Weighted Sobolev-Lieb-Thirring inequalities

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Abstract. We give a weighted version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions. In the proof of our result we use \( \varphi \)-transform of Frazier-Jawerth.

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1 Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring inequality.

Theorem 1.1 ([2]). Let \( n \in \mathbb{N}, s > 0 \) and \( p \) with

\[
\max \left( 1, \frac{n}{2s} \right) < p \leq 1 + \frac{n}{2s}.
\]

Then there exists a positive constant \( c = c(p, n, s) \) such that for every family \( \{\phi_i\}_{i=1}^N \) in \( H^s(\mathbb{R}^n) \) which is orthonormal in \( L^2(\mathbb{R}^n) \), we have

\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \|(-\Delta)^{s/2}\phi_i\|^2
\]

where

\[
\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.
\]

In this theorem \( H^s(\mathbb{R}^n) \) denotes the Sobolev space of order \( s \) and \( \| \cdot \| \) is the norm of \( L^2(\mathbb{R}^n) \).

In [8] Lieb and Thirring proved this theorem for \( s = 1 \) and applied it to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved (1) for \( s \in \mathbb{N} \) under
the suborthonormal condition on \( \{ \phi_i \} \), where \( \{ \phi_i \}_{i=1}^N \) in \( L^2(\mathbb{R}^n) \) is called suborthonormal if the inequality
\[
\sum_{i,j=1}^N \xi_i \overline{\xi}_j (\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2
\]
holds for all \( \xi_i \in \mathbb{C}, i = 1, \ldots, N \) ([4]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations (c.f. [13]). In this paper we shall give a weighted version of (1) under suborthonormal condition on \( \{ \phi_i \} \). In the proof of our theorem we shall use Frazier-Jawerth’s \( \varphi \)-transform ([3]).

For the statement of our result we need to recall the definition of \( A_p \)-weights (c.f. [5], [10]). By a cube in \( \mathbb{R}^n \) we mean a cube which sides are parallel to coordinate axes. Let \( w \) be a non-negative, locally integrable function on \( \mathbb{R}^n \). We say that \( w \) is an \( A_p \)-weight for \( 1 < p < \infty \) if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C
\]
for all cubes \( Q \subset \mathbb{R}^n \). The infimum of the constant \( C \) is called the \( A_p \)-constant of \( w \).

For example, \( w(x) = |x|^\alpha \) is an \( A_p \)-weight when \( -n < \alpha < n(p-1) \).

We say that \( w \) is an \( A_1 \)-weight if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(y) \, dy \leq Cw(x) \quad \text{a.e. } x \in Q
\]
for all cubes \( Q \subset \mathbb{R}^n \). The infimum of the constant \( C \) is called the \( A_1 \)-constant of \( w \).

Let \( A_p \) be the class of \( A_p \)-weights. The inclusion \( A_p \subset A_q \) holds for \( p < q \). For a nonnegative, locally integrable function \( w \) on \( \mathbb{R}^n \) we define
\[
L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.
\]

For \( \nu \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) the cube \( Q \) defined by
\[
Q = Q_{\nu k} = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1, i = 1, \ldots, n\}
\]
is called a dyadic cube in \( \mathbb{R}^n \). Let \( Q \) be the set of all dyadic cubes in \( \mathbb{R}^n \). For any \( Q \in Q \) there exists a unique \( Q' \in Q \) such that \( Q \subset Q' \) and the side-length of \( Q' \) is double of that of \( Q \). We call \( Q' \) the parent of \( Q \).

For \( s > 0 \) and \( f \in C^\infty_0(\mathbb{R}^n) \) we define via inverse Fourier transform
\[
(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).
\]
Let \( w \in A_2 \) and \( \mathcal{H}^s(w) \) be the completion of \( C^\infty_0(\mathbb{R}^n) \) with respect to the norm
\[
\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx + \|f\|^2 \right\}^{1/2}.
\]

2
We remark that for \( f \in C_0^\infty(\mathbb{R}^n) \) we have
\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx < \infty
\]
because
\[
|(-\Delta)^{s/2} f(x)| \leq \frac{c}{(1 + |x|)^n} \quad (x \in \mathbb{R}^n)
\]
and
\[
\int_{\mathbb{R}^n} \frac{w(x)}{(1 + |x|)^{2n}} \, dx < \infty
\]
(c.f. [10, p.209]).

