Singular diffusivity–facets, shocks and more

Yoshikazu Giga*
Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan

Abstract: There is a class of nonlinear evolution equations with singular diffusivity, so that diffusion effect is nonlocal. A simplest one-dimensional example is a diffusion equation of the form $u_t = \delta(u_x)u_{xx}$ for $u = u(x,t)$, where δ denotes Dirac's delta function. This lecture is intended to provide an overview of analytic aspects of such equations, as well as various applications. Equations with singular diffusivity are applied to describe several phenomena in the applied sciences, and to provide several devices in technology, especially image processing. A typical example is a gradient flow of the total variation of a function, which arises in image processing, as well as in material science to describe the motion of grain boundaries. In the theory of crystal growth the motion of a crystal surface is often described by an anisotropic curvature flow equation with a driving force term. At low temperature the equation includes a singular diffusivity, since the interfacial energy is not smooth. Another example is a crystalline algorithm to calculate curvature flow equations in the plane numerically, which is formally written as an equation with singular diffusivity.

Because of singular diffusivity, the notion of solution is not a priori clear, even for the above one-dimensional example. It turns out that there are two systematic approaches. One is variational, and applies to divergence type equations. However, there are many equations like curvature flow equations which are not exactly of divergence type. Fortunately, our approach based on comparision principles turns out to be succesful in several interesting problems. It also asserts that a solution can be considered as a limit of solution of an approximate equation. Since the equation has a strong diffusivity at a particular slope of a solution, a flat portion with this slope is formed. In crystal growth ploblems this flat portion is called a facet. The discontinuity of a solution (called a shock) for a scalar conservation law is also considered as a result of singular diffusivity in the vertical direction.

*Partly supported by the Grant-in-Aid for Scientific Research, No.14204011, No.15634008, Japan Society for the Promotion of Science.

1 Introduction

This is a review paper on nonlinear evolution equations with singular diffusivity, so that the diffusion effect is nonlocal. Recently, such equations become popular in material sciences and image analysis, although its mathematical analysis is far from being satisfactory to cover all important problems. Our goal is to provide an overview of mathematical analysis on this type of problems especially,

(i) unique solvability of the initial value problem;

(ii) stability under approximation of equations as well as data

for various but typical problems.

To see the difficulty of the problem we give a simplest one-dimensional example of the form

$$u_t = \delta(u_x)u_{xx},\ x \in \mathbf{R},\ t > 0 \tag{1.1}$$

for $u = u(x,t)$, where δ denotes Dirac's delta function and $u_t = \partial u/\partial t, u_x = \partial u/\partial x, u_{xx} = \partial^2 u/\partial x^2$. The multiplier $\delta(u_x)$ is not well-defined as a distribution even if u is smooth. So even the notion of solution is unclear in naive sense. Fortunately, this example can be regarded as the gradient flow of (the half of) the total variation $\int |u_x| dx$. So we are able to apply the nonlinear semigroup theory [10], [5] for maximal monotone operators (initiated by Kōmura [56]) to conclude unique global solvability under suitable boundary conditions [49], [53], [31]. This example has also stability under approximation of the functionals [11], [71]. An explicit form of a solution is often computable. For example if the initial data is $\cos x$, then a flat portion (ofter called a facet) instanteneously formed at the maximum and minimum point of $\cos x$. In fact, as in [53] we have

$$\begin{aligned} u(x,t) &= \min(\cos x, h(t)), \quad |x| \le \pi/2, \\ dh/dt &= -L(t)^{-1}, \quad h(0) = 1, \end{aligned}$$

where $L(t)$ is the length of a facet (i.e. the length of the interval on which $h(t) < \cos x$), until h becomes zero. (By an elemetary observation this gives an explicit formula of u [22], [53]). It is not difficult to see that $h(t)$ becomes zero in finite time. From that time $u(x,t) \equiv 0$. As we learn from this example, the speed on the facet is determined by its length so the speed is nonlocal. It is formally obtained in the following way. Let $(a(t), b(t))$ be the interval corresponding to the facet. Assume that u_t is constant on $(a(t), b(t))$. (This assumption is called facet-stay-as-facet hypothesis.) Since our equation (1.1) can be written as

$$u_t = \frac{1}{2}(\text{sgn } u_x)_x, \tag{1.2}$$

integrating over $(a(t)-\varepsilon, b(t)+\varepsilon)$ and sending $\varepsilon(>0)$ to zero yields

$$u_t(b(t)-a(t)) = \frac{1}{2}(\mathrm{sgn}(-0) - \mathrm{sgn}(+0)) = -1$$

if u is concave near $(a(t), b(t))$. This yields the desired speed.

One may write the equation (1.2) in the form

$$u_t = \frac{1}{2}\xi_x, \quad \xi = \mathrm{sgn}\ u_x.$$

However, sgn p should be regarded multivalued at $p=0$ with values [-1,1] as a maximal monotone function. Thus, instead of $\xi = \mathrm{sgn}\ u_x$ one has to write

$$\xi(x,t) \in \mathrm{sgn}\ u_x(x,t).$$

At the first glance there should be a huge ambiguity. However, a honest regularity of u implies the uniqueness of a solution thanks to the theory of nonlinear semigroups. Although the equation may have ambiguity, the solution knows how to evolve.

Unfortunately, many important equations are not of the form of a gradient flow. A typical example is a level-set equation of an anisotropic curvature flow, which is of the form

$$\psi_t + |\nabla\psi|\mathrm{div}\xi(-\nabla\psi/|\nabla\psi|) = 0,\ x \in \mathbf{R}^n,\ t>0 \tag{1.3}$$

for $\psi = \psi(x,t)$, where $\nabla\psi = (\partial\psi/\partial x_1, \ldots, \partial\psi/\partial x_n)$ and $n \geq 2$. Here ξ is the gradient of the *interfacial energy density* $\gamma : \mathbf{R}^n \to [0,\infty)$ which is convex and positively homogeneous of degree one. If $\gamma(p) = |p|$ so that γ is isotropic, then (1.3) becomes the level-set equation

$$\psi_t = |\nabla\psi|\mathrm{div}(\nabla\psi/|\nabla\psi|) \tag{1.4}$$

of the mean curvature flow equation $V = H$, i.e., each level-set of ψ moves by its mean curvature H at least formally. Here V denotes the normal velocity in the direction of the normal $\mathbf{n} = -\nabla\psi/|\nabla\psi|$. In general, (1.3) is the level-set equation of an anisotropic curvature flow equation:

$$V = \Lambda_\gamma(\mathbf{n}) \quad \text{with} \quad \Lambda_\gamma(\mathbf{n}) := -\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) \quad \text{on} \quad \Gamma_t \tag{1.5}$$

for an evolving hypersurface $\{\Gamma_t\}$. The quantity $\Lambda_\gamma(\mathbf{n})$ is called a weighted mean curvature. The level-set equation is important to solve equations like (1.5) globally in time beyond singularities both analytically [15], [20] and numerically [60]; see also books [64], [36], [59] and review articles [32], [33]. A classical result based on the theory of viscosity solutions implies unique global solvability for any continuous (periodic) initial data for

