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Invariant Subspaces Of Finite Codimension And Uniform Algebras

by

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Abstract. Let A be a uniform algebra on a compact Hausdorff space X and m a probability measure on X. Let $H^p(m)$ be the norm closure of A in $L^p(m)$ with $1 \leq p < \infty$ and $H^\infty(m)$ the weak $*$ closure in $L^\infty(m)$. In this paper, we describe a closed ideal of A and a closed invariant subspace of $H^p(m)$ which is of finite codimension.

§1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X. $M(A)$ denotes the maximal ideal space of A. Let m be a probability measure on X. $H^p(m)$ denotes the norm closure of A in $L^p(m)$ with $1 \leq p < \infty$ and $H^\infty(m)$ denotes the weak $*$ closure of A in $L^\infty(m)$. $H^p(m)$ is called an abstract Hardy space. When A is a disc algebra, if m is the normalized Lebesgue measure on the unit circle, $H^p(m)$ is the usual Hardy space and if m is the normalized area measure on the unit disc, $H^p(m)$ is the usual Bergman space.

Let I be a closed ideal of A. In this paper, we are interested in I with $\dim A/I < \infty$. Then A/I is called a Q-algebra. Two dimensional Q-algebras can be described easily, that is, $I = \{f \in A \; ; \; \phi_1(f) = \phi_2(f) = 0\}$ where $\phi_j \in M(A) \quad (j = 1, 2)$, or $I = \{f \in A \; ; \; \phi(f) = D_\phi(f) = 0\}$ where $\phi \in M(A)$ and D_ϕ is a bounded point derivation at ϕ. One of the authors [3] showed that a two dimensional operator algebra on a Hilbert space is a Q-algebra. It seems to be worthwile to describe a finite dimensional Q-algebra. In Section 2, we describe an ideal I with $\dim A/I < \infty$ using a theorem of T.W.Gamelin [2]. As a result, a finite dimensional Q-algebra is described.

When M is a closed subspace of $H^p(m)$ and $AM \subset M$, M is called an invariant subspace. In this paper, we are interested in M with $\dim H^p(m)/M < \infty$. When A is the polydisc algebra on T^n and m is the normalized Lebesgue measure on T^n, a finite codimensional invariant subspace M in $H^p(m)$ was described by P.Ahern and D.N.Clark [1] using the ideals in the polynomial ring $\mathbb{C}[z_1, \cdots, z_n]$ of finite codimension whose zero sets are contained in the polydisk, D^n. In Section 3, for arbitrary uniform algebra A we describe a finite condimensional invariant subspace M in $H^p(m)$ using the result in Section 2.

§2. Finite codimensional ideal

Let $\phi \in M(A)$. A closed subalgebra H of A is a (ϕ, k)-subalgebra if there is a sequence of closed subalgebras $A = A_0 \supset A_1 \supset \cdots \supset A_k = H$ such that A_j is the kernel of a continuous point derivation D_j of A_{j-1} at ϕ. If H is a (ϕ, k)-subalgebra of A, then H has finite codimension in A and $M(H) = M(A)$ by [2, Lemma 9.1].

If I is a closed ideal of A and A/I is of finite dimension, $B = \mathbb{C} + I$ is a closed subalgebra of A and A/B is of finite dimension. By a theorem of T. W. Gamelin [2, Theorem 9.8], we can describe B and so I. Since B is a special closed subalgebra of A we can describe I more explicitly.

Theorem 1. *If I is a closed ideal of A and A/I is of finite dimension, then there exists a closed subalgebra $E = E(I)$ of A such that $E = \{f \in A : \phi_1(f) = \cdots = \phi_n(f)\}$, $1 \leq n < \infty$, $\{\phi_j\} \subset M(A)$ and*

$$I = H_\phi^E \cap \ker \phi$$

where $\phi = \phi_j | E$, $1 \leq j \leq n$ and H_ϕ^E is a (ϕ, k)-subalgebra with respect to E for some k.

Proof. Put $H = I + \mathbb{C}$, then A/H is of finite dimension. By a theorem of T.W.Gamelin [2, Theorem 9.8], H can be obtained from A in two steps.

(i) There exist pairs of points ψ_j, ψ'_j, $1 \le j \le \ell$, in $M(A)$ such that if E consists of the $f \in A$ such that $\psi_j(f) = \psi'_j(f)$, $1 \le j \le \ell$, then $H \subset E \subset A$.

(ii) There exist distinct points $\theta_j \in M(E)$ and θ_j-subalgebras H_j of E, $1 \le j \le k$, such that $H = H_1 \cap \cdots \cap H_k$.