Let \( f \in \mathcal{H}^s(w) \) and \( \{f_i\}_{i=1}^\infty \) be a sequence in \( C_0^\infty(\mathbb{R}^n) \) such that \( \|f - f_i\|_{\mathcal{H}^s(w)} \to 0 \ (i \to \infty) \). This means that there exist \( g_1 \in L^2(\mathbb{R}^n) \) and \( g_2 \in L^2(w) \) such that
\[
\|g_1 - f_i\| \to 0 \quad \text{and} \quad \int_{\mathbb{R}^n} |g_2(x) - (-\Delta)^{s/2} f_i(x)|^2 w(x) \, dx \to 0
\]
as \( i \to \infty \). We denote \((-\Delta)^{s/2} f = g_2 \). We remark that \( g_1 \equiv 0 \) means \( g_2 \equiv 0 \). In fact, for any \( \varphi \in C_0^\infty(\mathbb{R}^n) \), we have
\[
\int_{\mathbb{R}^n} g_2 \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} f_i (-\Delta)^{s/2} \varphi \, dx = 0.
\]
Hence we have \( g_2 \equiv 0 \). This means that we can identify \( \mathcal{H}^s(w) \) as a subspace of \( L^2(\mathbb{R}^n) \).

The following is the main result of this paper.

**Theorem 1.2.** Let \( n \in \mathbb{N}, s > 0, \) and
\[
\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}.
\]
Let \( w \in A_2 \). If \( 2s < n \), then we assume that \( w^{-n/(2s)} \in A_{n/(2s)} \). If \( 2s \geq n \), then we assume that \( w^{-n/(2s)} \in A_p \) and
\[
\int_{Q'} w \, dx \leq 2^{2s} \int_Q w \, dx
\]
for all dyadic cubes \( Q \in Q \) and its parent \( Q' \).

Then there exists a positive constant \( c \) such that for every family \( \{\phi_i\}_{i=1}^N \) in \( \mathcal{H}^s(w) \) which is suborthonormal in \( L^2(\mathbb{R}^n) \), we have
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx,
\]

3
where
\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]
and \( c \) depends only on \( n, s, p, A_2 \)-constant of \( w \), and \( A_{n/(2s)} \) or \( A_p \)-constant of \( w^{-n/(2s)} \).

When \( 2s < n \), an example of weight function \( w \) is given by \( w(x) = |x|^\alpha \) for \( -n + 2s < \alpha < 2s \). When \( 2s > n \), an example of weight function \( w \) is given by \( w(x) = |x|^\alpha \) for \( 0 \leq \alpha < \min\{2s - n, n\} \) (c.f.[12, Section 4]). When \( 2s = n \), the condition (2) means \( w \) is equivalent to a constant almost everywhere (c.f.[12, Proposition 4.1]).

2 Preliminaries

Let \( \psi \) be a function which satisfies the following conditions.

(A1) \( \psi \in \mathcal{S}(\mathbb{R}^n) \).

(A2) \( \text{supp} \, \hat{\psi} \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \} \)

(A3) \( |\hat{\psi}(\xi)| \geq c > 0 \) if \( \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \).

(A4) \( \sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 = 1 \) for all \( \xi \neq 0 \).

For \( \nu \in \mathbb{Z}, k \in \mathbb{Z}^n \) and \( Q = Q_{\nu k} \), we set
\[ \psi_Q(x) = 2^{\nu n/2} \psi(2^\nu x - k) \quad (x \in \mathbb{R}^n) \]

Let \( M \) be the Hardy-Littlewood maximal operator, that is,
\[ M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \]
where \( f \) is a locally integrable function on \( \mathbb{R}^n \) and the supremum is taken over all cubes \( Q \) which contain \( x \).

**Proposition 2.1.**

(i) Let \( 1 < p < \infty \) and \( w \) be a non-negative locally integrable function on \( \mathbb{R}^n \). Then there exists a positive constant \( c \) such that
\[ \int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx \]
for all \( f \in L^p(w) \) if and only if \( w \in A_p \). The constant \( c \) depends only on \( n, p \) and \( A_p \)-constant of \( w \).
(ii) Let \(1 < p < \infty\) and \(w \in A_p\). Then there exists a \(q \in (1, p)\) such that \(w \in A_q\).

(iii) Let \(0 < \tau < 1\) and \(f\) be a locally integrable function on \(\mathbb{R}^n\) such that \(M(f)(x) < \infty\) a.e.. Then \((M(f))^\tau \in A_1\) and the \(A_1\)-constant of \((M(f))^\tau\) depends only on \(n\) and \(\tau\).

(iv) Let \(1 \leq p < \infty\) and \(w \in A_p\). Then there exists a positive constant \(c\) such that
\[
\int_{2Q} w \, dx \leq c \int_Q w \, dx
\]
for all cubes \(Q \in \mathbb{R}^n\), where \(2Q\) denotes the double of \(Q\) and \(c\) depend only on \(n\) and \(A_p\)-constant of \(w\).