(1.3) provided that γ is smooth enough, say C^2 [15]. However, in crystal growth problems γ may not be C^1 [29]. A typical example is that

$$\text{Frank } \gamma = \{p \in \mathbf{R}^n | \gamma(p) \leq 1\} \tag{1.6}$$

is a polyhedra. In this case γ is called a *crystalline energy* and (1.5) is called a *crystalline curvature flow* (equation). This problem has a feature which is similar to (1.1). The diffusion effect is very strong in the direction corresponding to verteces of Frank γ. Indeed, if $n = 2$ and $\gamma(p) = (|p_1| + |p_2|)/2$ for $p = (p_1, p_2)$, then (1.5) becomes (1.1) when Γ_t is represented as the graph of function $u(x_1, t)$ of one variable x_1.

Crystalline flow problems are first proposed by [4] and [68] independently for planar curve evolutions, i.e. $n = 2$. They restricted a class of solution into special polygonal evolution and reduced the problem to a system of ordinary differential equations (ODEs) under facet-stay-as-facet hypothesis. Their solution is called a crystalline flow (for (1.5)) [26]. However, it was not clear that this facet-stay-as-facet hypothesis is natural at that time. This problem has been solved affirmatively in [22], [25], [27] by extending the notion of viscosity solutions [17] so that it applies to nonlocal problem. In particular, facet-stay-as-facet hypothesis is actually obtained as a limit of smoother problem. Moreover, the level-set method for (1.3) for singular γ has been established in [26], [27]. The key observation is that the evolution is order-preserving despite the fact that diffusion effect is nonlocal. In section 2 we highlight these results more precisely. For a higher dimensional problem [9] the facet-stay-as-facet hypothesis contradicts the comparison principle so such a hypothesis is no longer natural [73]. It is even not clear what is the reasonable notion so that (1.3) is solvable globally-in-time for crystalline γ when $n \geq 3$. (Of course, if one admits the facet-stay-as-facet hypothesis, the problem is solvable [38]). Several notions of solutions are proposed [8], [41]. However, it is not known that the initial value problem is solvable even locally in time.

Instead of (1.5) one has to consider the equation with driving force like

$$V = \Lambda_\gamma(\mathbf{n}) + C(x),$$

where C is a given function. Such an equation is important to study the Stefan type problem with Gibbs-Thomson and kinetic supercooling effect. If $C(x)$ is not a constant so that the problem is spatially inhomogeneous, again the facet-stay-as-facet hypothesis may contradict the comparison principle [23]. An expected speed is computable [24]. However, local existence of a solution is not known also for this problem. A Stefan type problem with crystalline γ has been studied [42], [43]. Local existence of solution is obtained under the facet-stay-as-facet hypothesis for a cylindrical crystal [42]. It is

expected that the facet-stay-as-facet hypothesis should be true for a small crystal but the work is still under progress.

Besides these rather classical fields the singular diffusivity is important to understand shocks for scalar first order equations. Let us start with an example

$$\varphi_t + y\varphi_x = M|\nabla_{x,y}\varphi|\frac{\partial}{\partial y}(\varphi_y/|\varphi_y|),\ x \in \mathbf{R},\ y \in \mathbf{R},\ t > 0 \tag{1.7}$$

for $\varphi = \varphi(x, y, t)$, where $M > 0$ is a constant. Without M-term this is the level-set equation for the graph of a solution of the Burgers equation

$$u_t + uu_x = 0. \tag{1.8}$$

However, the level-set of φ may overturn and it does not represent the graph of an entropy solution after it develops jump discontinuities called shocks. It turns out that for sufficiently large M the level-set of φ of (1.6) represents the graph of an entropy solution. This is analytically proved in [28] and numerically confirmed in [70]. The right hand side of (1.7) plays a role if $\varphi_y = 0$ otherwise it does not play any role. So this term represents singuler vertical diffusion to prevent overturning. The method is not limited to conservation laws but equations with nonconservative type, for which the level-set is expected to represent a proper viscosity solution introduced by [34]. Since the problem is not spatially homogeneous, it might be a chance that the level-set may still overturn so facet-stay-as-facet hypothesis is not expected to hold for small M. The notion of solution for (1.7) has not yet been fully established. In section 3 we highlight a level-set method to track the graph of solutions with shocks by vertical singular diffusivity.

We are now going back to the gradient flow of total variation, its one-dimensional version is (1.1). To remove noises from image it has been proposed in [62] to use the gradient flow of total variation of a grey-level function under constraint $\int |u - u_0|^2 dx =$ const, where u_0 as a given image. There are several models related to this problem. For example, the total variation flow with values in a sphere is important to remove noise from direction fields of color grey-level mappings $u = (u_1, u_2, u_3)$ keeping its strength [67]. Its explicit form is

$$u_t = \mathrm{div}\left(\frac{\nabla u}{|\nabla u}\right) + |\nabla u|u. \tag{1.9}$$

These problems do not have order-preserving structure and it does not apply nonlinear semigroup theory since the nonlinear operator is not necessarily maximal monotone. For a background of problems related to image analysis see [63]. A similar model for a direction field is also important to model the evolution of multigrain [55], [72], where u represents the orientation of the grain. In section 5 we review several analytic results for a gradient flow of total variation.

A fourth order parabolic equation with singular diffusivity is proposed for example in [66] to model an evolution by surface diffusion with facet. Its analysis is just started by Y. Kashima [51]. For more references see papers cited in [51].

2 Anysotropic curvature flow with singular interfacial energy

A curvature flow equation is by now very popular to describe the motion of phase-boundaries such as the motion of grain boundaries in material sciences (see e.g.[47]). It is also used in image processing to remove noises from images (see [63], [12]). Anisotropic effect is often important for a crystal growth problem [54]. In image processing it is also important to use an equation with anisotropy depending on local feature of images (e.g.[61]).

