta - a_j)(1 - \bar{a}_j \zeta) \in S \ ; \ |a_j| \leq 1 \ (j = 1, \cdots, \alpha) \text{ and } \lambda > 0 \right\}$ (see [D, p143]).

Problem 1. *If S_ϕ is a non-empty weak $*$ compact set, then is $\langle S_\phi \rangle$ of finite dimension ?*

Problem 2. *If $\phi = \bar{\zeta}^\alpha |f_0|/f_0$, $\alpha \in \mathbf{Z}_+^2$ and f_0 is a strongly outer function in H^1, then is $\langle S_\phi \rangle$ of finite dimension ?*

Problem 3. *If $\langle S_\phi \rangle$ is of finite dimension, then*

$$S_\phi = (\{\gamma \ ; \ \gamma > 0\} \times S_{\bar{\zeta}^\alpha} \times f_0) \cap S$$

where $\alpha \in \mathbf{Z}_+^2$ and f_0 is a strongly outer function ?

In this paper, we solve these problems in some special cases.

3. Rational extremal kernel

If ϕ is a rational function in L^∞, then ϕ is continuous on T^2 and so by Proposition 1 S_ϕ is not empty. When ϕ is a rational function, unlike the one variable case. We don't know whether the extremal kernel is also rational. In this section, we show that $\langle S_\phi \rangle$ is of finite dimension when ϕ is a rational extremal kernel. If q is a rational inner function, then on T^2 $q = \zeta^\alpha \bar{g}/g$ where $\alpha = (\alpha_1, \alpha_2) \in Z_+^2$, $\zeta^\alpha = z^{\alpha_1} w^{\alpha_2}$, g is an analytic polynomial without zeros in the bidisc D^2. If q is continuous, then g is an invertible function in H^∞ (see [R, p112]).

Lemma 2. *If ψ is a rational function in L^∞, then $H^1 \cap \psi \bar{H}^1$ is of finite dimension.*

Proof : Put $\psi = P/Q$ where P and Q are analytic polynomials, then

$$H^1 \cap \psi \bar{H}^1 \subseteq Q^{-1}(H^1 \cap P\bar{H}^1).$$

If $P = P(\zeta) = \sum_{|\alpha|=0}^{n} a_\alpha \zeta^\alpha$ where $\alpha \in Z_+^2$ and $\zeta \in \mathbf{C}^2$, then

$$H^1 \cap P\bar{H}^1 \subset \bigcup_{|\alpha|=0}^{n} H^1 \cap \zeta^\alpha \bar{H}^1 = \bigcup_{|\alpha|=n} H^1 \cap \zeta^\alpha \bar{H}^1.$$

Since $H^1 \cap \zeta^\alpha \bar{H}^1$ is of finite dimension, $H^1 \cap \psi \bar{H}^1$ is of finite dimension. □

Theorem 2. *Let ϕ be a rational function and an extremal kernel. Then $\langle S_\phi \rangle$ is of finite dimension.*

Proof : We may assume that $\|\phi + K_0^\infty\| = 1$. Since ϕ is an extremal kernel, ϕ is unimodular rational function and $S_\phi = \{f \in H^1 \ ; \ \|f\|_1 = 1$ and ϕf is non-negative on $T^2\}$ (see Introduction). Hence $S_\phi \subset \mathcal{P}_\phi$. By Lemma 1 and Proposition 2, $S_\phi \subset H^1 \cap \bar{\phi}^2 \bar{H}^1$. Lemma 2 implies that $\langle S_\phi \rangle$ is of finite dimension. □

Theorem 3. *Let q be a rational inner function. That is, suppose $q = \zeta^\alpha \bar{g}/g$, $\alpha \in Z_+^2$ and g is an analytic polynomial with no zeros in D^2. If $\phi = \bar{q}$, then*

$$S_\phi = (\{\gamma \ ; \ \gamma > 0\} \times S_{\bar{\zeta}^\alpha} \times g^{-2}) \cap S$$

and

$$\left\{ \left(\sum_{0 \le \beta \le \alpha} b_\beta \zeta^\beta \right)^2 \in S \ ; \ b_{\alpha-\beta} = \bar{b}_\beta \right\} \subset S_{\bar{\zeta}^\alpha} \subset \left\{ \sum_{0 \le \beta \le 2\alpha} b_\beta \zeta^\beta \in S \ ; \ b_{2\alpha-\beta} = \bar{b}_\beta \right\}.$$