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3 Proof of Theorem 1.2

The suborthonormal condition on \(\{\phi_i\}\) is equivalent to the inequality
\[
\sum_{i=1}^{N} |(\phi_i, f)|^2 \leq \|f\|^2
\]
for all \(f \in L^2(\mathbb{R}^n)\) (c.f.[1, p57]).

We shall prove the inequality
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq cK^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx
\]
under the assumption
\[
\sum_{i=1}^{N} |(\phi_i, f)|^2 \leq K \|f\|^2
\]
for all \(f \in L^2(\mathbb{R}^n)\) where \(K\) is a positive constant. This is equivalent to the statement of Theorem 1.2. We remark that \(K\) may depend on \(\{\phi_i\}\). For example, the inequality (3) says that
\[
\left\{ \int_{\mathbb{R}^n} |\phi|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi|^2 w \, dx
\]
holds for all $\phi \in H^s(w)$ under suitable condition on $s, p, n$ and $w$ because

$$|(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C^\infty_0(\mathbb{R}^n), i = 1, \ldots, N$. Let

$$V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2s(p-1))}$$

where the value of the constant $\delta_1 > 0$ will be given later. Since $\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$ is a bounded function with compact support and $w^{n/(2s(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have

$$\int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty.$$  

We may also assume that

$$0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx.$$  

By (ii) of Proposition 2.1 there exists a constant $\kappa$ such that $1 < \kappa < p$ and $w^{-n/(2s)} \in A_p/\kappa$. We set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty.$$  

Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 2.1.

We have the following lemmas.

**Lemma 3.1.** For $s > 0$ and $w \in A_2$ there exists a positive constant $\alpha$ such that

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx$$

for all $f \in C^\infty_0(\mathbb{R}^n)$, where $\alpha$ is given by

$$\alpha^{-1} = c \max_{|\sigma| \leq n} \| \partial^\sigma \hat{\psi} \|^2_\infty$$

and $c$ is a constant depending only on $n, s$ and $A_2$-constant of $w$.

**Lemma 3.2.** For $v \in A_2$ there exist positive constants $\beta$ and $\beta'$ such that

$$\beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \leq \int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx$$

for all $f \in C^\infty_0(\mathbb{R}^n)$, where $\beta$ is given by

$$\beta = c \max_{|\sigma| \leq n} \| \partial^\sigma \hat{\psi} \|^2_\infty$$

and $c$ is a constant depending only on $n$ and $A_2$-constant of $v$.  

6
The proof of Lemmas 3.1 and 3.2 are in [11, Proposition 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader’s convenience because the dependence of $\psi$ in $\alpha$ and $\beta$ is not explained in [11].

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f|^2 V\, dx \leq \int_{\mathbb{R}^n} |f|^2 v\, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v\, dx,$$

where we used Lemma 3.2. Hence by Lemma 3.1

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 \, dx - \int_{\mathbb{R}^n} V|f|^2 \, dx \geq \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w\, dx - \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v\, dx.
$$

(7)

Now we set

$$\mathcal{I} = \left\{ Q \in \mathcal{Q} : \beta \int_Q v\, dx > \alpha |Q|^{-2s/n} \int_Q w\, dx \right\}.
$$

(8)

Let $\{\mu_k\}_{1 \leq k}$ be the non-decreasing rearrangement of

$$\left\{ \alpha |Q|^{-2s/n-1} \int_Q w\, dx - \beta |Q|^{-1} \int_Q v\, dx \right\}_{Q \in \mathcal{I}}.
$$

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When

$$\mu_k = \alpha |Q|^{-2s/n-1} \int_Q w\, dx - \beta |Q|^{-1} \int_Q v\, dx,$$

we define $\Psi_k = \psi_Q$. Then we have by (7)

$$\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 \, dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V|\phi_i|^2 \, dx \geq \sum_{i=1}^N \sum_{Q \in \mathcal{Q}} |(\phi_i, \psi_Q)|^2 \left\{ \alpha |Q|^{-2s/n-1} \int_Q w\, dx - \beta |Q|^{-1} \int_Q v\, dx \right\}
$$

$$\geq \sum_{i=1}^N \sum_{k} \mu_k |(\phi_i, \Psi_k)|^2 = \sum_{k} \mu_k \sum_{i=1}^N |(\phi_i, \Psi_k)|^2
$$

(9)

$$\geq -K\|\psi\|^2 \sum_k |\mu_k| \geq -K\|\psi\|^2 \left( \sum_k |\mu_k|^{\gamma} \right)^{1/\gamma},$$

where $\gamma = p - n/(2s) \in (0, 1]$ and we used (4).

