Weighted Sobolev-Lieb-Thirring inequalities

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Abstract. We give a weighted version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions. In the proof of our result we use \( \varphi \)-transform of Frazier-Jawerth.

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1 Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring inequality.

**Theorem 1.1 ([2]).** Let \( n \in \mathbb{N}, s > 0 \) and \( p \) with

\[
\max \left( 1, \frac{n}{2s} \right) < p \leq 1 + \frac{n}{2s}.
\]

Then there exists a positive constant \( c = c(p, n, s) \) such that for every family \( \{ \phi_i \}_{i=1}^N \) in \( H^s(\mathbb{R}^n) \) which is orthonormal in \( L^2(\mathbb{R}^n) \), we have

\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \|(-\Delta)^{s/2} \phi_i\|^2
\]

where

\[
\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.
\]

In this theorem \( H^s(\mathbb{R}^n) \) denotes the Sobolev space of order \( s \) and \( \| \cdot \| \) is the norm of \( L^2(\mathbb{R}^n) \).

In [8] Lieb and Thirring proved this theorem for \( s = 1 \) and applied it to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved (1) for \( s \in \mathbb{N} \) under
the suborthonormal condition on \( \{ \phi_i \} \), where \( \{ \phi_i \}_{i=1}^{N} \) in \( L^2(\mathbb{R}^n) \) is called suborthonormal if the inequality
\[
\sum_{i,j=1}^{N} \xi_i \overline{\xi_j} (\phi_i, \phi_j) \leq \sum_{i=1}^{N} |\xi_i|^2
\]
holds for all \( \xi_i \in \mathbb{C}, i = 1, \ldots, N \) ([4]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations (c.f. [13]). In this paper we shall give a weighted version of (1) under suborthonormal condition on \( \{ \phi_i \} \). In the proof of our theorem we shall use Frazier-Jawerth’s \( \varphi \)-transform ([3]).

For the statement of our result we need to recall the definition of \( A_p \)-weights (c.f. [5], [10]). By a cube in \( \mathbb{R}^n \) we mean a cube which sides are parallel to coordinate axes. Let \( w \) be a non-negative, locally integrable function on \( \mathbb{R}^n \). We say that \( w \) is an \( A_p \)-weight for \( 1 < p < \infty \) if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \quad x \in Q
\]
for all cubes \( Q \subset \mathbb{R}^n \). The infimum of the constant \( C \) is called the \( A_p \)-constant of \( w \).

For a nonnegative, locally integrable function \( w \) on \( \mathbb{R}^n \) we define
\[
L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.
\]

For \( \nu \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) the cube \( Q \) defined by
\[
Q = Q_{\nu k} = \{(x_1, \ldots, x_n) : k_i \leq 2^{\nu} x_i < k_i + 1, \, i = 1, \ldots, n\}
\]
is called a dyadic cube in \( \mathbb{R}^n \). Let \( \mathcal{Q} \) be the set of all dyadic cubes in \( \mathbb{R}^n \). For any \( Q \in \mathcal{Q} \) there exists a unique \( Q' \in \mathcal{Q} \) such that \( Q \subset Q' \) and the side-length of \( Q' \) is double of that of \( Q \). We call \( Q' \) the parent of \( Q \).

For \( s > 0 \) and \( f \in C_0^\infty (\mathbb{R}^n) \) we define via inverse Fourier transform
\[
(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).
\]
Let \( w \in A_2 \) and \( \mathcal{H}^s(w) \) be the completion of \( C_0^\infty (\mathbb{R}^n) \) with respect to the norm
\[
\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx + \|f\|^2 \right\}^{1/2}.
\]
We remark that for $f \in C_0^\infty(\mathbb{R}^n)$ we have
\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx < \infty
\]
because
\[
|(-\Delta)^{s/2} f(x)| \leq \frac{c}{(1 + |x|)^n} \quad (x \in \mathbb{R}^n)
\]
and
\[
\int_{\mathbb{R}^n} \frac{w(x)}{(1 + |x|)^{2n}} \, dx < \infty
\]
(c.f. [10, p.209]).