Here is a general form of an anisotropic curvature flow equation for evolving curves $\{\Gamma_t\}$ in $\mathbf{R}^2$ when it is spatially homogeneous and time independent:

$$V = g(\mathbf{n}, \Lambda_\gamma(\mathbf{n})) \quad \text{on} \quad \Gamma_t, \tag{2.1}$$

where g is nondereasing in the second variable so that the problem in parabolic. Recently, for nondifferentiable γ including crystalline energy we are able to establish a level-set method. First let us write a level-set equation of (2.1). It is of the form

$$\varphi_t - |\nabla\varphi| g(-\nabla\varphi/|\nabla\varphi|, \Lambda_\gamma(-\nabla\varphi/|\nabla\varphi|)) = 0. \tag{2.2}$$

Let $\mathcal{I}$ denote the set of all convex interfacial energy density $\gamma(\geq 0)$ such that the boundary curve of Frank γ is a closed piecewise C^2 curve so that it has at most finitely many singularities. Let $\mathcal{G}_0$ denote the set of all continuous functions on $S^1 \times \mathbf{R}$ such that $\lambda \mapsto g(p, \lambda)$ is nondereasing and

$$||g||_1 = \sup\{|g(p,\lambda)|/(|\lambda|+1),\ p \in S^1,\ x \in \mathbf{R}\} < \infty.$$

We state a periodic version of main results of [27]. Let $\mathbf{T}^n$ denote a flat torus defined by $\mathbf{T}^n = \prod_{i=1}^{n} (\mathbf{R}/\omega_i\mathbf{Z})$ with $\omega_i > 0$.

Theorem 2.1. *Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. There is an explicit notion of viscosity-like solutions (consistent with usual viscosity solutions) such that following properties are valid.*

*(***Global Unique Solvability***). For $\varphi_0 \in C(\mathbf{T}^2)$ there exists a unique solution $\varphi \in C(\mathbf{T}^2 \times [0,\infty))$ of (2.2) with $\varphi|_{t=0} = \varphi_0$.*

(**Convergence Property**). *Assume that* $\gamma_\varepsilon \in \mathcal{I}$ *and* $g_\varepsilon \in \mathcal{G}_0$ *for* $\varepsilon \in$ *(0,1). Assume that* $\gamma_\varepsilon \to \gamma$, $g_\varepsilon \to g$ *locally uniformly as* $\varepsilon \to 0$. *Assume that* $\sup_{0<\varepsilon<1} ||g_\varepsilon||_1 < \infty$. *Let* φ_ε *be the solution of*

$$\varphi_t - |\nabla\varphi| g_\varepsilon(-\nabla\varphi/|\nabla\varphi|, \Lambda_{\gamma_\varepsilon}(-\nabla\varphi/|\nabla\varphi|)) = 0$$

with $\varphi|_{t=0} = \varphi_{0\varepsilon} \in C(\mathbf{T}^2)$. *If* $\varphi_{0\varepsilon} \to \varphi_0$ *in* $C(\mathbf{T}^2)$, *then* $\varphi_\varepsilon \to \varphi$ *locally uniformly in* $\mathbf{T}^2 \times [0,\infty)$ *as* $\varepsilon \to 0$.

As expected the (super) level-set $\{\varphi \geq c\}$ is uniquely determined from the initial level-set $\{\varphi_0 \geq c\}$ and the comparison principle still holds. However, one should notice that the proof is not easy since the problem has nonlocal nature. We refer to [27] for the proof.

We shall discuss a few applications of these powerful results. For this purpose following [26] we recall a crystalline flow [4], [68] of (2.1) when γ is crystalline, i.e., Frank γ is a convex , m-polygon. Let $q_i (i = 1, 2, ..., m)$ be its verteces. Let $\mathcal{N} \subset S^1 = \{p \in \mathbf{R}^2 | |p| = 1\}$ be the set of all unit vectors of form $q_i/|q_i| (i = 1, 2, ..., m)$. A simple polygonal curve S in $\mathbf{R}^2$ is called an *admissible crystal* if all outward normal (orientation) belongs to $\mathcal{N}$ and the orientations of adjacent segments (facet) point to verteces adjacent in Frank γ. A family of polygon $\{S_t\}_{t\in J}$ is an *admissible evolving crystal* if S_t moves at least C^1 in time, where J is a time interval. The last requirement implicitly assumes that the numbers of facets and the orientation at each facet are independent of time. In other words, S_t is of form $S_t = \cup_{j=1}^r S_j(t)$ and $S_j(t)$ is a maximal nontrivial closed segment (facet) of S_t and the orientation $\mathbf{n}_j$ of $S_j(t)$ is independent of time. To fix the idea we number facets clockwise.

Let $\{S_t\}_{t\in J}$ be an admissible evolving crystal with $J = [0, T)$. We say that $\{S_t\}_{t\in J}$ is a γ-*regular flow* of (2.1) if

$$V_j = g(\mathbf{n}_j \ ; \ \chi_j \Delta(\mathbf{n}_j)/L_j(t)) \quad \text{on} \quad S_j(t) \tag{2.3}$$

for $j = 1, 2, ..., r$, where V_j denotes the normal velocity of $S_j(t)$. The quantity $\chi_j \Delta(\mathbf{n}_j)/L_j(t)$ is a nonlocal weighted curvature $\Lambda_\gamma(\mathbf{n}_j)$, where $L_j(t)$ denotes the length of $S_j(t)$ and $\Delta(\mathbf{m}_i) = \tilde{\gamma}'(\theta_i + 0) - \tilde{\gamma}'(\theta_i - 0), \mathbf{m}_i = (\cos\theta_i, \sin\theta_i) \in \mathcal{N}$ with $\tilde{\gamma}(\theta) = \gamma(\cos\theta, \sin\theta)$. The quantity χ_j is a transition number. It takes +1 (resp.-1) if S_t is concave (resp. convex) in the direction of $\mathbf{n}_j$ near S_j; we use the convention that $\chi_j = -1$ for all $j = 1, ..., r$ if S_t is a convex polygon. Otherwise we set $\chi_j = 0$. As well-known that $\Delta(\mathbf{m}_i)$ is the length of a facet of the Wulff shape

$$W_\gamma = \{x \in \mathbf{R}^2 \quad | \quad x \cdot p \leq \gamma(p) \quad \text{for all} \quad p \in \mathbf{R}^2\}$$

with outward normal $\mathbf{m}_i \in \mathcal{N}$. By this convention the weighted curvature Λ_γ of Wulff shape is -1, independent of the facet. For smooth γ the curvature Λ_γ of the corresponding Wulff shape is always -1 so it is a substitute of a circle for isotropic γ. So the definition of nonlocal curvature is quite natural as far as one admits the facet-stay-as-facet hypothesis as derived from (1.1).