Now the following lemma holds.
Lemma 3.3.

\[ \sum_k |\mu_k|^\gamma \leq c \int_{\mathbb{R}^n} \rho^p w^{-(n/(2s))} \, dx, \]

where \( c \) is given by

\[ c = c' \max_{|\sigma| \leq n} \| \partial^\sigma \hat{\psi} \|^{n/s+2p}_\infty \]

and \( c' \) depends only on \( n, s, p \) and \( w \).

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (6) the last quantity in (10) is estimated from below by

\[ -cK \left( \int_{\mathbb{R}^n} V^p w^{-(n/(2s))} \, dx \right)^{1/\gamma} \]

\[ = -cK \delta_1^{p/(p-n/(2s))} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{1/(p-n/(2s))}, \]

where

\[ c = c' \| \psi \|^2 \max_{|\sigma| \leq n} \| \partial^\sigma \hat{\psi} \|^{(4ps+2n)/(2ps-n)}_\infty \]

and \( c' \) depends only on \( n, s, p \) and \( w \). We may take the infimum of the above constant with respect to possible \( \psi \) and replace \( c \) by this infimum.

Let

\[ \delta_1 = \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-n/(2s)-1)/n}, \]

where \( \delta_2 \) is a positive constant. Then we have by (9)

\[ \sum_{i=1}^N \int_{\mathbb{R}^n} \big| (-\Delta)^{s/2} \phi_i \big|^2 w \, dx \]

\[ \geq \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ - cK \delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ = \{ \delta_2 - c\delta_2^{p/(p-n/(2s))} \} K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n}. \]

If we take \( \delta_2 \) small enough, then we get the inequality (3) because \( 1 < p/(p-n/(2s)) \).

Next we shall show (3) for \( \phi_i \in \mathcal{H}^s(w), i = 1, \ldots, N \). First we show

\[ \mathcal{H}^s(w) \subset L^{2p/(p-1)}(w, n/(2s(p-1))). \]
Let $h \in \mathcal{H}^s(w)$. Then there exists a sequence $\{h_m\}_{m=1}^{\infty} \subset C^\infty_0(\mathbb{R}^n)$ such that 
\[ \|h - h_m\|_{\mathcal{H}^s(w)} \to 0 \ (m \to \infty). \]
Since we proved that (5) holds for $h_m \in C^\infty_0(\mathbb{R}^n)$, we get 
\[ \left\{ \int_{\mathbb{R}^n} |h_m|^{2p/(p-1)}w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c\|h_m\|^{4sp/n - 2}\int_{\mathbb{R}^n} |(-\Delta)^{s/2}h_m|^2 w \, dx, \]
where $c$ does not depend on $h_m$. Since $4sp/n - 2 > 0$ and $\{h_m\}$ is a Cauchy sequence in $\mathcal{H}^s(w)$, the above inequality says that $\{h_m\}$ is a Cauchy sequence in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. Let $g$ be the limit of $\{h_m\}$ in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. For any compact set $E$ in $\mathbb{R}^n$ we have 
\[ \int_E |g - h_m| \, dx \leq \left( \int_E |g - h_m|^{2p/(p-1)}w^{n/(2s(p-1))} \, dx \right)^{(p-1)/(2p)} \times \left( \int_E w^{-n/(2s(p+1))} \, dx \right)^{(p+1)/(2p)}. \]
Since $w^{-n/(2s)}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ or $w^{-n/(2s)} \in A_p$, we get $h_m \to g$ in $L^1_{loc}(\mathbb{R}^n)$ as $m \to \infty$. Hence we have $g = h$ and (11). Furthermore we have 
\[ \left\{ \int_{\mathbb{R}^n} |h|^{2p/(p-1)}w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c\|h\|^{4sp/n - 2}\int_{\mathbb{R}^n} |(-\Delta)^{s/2}h|^2 w \, dx. \]
We fix a positive number $\varepsilon$. Let $\chi_1, \ldots, \chi_N$ be functions in $C^\infty_0(\mathbb{R}^n)$ such that 
\[ \sum_{i=1}^{N} \|\phi_i - \chi_i\|_{\mathcal{H}^s(w)}^2 < \varepsilon. \]
Now the inequalities 
\[ \sum_{i=1}^{N} |(\chi_i, f)|^2 \leq 2\sum_{i=1}^{N} |(\chi_i - \phi_i, f)|^2 + 2\sum_{i=1}^{N} |(\phi_i, f)|^2 \]
\[ \leq 2\sum_{i=1}^{N} \|\chi_i - \phi_i\|^2 \|f\|^2 + 2K\|f\|^2 \leq 2(K + \varepsilon)\|f\|^2 \]
hold for all \( f \in L^2(\mathbb{R}^n) \). On the other hand

