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# On solutions of the wave equation with homogeneous Cauchy data

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## Abstract

In this article, the behavior of solutions to the free wave equation with homogeneous Cauchy data are considered. In particular, the propagation of singularities are observed explicitly. Such Cauchy data are of special interest in view of applications to self-similar solutions to nonlinear wave equations.

## 1 Introduction

We consider the Cauchy problem of the free wave equation

$$(FW) \quad \begin{cases} u_{tt} - \Delta u = 0, & (t, x) \in (0, \infty) \times \mathbf{R}^n \equiv \mathbf{R}_+^{1+n}, \\ u|_{t=0} = \phi, \quad u_t|_{t=0} = \psi, & x \in \mathbf{R}^n, \end{cases}$$

with data given by homogeneous functions such as

$$\phi(x) = |x|^{-p}, \quad \psi(x) = |x|^{-p-1}. \quad (1.1)$$

These initial data are of special interest in view of the applications to self-similar solutions of wave equations with power type nonlinearity [2, 7, 8, 9] (see also [1]):

$$u_{tt} - \Delta u = |u|^\alpha, \quad (t, x) \in \mathbf{R}_+^{1+n}. \quad (1.2)$$

In this paper we study explicit behavior of solutions to  $FW$  with special attention on the propagation of singularity.

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More precisely, a typical estimate to be shown takes the form

$$|u(t, x)| \leq C(t + |x|)^{-\frac{n-1}{2}} |t - |x||^{-p+\frac{n-1}{2}}, \quad (t, x) \in \mathbf{R}_+^{1+n}, \quad (1.3)$$

for the initial data such as (1.1), where  $p > \frac{n-1}{2}$ . This estimate shows that the singularity of the initial data at the origin propagates along the light cone with specific order there. Moreover, we observe that the order of singularity of solutions is less than that of the corresponding initial data for each  $t > 0$ . Estimate (1.3) also implies the following integrability

$$u(t, \cdot) \in L^r(\mathbf{R}^n), \quad t > 0 \quad (1.4)$$

for some  $r$ . This fact has an interest because  $u(0, \cdot) = \phi \notin L^r(\mathbf{R}^n)$  for any  $r$ .

Estimates of the form (1.3), (1.4) for solutions of  $FW$  with homogeneous initial data such as (1.1) have been used to construct self-similar solutions to (1.2). We call  $u$  a self-similar solution of (1.2) if  $u$  satisfies

$$u(t, x) = \lambda^{\frac{2}{\alpha-1}} u(\lambda t, \lambda x), \quad (t, x) \in \mathbf{R}_+^{1+n} \quad (1.5)$$

for all  $\lambda > 0$ . By the condition (1.5) initial data of self-similar solutions must be homogeneous functions such as (1.1).

In fact, Pecher [7] proved (1.4) for  $n = 3$  to construct self-similar solutions satisfying

$$\sup_{t>0} t^\mu \|u(t)\|_{L^r} < \infty$$

for suitable  $\mu, r$ . The case  $n \geq 2$  is treated by Ribaud-Youssfi [9], where they showed (1.4) for a class of initial data in terms of some homogeneous Besov spaces containing homogeneous functions such as (1.1). Pecher [8] also proved (1.3) for  $n = 3$  to construct self-similar solutions satisfying

$$\sup_{|x| \neq t} (t + |x|) |t - |x||^{\frac{2}{p-1}-1} |u(t, x)| < \infty.$$

Our estimate (1.3) seems to give a foundation to generalize the last result for higher dimensions.

Our method to obtain the estimate such as (1.3) is based on Fourier representation of solutions of  $FW$  and some results on oscillatory integrals. More precisely, we divide the representation into high frequency part and low frequency part. As we shall see below, the high frequency part contributes to the formation of singularity along the light cone and the low frequency part to the decay rate as  $|x| \rightarrow \infty$ .

To investigate the behavior of high frequency part of solutions we use the asymptotic expansion of oscillatory integrals over the unit sphere. The method here is essentially due to Miyachi [5], where the boundedness, together with unboundedness,

of some Fourier multipliers associated with the wave equation is proved. Concerning the low frequency part, we also consider an oscillatory integral over the sphere as above and derive its decay estimate via stationary phase method.

We remark that the estimate (1.3) is connected with the *dispersive inequality*

$$|u(t, x)| \leq C(1 + t + |x|)^{-\frac{n-1}{2}} \|\widehat{f}\|_{W^{n-1,1}(\mathbf{R}^n)}$$

for the data  $f$  satisfying  $\text{supp } \widehat{f} \subset \{1/2 < |\xi| < 2\}$ , which is also based on the estimate by the stationary phase method. (See [4] and references there in.) It is known that the Strichartz type inequalities, which play a crucial role in many recent advances of the theory of nonlinear wave equations, are obtained on the basis of the dispersive inequality.

The contents of the paper are as follows. The main results are presented and proved in the next section. Results of the types (1.3) and (1.4) are stated in Theorems 2.2 and 2.3, respectively. Theorems 2.2 and 2.3 follow from Theorem 2.1. The basic estimates on oscillatory integrals to be used in the proof of Theorem 2.1 are summarized in the appendix without proof (see [5, 6, 10] for details). Part of the contents of this paper has been announced in [3].

## 2 Main Results

In this section we give the estimates such as (1.3), (1.4) for solutions of  $FW$  with homogeneous functions as initial data. We consider the solution of  $FW$  given by

$$u(t) = U(t)\phi + V(t)\psi,$$

where

$$U(t)\phi = \mathcal{F}^{-1}[\cos t|\xi| \widehat{\phi}(\xi)], \quad V(t)\psi = \mathcal{F}^{-1}[|\xi|^{-1} \sin t|\xi| \widehat{\phi}(\xi)].$$

Here  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the Fourier transform and its inverse, respectively and  $\widehat{\phi} = \mathcal{F}\phi$ . Our main results are as follows.

**Theorem 2.1.** *Let  $n \geq 2$  and let  $0 < p < \min\{\frac{n+1}{2}, n-1\}$ . We set  $m = \max\{2N + 2, [\frac{n-1}{2}] + N + 2\}$ , where  $N = \max\{[p - \frac{n+1}{2}] + l + 1, -1\}$  for  $l \in \mathbf{N} \cup \{0\}$ . We assume that*

$$\phi, \psi \in C^{n+m-[p]-}(\mathbf{R}^n \setminus \{0\})$$

*are homogeneous of degree  $-p$ ,  $-p-1$ , respectively. Then, for any  $t > 0$*

$$U(t)\phi, V(t)\psi \in C^l(\mathbf{R}^n \setminus \{|x| = t\}),$$

with the decay order

$$U(t)\phi = O(|x|^{-p}), \quad V(t)\psi = O(|x|^{-p-1}) \quad \text{as } |x| \rightarrow \infty.$$

Moreover, the following asymptotic behavior holds.