Let $f \in \mathcal{H}^s(w)$ and \{f_i\}_{i=1}^\infty be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\|f - f_i\|_{\mathcal{H}^s(w)} \to 0$ ($i \to \infty$). This means that there exist $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(w)$ such that $\|g_1 - f_i\| \to 0$ and
\[
\int_{\mathbb{R}^n} |g_2(x) - (-\Delta)^{s/2} f_i(x)|^2 w(x) \, dx \to 0
\]
as $i \to \infty$. We denote $(-\Delta)^{s/2} f = g_2$. We remark that $g_1 \equiv 0$ means $g_2 \equiv 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have
\[
\int_{\mathbb{R}^n} g_2 \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} f_i (-\Delta)^{s/2} \varphi \, dx = 0.
\]
Hence we have $g_2 \equiv 0$. This means that we can identify $\mathcal{H}^s(w)$ as a subspace of $L^2(\mathbb{R}^n)$.

The following is the main result of this paper.

**Theorem 1.2.** Let $n \in \mathbb{N}, s > 0$, and
\[
\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}.
\]
Let $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{p/(2s)}$. If $2s \geq n$, then we assume that $w^{-n/(2s)} \in A_p$ and
\[
\int_{Q'} w \, dx \leq 2^{2s} \int_Q w \, dx
\]
for all dyadic cubes $Q \in Q$ and its parent $Q'$.

Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $\mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$, we have
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx,
\]
where 
\[
\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2
\]
and \(c\) depends only on \(n, s, p, A_2\)-constant of \(w\), and \(A_{n/(2s)}\) or \(A_p\)-constant of \(w^{-n/(2s)}\).

When \(2s < n\), an example of weight function \(w\) is given by \(w(x) = |x|^\alpha\) for \(-n+2s < \alpha < 2s\). When \(2s > n\), an example of weight function \(w\) is given by \(w(x) = |x|^\alpha\) for \(0 \leq \alpha < \min\{2s-n, n\}\) (c.f.\cite[Section 4]{12}). When \(2s = n\), the condition (2) means \(w\) is equivalent to a constant almost everywhere (c.f.\cite[Proposition 4.1]{12}).

2 Preliminaries

Let \(\psi\) be a function which satisfies the following conditions.

(A1) \(\psi \in \mathcal{S}(\mathbb{R}^n)\).

(A2) \(\text{supp} \hat{\psi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}\)

(A3) \(|\hat{\psi}(\xi)| \geq c > 0\) if \(\frac{2}{5} \leq |\xi| \leq \frac{5}{3}\).

(A4) \(\sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 = 1\) for all \(\xi \neq 0\).

For \(\nu \in \mathbb{Z}, k \in \mathbb{Z}^n\) and \(Q = Q_{\nu k}\), we set

\[
\psi_Q(x) = 2^{n\nu/2} \psi(2^\nu x - k) \quad (x \in \mathbb{R}^n).
\]

Let \(M\) be the Hardy-Littlewood maximal operator, that is,

\[
M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
\]

where \(f\) is a locally integrable function on \(\mathbb{R}^n\) and the supremum is taken over all cubes \(Q\) which contain \(x\).

**Proposition 2.1.**

(i) Let \(1 < p < \infty\) and \(w\) be a non-negative locally integrable function on \(\mathbb{R}^n\). Then there exists a positive constant \(c\) such that

\[
\int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx
\]

for all \(f \in L^p(w)\) if and only if \(w \in A_p\). The constant \(c\) depends only on \(n, p\) and \(A_p\)-constant of \(w\).
(ii) Let \( 1 < p < \infty \) and \( w \in A_p \). Then there exists a \( q \in (1,p) \) such that \( w \in A_q \).

(iii) Let \( 0 < \tau < 1 \) and \( f \) be a locally integrable function on \( \mathbb{R}^n \) such that \( M(f)(x) < \infty \) a.e. Then \( (M(f))^{\tau} \in A_1 \) and the \( A_1 \)-constant of \( (M(f))^{\tau} \) depends only on \( n \) and \( \tau \).

(iv) Let \( 1 \leq p < \infty \) and \( w \in A_p \). Then there exists a positive constant \( c \) such that
\[
\int_{2Q} w \, dx \leq c \int_{Q} w \, dx
\]
for all cubes \( Q \in \mathbb{R}^n \), where \( 2Q \) denotes the double of \( Q \) and \( c \) depend only on \( n \) and \( A_p \)-constant of \( w \).