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq \left\{ \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{(p-1)/p} \right\}^{2s/p/n} \\
\leq N^{2sp/n-1} \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{(p-1)/p} \\
\leq cN^{2sp/n-1} \sum_{i=1}^{N} \|\phi_i - \chi_i\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w \, dx \\
\leq cN^{2sp/n-1} e^{2sp/n-1} N \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w \, dx \\
\leq cN^{2sp/n-1} e^{2sp/n} ,
\]

where we used (12).

Therefore

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq \left\{ \int_{\mathbb{R}^n} \left( 2 \sum_{i=1}^{N} |\phi_i - \chi_i|^2 + 2 \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq 2^{2sp/n} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{(p-1)/p} + \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{(p-1)/p} \\
\leq 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} + 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq c^{2sp/n-1} N^{2sp/n-1} e^{2sp/n} + c^{2sp/n-2} (K + e)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 w \, dx ,
\]
where we used (13) and (3) for \( \chi_i \). Since
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |(\Delta)^{s/2} \chi_i|^2 w \, dx
\]
\[
\leq 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(\Delta)^{s/2} \chi_i - (\Delta)^{s/2} \phi_i|^2 w \, dx + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(\Delta)^{s/2} \phi_i|^2 w \, dx
\]
\[
\leq 2 \varepsilon + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(\Delta)^{s/2} \phi_i|^2 w \, dx,
\]
we have by (14)
\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n}
\]
\[
\leq c^{2s\varepsilon_{p-1}n-1} \alpha^{2s\varepsilon_{p-1}n-1} \varepsilon^{2s\varepsilon_{p-1}} + c^{2s\varepsilon_{p-1}n-1} (K + \varepsilon)^{2s\varepsilon_{p-1}n-1} \varepsilon
\]
\[
+ c^{2s\varepsilon_{p-1}n-1} (K + \varepsilon)^{2s\varepsilon_{p-1}n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(\Delta)^{s/2} \phi_i|^2 w \, dx.
\]
Since we can take \( \varepsilon \) arbitrary small, we conclude
\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n}
\]
\[
\leq c^{2s\varepsilon_{p-1}n-1} K^{2s\varepsilon_{p-1}n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(\Delta)^{s/2} \phi_i|^2 w \, dx.
\]
Hence we get (3).

### 4 Proof of Lemma 3.3

The arguments of the proof is similar to those in [11] and [12]. First we consider the case \( n > 2s \). For \( \lambda > 0 \) we set
\[
\mathcal{I}_\lambda = \{ Q \in Q : \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx < -\lambda \}.
\]
Then we have for \( Q \in \mathcal{I}_\lambda \)
\[
\alpha |Q|^{-2s/n-1} \int_Q w \, dx < |Q|^{-1} \int_Q (\beta v - \lambda)_+ \, dx,
\]
where
\[
(\beta v - \lambda)_+(x) = \max\{0, \beta v(x) - \lambda\}.
\]
Since \( p = n/(2s) + \gamma, \gamma \in (0, 1] \), and
\[
\beta^{-p}\gamma \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx \, \lambda^{\gamma-1} \, d\lambda \leq \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx < \infty,
\]
we have
\[
\int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx < \infty
\]
for all \( \lambda > 0 \). By the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) and (ii) of Proposition 2.1, there exists a \( \kappa' \in (1, n/(2s)) \) such that \( w^{-n/(2s)} \in A_{n/(2s\kappa')} \). We set
\[
v_\lambda^\star(x) = M((\beta v - \lambda)^{\kappa'})_+(x)^{1/\kappa'}.
\]
Then
\[
\int_{\mathbb{R}^n} (v_\lambda^\star)^{n/(2s)} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx < \infty
\]
and \( v_\lambda^\star \in A_1 \) by (iii) of Proposition 2.1, where \( c_1 \) depends only on \( n, s \) and \( A_{n/(2s)} \) constant of \( w^{-n/(2s)} \).