(1) For  $\frac{n-1}{2} < p < \frac{n+1}{2}$ ,

$$\begin{aligned} (U(t)\phi)(x) &= C_p t^{-\frac{n-1}{2}} \left\{ e^{\frac{p\pi}{2}i} \widehat{\phi}(-x/|x|)(t - |x| + i0)^{-p+\frac{n-1}{2}} \right. \\ &\quad \left. + e^{-\frac{p\pi}{2}i} \widehat{\phi}(x/|x|)(t - |x| - i0)^{-p+\frac{n-1}{2}} \right\} + o(|t - |x||^{-p+\frac{n-1}{2}}), \end{aligned}$$

$$\begin{aligned} (V(t)\psi)(x) &= -i C_p t^{-\frac{n-1}{2}} \left\{ e^{\frac{p\pi}{2}i} \widehat{\psi}(-x/|x|)(t - |x| + i0)^{-p+\frac{n-1}{2}} \right. \\ &\quad \left. - e^{-\frac{p\pi}{2}i} \widehat{\psi}(x/|x|)(t - |x| - i0)^{-p+\frac{n-1}{2}} \right\} + o(|t - |x||^{-p+\frac{n-1}{2}}), \end{aligned}$$

as  $|x| \rightarrow t$ , where  $C_p = (8\pi)^{-1/2} \Gamma(p - \frac{n-1}{2})$ .

(2) For  $p = \frac{n-1}{2}$ ,

$$\begin{aligned} (U(t)\phi)(x) &= (8\pi)^{-\frac{1}{2}} t^{-\frac{n-1}{2}} \left\{ e^{\frac{n-1}{4}\pi i} \widehat{\phi}(-x/|x|) \log t / (t - |x| + i0) \right. \\ &\quad \left. + e^{-\frac{n-1}{4}\pi i} \widehat{\phi}(x/|x|) \log t / (t - |x| - i0) \right\} + O(1), \end{aligned}$$

$$\begin{aligned} (V(t)\psi)(x) &= -i (8\pi)^{-\frac{1}{2}} t^{-\frac{n-1}{2}} \left\{ e^{\frac{n-1}{4}\pi i} \widehat{\psi}(-x/|x|) \log t / (t - |x| + i0) \right. \\ &\quad \left. - e^{-\frac{n-1}{4}\pi i} \widehat{\psi}(x/|x|) \log t / (t - |x| - i0) \right\} + O(1), \end{aligned}$$

as  $|x| \rightarrow t$ .

(3) For  $0 < p < \frac{n-1}{2}$ ,

$$U(t)\phi, V(t)\psi \in C^{[\frac{n-1}{2}-p]_-}(\mathbf{R}^n) \cap C^l(\mathbf{R}^n \setminus \{|x| = t\}),$$

where  $[s]_- = [s]$  if  $s \in \mathbf{R}_+ \setminus \mathbf{N}$ ,  $[s]_- = [s] - 1$  if  $s \in \mathbf{N}$ , and  $[s]$  is an integer part of  $s$ .

**Remark 2.1.** (1) The condition  $p < \frac{n+1}{2}$  is related to local integrability in space of solutions.

**Theorem 2.2.** Let  $n \geq 2$  and let  $0 < p < \min\{\frac{n+1}{2}, n-1\}$ . For  $m = [\frac{n-1}{2}] + N + 2$  with  $N = \max\{[p - \frac{n+1}{2}] + 1, -1\}$ , we assume that

$$\phi, \psi \in C^{n+m-[p]_-}(\mathbf{R}^n \setminus \{0\})$$

are homogeneous of degree  $-p$ ,  $-p-1$ , respectively. Then, for any  $t > 0$  we have the following estimates.

(1) For  $\frac{n-1}{2} < p < \frac{n+1}{2}$ ,

$$\begin{aligned} |(U(t)\phi)(x)| &\leq C(t+|x|)^{-\frac{n-1}{2}} |t-|x||^{-p+\frac{n-1}{2}}, \\ |(V(t)\psi)(x)| &\leq Ct(t+|x|)^{-\frac{n+1}{2}} |t-|x||^{-p+\frac{n-1}{2}}. \end{aligned}$$

(2) For  $p = \frac{n-1}{2}$ ,

$$\begin{aligned} |(U(t)\phi)(x)| &\leq C(t+|x|)^{-\frac{n-1}{2}} (1 + \log^+ t/|t-|x||), \\ |(V(t)\psi)(x)| &\leq Ct(t+|x|)^{-\frac{n+1}{2}} (1 + \log^+ t/|t-|x||), \end{aligned}$$

where  $\log^+ s = \max(\log s, 0)$ .

(3) For  $0 < p < \frac{n-1}{2}$ ,

$$|(U(t)\phi)(x)| \leq C(t+|x|)^{-p}, \quad |(V(t)\psi)(x)| \leq Ct(t+|x|)^{-p-1},$$

**Remark 2.2.** As for the application to self-similar solutions for nonlinear wave equations (1.2), the most interesting case is (1) in Theorem 2.2 with  $p = 2/(\alpha - 1)$ , see [7] for example. In this case the required regularity is

$$\phi, \psi \in C^{n+2}(\mathbf{R}^n \setminus \{0\}).$$

**Theorem 2.3.** Let  $n \geq 2$  and let  $0 < p < \frac{n}{2}$ . We assume that  $\phi$  and  $\psi$  satisfy the assumptions in Theorem 2.2. Then, for any  $t > 0$  we have

$$\begin{aligned} U(t)\phi &\in L^r(\mathbf{R}^n) \quad \text{if} \quad \max(p - \frac{n-1}{2}, 0) < \frac{1}{r} < \frac{p}{n}, \\ U(t)\phi &\in L^\infty(\mathbf{R}^n) \quad \text{if} \quad 0 < p < \frac{n-1}{2}, \\ V(t)\psi &\in L^r(\mathbf{R}^n) \quad \text{if} \quad \max(p - \frac{n-1}{2}, 0) < \frac{1}{r} < \frac{p+1}{n}, \\ V(t)\psi &\in L^\infty(\mathbf{R}^n) \quad \text{if} \quad 0 < p < \frac{n-1}{2}. \end{aligned}$$

Moreover, if  $\max(p - \frac{n-1}{2}, 0) < \frac{1}{r} < \frac{p}{n}$ , then

$$\sup_{t>0} t^{p-n/r} \|u(t)\|_{L^r(\mathbf{R}^n)} < \infty,$$

where  $u(t) = U(t)\phi + V(t)\psi$ .