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3 Proof of Theorem 1.2

The suborthonormal condition on \( \{\phi_i\} \) is equivalent to the inequality
\[
\sum_{i=1}^{N} |(\phi_i, f)|^2 \leq \|f\|^2
\]
for all \( f \in L^2(\mathbb{R}^n) \) (c.f. [1, p57]).

We shall prove the inequality
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq cK^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx
\]
under the assumption
\[
\sum_{i=1}^{N} |(\phi_i, f)|^2 \leq K\|f\|^2
\]
for all \( f \in L^2(\mathbb{R}^n) \) where \( K \) is a positive constant. This is equivalent to the statement of Theorem 1.2. We remark that \( K \) may depend on \( \{\phi_i\} \). For example, the inequality (3) says that
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c\|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi|^2 w \, dx
\]
holds for all $\phi \in H^s(w)$ under suitable condition on $s, p, n$ and $w$ because

$$|(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, \ldots, N$. Let

$$V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2s(p-1))}$$

where the value of the constant $\delta_1 > 0$ will be given later. Since $\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$ is a bounded function with compact support and $w^{n/(2s(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have

$$\int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty.$$ 

We may also assume that

$$0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx.$$ 

By (ii) of Proposition 2.1 there exists a constant $\kappa$ such that $1 < \kappa < p$ and $w^{-n/(2s)} \in A_{p/\kappa}$. We set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty.$$ 

Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 2.1.

We have the following lemmas.

**Lemma 3.1.** For $s > 0$ and $w \in A_2$ there exists a positive constant $\alpha$ such that

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} w \, dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where $\alpha$ is given by

$$\alpha^{-1} = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and $c$ is a constant depending only on $n, s$ and $A_2$-constant of $w$.

**Lemma 3.2.** For $v \in A_2$ there exist positive constants $\beta$ and $\beta'$ such that

$$\beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} v \, dx \leq \int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} v \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where $\beta$ is given by

$$\beta = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and $c$ is a constant depending only on $n$ and $A_2$-constant of $v$. 

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The proof of Lemmas 3.1 and 3.2 are in [11, Proposition 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader’s convenience because the dependence of \( \psi \) in \( \alpha \) and \( \beta \) is not explained in [11].

For \( f \in C_0^\infty(\mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}^n} |f|^2 V \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx,
\]

where we used Lemma 3.2. Hence by Lemma 3.1

\[
\int_{\mathbb{R}^n} |\nabla f|^2 w \, dx - \int_{\mathbb{R}^n} V |f|^2 \, dx \\
\geq \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx - \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx.
\]

Now we set

\[
\mathcal{I} = \left\{ Q \in \mathcal{Q} : \beta \int_Q v \, dx > \alpha |Q|^{-2s/n} \int_Q w \, dx \right\}.
\]

Let \( \{\mu_k\}_{1 \leq k} \) be the non-decreasing rearrangement of

\[
\left\{ \alpha |Q|^{-2s/n} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx \right\}_{Q \in \mathcal{I}}.
\]

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When

\[
\mu_k = \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx,
\]

we define \( \Psi_k = \psi_Q \). Then we have by (7)

\[
\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V |\phi_i|^2 \, dx \\
\geq \sum_{i=1}^N \sum_{Q \in \mathcal{Q}} |(\phi_i, \psi_Q)|^2 \left\{ \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx \right\} \\
\geq \sum_{i=1}^N \sum_k \mu_k |(\phi_i, \Psi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\phi_i, \Psi_k)|^2 \\
\geq -K \|\psi\|^2 \sum_k |\mu_k| \geq -K \|\psi\|^2 \left( \sum_k |\mu_k| \right)^{1/\gamma},
\]

where \( \gamma = p - n/(2s) \in (0, 1] \) and we used (4).

Now the following lemma holds.
Lemma 3.3.

\[ \sum_k |\mu_k|^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx, \]

where \( c \) is given by

\[ c = c' \max_{|\sigma| \leq n} \| \partial^\sigma \hat{\psi} \|_{\infty}^{n/s + 2p} \]

and \( c' \) depends only on \( n, s, p \) and \( w \).

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (6) the last quantity in (10) is estimated from below by

\[ -cK \left( \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx \right)^{1/\gamma} \]

\[ = -cK \delta_1^{p/(p-n/(2s))} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{1/(p-n/(2s))}, \]

where

\[ c = c' \| \psi \|^2 \max_{|\sigma| \leq n} \| \partial^\sigma \hat{\psi} \|_{4ps+2n}/(2ps-n) \]

and \( c' \) depends only on \( n, s, p \) and \( w \). We may take the infimum of the above constant with respect to possible \( \psi \) and replace \( c \) by this infimum.