Proof : If $f \in \mathcal{P}_\phi$, then $\bar{\zeta}^\alpha \dfrac{g}{\bar{g}} f \ge 0$ and so $\bar{\zeta}^\alpha g^2 f \ge 0$. Hence $g^2 \mathcal{P}_\phi \subseteq \mathcal{P}_{\bar{\zeta}^\alpha}$. If $F \in \mathcal{P}_{\bar{\zeta}^\alpha}$, then $\bar{\zeta}^\alpha F \ge 0$ and so $\bar{\zeta}^\alpha \dfrac{g}{\bar{g}} \, g^{-2} F \ge 0$. Hence $\mathcal{P}_\phi = (g^{-2} \mathcal{P}_{\bar{\zeta}^\alpha}) \cap H^1$ and so $S_\phi = \{\gamma F g^{-2} \in S \ ; \ \gamma > 0, \ F \in S_{\bar{\zeta}^\alpha}\}$. By Lemma 1

$$\mathcal{R}_{\bar{\zeta}^\alpha} + i\mathcal{R}_{\bar{\zeta}^\alpha} = H^1 \cap \zeta^{2\alpha} \bar{H}^1 = \left\{ \sum_{0 \le \beta \le 2\alpha} a_\beta \zeta^\beta \right\}$$

and

$$\mathcal{R}_{\bar{\zeta}^\alpha} = \left\{ \frac{1}{2} \left(\sum_{0 \le \beta \le 2\alpha} a_\beta \zeta^\beta + \zeta^{2\alpha} \sum_{0 \le \beta \le 2\alpha} \overline{a_\beta} \bar{\zeta}^\beta \right) \right\} = \left\{ \sum_{0 \le \beta \le 2\alpha} b_\beta \zeta^\beta \ ; \ b_{2\alpha-\beta} = \bar{b}_\beta \right\}.$$

Hence by Proposition 2 $S_{\bar{\zeta}^\alpha} \subset \left\{ \sum_{0\le\beta\le 2\alpha} b_\beta \zeta^\beta \in S \; ; \; b_{2\alpha-\beta} = \bar{b}_\beta \right\}$. Observe that $\mathcal{R}_{\bar{\zeta}^{\alpha/2}} = \mathcal{R}_{\bar{\zeta}^{\alpha/2}} \cap H^2$ and by Lemma 1

$$\mathcal{R}_{\bar{\zeta}^{\alpha/2}} + i\mathcal{R}_{\bar{\zeta}^{\alpha/2}} = H^2 \cap \zeta^\alpha \bar{H}^2 = \left\{ \sum_{0\le\beta\le\alpha} a_\beta \zeta^\beta \right\}.$$

Since $\mathcal{R}_{\bar{\zeta}^{\alpha/2}} = \left\{ \sum_{0\le\beta\le\alpha} b_\beta \zeta^\beta \; ; \; b_{\alpha-\beta} = \bar{b}_\beta \right\}$, by Proposition 2 $\left\{ \left(\sum_{0\le\beta\le\alpha} b_\beta \zeta^\beta \right)^2 \in S \; ; \; b_{\alpha-\beta} = \bar{b}_\beta \right\}$ $\subset S_{\bar{\zeta}^\alpha}$. $\square$

Corollary 1. *Let $q = \zeta^\alpha \bar{g}/g$ be a rational inner function. If $\phi = \bar{q}$, then the dimension of $\langle S_\phi \rangle$ is less than or equal to $2|\alpha| + 1$. If ϕ is continuous, then the dimension of $\langle S_\phi \rangle$ is $2|\alpha| + 1$.*