**Remark 2.3.** (1) The condition  $p < \frac{n}{2}$  is necessary for the existence of  $1/r$  over the intervals above.

(2) Below we prove Theorem 2.3 using Theorem 2.2. In the case  $\frac{n-1}{2} < p < \frac{n}{2}$ , a direct proof works under the the following weaker assumption

$$\phi \in C^{n+2}(\mathbf{R}^n \setminus \{0\}), \quad \psi \in C^{n+1}(\mathbf{R}^n \setminus \{0\}).$$

The proof proceeds as follows.

We decompose the Fourier multipliers  $U(t)\phi$  and  $V(t)\psi$  into high frequency part and low frequency part as in the proof of Theorem 2.1 below. As for high frequency part, an application of Hörmander-Mihlin multiplier theorem to the multiplier  $\widehat{\phi}(\xi/|\xi|)$ ,  $\widehat{\psi}(\xi/|\xi|)$  reduces the problem to the radially symmetric case, where necessary calculations are carried out explicitly and everything is smooth. On the regularity of  $\widehat{\phi}$ ,  $\widehat{\psi}$ , see Lemma 2.1. As for low frequency part, the Hardy-Littlewood-Sobolev inequality enables us to obtain the desired result.

(3) Theorem 2.3 is optimal in the sense that regarding the exponent  $r$  the converses in the statements of the theorem hold. See [3].

We first collect the elementary lemmas.

**Lemma 2.1.** *Let  $0 < p < n$  and let  $k \in \mathbf{N}$  satisfy  $k - n + p > 0$ . If  $f \in C^k(\mathbf{R}^n \setminus \{0\})$  is homogeneous of degree  $-p$ , then  $\widehat{f}$ , Fourier transform of  $f$  in  $\mathcal{S}'(\mathbf{R}^n)$ , is homogeneous of degree  $-n + p$  and*

$$\widehat{f} \in C^{k-n+[p]-}(\mathbf{R}^n \setminus \{0\}),$$

where  $\mathcal{S}'$  is the space of tempered distributions.

*Proof.* It is well known that  $\widehat{f}$  is homogeneous of degree  $-n + p$ . So we show the regularity of  $\widehat{f}$  away from the origin. We set  $\rho$  be a smooth cut-off function such that  $0 \leq \rho \leq 1$ ,  $\rho(\xi) = 1$  if  $|\xi| \leq 1$ ,  $\rho(\xi) = 0$  if  $|\xi| \geq 2$ , and divide  $\widehat{f}$  using  $\rho$ :

$$\widehat{f} = \mathcal{F}[\rho f] + \mathcal{F}[(1 - \rho)f].$$

It is clear that  $\mathcal{F}[\rho f] \in C^\infty(\mathbf{R}^n)$ , so we consider the regularity of  $\mathcal{F}[(1 - \rho)f]$ .

We first consider the case  $p \notin \mathbf{N}$ . For multi-indices  $\alpha, \beta$  with

$$|\alpha| \leq k - n + [p], \quad |\beta| = k,$$

we have

$$\xi^\beta \partial^\alpha ((1 - \rho)f)^\widehat{=} = C_{\alpha\beta} \{(1 - \rho) \partial^\beta (x^\alpha f)\} + g_{\alpha\beta},$$

for a suitable constant  $C_{\alpha\beta}$  and  $g_{\alpha\beta} \in \mathcal{S}(\mathbf{R}^n)$ , the space of smooth functions of rapid decrease. Then, we observe that  $\partial^\beta (x^\alpha f)$  is a homogeneous function of degree  $-p + |\alpha| - k$ . This implies that  $(1 - \rho) \partial^\beta (x^\alpha f) \in L^1(\mathbf{R}^n)$ , since

$$-p + |\alpha| - k \leq -p + [p] - n < -n.$$

Therefore, we conclude that  $\xi^\beta \partial^\alpha \mathcal{F}[(1 - \rho)f]$  is identified with a continuous function on  $\mathbf{R}^n$  as long as  $|\alpha| \leq k - n + [p]$  and thus  $\widehat{f} \in C^{k-n+[p]}(\mathbf{R}^n \setminus \{0\})$ .

In the case  $p \in \mathbf{N}$ , we conclude that  $\widehat{f} \in C^{k-n+p-1}(\mathbf{R}^n \setminus \{0\})$  in the same manner as above, where the regularity is determined by the range of  $\alpha$  that is to be  $|\alpha| \leq k - n + p - 1$  so that  $-p + |\alpha| - k \leq -n - 1$ .  $\square$

Now we define the dilation operator  $D_{\lambda,p}$  by

$$(D_{\lambda,p}u)(t,x) = \lambda^p u(\lambda t, \lambda x), \quad t > 0, x \in \mathbf{R}^n$$

for  $\lambda, p > 0$ .

**Lemma 2.2.** *Let  $n \geq 2$  and let  $0 < p < n - 1$ . We assume that  $\phi, \psi \in \mathcal{S}'(\mathbf{R}^n)$  are homogeneous of degree  $-p, -p - 1$ , respectively. Then we have in  $\mathcal{S}'(\mathbf{R}^n)$ ,*

$$D_{\lambda,p}U(t)\phi = U(t)\phi, \quad D_{\lambda,p}V(t)\psi = V(t)\psi \quad (2.1)$$

for each  $t > 0$ .

**Remark 2.4.** *In particular if we take  $\lambda = 1/t$  in (2.1), then we obtain*

$$(U(t)\phi)(x) = t^{-p}(U(1)\phi)(x/t), \quad (V(t)\psi)(x) = t^{-p}(V(1)\psi)(x/t).$$

*Proof of Lemma 2.2.* This lemma follows by the relation with dilation operators and Fourier transform, together with the homogeneity of  $\phi, \psi$ . We only prove the second equality of (2.1), since the first one follows similarly. We first notice that

$$D_{\lambda,p}V(t)\psi = \lambda^p D_{\lambda} \mathcal{F}^{-1} [|\xi|^{-1} \sin(\lambda t|\xi|) \widehat{\psi}(\xi)], \quad (2.2)$$

where  $D_{\lambda}f(x) = f(\lambda x)$ . The right hand side of (2.2) equals to

$$\lambda^p \mathcal{F}^{-1} [\lambda^{-n} D_{1/\lambda} \{|\xi|^{-1} \sin(\lambda t|\xi|) \widehat{\psi}(\xi)\}],$$

by the relation with dilation operators and Fourier transform. This completes the proof, since  $\widehat{\psi}$  is homogeneous of degree  $-n + p + 1$ .  $\square$