We can show that \( \mathcal{I}_\lambda \) is a finite set as follows. Let \( Q \in \mathcal{I}_\lambda \). Then we have
\[
\alpha |Q|^{-2s/n} \int_Q w \, dx \leq \int_Q v_\lambda^\star \, dx
\]
\[
\leq \left\{ \int_Q (v_\lambda^\star)^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} \left\{ \int_Q w^{n/(n-2s)} \, dx \right\}^{(n-2s)/n}.
\]
Since \( w^{-n/(2s)} \in A_{n/(2s)} \), the last quantity is bounded by
\[
c_2 \left\{ \int_Q (v_\lambda^\star)^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} |Q| \left( \int_Q w^{-n/(2s)} \, dx \right)^{-2s/n}
\]
\[
\leq c_2 \left\{ \int_Q (v_\lambda^\star)^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} |Q|^{-2s/n} \int_Q w \, dx,
\]
where we used the inequality
\[
1 \leq \frac{1}{|Q|} \int_Q w \, dx \left( \frac{1}{|Q|} \int_Q w^{-n/(2s)} \, dx \right)^{2s/n}.
\]
The above calculation says
\[
1 \leq c_3 \int_Q (v_\lambda^\star)^{n/(2s)} w^{-n/(2s)} \, dx,
\]
where \( c_3 = c' \alpha^{-n/(2s)} \) and \( c' \) is the \( A_{n/(2s)} \)-constant of \( w^{-n/(2s)} \).
First we assume that $\mathcal{I}_\lambda$ includes infinite disjoint cubes $\{Q_i\}_{i=1}^\infty$. Then we have
\[ \infty = \sum_{i=1}^{\infty} 1 \leq \sum_{i=1}^{\infty} c_3 \int_{Q_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \leq c_3 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx < \infty. \]
This is a contradiction. Hence $\mathcal{I}_\lambda$ does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes $\{Q_i\}_{i=1}^\infty \subset \mathcal{I}_\lambda$ such that $Q_i \neq Q_j$ ($i \neq j$) and $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$. Let $\hat{Q}_i$ be a half size dyadic sub-cube of $Q_{i+1}$ such that $Q_i \cap \hat{Q}_i = \emptyset$. Since $Q_{i+1} \in \mathcal{I}_\lambda$, we have
\[ \alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \leq \int_{Q_{i+1}} v_\lambda^* \, dx. \]
Now we get
\[ \int_{Q_{i+1}} v_\lambda^* \, dx \leq \int_{3\hat{Q}_i} v_\lambda^* \, dx \leq c_4 \int_{\hat{Q}_i} v_\lambda^* \, dx, \]
where we used the doubling property of $v_\lambda^*$. Since
\[ |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \geq 2^{-2s}|\hat{Q}_i|^{-2s/n} \int_{\hat{Q}_i} w \, dx, \]
we get
\[ c_5 |\hat{Q}_i|^{-2s/n} \int_{\hat{Q}_i} w \, dx \leq \int_{\hat{Q}_i} v_\lambda^* \, dx. \]
The similar calculation as before leads to
\[ 1 \leq c_6 \int_{\hat{Q}_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx, \]
where $c_6 = c'' \alpha^{-n/(2s)}$ and $c''$ depends only on $n, s,$ and $w$. Since $\{\hat{Q}_i\}_{i=1}^\infty$ is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$ has a maximal element. Similarly we can show that any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in $\mathcal{I}_\lambda$ with respect to the inclusion relation is finite. Hence $\mathcal{I}_\lambda$ is a finite set. We remark that the non-decreasing rearrangement of $\mathcal{I}$ in (8) is possible because $\mathcal{I}_\lambda$ is a finite set for every $\lambda > 0$.

Let $N(\lambda) = |\mathcal{I}_\lambda|$, that is, the number of elements of $\mathcal{I}_\lambda$. Let $\hat{\mathcal{I}}_\lambda$ be the set of all $Q \in \mathcal{I}_\lambda$ which satisfy the following condition: there exists a half size dyadic sub-cube $\hat{Q} \subset Q$ such that $\hat{Q} \not\in \mathcal{I}_\lambda$ and $\hat{Q}$ does not contain any dyadic cube in $\mathcal{I}_\lambda$. Then we have the following lemma.
Lemma 4.1. \( \| I_\lambda \| \leq 2 \| \tilde{I}_\lambda \|. \)

Lemma 4.1 is proved in Rochberg and Taibleson’s paper([9, Lemma 1]).