Proof : The first statement is clear by Theorem 3 if the dimension of $\langle S_{\bar{\zeta}^\alpha} \rangle = 2|\alpha| + 1$. For the second statement, if ϕ is continuous, by [R, p112] g is invertible in H^∞ and so the dimension of $\langle S_\phi \rangle$ is equal to the dimension of $\langle S_{\bar{\zeta}^\alpha} \rangle$. Now we will prove that the dimension of $\langle S_{\bar{\zeta}^\alpha} \rangle = 2|\alpha| + 1$. For any $a \in \mathbf{C}$ with $|a| \le 1$,

$$\zeta^\beta(\zeta^\gamma + a)(1 + \bar{a}\zeta^\gamma) \in \mathcal{P}_{\bar{\zeta}^\alpha}$$

where $\beta + \gamma = \alpha$, $\beta \in Z_+^2$ and $\gamma \in Z_+^2$. An elementary calculation implies that

$$\zeta^{\alpha+\beta} \in \langle \mathcal{P}_{\bar{\zeta}^\alpha} \rangle \quad \text{and} \quad \zeta^{\alpha-\beta} \in \langle \mathcal{P}_{\bar{\zeta}^\alpha} \rangle$$

for any β with $\alpha - \beta \in Z_+^2$. Hence the dimension of $\langle \mathcal{P}_{\bar{\zeta}^\alpha} \rangle$ is $2|\alpha| + 1$ and $\langle S_{\bar{\zeta}^\alpha} \rangle = \langle \mathcal{R}_{\bar{\zeta}^\alpha} \rangle$. $\square$

In one variable case, if q is an irreducible inner function (that is, $q = z$) then $\langle S_{\bar{q}} \rangle$ is of dimension three. In the two variable case, this is not true. In fact, if $q(z,w) = \dfrac{zw - a}{1 - \bar{a}zw}$ for some $a \in D$, then $S_{\bar{q}} = S_{\bar{z}\bar{w}} \times (1 - \bar{a}zw)^{-2}$ and so $\langle S_q \rangle$ is of dimension nine because $\langle S_{\bar{z}\bar{w}} \rangle = H^1 \cap z^2 w^2 \bar{H}^1$.

4. Kernels of products of one variable functions

In general, even if ϕ is a rational function, ϕ can not be factorized to $\phi_1\phi_2$ where $\phi_1 = \phi_1(z)$ and $\phi_2 = \phi_2(w)$. In this section, we assume that ϕ can be factorized to $\phi_1\phi_2$.

Theorem 4. *Suppose $\phi = \phi_1\phi_2$ where $\phi_1 = \phi_1(z) \in L^\infty(T_z)$ and $\phi_2 = \phi_2(w) \in L^\infty(T_w)$. If S_{ϕ_j} is not empty for $j = 1, 2$, then S_ϕ is not empty and $S_\phi \supset S_{\phi_1} \times S_{\phi_2}$.*

Proof : We may assume that $\|B_{\phi_j}\| = 1$ for $j = 1, 2$. Then $\|\phi_1 + zH^\infty(T_z)\| = \|\phi_2 + wH^\infty(T_w)\| = 1$. By the one variable theory, there exist unimodular functions $\psi_1 \in L^\infty(T_z)$ and $\psi_2 \in L^\infty(T_w)$ such that $\psi_1 = \phi_1 + g_1$ and $\psi_2 = \phi_2 + g_2$ where $g_1 \in zH^\infty(T_z)$ and $g_2 \in wH^\infty(T_w)$. Put $\psi = \psi_1\psi_2$, then ψ is a unimodular function in L^∞ and $B_\psi = B_\phi$ because

$$\psi = (\phi_1 + g_1)(\phi_2 + g_2) = \phi + g_1\phi_2 + g_2\phi_1 + g_1g_2$$

belongs to $\phi + K_0^\infty$. It is easy to see that $S_\psi \supseteq S_{\psi_1} \times S_{\psi_2} = S_{\phi_1} \times S_{\phi_2}$, and so $S_\psi \neq \emptyset$.

Lemma 4. *Suppose $q = q_1q_2$ is an inner function where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one-variable inner functions. If q_1 is a finite Blaschke product, then*

$$H^1 \cap q\bar{H}^1 = (H^1 \cap q_1\bar{H}^1) \times (H^1 \cap q_2\bar{H}^1),$$

and $H^1 \cap q_1\bar{H}^1 = H^1(T_z) \cap q_1\bar{H}^1(T_z)$ and $H^1 \cap q_2\bar{H}^1 = H^1(T_w) \cap q_2\bar{H}^1(T_w)$.