Now we consider the behavior of the Fourier transform of functions associated with solutions to  $FW$ . Let  $a$  be a homogeneous function of degree  $-\lambda$ . We set

$$\begin{aligned} K_{\varepsilon}^{\pm} a &= \mathcal{F}^{-1} [e^{-\varepsilon|\xi|} \eta(\xi) a(\xi) e^{\pm i|\xi|}], \\ K_s a &= \mathcal{F}^{-1} \left[ \rho(\xi) a(\xi) \frac{\sin |\xi|}{|\xi|} \right], \\ K_c a &= \mathcal{F}^{-1} [\rho(\xi) a(\xi) \cos |\xi|], \end{aligned}$$

where  $\eta \in C^{\infty}(\mathbf{R}^n)$  is a radial function satisfying  $0 \leq \eta \leq 1$ ,  $\eta(\xi) = 0$  if  $|\xi| \leq 1$ ,  $\eta(\xi) = 1$  if  $|\xi| \geq 2$ , and  $\rho = 1 - \eta$ . We notice that  $K_0^{\pm} a = \lim_{\varepsilon \downarrow 0} K_{\varepsilon}^{\pm} a$  corresponds to a high frequency part of solutions to  $FW$  and  $K_s a, K_c a$  a low frequency part.



**Proposition 2.1.** For  $\lambda > 0$  and  $k, l \in \mathbf{N} \cup \{0\}$ , we set  $m = \max\{2N + 2, [\frac{n-1}{2}] + N + k + 2\}$ , where  $N = \max\{[\frac{n-1}{2} - \lambda] + l + 1, -1\}$ . If  $a \in C^m(\mathbf{R}^n \setminus \{0\})$  is homogeneous of degree  $-\lambda$ , then

$$K_0^\pm a \equiv \lim_{\varepsilon \downarrow 0} K_\varepsilon^\pm a \in C^l(\mathbf{R}^n \setminus S^{n-1})$$

with the decay order

$$K_0^\pm a = O(|x|^{-\frac{n+1}{2} - N - k}) \quad \text{as } |x| \rightarrow \infty,$$

where the convergence is uniform in  $\{x \in \mathbf{R}^n; ||x| - 1| \geq \delta\}$  for any  $\delta > 0$ .

Moreover, the behavior of  $K_0^\pm a$  near the unit sphere is given as follows.

(1) If  $0 < \lambda < \frac{n+1}{2}$ , then

$$K_0^\pm a(x) = A_\lambda^\pm a(\mp x/|x|)(1 - |x| \pm i0)^{-\frac{n+1}{2} + \lambda} + o(|1 - |x||^{-\frac{n+1}{2} + \lambda}) \quad (2.3)$$

as  $|x| \rightarrow 1$ , where  $A_\lambda^\pm = (2\pi)^{-\frac{1}{2}} e^{\pm \frac{n-\lambda}{2} \pi i} \Gamma(\frac{n+1}{2} - \lambda)$ .

(2) If  $\lambda = \frac{n+1}{2}$ , then

$$K_0^\pm a(x) = A^\pm a(\mp x/|x|) \log 1/(1 - |x| \pm i0) + O(1), \quad (2.4)$$

as  $|x| \rightarrow 1$ , where  $A^\pm = (2\pi)^{-\frac{1}{2}} e^{\pm \frac{(n-1)\pi}{4} i}$ .

(3) If  $\lambda > \frac{n+1}{2}$ , then

$$K_0^\pm a \in C^{[\lambda - \frac{n+1}{2}]_-}(\mathbf{R}^n) \cap C^l(\mathbf{R}^n \setminus S^{n-1}).$$

This proposition is essentially due to Miyachi [4; Proposition 2], and therefore we describe the outline of the proof here using lemmas in the appendix.

*Outline of Proof.* We first consider the case  $0 \leq \lambda \leq \frac{n+1}{2}$ . Then  $N = [\frac{n-1}{2} - \lambda] + l + 1$ , and for  $|\alpha| \leq l$ ,  $|x| \geq 1/4$ ,

$$\begin{aligned} \partial^\alpha K_\varepsilon^+ a(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{ix \cdot \xi + i|\xi| - \varepsilon|\xi|} \eta(\xi) (i\xi)^\alpha a(\xi) d\xi \\ &= (2\pi)^{-\frac{n}{2}} e^{\frac{|\alpha|}{2} \pi i} \int_0^\infty e^{-\varepsilon s + is} \eta(s) s^{n-\lambda+|\alpha|-1} \left( \int_{S^{n-1}} e^{isx \cdot \theta} \theta^\alpha a(\theta) d\sigma(\theta) \right) ds \\ &= (2\pi)^{-\frac{n}{2}} e^{\frac{|\alpha|}{2} \pi i} \left\{ \sum_{j=0}^N a_j^+(x/|x|) |x|^{-\frac{n-1}{2}-j} \int_0^\infty e^{-\varepsilon s + i(1+|x|)s} \eta(s) s^{-\lambda + \frac{n-1}{2} + |\alpha| - j} ds \right. \\ &\quad + \sum_{j=0}^N a_j^-(-x/|x|) |x|^{-\frac{n-1}{2}-j} \int_0^\infty e^{-\varepsilon s + i(1-|x|)s} \eta(s) s^{-\lambda + \frac{n-1}{2} + |\alpha| - j} ds \\ &\quad \left. + \int_0^\infty e^{-\varepsilon s + is} \eta(s) s^{n-\lambda+|\alpha|-1} R_N(|x|s, x/|x|) ds \right\}, \end{aligned}$$

where we have used the asymptotic expansion of order  $N$  for oscillatory integrals on the unit sphere, see Lemma A.1 in the appendix. Letting  $\varepsilon \downarrow 0$ , we obtain

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \partial^\alpha K_\varepsilon^+ a(x) &= (2\pi)^{-\frac{n}{2}} e^{\frac{|\alpha|}{2}\pi i} \left\{ \sum_{j=0}^N a_j^+(x/|x|) |x|^{-\frac{n-1}{2}-j} I_{-\lambda+\frac{n+1}{2}+|\alpha|-j}(1+|x|) \right. \\ &\quad + \sum_{j=0}^N a_j^-(x/|x|) |x|^{-\frac{n-1}{2}-j} I_{-\lambda+\frac{n+1}{2}+|\alpha|-j}(1-|x|) \\ &\quad \left. + \int_0^\infty e^{is} \eta(s) s^{n-\lambda+|\alpha|-1} R_N(|x|s, x/|x|) ds \right\}, \end{aligned} \quad (2.5)$$

where we have set

$$I_\mu(\tau) = \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{-\varepsilon s + is\tau} \eta(s) s^{\mu-1} ds.$$