Let \( Q \in \tilde{I}_\lambda \) and \( \tilde{Q} \) be a dyadic cube which satisfies the condition in the definition of \( \tilde{I}_\lambda \). Then by similar calculations as before we get

\[
1 \leq c_6 \int_Q (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} \, dx.
\]

For every \( Q \in \tilde{I}_\lambda \) we choose a \( \tilde{Q} \) as above. Let \( \{ \tilde{Q}_j \}_{j \in J} \) be the set of all such cubes \( \tilde{Q} \). Then the cubes in \( \{ \tilde{Q}_j \}_{j \in J} \) are mutually disjoint. Therefore we get

\[
\| \tilde{I}_\lambda \| = \| J \| \leq \sum_{j \in J} c_6 \int_{\tilde{Q}_j} (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} \, dx \leq c_7 \int_{\mathbb{R}^n} (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} \, dx,
\]

where we used (16). Hence we have

\[
N(\lambda) \leq 2c_7 \int_{\mathbb{R}^n} (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} \, dx.
\]

Therefore we conclude

\[
\sum_k |\mu_k|^{\gamma} = \int_0^{\infty} \gamma \lambda^{\gamma-1} N(\lambda) \, d\lambda
\]

\[
\leq 2c_7 \int_{\beta v > \lambda}^{\infty} \int_{\beta v - \lambda}^{\lambda} (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} \, dx \gamma \lambda^{\gamma-1} \, d\lambda
\]

\[
\leq c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,
\]

where \( c_8 = c'' \alpha^{n/(2s)} \beta^p \) and \( c'' \) depends only on \( n, s, p \) and \( w \).

Next we consider the case \( n \leq 2s \). We remark that \( v(x) > 0 \) for all \( x \in \mathbb{R}^n \). In fact if \( v(x_0) = 0 \) at some point \( x_0 \), then by the definition of the maximal operator we have \( V \equiv 0 \), that is, \( \phi_i \equiv 0, i = 1, \ldots, N \).

We also remark that \( I \) in (8) is not empty. In fact if \( I \) is empty, then we have

\[
\beta \int_Q v \, dx \leq \alpha |Q|^{2s/n} \int_Q w \, dx
\]

for all \( Q \in \mathcal{Q} \). Let \( Q_0 \in \mathcal{Q} \) and \( Q_0 \subset Q_1 \subset Q_2 \subset \cdots \) be the infinite sequence of dyadic cubes such that \( Q_{i+1} \) is the parent of \( Q_i \) for all \( i = 0, 1, 2, \ldots \). By (2) we have

\[
|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \leq |Q_i|^{-2s/n} \int_{Q_i} w \, dx
\]

for all \( i \). Hence we have
(17) \[ \beta \int_{Q_i} v \, dx \leq \alpha |Q_0|^{-2s/n} \int_{Q_0} w \, dx \]

for all \( i \). On the other hand, since \( v \in A_1 \), there exists a constant \( d > 1 \) such that

\[ d \int_{Q_i} v \, dx \leq \int_{Q_{i+1}} v \, dx \]

for all \( i \) (c.f.[5, p141]). Hence we have

\[ d^i \int_{Q_0} v \, dx \leq \int_{Q_i} v \, dx \]

and

\[ \lim_{i \to \infty} \int_{Q_i} v \, dx = \infty, \]

which contradicts to (17). Therefore \( \mathcal{I} \) is not empty.

Let \( Q \in \mathcal{I} \) and \( Q' \) be the parent of \( Q \). Then we have

\[ \alpha |Q'|^{-2s/n} \int_{Q'} w \, dx \leq \alpha |Q|^{-2s/n} \int_{Q} w \, dx < \beta \int_{Q} v \, dx \leq \beta \int_{Q'} v \, dx, \]

where we used the assumption (2). Hence we have \( Q' \in \mathcal{I} \), which means that \( \mathcal{I} \) is an infinite set.

**Lemma 4.2.** There exists a \( c > 0 \) such that

\[ \sum_{Q \in \mathcal{I}} \left( \frac{1}{|Q|} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx, \]

where \( c = c' \alpha^{-n/(2s)} \beta^p \) and \( c' \) depends only on \( n, p, s \) and \( w \).

This lemma is proved in [12, Lemma 3.3]. Let \( \mathcal{I}_\lambda \) be the set defined by (15).

**Lemma 4.3.** For each \( \lambda > 0 \), \( \mathcal{I}_\lambda \) is a finite set.

Lemma 4.3 is easily proved by Lemma 4.2 (c.f.[12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of \( \mathcal{I} \) is possible.