Proof : Since q_1 is a one variable finite Blaschke product, $q_1(z) = z^n\bar{g}/g$ where g is an invertible function in H^∞. Hence

$$H^1 \cap q_1q_2\bar{H}^1 = g^{-1}(H^1 \cap z^nq_2\bar{H}^1).$$

If $f = z^nq_2\bar{h} \in H^1 \cap z^nq_2\bar{H}^1$, then $q_2\bar{h}$ is conjugate analytic in z and $z^nq_2\bar{h}$ is analytic in z. Hence $h = \sum_{j=0}^{n} h_j(w)z^j$ and so $f = z^nq_2\bar{h} = \sum_{j=0}^{n} q_2\bar{h}_jz^{n-j}$. Therefore

$$\begin{aligned} H^1 \cap z^nq_2\bar{H}^1 &= \sum_{j=0}^{n}(H^1 \cap q_2\bar{H}^1)z^j = \sum_{j=0}^{n}(H^1(T_w) \cap q_2\bar{H}^1(T_w))z^j \\ &= (H^1 \cap z^n\bar{H}^1) \times (H^1 \cap q_2\bar{H}^1) \end{aligned}$$

because $H^1 \cap z^n\bar{H}^1 = H^1(T_z) \cap z^n\bar{H}^1(T_z)$ and $H^1 \cap q_2\bar{H}^1 = H^1(T_w) \cap q_2\bar{H}^1(T_w)$. This implies the lemma. □

Proposition 5. *Suppose $q = q_1q_2$ is an inner function where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one-variable finite Blaschke products. If $\phi = \bar{q}$, $\phi_1 = \bar{q}_1$ and $\phi_2 = \bar{q}_2$, then*

$$\langle S_\phi \rangle = \langle S_{\phi_1} \rangle \times \langle S_{\phi_2} \rangle$$

and $S_{\phi_1} = S_{\phi_1} \cap H^1(T_z)$ and $S_{\phi_2} = S_{\phi_2} \cap H^1(T_w)$.

Proof : By Lemma 1,

$$S_\phi \subset \mathcal{P}_\phi \subset \mathcal{R}_\phi \subset H^1 \cap \bar{\phi}^2\bar{H}^1$$

and by Lemma 4,

$$\begin{aligned} H^1 \cap \bar\phi^2 \bar H^1 &= H^1 \cap q_1^2 q_2^2 \bar H^1 \\ &= (H^1 \cap q_1^2 \bar H^1) \times (H^1 \cap q_2^2 \bar H^1) \\ &= (H^1(T_z) \cap q_1^2 \bar H^1(T_z)) \times (H^1(T_w) \cap q_2^2 \bar H^1(T_w)). \end{aligned}$$

It is easy to see that $\langle S_{\phi_1}\rangle = H^1(T_z) \cap q_1^2 \bar H^1(T_z)$ and $\langle S_{\phi_2}\rangle = H^1(T_w) \cap q_2^2 \bar H^1(T_w)$ because q_1 and q_2 are finite Blaschke products. Thus $\langle S_\phi\rangle = \langle S_{\phi_1}\rangle \times \langle S_{\phi_2}\rangle$. □

5. $\phi = \bar\zeta^\alpha |f_0|/f_0$ for $f_0^{-1} \in H^1$

If q is a rational inner function and $\phi = \bar q$, then $\phi = \bar\zeta^\alpha |g^{-2}|/g^{-2}$ where $\alpha \in \mathbf{Z}_+^2$ and g is an analytic polynomial with no zeros in D^2. Then g^{-2} belongs to the class N_* (see [R, p44]) but we don't know whether g^{-2} is in H^1. If q is continuous, it is known that g^{-2} belongs to H^∞ (see [R, p112]). In general, if $\phi = \bar\zeta^\alpha |f_0|/f_0$, $f_0 \in H^1$ and $f_0^{-1} \in H^\infty$, then by the proof of Theorem 3 it is easy to see that $S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar\zeta^\alpha} \times f_0) \cap S$. In this section, we assume that f_0^{-1} belongs to H^1 but we don't assume that $f_0^{-1} \in H^\infty$.