Note that L_j always fulfills a transport equation

$$\frac{dL_j}{dt}(t) = (\cot\sigma_{j+1} + \cot\sigma_j)V_j - (\sin\sigma_j)^{-1}V_{j-1} - (\sin\sigma_{j+1})^{-1}V_{j+1} \tag{2.4}$$

as observed in [4], [68]. Here $\sigma_j = \theta_j - \theta_{j-1}$ for $\mathbf{n}_j = (\cos\theta_j, \sin\theta_j)$ and V_j denotes the normal velocity of $S_j(t)$; the index j is considered modulo r. Combining (2.3) and (2.4) we get an r-system of ordinary differential equations of L_j's. A local existence theorem of ODEs yields a local existence of γ-regular flow.

Proposition 2.2. *Assume that $\lambda \mapsto g(\mathbf{m}_i, \lambda)(\mathbf{n}_i \in \mathcal{N})$ is locally Lipschitz continuous on $\mathbf{R}\setminus\{0\}$. Let S_0 be an admissible crystal. Then there is a constant $T > 0$ and a unique γ-regular flow $\{S_t\}_{t\in J}$ of (2.1) with initial data S_0 where $J = [0, T)$.*

During evolution some facets may disappear at the maximal existence time T of γ-regular flow. As discussed in [26] for a wide class of equations (2.3) (or (2.1)) S_t becomes still admissible at T so one can construct γ-regular flow starting from S_T. A *crystalline flow* $\{S_t\}$ of (2.1) is obtained by repeating this procedure. For simplicity we consider (1.5) with symmetric crystalline γ i.e. $\gamma(p) = \gamma(-p)$. For this equation a unique crystalline flow exists until it shrinks to a point without self-intersection if initially S_0 is an admissible crystal. Moreover, this crystalline flow is consistent with a level-set flow defined by $\{\varphi = c\}$ for φ solving (1.3) [26].

We now mention two typical applications of our convergence theorem (Theorem 2.1). Let $\mathcal{I}_s$ denote the set of $\gamma \in \mathcal{I}$ such that Frank γ has a smooth boundary with nonzero curvature and that γ is symmetric, i.e. $\gamma(p) = \gamma(-p)$. For $\gamma \in \mathcal{I}_s$ it is well-known [16] that (1.5) admits a smooth solution $\{\Gamma_t\}$ starting from a simple, closed, smooth curve Γ_0 until it shrinks to a point. As well-known this $\{\Gamma_t\}$ is also a level-set flow of (1.3), in particular, no fattening occurs [37], [36]. It is easy to see that a symmetric crystalline γ there is a family $\{\gamma_\varepsilon\} \subset \mathcal{I}_s$ such that $\gamma_\varepsilon \to \gamma$ locally uniformly as $\varepsilon \to 0$. It is also easy to see that for $\gamma \in \mathcal{I}_s$ there is a family $\{\gamma_\varepsilon\}$ of crystalline energies such that $\gamma_\varepsilon \to \gamma$. Theorem 2.1 (with help of [27, Corollary 8.3]) provides various approximation results; see e.g. [26]. We give a simple example.

Theorem 2.3 (**Convergence of crystalline algorithm**). *For $\gamma \in \mathcal{I}_s$ let γ_ε be a symmetric crystalline energy such that $\gamma_\varepsilon \to \gamma$ locally uniformly as $\varepsilon \to 0$. Let $\{S_t^\varepsilon\}_{t\in J}$*

be the crystalline flow of (1.5) with γ replaced by γ_ε starting from S_0^ε. For a smooth, closed, simple curve Γ_0 let $\{\Gamma_t\}_{t\in J}$ be the solution of (1.5) with $\Gamma_t|_{t=0} = \Gamma_0$. If $d_H(S_0^\varepsilon, \Gamma_0) \to 0$ as $\varepsilon \to 0$, then $\sup_{0\le t\le T'} d_H(\Gamma_t, S_t^\varepsilon) \to 0$ for $T' < T$, where $J = [0, T)$. Here d_H denotes the Haussdroff distance and T is the time when Γ_t shrinks to a point.

Theorem 2.4 (**Approximation by a smooth problem**). *For a symmetric crystalline energy let $\gamma_\varepsilon \in \mathcal{I}_s$ satisfy $\gamma_\varepsilon \to \gamma$ as $\varepsilon \to 0$ locally uniformly. Let $\{\Gamma_t^\varepsilon\}_{t\in J}$ be the solution of (1.5) with γ replaced by γ_ε starting from a smooth, closed, simple curve Γ_0^ε. Let $\{S_t\}_{t\in J}$ be the crystalline flow of (1.5) starting from an adrissible crystal S_0. If $d_H(\Gamma_0^\varepsilon, S_0) \to 0$ as $\varepsilon \to 0$, then $\sup_{0\le t\le T'} d_H(\Gamma_t^\varepsilon, S_t) \to 0$ for $T' < T$, where $J = [0, T)$. Here T is the time when S_t shrinks to a point.*

There are several preceding work on convergence of crystalline algorithm [21], [46], [25] for graph-like curves. For isotropic energy i.e. for a curve shortening equation the convergence is shown for a convex curve [45] and a general curve [50]. Theorem 2.3 is a generalization of these results. For more general statement and more references see [26] and references cited there.

Theorem 2.4 justified the crystalline flow as a limit of a smoother problem. There are several preceeding works including [21], [25] for graph-like curves. This result is 'essentially' known from [27, Corollary 8.3] and consistency results; however, it seems that this was not explicitly stated in the literature.

Our level-set flow for a crystalline energy provides a solution starting from nonadmissible polygon. It turns out it evolves as a polygon and is still computable. We do not touch this problem here and leave it to [30] and its application to image analysis to [48].

Finally, we mention that a singular interfacial energy really arises when one considers a crystal at low temperature [29], where it is shown that an evaporation dynamics (with a facet) proposed by [66] is approximated by a smoother problem as an application of a convergence result in [25].

3 Shocks and vertical diffusion

We consider the initial value problem for a nonlinear first order equation of the form

$$u_t + H(u, \nabla u) = 0, \quad x \in \mathbf{R}^n, \quad t > 0 \tag{3.1}$$

$$u(x, 0) = u_0(x), \tag{3.2}$$

where $u = u(x, t)$ is a real-valued function and $H = H(r, p)$ is continuous. Let us derive a level-set equation for the graph of u. Let $\varphi = \varphi(x, y, t)$ be a function such that