Put $\tilde{\psi}_j = \psi_j|E = \psi'_j|E$ for $1 \le j \le \ell$, then $\tilde{\psi}_j$ belongs to $M(E)$. Since I is an ideal of A, $I \subset \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j$. For let $f \in A$ such that $\psi_j(f) \ne \psi'_j(f)$. If $g \in I$, then $fg \in I$ but $\psi_j(fg) \ne \psi'_j(fg)$ when $\tilde{\psi}_j(g) \ne 0$. This contradicts that $fg \in E$. Thus $\tilde{\psi}_j(g) = 0$. Hence $I \subset \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j$ and so $H \subseteq \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j + \mathbb{C}$. By definition of E, $\tilde{\psi}_1 = \cdots = \tilde{\psi}_\ell$. Therefore E has the form : $E = \{f \in A \ ; \ \phi_1(f) = \cdots = \phi_n(f)\}$, $1 \le n < \infty$ and $\{\phi_j\} \subset M(A)$.

For each j with $1 \le j \le k$, H_j is a θ_j-subalgebra of E for $\theta_j \in M(E)$. Hence there is a sequence of closed subalgebras $E = E_{j0} \supset E_{j1} \supset \cdots \supset E_{j\ell_j} = H_j$ such that E_{jt} is the kernel of a continuous point derivation D_{jt} of E_{jt-1} at θ_j. We will write $E_{j\ell_j} = \ker D_{\theta_j}$ where D_{θ_j} is a derivation on $E_{j(\ell_j-1)}$. Then $H = \bigcap_{j=1}^{k} \ker D_{\theta_j}$ and so $I = \left\{ \bigcap_{j=1}^{k} \ker D_{\theta_j} \right\} \cap \ker \theta$ for some $\theta \in M(H)$. Suppose g is an arbitrary function in I. For any $j (1 \le j \le k)$, there exists a function $f \in E_{j(\ell_j-1)}$ such that $f \notin E_{j\ell_j} = \ker D_{\theta_j}$. Since $fg \in I$ and $D_{\theta_j}(g) = 0$, $D_{\theta_j}(fg) = \theta_j(g) D_{\theta_j}(f) = 0$ because D_{θ_j} is a derivation on $E_{j(\ell_j-1)}$. This implies that $\theta_j(g) = 0$. Hence $I \subset \bigcap_{j=1}^{k} \ker \theta_j$. Therefore by definition of E, $\theta_1 = \cdots = \theta_k \in M(E)$ and so $H_1 = \cdots = H_k$. Thus $\theta_1|H = \theta$ and $I = (\ker D_{\theta_1}) \cap \ker \theta_1$. Since $I \subset \bigcap_{j=1}^{n} \ker \phi_j$, $I \subset (\ker \phi_1) \cap (\ker D_{\theta_1} \cap \ker \theta_1)$ and so $\phi_1|E = \theta_1$.

Corollary 1. *If I is a closed ideal of A and A/I is of finite dimension 2, then $I = \{f \in A \ ; \ \phi_1(f) = \phi_2(f) = 0\}$ where $\phi_j \in M(A)$ $(j = 1, 2)$ and $\phi_1 \ne \phi_2$, or $I = \{f \in A \ ; \ \phi(f) = D_\phi(f) = 0\}$ where $\phi \in M(A)$ and D_ϕ is a bounded point derivation at ϕ.*

Proof. When $\dim A/I = 2$, by Theorem 1 $E = A$ or $E = \{f \in A \ ; \ \phi_1(f) = \phi_2(f)\}$. If $E = A$, then $H^E_\phi = \{f \in A \ ; \ D_\phi(f) = 0\}$ and if $E = \{f \in A \ ; \ \phi_1(f) = \phi_2(f)\}$, then $H^E_\phi = E$. For $\dim A/H^E_\phi = 1$ because $H^E_\phi = I + \mathbb{C}$. This implies the corollary.

Corollary 2. *If B is a finite dimensional Q-algebra and $B_0 =$* rad B *is its radial, then there exist subalgebras $B_1, B_2, \cdots, B_{k+1}$ in B_0 such that $B_{k+1} = \{0\}$,* $\dim B_j/B_{j+1} =$ *1 and B_{j+1} is an ideal of B_j for $j = 0, 1, \cdots, k$. Hence* rad B *has a basis $\{f_0, f_1, \cdots, f_k\}$ such that $(f_j)^{2((k+1)-j)} = 0$ for $j = 0, 1, \cdots, k$.*

Proof. Since B is a Q-algebra, $B = A/I$ for some uniform algebra A and some closed ideal I of A. Since B is of finite dimension, we can apply Theorem 1 to A and I. In the notation of Theorem 1, rad $B = \{f \in E \ ; \ \phi(f) = 0\}/I$. Since H_ϕ^E is a ϕ-subalgebra with respect to E, there exists a sequence of closed subalgebras $E = E_0 \supsetneq E_2 \supsetneq \cdots \supsetneq E_{k+1} = H_\phi^E$ such that E_j is the kernel of a continuous point derivation D_j of E_{j-1} at ϕ. Hence $E_{j+1} \cap \ker\phi$ is an ideal of $E_j \cap \ker\phi$ and $\dim\{E_j \cap \ker\phi / E_{j+1} \cap \ker\phi\} = 1$. Put $B_j = (E_j \cap \ker\phi)/I$. Then $\dim B_j/B_{j+1} = 1$ and B_{j+1} is an ideal of B_j for $j = 0, 1, \cdots, k$ and $B_{k+1} = \{0\}$. For each j, there exists f_j such that $B_j = \langle f_j \rangle + B_{j+1}$ and then $\{f_0, f_1, \cdots, f_k\}$ is a basis of rad $B = B_0$. Since f_j^2 belongs to B_{j+1} because $E_{j+1} = \ker D_{j+1}$. Thus $(f_j)^{2(k+1-j)} = 0$.