By lemma A.4 the convergence in (2.5) is uniform for each  $\alpha$  in  $\{x; |1-|x|| \geq \delta, |x| \geq 1/4\}$  for any  $\delta > 0$ , and thus the regularity of  $K_0^+$  follows away from the unit sphere and the origin. The regularity near the origin follows from the following representation

$$\lim_{\varepsilon \downarrow 0} \partial^\alpha K_\varepsilon^+ a(x) = (2\pi)^{-\frac{n}{2}} \int_{S^{n-1}} (i\theta)^\alpha a(\theta) I_{n-\lambda+|\alpha|}(1+x \cdot \theta) d\sigma(\theta),$$

since the above convergence is uniform in  $\{|x| \leq 1/2\}$ . In fact, we have  $|1-x \cdot \theta| \geq 1/2$  for  $\theta \in S^{n-1}$ ,  $|x| \leq 1/2$ .

The asymptotic behavior of  $K_0^+ a$  also follows from (2.5) with  $\alpha = 0$ . In fact, the behavior as  $|x| \rightarrow 1$  and as  $|x| \rightarrow \infty$  is dominated by the term involving  $I_{-\lambda+\frac{n+1}{2}}(1-|x|)$  and  $R_N(|x|s, x/|x|)$ , respectively. See Lemma A.1, A.4. In particular, integration by parts works well to derive more decay, see Remark A.1(1).

We next consider the case  $\lambda > \frac{n+1}{2}$ . In this case,

$$K_0^+ a(x) = (2\pi)^{-\frac{n}{2}} \int_0^\infty e^{is} \eta(s) s^{n-\lambda-1} \left( \int_{S^{n-1}} e^{isx \cdot \theta} a(\theta) d\sigma(\theta) \right) ds,$$

from which we obtain  $K_0^+ a \in C^{[\lambda-\frac{n+1}{2}]_-}(\mathbf{R}^n)$ , since

$$\left| \int_{S^{n-1}} e^{isx \cdot \theta} a(\theta) d\sigma(\theta) \right| \leq C \|a\|_{C^{[\frac{n-1}{2}]_+1}(S^{n-1})} (s|x|)^{-\frac{n-1}{2}},$$

by Lemma A.2, A.3. Moreover, if  $l$  satisfies  $N = [\frac{n-1}{2} - \lambda] + l + 1 \geq 0$ , the regularity and the decay are obtained as in the first case. In the case  $N = -1$ , to derive the decay we decompose  $K_0^+ a$  into three parts as

$$\begin{aligned} K_0^+ a(x) &= (2\pi)^{-\frac{n}{2}} \left\{ \int_0^\infty e^{i(1+|x|)s} \eta(s) s^{n-\lambda-1} A^+(|x|s, x/|x|) ds \right. \\ &\quad + \int_0^\infty e^{i(1-|x|)s} \eta(s) s^{n-\lambda-1} A^-(|x|s, x/|x|) ds \\ &\quad \left. + \int_0^\infty e^{is} \eta(s) s^{n-\lambda-1} A^0(|x|s, x/|x|) ds \right\}, \end{aligned}$$

where

$$A^\pm(r, \omega) = \int_{S^{n-1}} e^{ir(\omega \cdot \theta \mp 1)} \varphi^\pm(\theta) a(\theta) d\sigma(\theta),$$

$$A^0(r, \omega) = \int_{S^{n-1}} e^{ir\omega \cdot \theta} \varphi^0(\theta) a(\theta) d\sigma(\theta),$$

for  $r > 0$ ,  $\omega \in S^{n-1}$ . Here  $\varphi^\pm \in C_0^\infty(\mathbf{R}^n)$  satisfy  $0 \leq \varphi^\pm \leq 1$  with  $\varphi^\pm \equiv 1$  near  $\pm\omega$  and are supported in small neighborhoods of  $\pm\omega$ , respectively, and we set  $\varphi^0(\theta) = 1 - \varphi^+(\theta) - \varphi^-(\theta)$ . This decomposition is due to the fact that the stationary points of the phase function  $S^{n-1} \ni \theta \mapsto \omega \cdot \theta \in \mathbf{R}$  are  $\pm\omega$ , and we have decay estimates such as

$$\left| \left( \frac{\partial}{\partial s} \right)^j A^\pm(r, \omega) \right| \leq C \|a\|_{C^{[\frac{n-1}{2}] + k + 1}(S^{n-1})} (1+r)^{-\frac{n-1}{2} - j}, \quad (2.6)$$

$$|A^0(r, \omega)| \leq C \|a\|_{C^{[\frac{n-1}{2}] + k + 1}(S^{n-1})} (1+r)^{-[\frac{n-1}{2}] - k - 1}, \quad (2.7)$$

for  $0 \leq j \leq k$  by Lemma A.2, A.3. Thus, we obtain  $K_0^+ a = O(|x|^{-\frac{n-1}{2} - k})$  as  $|x| \rightarrow \infty$  by integration by parts for the first and second terms on the right hand side of (2.5) and by using (2.7) directly for the third term.

Finally, the results on  $K_0^- a$  are obtained similarly.  $\square$

**Proposition 2.2.** *For  $0 < \lambda < n$ , we set  $m = [\frac{n-1}{2}] + N + 1$ , where  $N = \max\{[\frac{n+1}{2} - \lambda] + 1, 0\}$ . If  $a \in C^m(\mathbf{R}^n \setminus \{0\})$  is homogeneous of degree  $-\lambda$ , then*

$$K_s a, K_c a \in C^\infty(\mathbf{R}^n)$$

and we have

$$K_s a, K_c a = O(|x|^{-n+\lambda}), \quad \text{as } |x| \rightarrow \infty.$$

*Proof.* The regularity of  $K_s a, K_c a$  follows from the fact that they are the inverse Fourier transforms of integrable functions with compact support. The decay order of  $K_s a, K_c a$  is derived in a similar way, so we only consider  $K_s a$  here.