$\varphi(x, u(x,t), t) = 0$ for $x \in \mathbf{R}^n, t > 0$. To fix the idea we assume that $\varphi_y \leq 0$. Then $u_t = -\varphi_t/\varphi_y,\ \partial u/\partial x_i = -(\partial\varphi/\partial x_i)/\varphi_y$ so (3.1) yields

$$\varphi_t - \varphi_y H(y, -\nabla_x \varphi/\varphi_y) = 0 \tag{3.3}$$

on the zero level-set of φ. We consider (3.2) in $\mathbf{R}^n \times \mathbf{R} \times (0,T)$ rather than on the zero level-set of φ. To solve the initial value problem (3.1)-(3.2) we rather solve φ with initial data φ_0 such that the zero level-set of φ_0 agrees with the graph of u_0. One advantage of (3.3) over (3.1) is that (3.3) does not contain unknown φ while (3.1) depends on unknown u explicitly. This idea goes back to Jacobi [13] to solve (3.1)-(3.2) by a method of characteristic. Such a level-set formulation is studied by Evans [19] for the heat equation. A related level-set approach is proposed by Osher [58] for a stationary Hamilton-Jacobi equation. The author with M.-H. Sato [44] applies the above level-set method to solve (3.1) globally-in-time in viscosity sense when u_0 is not necessarily continuous assuming that $r \mapsto H(r,p)$ is nondecreasing (under some linear growth condition in p of $H(r,p)$). Since the solution is not expected to be continuous, the uniqueness of a solution is not generally valid. It turns out that this problem is related to 'fattening' in the level-set method for a curvature flow equation [6]. The level-set method [44] provides the largest and the smallest solutions of (3.1)-(3.2).

If the monotonicity condition on $H(r,p)$ w.r.t. r is removed, the situation is quite different. A typical example is the a conservation law

$$u_t + \text{div } \mathbf{F}(u) = 0$$

with $\mathbf{F} = (F_1, ..., F_n)$, which includes the Burgers equation as a special example. The solution of the initial value problem (3.1) may develop discontinuity (called shock) in finite time even if initial data is smooth. (Under the monotonicity condition on H a solution stays continuous if initial data is (uniformly) continuous.) For conservation laws there is a unique way to extend a solution globally-in-time after it develop singularities. This special weak solution is called an *entropy solution* [18], [57]. If one tracks the zero level-set of (3.3) it may 'overturn' and it cannot be viewed as the graph of single-valued function after the solution u develops shocks. In what way one should track the graph of entropy solution ?

In [34], [35] we propose to consider

$$\varphi_t - \varphi_y H(y, -\nabla_x \varphi/\varphi_y) = M|\nabla_{x,y}\varphi| \frac{\partial}{\partial y}\left(\frac{\varphi_y}{|\varphi_y|}\right) \tag{3.4}$$

with some $M > 0$ instead of (3.3). This is again the equation with singular diffusivity and the notion of solution is unclear. Each level-set $\{\Gamma_t\}$ of φ is moved by

$$V = -n_{n+1} H(y, -(n_1, .., n_n)/n_{n+1}) - M \text{ div}_{\Gamma_t} \xi(\mathbf{n}) \quad \text{on} \quad \Gamma_t, \tag{3.5}$$

where $\mathbf{n} = (n_1, ..., n_{n+1})$ is the upward normal of Γ_t and $\xi = \nabla\gamma$ with $\gamma(p) = |p_{n+1}|$ for $p = (p_1, ..., p_{n+1})$. The equation (3.4) is the level-set equation of anisotropic curvature flow equation (3.5) with singular interfacial energy. However, the theory in Section 2 does not apply even for $n = 1$ since (3.5) is spatially inhomogeneous.

Instead of developing general theory for (3.4) or (1.7) we rather study its approximation. We consider a simple example (1.8) with initial data

$$u(x,0) = u_0(x) = \begin{cases} a & , \quad x > 0 \\ a + d & , \quad x < 0 \end{cases}$$

for $a \in \mathbf{R}, d > 0$. Then it is well-known that the entropy solution of (1.8) is of the form

$$u_E(x,t) = \begin{cases} a & , \quad x > ct \\ a + d & , \quad x < ct \end{cases}$$

with $c = (f(a+d) - f(a))/d$, where $f(r) = r^2/2$. Consider

$$\varphi_t + y\varphi_x = M|\nabla_{x,y}\varphi|\mathrm{div}\xi_\varepsilon(-\nabla\varphi/|\nabla\varphi|) \tag{3.6}$$

with $\xi_\varepsilon = \nabla\gamma_\varepsilon$ such that γ_ε approximates γ. Initial data φ_0 of φ is taken so that it is uniformly continuous with

$$\{(x,y)|\varphi_0(x,y) \le 0\} = \{(x,y)|y \le u_0(x)\} \tag{3.7}$$

and that $y \mapsto \varphi_0(x,y)$ is nonincreasing. If γ_ε is convex and C^2, it is known that (3.6) admits a unique continuous solution φ_ε [36]. The next result indicates the role of vertical diffusion term.

Let φ_ε be the solution (3.6) satistfying (3.7). For technical reasons we assume that $M < d^2/8$.

Theorem 3.1. [28] *There is a sequence of convex, C^2 functions $\{\gamma_\varepsilon\}$ converging to $\gamma(p) = |p_{n+1}|$ locally uniformly such that*

$$E_\varepsilon = \{(x,y,t) \in \mathbf{R}^2 \times [0,\alpha)|\varphi_\varepsilon(x,y,t) \le 0\}$$

converges to

$$\{(x,y,t) \in \mathbf{R}^2 \times [\partial,\infty)|y \le u_E(x,t)\}$$

in the Haussdorff distance as $\varepsilon \to 0$ if and only if $M \ge d^2/16$. If $M < d^2/16$, then the limit E_ε cannot be viewed as the graph of a single-valued function.

The class of approximation so that the convergence is valid is given explicitly in [28]. For technical reasons we assume that γ_ε is positively homogeneous of degree one and

$M < d^2/8$ but these assumptions are expected to be removed by further development of the theory. The threshold value $d^2/16$ is consistent with one obtained by the theory of nonlinear semigroups [35], [28]. It does not depend on the way of our approximation. The limit of E_ε for $M < d^2/16$ is also explicitly described in [28].

If the equation (3.1) is not a conservation law, it was not clear a priori what is a reasonable notion of a weak solution after u develops shocks. A typical example is

$$u_t - u(1 + |\nabla u|^2)^{1/2} = 0 \tag{3.8}$$

It controls the speed V of the graph Γ_t of $u(\cdot, t)$ as $V = y$ in xy-space. Evidently, u may develop jump discontinuity in finite time for some smooth initial data. Classical theory for viscosity solutions does not apply for such a problem. In [34] we introduced a notion of a *proper viscosity solution* which is a special viscosity solution [17] with a control of the speed of shock. It turns out this is a suitable notion to solve the initial value problem globally in time and it is consistent with the notion of an entropy solution when the equation is a conservation law. A proper viscosity solution is obtained by a vanishing viscosity method like an entropy solution [34].