§3. Finite codimensional invariant subspace

Let m be a probability measure on X. For $1 \leq p < \infty$, $H^p(m)$ denotes the norm closure of A in $L^p(m)$ and $H^\infty(m)$ denotes the weak $*$ closure of A in $L^\infty(m)$. When M is a closed subspace of $H^p(m)$ and $AM \subset M$, M is called an invariant subspace. For a subset S of $H^p(m)$, $[S]_p$ denotes the closure of S in $H^p(m)$.

Theorem 2. *If M is an invariant subspace of H^p with $\dim H^p/M = n < \infty$, then there exists a closed ideal of A such that $\dim A/I = n$, $[I]_p = M$ and $I = M \cap A$. If H_ϕ^E is a (ϕ, k)-subalgebra with respect to $E = E(I)$, then $[E_j]_p \supsetneq [E_{j+1}]_p$ for any $j (0 \leq j \leq k-1)$ and $\dim H^p/[E]_p = \dim A/E$. Conversely if $\dim A/I = n < \infty$ then $\dim H^p/[I]_p \leq n$. If $[E_j]_p \supsetneq [E_{j+1}]_p$ for any $j (0 \leq j \leq k-1)$ and $\dim H^p/[E]_p = \dim A/E$, then $\dim H^p/[I]_p = n$ and $[I]_p \cap A = I$.*

Proof. Suppose that M is an invariant subspace of $H^p(m)$ and $\dim H^p(m)/M = n < \infty$. Then there exist n linear independent linear functionals $\psi_1, \cdots, \psi_n$ in $(H^p)^*$ such that $\psi_j = 0$ on M for $1 \leq j \leq n$. Put $\phi_j = \psi_j|A$ for $1 \leq j \leq n$ and $I = M \cap A$, then $I = \bigcap_{j=1}^{n} \ker\phi_j$ and so $\dim A/I = n$. For $\phi_1, \cdots, \phi_n$ are independent linear functionals in A^* because A is dense in $H^p(m)$. If $M \supsetneq [I]_p$, then there exists $\psi_{n+1} \in H^p(m)^*$ such that $\psi_{n+1} = 0$ on $[I]_p$ and $\psi_1, \cdots, \psi_n, \ \psi_{n+1}$ are independent linear functionals in $(H^p)^*$. If put $\phi_{n+1} = \psi_{n+1}|A$, then $\phi_1, \cdots, \phi_n, \ \phi_{n+1}$ are independent linear functionals in A^* and $I \subseteq \bigcap_{j=1}^{n+1} \ker\phi_j$. This contradiction implies that $M = [I]_p$. Note that $\dim H^p/[E_k] = \dim H^p/[I]_p - 1 = \dim A/I - 1 = \dim A/E_k$. If $\dim H^p/[E_0]_p \lneq \dim A/E_0$ where $E_0 = E$ or $[E_j]_p = [E_{j+1}]_p$ for some $j \ (0 \leq j \leq k-1)$, then this contradicts that $\dim H^p/[E_k]_p = \dim A/E_k$. The converse is clear.

Corollary 3. *If M is an invariant subspace of H^p with $\dim H^p/M = 2$, then $M = \{f \in H^p \ ; \ \Phi_1(f) = \Phi_2(f) = 0\}$ where $\Phi_j \in (H^p)^*$, and $\Phi_j(fg) = \Phi_j(f)\Phi_j(g)$ for*

$f \in H^p$ and $g \in A$, or $M = \{f \in H^p \ ; \ \Phi(f) = D_\phi(f) = 0\}$ where $\Phi, D_\Phi \in (H^p)^*$, and $\Phi(fg) = \Phi(f)\Phi(g)$ and $D_\Phi(fg) = \Phi(f)D_\Phi(g) + \Phi(g)D_\Phi(f)$ for $f \in H^p$ and $g \in A$.

Proof. This is a result of Corollary 1 and Theorem 2.

Corollary 4. *If M is an invariant subspace of H^p with $\dim H^p/M = n < \infty$, then there exist $f_1, \cdots, f_n$ in A such that $\{f_j + M\}_{j=1}^n$ is a basis in H^p/M.*

Proof. By Theorem 2, if $I = M \cap A$ then $\dim A/I = n$ and $M = [I]_p$. Hence there exist $f_1, \cdots, f_n$ in A such that $\{f_j + I\}_{j=1}^n$ is a basis in A/I. If f_j belongs to M, then f_j also belongs to $M \cap A = I$ and so f_j does not belong to M. This shows the corollary.

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