Representing the integral by polar coordinates, we have

$$\begin{aligned} K_s a(x) &= (2\pi)^{-\frac{n}{2}} \int_0^2 \rho(s) s^{n-\lambda-1} \frac{\sin s}{s} \left( \int_{S^{n-1}} e^{isr\omega \cdot \theta} a(\theta) d\sigma(\theta) \right) ds \\ &= (2\pi)^{-\frac{n}{2}} r^{-n+\lambda} \int_0^{2r} \rho(s/r) s^{n-\lambda-1} \frac{\sin(s/r)}{s/r} \left( \int_{S^{n-1}} e^{is\omega \cdot \theta} a(\theta) d\sigma(\theta) \right) ds \\ &= (2\pi)^{-\frac{n}{2}} r^{-n+\lambda} B(r, \omega), \end{aligned}$$

where we set  $x = r\omega$  with  $r > 0$ ,  $\omega \in S^{n-1}$ , and denote by  $d\sigma$  the surface measure on  $S^{n-1}$ . Thus, it is sufficient to show

$$\sup_{\omega \in S^{n-1}, r > 1} |B(r, \omega)| < \infty. \quad (2.8)$$

For the proof of (2.8), the behavior of the oscillatory integral

$$\int_{S^{n-1}} e^{is\omega \cdot \theta} a(\theta) d\sigma(\theta) \quad (2.9)$$

as  $s \rightarrow \infty$  plays an important role. To derive the decay precisely we divide (2.9) into three parts to obtain

$$\begin{aligned} B(r, \omega) &= \int_0^{2r} e^{is} \tilde{\rho}(s/r) s^{n-\lambda-1} A^+(s, \omega) ds \\ &\quad + \int_0^{2r} e^{-is} \tilde{\rho}(s/r) s^{n-\lambda-1} A^-(s, \omega) ds \\ &\quad + \int_0^{2r} \tilde{\rho}(s/r) s^{n-\lambda-1} A^0(s, \omega) ds, \end{aligned} \quad (2.10)$$

where  $\tilde{\rho}(t) = \rho(t) \sin t/t \in C_0^\infty(\mathbf{R})$  and  $A^\pm$  and  $A^0$  are those given by (2.6), (2.7). We recall that  $A^\pm$  and  $A^0$  have decay estimates such as

$$\left| \left( \frac{\partial}{\partial s} \right)^k A^\pm(s, \omega) \right| \leq C \|a\|_{C^m(S^{n-1})} (1+s)^{-\frac{n-1}{2}-k}, \quad (2.11)$$

$$|A^0(s, \omega)| \leq C \|a\|_{C^m(S^{n-1})} (1+s)^{-m}, \quad (2.12)$$

for  $0 \leq k \leq N$  by Lemma A.2, A.3, where  $m = [\frac{n-1}{2}] + N + 1$  and  $N = \max\{[\frac{n+1}{2} - \lambda] + 1, 0\}$ .

Thus, the third term on the right hand side of (2.10) is bounded by

$$C \|a\|_{C^m(S^{n-1})} \int_0^\infty s^{n-\lambda-1} (1+s)^{-m} ds$$

uniformly on  $r > 1$  and  $\omega \in S^{n-1}$ , and therefore the above integral converges. For example, if  $\lambda < \frac{n+3}{2}$ , then

$$n - \lambda - 1 - m = \begin{cases} \frac{n+1}{2} - \lambda - [\frac{n+1}{2} - \lambda] - 2, & \text{if } n \text{ is odd,} \\ \frac{n+1}{2} - \lambda - [\frac{n+1}{2} - \lambda] - 3/2, & \text{if } n \text{ is even,} \end{cases}$$

and less than  $-1$  in both cases. When  $\frac{n+1}{2} < \lambda < n$ , the first and second term on the right hand side of (2.10) are similarly estimated as above by using (2.11) with  $k = 0$ . Meanwhile, when  $0 < \lambda \leq \frac{n+1}{2}$ , we divide their integral domain into two

parts  $(0, 1)$ ,  $(1, \infty)$  and apply integration by parts. Then, the first and second terms on the right hand side of (2.10) are bounded by

$$C \int_0^1 s^{n-\lambda-1} ds + \sum_{j=0}^{N-1} C_j r^{-j} + \left| \int_1^{2r} i^{-N} e^{is} \left( \frac{d}{ds} \right)^N \{ \tilde{\rho}(s/r) s^{n-\lambda-1} A^\pm(s, \omega) \} ds \right|, \quad (2.13)$$

where the second term corresponds to the contribution of the boundary  $s = 1$ . By (2.11) we have

$$\left| \left( \frac{d}{ds} \right)^N \{ \tilde{\rho}(s/r) s^{n-\lambda-1} A^\pm(s, \omega) \} \right| \leq C \|a\|_{C^m} s^{-\lambda + \frac{n-1}{2} - N}, \quad 0 < s < 2r,$$

and thus we observe that (2.13) is bounded independently of  $r > 1$ ,  $\omega \in S^{n-1}$ , since  $N = \lfloor \frac{n+1}{2} - \lambda \rfloor + 1$  in this case. This completes the proof.  $\square$

*Proof of Theorem 2.1.* Since  $\phi$  is homogeneous of degree  $-p$ , we observe that

$$(U(t)\phi)(x) = t^{-p}(U(1)\phi)(x/t) \quad (2.14)$$

by Lemma 2.2. Thus, it is sufficient to consider  $U(1)\phi$ , and we have

$$\begin{aligned} U(1)\phi &= \mathcal{F}^{-1}[\cos|\xi| \widehat{\phi}(\xi)] \\ &= \mathcal{F}^{-1}[\eta(\xi) \cos|\xi| \widehat{\phi}(\xi)] + \mathcal{F}^{-1}[\rho(\xi) \cos|\xi| \widehat{\phi}(\xi)] \\ &= \lim_{\varepsilon \downarrow 0} \mathcal{F}^{-1}[e^{-\varepsilon|\xi|} \eta(\xi) (e^{i|\xi|} + e^{-i|\xi|}) \widehat{\phi}(\xi)] / 2 + K_c \widehat{\phi} \\ &= (K_0^+ \widehat{\phi} + K_0^- \widehat{\phi}) / 2 + K_c \widehat{\phi}, \end{aligned} \quad (2.15)$$

where  $\eta$  and  $\rho$  are cut-off functions stated before. By Lemma 2.1 we also observe that  $\widehat{\phi} \in C^m(\mathbf{R}^n \setminus \{0\})$  with the homogeneity of degree  $-n + p$ . Because the regularity of  $\widehat{\phi}$  is sufficient to apply Proposition 2.1 and Proposition 2.2, we obtain  $K_0^\pm \widehat{\phi} \in C^l(\mathbf{R}^n \setminus S^{n-1})$ ,  $K_c \widehat{\phi} \in C^\infty(\mathbf{R}^n)$  with the asymptotic behavior