The equation (3.4) is useful to track the evolution of the graph of a proper viscosity solution of (3.1). Although it is still work in progress, it is expected that the set like E_ε converges to the subgraph set of a (maximal) proper viscosity solution for sufficiently large M at least for bounded solutions. Thus it is reasonable to define the notion of a solution of the level-set equation of the form

$$\varphi_t - \varphi_y H(y, -\nabla_x \varphi/\varphi_y) = \infty |\nabla_{x,y} \varphi| \frac{\partial}{\partial y} \left(\frac{\varphi_y}{|\varphi_y|} \right). \tag{3.9}$$

For simplicity we consider a periodic function in x.

Definition 4.2. Assume that $\varphi = \varphi(x, y, t)$ is nonincreasing in y and φ is upper semicontinuous in $Q = \mathbf{T}^n \times \mathbf{R} \times (0, T)$. We say that φ is a viscosity subsolution of (3.9) if the height function

$$u_c^\sharp(x, t) = \sup\{y | \varphi(x, y, t) \geq c\}$$

of each super level-set $\{\varphi \geq c\}$ is a proper viscosity subsolution [34] of (3.1). The definition of supersolution for a lower semicontinuous function is obtained by replacing sup by inf, $\{\varphi \geq c\}$ by $\{\varphi \leq c\}$, and subsolution by supersolution.

Since proper viscosity solutions enjoy a weak comparison principle [34, Theorem 4.1], it is not difficult prove a comparison principle for (3.9).

Theorem 4.3. *Assume that H is continuous and that H satisfies*

$$|H(r, p) - H(r', p)| \leq C|r - r'|(|p| + 1)$$

for all $r, r' \in \mathbf{R}$ satisfying $|r|, |r'| \leq K$ and for all $p \in \mathbf{R}^n$ with C depending only on K. Let φ_1 and $-\varphi_2$ be upper semicontinuous in $\bar{Q}$. Assume that $\lambda H(r, p/\lambda)$ converges locally uniformly in $(r, p) \in \mathbf{R}^n \times \mathbf{R}$ as $\lambda \to 0$. Assume that $y \mapsto \varphi_i(x, y, t)(i = 1, 2)$ is nonincreasing. Assume that φ_1 and φ_2 are, respectively, a sub- and supersolution of (3.9) in Q. Then $\varphi_1 \leq \varphi_2$ in Q if $\varphi_1 \leq \varphi_2$ at $t = 0$ (provided that each super and sub level-set of φ_1 and φ_2 are bounded in y or $-y$ direction.)

Proof. Suppose that the conclusion were false. Then there is $c \in \mathbf{R}$ such that

$$\{\varphi_1 \geq c\} \cap \{\varphi_2 \leq c\}$$

contains an interior point in Q, where we abbreviate $\{(x, y, t) \in Q | \varphi_1(x, y, t) \geq c\}$ by $\{\varphi_1 \geq c\}$ e.t.c. as before. Thus there is $c' < c$ close to c such that

$$\{\varphi_1 \geq c\} \cap \{\varphi_2 \leq c'\} \neq \emptyset. \tag{3.10}$$

Since $\varphi_1 \leq \varphi_2$ at $t = 0$, then

$$\{\varphi_1|_{t=0} \geq c\} \cap \{\varphi_2|_{t=0} \leq c'\} = \emptyset \quad \text{in} \quad \mathbf{T}^n \times \mathbf{R}. \tag{3.11}$$

By definition

$$\begin{aligned} u^\sharp(x, t) &= \sup\{y | \varphi_1(x, y, t) \geq c\} \\ v_\sharp(x, t) &= \inf\{y | \varphi_2(x, y, t) \leq c'\} \end{aligned}$$

are, respectively, a proper viscosity sub and supersolution of (3.1).

Both functions are bounded. By (3.11) $u^\sharp < v_\sharp$ at $t = 0$. We now apply the weak comparison principle [34, Theorem 4.1] to conclude that $u^\sharp < v_\sharp$ for all $t \in [0, T)$. However, this contradicts (3.10). So we conclude that $\varphi_1 \leq \varphi_2$ in Q. (A similar idea is found in [36, Chapter 5] to discuss the relation of comparison principles of solutions of a level-set equation and of their slices.) □

If each level-set of φ is bounded in y direction and the difference of upper bound and lower bound is bounded independent of levels, it is expected that for a sufficiently large M equation (3.4) and (3.9) are the same for such φ's although the definition of a solution for (3.4) is not yet established.

The equation (3.4) is also convenient to calculate a proper viscosity solution numerically as developed in [70], [69].

4 Gradient flow of total variation

The gradient flow of total variation is of the form

$$u_t = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \tag{4.1}$$

for $u = u(x,t),\ x \in \Omega \subset \mathbf{R}^n$ with a bounded domain Ω. As mentioned for (1.1) under the Dirichlet (or Neumann) boundary condition for this problem classical nonlinear semigroup theory yields a global unique solution [53], [31] for the initial-boundary value problem for (4.1). In [49] the global unique solvability is established for equations including (4.1) with time dependent Dirichlet boundary data. It is also shown in [49] that the solution tends to a minimizer of total variation as time tends infinity when the boudary data is time independent. Note that (4.1) is different from the level-set mean curvature flow equation (1.4) [15], [20] where the singularity at $\nabla\varphi = 0$ is weaker than (4.1) so that equation (1.4) is still a local equation. Other than L^2 theory, L^1-theory has been esteblished in [3], [1], [2] and concludes that solution semigroup is L^1-contraction as well as L^2 and L^∞-contraction. However, a detailed behaviour of a facet is not well studied for $n \geq 2$ except [7]. What is known is that the facet-stay-as-facet hypothesis is no longer valid. In other words u_t may not be a constant on a facet of slope zero [8], [9] when the space dimension $n \geq 2$.

For applications to both image processing and multigrain motion [55], [72] it is important to consider spatially inhomogeneous equation of the form

$$b(x)u_t = \mathrm{div}(a(x)\nabla u/|\nabla u|) \tag{4.2}$$

with $a(x) > 0,\ b(x) > 0$. In [53], [31] one dimensional version of (4.2) with Dirchlet condition is studied. It provides a necessary and sufficient condition that a facet (called a plateau in [53], [31]) may break. Note that this problem is still in a realm of nonlinear semigroup theory for maximal monotone operators in L^2. Thus the notion of solution is a priori defined. When a is piecewise linear with $b \equiv 1$, the whole evolution for piecewise constant initial data with jumps included in the singular set of a is easily computed by solving ODEs; here, facet-stay-facet-hypothesis in still valid [31]. In [31] the evolution of piecewise constant functions is studied in detail. In the meanwhile the Allen-Cahn type equation with top order term $\mathrm{div}(\nabla u/|\nabla u|)$ and with obstacle type double well potential has been studied in detail. For example, classification of a stationary solution and its stability is discussed in [52], [65] for $n = 1$. Recently, a higher dimensional problem is also studied by M. Kimura and K. Shirakawa.