Lemma 5. *If $\alpha \in \mathbf{Z}_+^2$ and $f \in H^{1/2}$ such that $\bar\zeta^\alpha f$ is non-negative on T^2, then f belongs to H^1.*

Proof : If $\alpha = (\ell, m), \ell \geq 0$ and $m \geq 0$, then by a result of one variable Hardy space [HS] $\bar z^\ell \bar w^m f(z,w) = \sum_{j=-\ell}^{\ell} a_j(w) z^j$ a.e. w and $\bar z^\ell \bar w^m f(z,w) = \sum_{j=-m}^{m} b_j(z) w^j$ a.e. z. □

Proposition 6. *Suppose $\alpha \in \mathbf{Z}_+^2$ and f_0 is a function in H^1 whose inverse is in H^1. If $\phi = \bar\zeta^\alpha |f_0|/f_0$, then*

$$S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar\zeta^\alpha} \times f_0) \cap S.$$

Proof : It is clear that $(\{\gamma\ ;\ \gamma > 0\} \times S_{\bar\zeta^\alpha} \times f_0) \cap S \subseteq S_\phi$. If $f \in S_\phi$, then $\bar\zeta^\alpha f/f_0 \geq 0$ a.e. on T^2 and $f f_0^{-1} \in H^{1/2}$. Lemma 5 implies that $f f_0^{-1}$ belongs to H^1 and so $f f_0^{-1} \in \{\gamma\ ;\ \gamma > 0\} \times S_{\bar\zeta^\alpha}$. Thus $f \in (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar\zeta^\alpha} \times f_0) \cap S$. □

6. $\phi = \bar{\zeta}^\alpha |f_0|/f_0$ and $\mathrm{Re} f_0 \geq 0$

We call a function f in H^1 w-strongly outer when $f(z,w)$ is strongly outer with respect to w for a.e. z in T. It is easy to see that if f is w-strongly outer and z-strongly outer, then f is strongly outer [Ha]. The converse is not true [Ha]. When f is a function in H^1 with $f^{-1} \in H^1$ or with $\mathrm{Re} f \geq 0$, f is w-strongly outer and z-strongly outer (see [N]). In this section, we consider an easier problem than Problems 2 and 3 in §2. Suppose f_0 is w-strongly outer and z-strongly outer, $\alpha \in Z_+^2$ and $\phi = \bar{\zeta}^\alpha |f_0|/f_0$. We can ask whether

$$S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar{\zeta}^\alpha} \times f_0) \cap S.$$

In this section, we show that it is true when $\alpha = (\alpha_1, 0)$ or $\alpha = (0, \alpha_2)$. In particular, if $\mathrm{Re} f_0 \geq 0$, then it is true. ϕ_z is defined by $\phi_z(w) = \phi(z,w)$ for a.e. z and ϕ_w is defined similarly.

Theorem 6. *Suppose f is a weakly outer function in H^1 and a w-strongly outer function, and $\phi = |f|/f$.*

(1) $\langle S_{\phi_z} \rangle$ is of finite dimension one for a.e. $z \in T$.

(2) If $\langle S_{\phi_w} \rangle$ is of finite dimension for a.e. w, *then there exist a non-negative integer n and a strongly outer function f_0 in H^1 such that*

$$S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar{z}^n} \times f_0) \cap S.$$

Proof : If $g \in S_\phi$ and $\alpha_g = g/f$, then α_g is a non-negative function in the class N_* (see [R, p112]) because f is weakly outer and $\alpha_g = \alpha_g(z)$ because f is w-strongly outer. The first statement is clear. We will show the second statement. If $\langle S_{\phi_w} \rangle$ is of finite dimension for all $w \in E$ with $m_w(T \backslash E) = 0$, put $E_k = \{w \in E\ ;\ \langle S_{\phi_w} \rangle$ has dimension $k\}$ for $1 \leq k < \infty$. By [N, Theorem 2], we may assume that $k = 2n+1$ and $0 \leq n < \infty$. Then there exist the smallest integer n such that $k = 2n+1$ and $E_k \neq \emptyset$. Since $\langle S_{\phi_w} \rangle$ is of finite dimension $k = 2n+1$ for all $w \in E_k$, by [N] for any $g \in S_\phi$

$$\alpha_g(z) = \frac{\prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)}{\prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z)}$$