$$K_0^\pm \widehat{\phi} \sim \begin{cases} A_{n-p}^\pm \widehat{\phi}(\mp x/|x|) (1 - |x| \pm i0)^{-p + \frac{n-1}{2}} + o(|1 - |x||^{-p + \frac{n-1}{2}}) & \text{if } p > \frac{n-1}{2}, \\ A^\pm \widehat{\phi}(\mp x/|x|) \log 1/(1 - |x| \pm i0) + O(1) & \text{if } p = \frac{n-1}{2}, \end{cases}$$

as  $|x| \rightarrow 1$ , and  $K_0^\pm \widehat{\phi} \in C^{\lfloor \frac{n-1}{2} - p \rfloor}(\mathbf{R}^n)$  if  $0 < p < \frac{n-1}{2}$ . We also obtain

$$K_0^\pm \widehat{\phi} = O(|x|^{-\frac{n+1}{2} - N}) = o(|x|^{-p}), \quad K_c \widehat{\phi} = O(|x|^{-p})$$

as  $|x| \rightarrow \infty$ , since  $N \geq 0$  if  $p \geq \frac{n-1}{2}$  and  $N \geq -1$  if  $0 < p < \frac{n-1}{2}$ .

By (2.14) and (2.15) we have

$$(U(t)\phi)(x) = t^{-p}\{K_0^+\widehat{\phi}(x/t) + K_0^-\widehat{\phi}(x/t)\}/2 + t^{-p}K_c\widehat{\phi}(x/t), \quad (2.16)$$

and thus we obtain  $U(t)\phi \in C^l(\mathbf{R}^n \setminus \{|x| = t\})$  with the asymptotic behavior

$$U(t)\phi = \begin{cases} C_p t^{-\frac{n-1}{2}} \{e^{\frac{p\pi}{2}i}\widehat{\phi}(-x/|x|)(t - |x| + i0)^{-p+\frac{n-1}{2}} \\ \quad + e^{-\frac{p\pi}{2}i}\widehat{\phi}(x/|x|)(t - |x| - i0)^{-p+\frac{n-1}{2}}\} + o(|t - |x||^{-p+\frac{n-1}{2}}), \\ (8\pi)^{-\frac{1}{2}} t^{-\frac{n-1}{2}} \{e^{\frac{n-1}{4}\pi i}\widehat{\phi}(-x/|x|) \log t/(t - |x| + i0) \\ \quad + e^{-\frac{n-1}{4}\pi i}\widehat{\phi}(x/|x|) \log t/(t - |x| - i0)\} + O(1), \end{cases}$$

as  $|x| \rightarrow t$ , if  $p > \frac{n-1}{2}$  and  $p = \frac{n-1}{2}$ , respectively, and  $U(t)\phi \in C^{[\frac{n-1}{2}-p]}(\mathbf{R}^n)$  if  $0 < p < \frac{n-1}{2}$ . Moreover, the decay order is

$$U(t)\phi = O(|x|^{-p}) \quad \text{as } |x| \rightarrow \infty.$$

We also observe that

$$(V(t)\psi)(x) = t^{-p}\{K_0^+(|\xi|^{-1}\widehat{\psi})(x/t) - K_0^-(|\xi|^{-1}\widehat{\psi})(x/t)\}/2i + t^{-p}K_c\widehat{\psi}(x/t),$$

and the results for  $V$  follows similarly.  $\square$

*Proof of Theorem 2.2.* From (2.16) it is sufficient to give estimates on  $K_0^\pm\widehat{\phi}$  and  $K_c\widehat{\phi}$ . Since the regularity assumption on  $\phi$  implies that  $K_0^\pm\widehat{\phi} \in C(\mathbf{R}^n \setminus S^{n-1})$ ,  $K_c\widehat{\phi} \in C^\infty(\mathbf{R}^n)$  with the asymptotic behavior stated in Propositions 2.1 and 2.2, we obtain

$$(U(1)\phi)(x) \leq \begin{cases} C(1+|x|)^{-p}|1-|x||^{-p+\frac{n-1}{2}} & \text{if } p > \frac{n-1}{2}, \\ C(1+|x|)^{-p}(1+\log^+1/|1-|x||) & \text{if } p = \frac{n-1}{2}, \\ C(1+|x|)^{-p} & \text{if } 0 < p < \frac{n-1}{2}. \end{cases}$$

Thus, by (2.14) we obtain the desired result. The estimates on  $V(t)\psi$  are derived similarly.  $\square$

*Proof of Theorem 2.3.* This theorem is easily derived from Theorem 2.2. In fact, if  $\frac{n-1}{2} < p < \frac{n}{2}$ , then by (2.16)

$$\begin{aligned} \int_{\mathbf{R}^n} |(U(t)\phi)(x)|^r dx &= t^{-pr+n} \int_{\mathbf{R}^n} |(U(1)\phi)(x)|^r dx \\ &\leq C t^{-pr+n} \left( \int_{|x| \leq 2} |1-|x||^{-(p-\frac{n-1}{2})r} dx + \int_{|x| > 2} |x|^{-pr} dx \right), \end{aligned}$$

where the last integrals converge if

$$-(p - \frac{n-1}{2})r > -1, \quad -pr + n - 1 < -1,$$

which implies that  $p - \frac{n-1}{2} < 1/r < p/n$ . This yields

$$\sup_{t>0} t^{p-n/r} \|U(t)\phi\|_{L^r(\mathbf{R}^n)} < \infty.$$

The other cases are derived similarly.  $\square$

## A Appendix

### A.1 Fourier Transforms of Measures on the Unit Sphere

For  $\lambda > 0$ ,  $\omega \in S^{n-1}$ ,  $h \in C^m(S^{n-1})$ , we set

$$A(\lambda, \omega) = \int_{S^{n-1}} e^{i\lambda\omega\cdot\theta} h(\theta) d\sigma(\theta).$$

The asymptotic behavior of  $A(\lambda, \omega)$  as  $\lambda \rightarrow \infty$  is given in the following lemma.