There are several analytic results reflecting constraint $\int |u-u_0|^2 dx = \text{const}$. For the equation

$$u_t = \text{div}(a(x)\nabla u/|\nabla u|) - b(u-u_0) \quad (b>0)$$

the global unique solvability has been established with Neumann data; see [14]. However, the gradient flow of total variation with the constraint $\int |u-u_0|^2 = \text{const}$ seems to be not studied from the mathematical analysis point of view. For the value constraint problem like (1.9) little is known. In [40] the initial value problem for (1.9) with the Dirichlet problem is studied when $n=1$ and the constraint is $u_1^2+u_2^2=1$ (with $u_3 \equiv 0$). For a piecewise constant initial data it is shown that the solution tends to some stationary solution in finite time [40]. It is expected that a solution exists globally-in-time for a general initial data by suitable interpretation of the equation (1.9). Recently, a local solvability has been proved in [39] when initial data is smooth and its total variation is small under periodic boundary conditions.

References

[1] F. Andreu, C. Ballester, V. Caselles and J. M. Mázon, *Minimizing total variation flow*, Diff. Int. Eq., **14** (2001), 321-360.

[2] F. Andreu, C. Ballester, V. Caselles and J. M. Mázon *The Dirichlet problem for the total variation flow*, J. Funct. Anal., **180** (2001), 347-403.

[3] F. Andreu-Vaillo, V. Caselles and J. M. Mázon, *Existence and uniqueness of a solution for a parabolic quasilinear problem for linear growth functionals with L^1 data*, Math. Ann., **322** (2002), 139-206.

[4] S. B. Angenent and M. E. Gurtin, *Multiphase thermomechanics with interfacial structre 2. Evolution of an isothermal interface*, Arch. Rational Mech. Anal., **108** (1989), 323-391.

[5] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach spaces*, Noordnoff, Groningen, 1976.

[6] G. Barles, H. M. Soner and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., **31** (1993), 439-469.

[7] G. Bellettini, V. Caselles and M. Novaga, *The total variation flow in $\mathbf{R}^N$*, J. Differential Equations., to appear.

[8] G. Bellettini and M. Novaga, *Approximation and comparison for non-smooth anisotropic motion by mean curvature in* $\mathbf{R}^N$, Math. Mod. Methods. Appl. Sci., **10** (2000), 1-10.

[9] G. Bellettini, M. Novaga and M. Paolini, *Facet-breaking for three dimensional crystal evolving by mean curvature*, Interfaces and Free Boundaries., **1** (1999), 39-55.

[10] H. Brezis, *Operateurs Maximaux Monotone*, North-Holland, Amsterdam, (1973).

[11] H. Brezis and A. Pazy, *Convergence and approximation of semigroups of nonlinear operators in Banach spaces*, J. Funct. Anal., **9** (1972), 63-74.

[12] F. Cao, *Geometric Curve Evolution and Image Processing*, Lecture Notes in Math., Springer 1805, 2003.

[13] C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, Part I, §49, Holden-Day, San Francisco, London, Amsterdam, 1965.

[14] Y. Chen and T. Wunderli, *Adaptive total variation for image restoration in BV space*, J. Math. Anal. Appl., **272** (2002), 117-137.

[15] Y. -G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom., **33** (1991), 749-786.

[16] K. -S. Chou and X. -P. Zhu, *The Curve Shortening Problem*, Chapman and Hall / CRC, 2001.

[17] M. G. Crandall, H. Ishii and P. -L. Lions, *User's guide to viscosity solutions of second order partial differential equation*, Bull. Amer. Math. Soc., **27** (1992), 1-67.

[18] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Grundlehren der Mathematischen Wissenschaften 325, Springer, Berlin 2000.

[19] L. C. Evans, *A geometric interpretation of the heat equation with multivalued initial data*, SIAM J. Math. Anal., **27** (1996), 932-958.

[20] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Diff. Geom., **33** (1991), 635-681.

[21] T. Fukui and Y. Giga, *Motion of a graph by nonsmooth weighted curvature*, in World Congress of Nonlinear Analysis '92, V. Lakshmikantham, ed., de Gruyter, Berlin, vol.I, 1996, pp.47-56.

[22] M.-H. Giga and Y. Giga, *Evolving graphs by singular weighted curvature*, Arch. Rational Mech. Anal., **141** (1998), 117-198.

[23] M.-H. Giga and Y. Giga, *Remark on convergence of evolving graphs by nonlocal curvature*, in Progress in Partial Differential Equations, vol.1, H. Amann et al. eds., Pitman Reseach Notes in Math. Ser., **383** (1998), pp.99-116.

[24] M.-H. Giga and Y. Giga, *A subdifferential interpretation of crystalline motion under nonuniform driving force*, in Proc. of the International Conference in Dynamical Systems and Differential Equations, Springfield Missouri, (1996), "Dynamical Systems and Differential Equations", W.-X. Chen and S.-C. Hu eds., Southwest Missouri Univ. vol.**1** (1998), pp.276-287.

[25] M.-H. Giga and Y. Giga, *Stability for evolving graphs by nonlocal weighted curvature*, Commum. in Partial Differential Equations., **24** (1999), 109-184.

[26] M.-H. Giga and Y. Giga, *Crystalline and level set flow - Convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane*, in Free Boundary Problems: Theory and Applications, (N. Kenmochi ed.), Gakuto International Series, Math. Sciences and Appl., vol **13** (2000), pp.64-79.

[27] M.-H. Giga and Y. Giga, *Generalized motion by nonlocal curvature in the plane*, Arch. Rational. Mech. Anal., **159** (2001), 295-333.

[28] M.-H. Giga and Y. Giga, *Minimal vertical singular diffusion preventing overturning for the Burges equation*, Contem. Math., to appear.

[29] M. -H. Giga and Y. Giga, *A PDE approach for motion of phase-boundaries by a singular interfacial energy*, Hokkaido Univ, Preprint Series in Math., ♯584 (2003).

[30] M.-H. Giga, Y. Giga and H. Hontani, *Selfsimilar expanding solutions in a sector for a crystalline flow*, in preparation.

[31] M.-H. Giga, Y. Giga and R. Kobayashi, *Very singular diffusion equations*, Proc. of Taniguchi Conf. on Math., Advanced Studies in Pure Mathematics., **31** (2001), pp. 93-125.

[32] Y. Giga, *A level set method for surface evolution equations*, Sūgaku **47** (1995), 321-340; English translation, Sugaku Exposition, **10** (1997), 217-241.