By Lemma 4.2 we conclude

\[ \sum_{k=1}^{\infty} |\mu_k|^\gamma = c \sum_{Q \in \mathcal{I}} \left( \frac{\beta}{|Q|} \int_{Q} v \, dx - \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx \right)^\gamma \]

\[ \leq c \sum_{Q \in \mathcal{I}} \left( \frac{\beta}{|Q|} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx, \]

where \( c = c'' \alpha^{-n/(2s)} \beta^p \) and \( c'' \) depends only on \( n, p, s \) and \( w \). This ends the proof of Lemma 3.3.
5 Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

**Lemma 5.1.** Let \( w \in A_2 \) and \( m \in C^n(\mathbb{R}^n \setminus \{0\}) \). Suppose that
\[
B = \max_{|\sigma| \leq n} \sup_{0 < r < R} r^{2|\sigma| - n} \left| \int_{|\xi| \leq 2r} \left( \frac{\partial}{\partial \xi} \right)^{\sigma} m(\xi) \right|^2 d\xi < \infty.
\]
Then the operator \( T \) defined by
\[
\hat{T}f(\xi) = m(\xi) \hat{f}(\xi)
\]
is bounded from \( L^2(w) \) to \( L^2(w) \) and the operator norm \( \|T\| \) is bounded by \( CB^{1/2} \) where \( C \) is a constant which depends only on \( n \) and \( w \).

The proof of Lemma 5.1 is in [6] or [7].

For \( \nu \in \mathbb{Z} \) we define \( \psi_\nu(x) = 2^{n\nu} \psi(2^\nu x) \). Let \( w \in A_2 \) and \( s \geq 0 \). Frazier and Jawerth proved that there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} (f, \psi_\nu)^2 \int_Q w dx \leq \sum_{\nu \in \mathbb{Z}} 2^{2n\nu} |f * \psi_\nu(x)|^2 \int_Q w dx
\]
\[
\leq c_2 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} (f, \psi_\nu)^2 \int_Q w dx
\]
for all \( f \in C^\infty_0(\mathbb{R}^n) \) where \( c_1 \) and \( c_2 \) depend only on \( n, s \) and \( w \) ([3, Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let \( \{r_\nu(t)\} \) be the Rademacher functions on \([0,1]\) indexed by \( \nu \in \mathbb{Z} \) and
\[
T_1 f(x) = \sum_{\nu \in \mathbb{Z}} r_\nu(t) f * \psi_\nu(x).
\]

Then \( T_1 \) satisfies the condition of Lemma 5.1. Hence
\[
\int_{\mathbb{R}^n} |T_1 f(x)|^2 w(x) dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx,
\]
for all \( f \in C^\infty_0(\mathbb{R}^n) \) where \( M = \max_{|\sigma| \leq n} \|\partial^\sigma \psi\|_\infty \) and \( C \) is a positive constant depending only on \( n \) and \( w \). By integrating from 0 to 1 with respect to \( t \), we get
\[
\int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx.
\]
By the duality argument and the fact \( w^{-1} \in A_2 \) we obtain
\[
\int_{\mathbb{R}^n} |f(x)|^2 w(x) dx \leq CM \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) dx.
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$. Hence we have
\[
    c_3 M^{-1} \int_{\mathbb{R}^n} |f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu|^2 \right\} w \, dx \leq c_4 M \int_{\mathbb{R}^n} |f|^2 w \, dx,
\]
where $c_3$ and $c_4$ are constants depending only on $n$ and $w$.

Therefore we get
\[
    c_3 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |(-\Delta)^{s/2} f * \psi_\nu|^2 \right\} w \, dx
\]
\[
    \leq c_4 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$ (c.f. [11]).

Let $\Phi \in S(\mathbb{R}^n)$ satisfy $\text{supp } \Phi \subset \{ \xi : 1/4 \leq |\xi| \leq 4 \}$ and $\Phi(\xi) = 1$ for $1/2 \leq |\xi| \leq 2$. For $\nu \in \mathbb{Z}$ the multiplier $m_\nu(\xi) = 2^{-s \nu} |\xi|^s \Phi(\xi/2^\nu)$ satisfies the condition of Lemma 5.1. Hence we have
\[
    \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2 w(x) \, dx \leq c_5 \int_{\mathbb{R}^n} 2^{2s \nu} |f * \psi_\nu(x)|^2 w(x) \, dx,
\]
where $c_5 = c_6 \inf \max_{\Phi, |\sigma| \leq n} \| \partial^\sigma \Phi \|_\infty^2$ and $c_6$ is a positive constant depending only on $n$, $s$ and $w$ and the infimum is taken over all possible $\Phi$.

Similarly there exists a positive constant $c_7$ depending only on $n$, $s$ and $w$ such that
\[
    \int_{\mathbb{R}^n} 2^{s \nu} |f * \psi_\nu(x)|^2 w(x) \, dx \leq c_7 \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2 w(x) \, dx.
\]

Hence we get
\[
    c_8 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \leq \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} w \, dx
\]
\[
    \leq c_9 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$, where $c_8$ and $c_9$ are positive constant depending only on $n$, $s$ and $w$. This ends the proof of Lemmas 3.1 and 3.2.

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References


17