**Lemma A.1.** *Let  $N \in \mathbf{N} \cup \{0\}$  and let  $m = \max\{2N + 2, [\frac{n-1}{2}] + N + 2\}$ . For  $h \in C^m(S^{n-1})$  the oscillatory integral  $A(\lambda, \omega)$  have the asymptotic expansion of the form*

$$A(\lambda, \omega) = e^{i\lambda} \left\{ \sum_{j=0}^N a_j^+(\omega) \lambda^{-\frac{n-1}{2}-j} \right\} + e^{-i\lambda} \left\{ \sum_{j=0}^N a_j^-(\omega) \lambda^{-\frac{n-1}{2}-j} \right\} + R_N(\lambda, \omega)$$

as  $\lambda \rightarrow \infty$ , where the remainder term  $R_N \in C((0, \infty) \times S^{n-1})$  satisfies

$$|R_N(\lambda, \omega)| \leq C \|h\|_{C^m(S^{n-1})} \lambda^{-\frac{n-1}{2}-N-1}, \quad \lambda > 0$$

and all coefficients  $a_j^\pm(\omega)$  appearing in the asymptotic expansion depend on the values of only finitely many derivatives of  $h$  at  $\pm\omega$ . In particular,

$$a_0^\pm(\omega) = (2\pi)^{\frac{n-1}{2}} e^{\mp \frac{n-1}{4}\pi i} h(\pm\omega).$$

**Remark A.1.** (1) More precisely, the remainder term is decomposed into three parts

$$R_N(\lambda, \omega) = e^{i\lambda} R_N^+(\lambda, \omega) + e^{-i\lambda} R_N^-(\lambda, \omega) + R_N^0(\lambda, \omega)$$

satisfying

$$\begin{aligned} \left| \left( \frac{\partial}{\partial s} \right)^k R_N^\pm(\lambda, \omega) \right| &\leq C \|h\|_{C^{m'}(S^{n-1})} \lambda^{-\frac{n-1}{2}-N-k-1}, \quad \lambda > 0, \\ |R_N^0(\lambda, \omega)| &\leq C \|h\|_{C^{m'}(S^{n-1})} \lambda^{-\frac{n-1}{2}-N-k-1}, \quad \lambda > 0, \end{aligned}$$

where  $m' = \max\{2N + 2, [\frac{n-1}{2}] + N + k + 2\}$ .

(2) Applying this lemma for  $N = 0$ , we obtain

$$A(\lambda, \omega) = (2\pi)^{\frac{n-1}{2}} \{e^{-\frac{n-1}{4}\pi i} e^{i\lambda} h(\omega) + e^{\frac{n-1}{4}\pi i} e^{-i\lambda} h(-\omega)\} \lambda^{-\frac{n-1}{2}} + R_0(\lambda, \omega)$$

as  $\lambda \rightarrow \infty$ , where

$$|R_0(\lambda, \omega)| \leq C \|h\|_{C^{[\frac{n-1}{2}]+2}(S^{n-1})} \lambda^{-\frac{n-1}{2}-1}, \quad \lambda > 0.$$

We next describe the decay estimates of  $A(\lambda, \omega)$  by the stationary phase method. The stationary points of the phase function  $S^{n-1} \ni \theta \mapsto \omega \cdot \theta \in \mathbf{R}$  are  $\pm\omega$ , so we first consider the case where the amplitude function  $h$  is identically zero near  $\pm\omega$ .

**Lemma A.2.** *Let  $l \in \mathbf{N} \cup \{0\}$ . We assume that  $h \in C^l(S^{n-1})$  satisfies  $\pm\omega \notin \text{supp } h$ . Then we have*

$$|A(\lambda, \omega)| \leq C \|h\|_{C^l(S^{n-1})} (1 + \lambda)^{-l}, \quad \lambda > 0.$$

When the amplitude functions are supported in small neighborhoods of  $\pm\omega$ , we especially consider the oscillatory integrals of the following form:

$$A^\pm(\lambda, \omega) = \int_{S^{n-1}} e^{i\lambda(\omega \cdot \theta \mp 1)} g^\pm(\theta) d\sigma(\theta),$$

where  $g^\pm$  are supported in small neighborhoods of  $\pm\omega$ , respectively.

**Lemma A.3.** *Let  $k \in \mathbf{N} \cup \{0\}$ . We assume that  $g^\pm \in C^{[\frac{n-1}{2}]+k+1}(S^{n-1})$  are supported in sufficiently small neighborhoods of  $\pm\omega$ . Then*

$$\left| \left( \frac{\partial}{\partial \lambda} \right)^k A^\pm(\lambda, \omega) \right| \leq C \|g^\pm\|_{C^{[\frac{n-1}{2}]+k+1}(S^{n-1})} (1 + \lambda)^{-\frac{n-1}{2}-k}, \quad \lambda > 0.$$

## A.2 Oscillatory Integrals on the Half Line

Let  $\eta \in C^\infty([0, \infty))$  with  $0 \leq \eta \leq 1$ ,  $\eta(s) = 0$  if  $0 \leq s \leq 1$ , and  $\eta(x) = 1$  if  $s \geq 2$ . For  $\mu \in \mathbf{R}$ , we define

$$I_\mu(\tau) = \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{-\varepsilon s + i s \tau} \eta(s) s^{\mu-1} ds, \quad \tau \in \mathbf{R}.$$

The asymptotic behavior of  $I_\mu(\tau)$  as  $\tau \rightarrow \infty$  and the behavior near  $\tau = 0$  are given in the following lemma.



**Lemma A.4.** For  $\mu \in \mathbf{R}$ ,  $I_\mu \in C^\infty(\mathbf{R} \setminus \{0\})$  and for any  $k \in \mathbf{N} \cup \{0\}$ ,  $M > 0$ ,

$$\left(\frac{\partial}{\partial \tau}\right)^k I_\mu(\tau) = o(|\tau|^{-M}), \quad \text{as } |\tau| \rightarrow \infty.$$

Moreover, we have

(1) For  $\mu > 0$  there exists  $r_\mu \in C^\infty(\mathbf{R})$  such that

$$I_\mu(\tau) = e^{\frac{\pi}{2}i\mu} \Gamma(\mu)(\tau + i0)^{-\mu} + r_\mu(\tau), \quad \tau \in \mathbf{R}. \quad (\text{A.1})$$

(2) For  $\mu = 0$  there exists  $r_0 \in C^\infty(\mathbf{R})$  such that

$$I_0(\tau) = \log 1/(\tau + i0) + r_0(\tau), \quad \tau \in \mathbf{R}. \quad (\text{A.2})$$

(3) For  $\mu < 0$  we have  $I_\mu \in C^{[-\mu]-, \nu}(\mathbf{R}) \cap C^\infty(\mathbf{R} \setminus \{0\})$ , where  $0 < \nu < -\mu - [-\mu]_-$ , where  $C^{m, \nu}$  is the Hölder space.

Moreover, if  $\mu \geq 0$ , the convergence to the right hand side of (A.1), (A.2) is uniform in  $\{|\tau| \geq \delta\}$  for any  $\delta > 0$ . If  $\mu < 0$ , the convergence of the right hand side in the definition of  $I_\mu(\tau)$  is uniform on  $\mathbf{R}$ .

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