[33] Y. Giga, *Anisotropic curvature effect in interface dynamics*, Sūgaku **52** (2000), 113-127; English translation, to appear.

[34] Y. Giga, *Viscosity solutions with shocks*, Comm. Pure Appl. Math., **55** (2002), 431-480.

[35] Y. Giga, *Shocks and very strong vertical diffusion*, Proc. of international conference on partial differential equations in celebration of the seventy fifth birthday of Professor Louis Nirenberg, Taiwan, 2001.

[36] Y. Giga, *Surfaces Evolution Equations - a level set method*, Hokkaido University Technical Report Series in Mathematics, ♯71, Sapporo, 2002, also Lipschitz Lecture Notes, **44**, University of Bonn, 2002.

[37] Y. Giga and S. Goto, *Geometric evolution of phase-boundaries*, in On the evolution of phase-boundaries, M. Gurtin and G. McFadden eds., IMA vol. Math. Appl. Springer **43** (1992), 51-65.

[38] Y. Giga, M. E. Gurtin and J. Matias, *On the dynamics of crystalline motions*, Japan J. Indust. Appl. Math., **15** (1998), 7-50.

[39] Y. Giga, Y. Kashima and N. Yamazaki, *Local solvability of a constrained gradient system of total variation*, preprint.

[40] Y. Giga and R. Kobayashi, *On constrained equations with singular diffusivity*, Hokkaido Univ. Preprint Series in Math., ♯588 (2003).

[41] Y. Giga, M. Paolini and P. Rybka, *On the motion by singular interfacial energy*, Japan J. Indust. Appl. Math., **18** (2001), 231-248.

[42] Y. Giga and P. Rybka, *Quasi-static evolution of 3-D crystal grown from supersaturated vapor*, Diff. Int. Eq., **15** (2002), 1-15.

[43] Y. Giga and P. Rybka, *Berg's effect*, Adv. Math. Sci. Appl., to appear.

[44] Y. Giga and M. -H. Sato, *A level set approach to semicontinuous viscosity solutions for Cauchy problems*, Comm. Partial Differential Equations., **26** (2001), 813-839.

[45] P. M. Girão, *Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature*, SIAM J. Numer. Anal., **32** (1995), 886-899.

[46] P. M. Girão and R. V. Kohn, *Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature,* Numer. Math., **67** (1994), 41-70.

[47] M. E. Gurtin, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Oxford, Clarendon Press., 1993.

[48] H. Hontani, Mi-Ho. Giga, Y. Giga and K. Deguchi, *A computation of a crystalline flow starting from non-admissible polygon using expanding selfsimilar solutions*, preprint.

[49] R. Hardt and X. Zhou, *An evolution problem for linear growth functionals*, Comm. Partial Differential Equations., **19** (1994), 1879-1907.

[50] K. Ishii and H. M. Soner, *Regularity and convergence of crystalline motion*, SIAM J. Math. Anal., **30** (1999), 19-37.

[51] Y. Kashima *A subdifferential formulation of fourth order singular diffusion equations*, Adv. Math. Sci Appl., to appear.

[52] N. Kenmochi and K. Shirakawa, *A variational inequality for total variation functional with constraint*, Nonlinear Anal., **46** (2001), 435-455.

[53] R. Kobayashi and Y. Giga, *Equations with singular diffusivity*, J. Stat. Phys., **95** (1999), 1187-1220.

[54] R. Kobayashi and Y. Giga, *On anisotropy and curvature effects for growing crystals*, Japan J. Indust. Appl. Math., **18** (2001), 207-230.

[55] R. Kobayashi, J. A. Warren and W. C. Carter, *A Continuum Model of Grain Boundaries*, Physica D, **140** (2000), 141-150.

[56] Y. Kōmura, *Nonlinear semi-groups in Hilbert space*, J. Math. Soc. Japan., **19** (1967), 493-507.

[57] P. D. Lax, *Hyperbolic Systems of Conservation Laws and Mathematical Theory of Shock Waves*, SIAM, Philadelphia, Pa., 1973.

[58] S. Osher, *A level set formulation for the solution of the Dirichlet problem for Hamilton-Jacobi equations*, SIAM J. Math. Anal., **24** (1993), 1145-1152.

[59] S. Osher and R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*, Applied Math. Ser. 153 Springer, 2003.

[60] S. Osher and J. A. Sethian, *Front propagation with curvature dependent speed: Algorithm based on Hamilton-Jacobi formulations*, J. Comput. Phys., **79** (1988), 12-49.

[61] T. Preusser and M. Rumpf, *A level set method for anisotropic geometric diffusion in 3D image processing*, SIAM J. Appl. Math., **62** (2002), 1772-1793.

[62] L. I. Rudin, S. Osher and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D., **60** (1992), 259-268.

[63] G. Sapiro, *Geometric Partial Differential Equations and Image Analysis*, Cambridge University Press, United Kingdom, 2001.

[64] J. A. Sethian, *Level Set Methods, Evolving Interfaces in Geomety, Fluid Mechanics, Computer Vision, and Materials Science*, Cambridge Univ. Press., 1996.

[65] K. Shirakawa, *Parabolic variational inequality associated with the total variation functional*, Nonlinear Anal., **47** (2001), 3195-3206.

[66] H. Spohn, *Surface dynamics below the roughening transition*, J. Phys. I France., **3** (1993), 69-81.

[67] B. Tang, G. Sapiro and V. Caselles, *Color image enhancement via chromaticity diffusion*, IEEE Trans. on Image Processing, **10** (2001), 701-707.

[68] J. Taylor, *Constructions and conjectures in crystalline nondifferential geometry*, in Differential Geometry, B. Lawson and K. Tanenblat, eds., Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monograph Surveys Pure Appl. Math., **52** (1991), 321-336.

[69] Y.-H. R. Tsai and Y, Giga, *A numerical study of anisotropic crystal growth with bunching under very singular vertical diffusion*, preprint.

[70] Y.-H. R. Tsai, Y. Giga and S. Osher, *A level set approach for computing discontinuous solutions of a class of Hamilton-Jacobi equations*, Math. Comp., **72** (2003), 159-181.

[71] J. Watanabe, *Approximation of nonlinear problems of a certain type, in Numerical Analysis of Evolution Equations*, H. Fujita and M. Yamaguti, eds., Lecture Notes in Num. Appl., 1, Kinokuniya, Tokyo, (1979), pp. 147-163.

[72] J. A. Warren, R. Kobayashi and W. C. Carter, *Modeling grain boundaries using a Phase-field technique*, J. Cryst. Growth., **211** (2000), 18-20.

[73] J. Yunger, *Facet stepping and motion by crystalline curvature*, PhD Thesis, Rutgers University, 1998.