where $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ are in $\bar{D}$. Moreover there exists a function $g \in S_\phi$ such that $\{a_j\}_{j=1}^n \cap \{b_j\}_{j=1}^n = \emptyset$ in the above representation of α_g. We will show that $E_k = E$. If $E_k \neq E$ and $k = 2n+1$, then there exists an integer ℓ such that $n \lneq \ell < \infty$ and $E_{2\ell+1} \neq \emptyset$. Hence, again by [N] there exists a function $h \in S_\phi$ such that

$$\alpha_h(z) = \frac{\prod_{j=1}^\ell (z - c_j)(1 - \bar{c}_j z)}{\prod_{j=1}^\ell (z - d_j)(1 - \bar{d}_j z)}$$

where $\{c_j\}_{j=1}^{\ell}$ and $\{d_j\}_{j=1}^{\ell}$ are in $\bar{D}$, and $\{c_j\}_{j=1}^{\ell} \cap \{d_j\}_{j=1}^{\ell} = \emptyset$. This contradicts that $\alpha_h(z) = \prod_{j=1}^{n}(z-a_j)(1-\bar{a}_j z)/\prod_{j=1}^{n}(z-b_j)(1-\bar{b}_j z)$ with $n < \ell$. Hence $E_k = E$. Thus $\langle S_{\phi_w}\rangle$ is of finite dimension $2n+1$ for a.e. w. By the above argument, there exists a function $g \in S_\phi$ such that

$$g(z,w) = \frac{\prod_{j=1}^{n}(z-a_j)(1-\bar{a}_j z)}{\prod_{j=1}^{n}(z-b_j)(1-\bar{b}_j z)} f(z,w)$$

where $\{a_j\}_{j=1}^{n}$ and $\{b_j\}_{j=1}^{n}$ are two disjoint sets in $\bar{D}$. Since g is in H^1,

$$f_0(z,w) = f(z,w)/\prod_{j=1}^{n}(z-b_j)(1-\bar{b}_j z)$$

belongs to H^1 because $\{a_j\}_{j=1}^{n} \cap \{b_j\}_{j=1}^{n} = \emptyset$. Then $f_0(z,w)$ is strongly outer. For, if not, there exists a function k in the unit ball of H^1 such that $\beta(z) = k(z,w)/f_0(z,w)$ is non-negative and non-constant on T^2 because $f_0(z,w)$ is w-strongly outer. Since $\langle S_{\phi_w}\rangle$ is of finite dimension for a.e. w, $\beta(z) = \prod_{j=1}^{s}(z-c_j)(1-\bar{c}_j z)/\prod_{j=1}^{s}(z-d_j)(1-\bar{d}_j z)$, where $1 \leq s < \infty$, $\{c_j\}_{j=1}^{s}$ and $\{d_j\}_{j=1}^{s}$ are two disjoint set in $\bar{D}$. Then $f_0(z,w)/\prod_{j=1}^{s}(z-d_j)(1-\bar{d}_j z)$ belongs to H^1. This contradicts that $\langle S_{\phi_w}\rangle$ is of finite dimension n for a.e. w. Hence $f_0(z,w)$ is strongly outer. Thus $S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar{z}^n} \times f_0) \cap S$.

Corollary 3. *Suppose $f_0 \in H^1$ is w-strongly outer and z-strongly outer and $\phi = \bar{z}^n|f_0|/f_0$ for some positive integer n. Then*

$$S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar{z}^n} \times f_0) \cap S.$$

Proof : Let $f = \gamma\prod_{j=1}^{n}(z-a_j)(1-\bar{a}_j z)f_0$ where $|a_j| = 1$ $(1 \leq j \leq n)$, $\gamma > 0$ and $\|f\|_1 = 1$. Then $\phi = |f|/f$, f is a weakly outer function in H^1 and w-strongly outer. Apply Theorem 6.

Corollary 4. *Suppose f_0 is a nonzero function in H^1 with* Re$f_0 \geq 0$ *and $\phi = \bar{z}^n|f_0|/f_0$ for some positive integer n. Then*

$$S_\phi = (\{\gamma\ ;\ \gamma > 0\} \times S_{\bar{z}^n} \times f_0) \cap S.$$

Proof : If Re$f_0 \geq 0$ and $f_0 \not\equiv 0$, then f_0 is w-strongly outer and z-strongly outer.

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