Let

\[ \delta_1 = \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-n/(2s)-1)/n}, \]

where \( \delta_2 \) is a positive constant. Then we have by (9)

\[ \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx \]

\[ \geq \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ - cK \delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ = \left\{ \delta_2 - c \delta_2^{p/(p-n/(2s))} \right\} K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n}. \]

If we take \( \delta_2 \) small enough, then we get the inequality (3) because \( 1 < p/(p-n/(2s)) \).

Next we shall show (3) for \( \phi_i \in \mathcal{H}^s(w), \ i = 1, \ldots, N \). First we show

\[ \mathcal{H}^s(w) \subset L^{2p/(p-1)}(w^{n/(2s(p-1))}). \]
Let \( h \in \mathcal{H}^s(w) \). Then there exists a sequence \( \{h_m\}_{m=1}^{\infty} \subset C_0^\infty(\mathbb{R}^n) \) such that
\[
\|h - h_m\|_{\mathcal{H}^s(w)} \to 0 \quad (m \to \infty).
\]
Since we proved that (5) holds for \( h_m \in C_0^\infty(\mathbb{R}^n) \), we get
\[
\left\{ \int_{\mathbb{R}^n} |h_m|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|h_m\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h_m|^2 w \, dx,
\]
where \( c \) does not depend on \( h_m \). Since \( 4sp/n-2 > 0 \) and \( \{h_m\} \) is a Cauchy sequence in \( \mathcal{H}^s(w) \), the above inequality says that \( \{h_m\} \) is a Cauchy sequence in \( L^{2p/(p-1)}(w^{n/(2s(p-1))}) \).

Let \( g \) be the limit of \( \{h_m\} \) in \( L^{2p/(p-1)}(w^{n/(2s(p-1))}) \). For any compact set \( E \) in \( \mathbb{R}^n \) we have
\[
\int_E |g - h_m| \, dx \leq \left( \int_E |g - h_m|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{(p-1)/(2p)} \times \left( \int_E w^{-n/(2s(p+1))} \, dx \right)^{(p+1)/(2p)}.
\]
Since \( w^{-n/(2s)} \) is locally integrable by the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) or \( w^{-n/(2s)} \in A_p \), we get \( h_m \to g \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( m \to \infty \). Hence we have \( g = h \) and (11). Furthermore we have
\[
\left\{ \int_{\mathbb{R}^n} |h|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|h\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h|^2 w \, dx.
\]

We fix a positive number \( \varepsilon \). Let \( \chi_1, \ldots, \chi_N \) be functions in \( C_0^\infty(\mathbb{R}^n) \) such that
\[
\sum_{i=1}^N \|\phi_i - \chi_i\|_{\mathcal{H}^s(w)}^2 < \varepsilon.
\]
Now the inequalities
\[
\sum_{i=1}^N |(\chi, f)|^2 \leq 2 \sum_{i=1}^N |(\chi_i - \phi_i, f)|^2 + 2 \sum_{i=1}^N |(\phi_i, f)|^2 \leq 2(2K + \varepsilon) \|f\|^2
\]

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\sum_{i=1}^N \|\phi_i - \chi_i\|_{\mathcal{H}^s(w)}^2 < \varepsilon.
\]
Now the inequalities
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\sum_{i=1}^N |(\chi, f)|^2 \leq 2 \sum_{i=1}^N |(\chi_i - \phi_i, f)|^2 + 2 \sum_{i=1}^N |(\phi_i, f)|^2 \leq 2(2K + \varepsilon) \|f\|^2
\]
hold for all \( f \in L^2(\mathbb{R}^n) \). On the other hand

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \right\}^{2s(p-1)/n} \\
\leq \left\{ \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \right)^{(p-1)/p} 2sp/n \right\}^{2s(p-1)/n} \\
\leq N^{2sp/n-1} \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i| |\phi_i|^{2p/(p-1)} w^{2n/(2s(p-1))} \right)^{2s(p-1)/n} \\
\leq cN^{2sp/n-1} \sum_{i=1}^{N} |\phi_i - \chi_i|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 \right| w \, dx \\
\leq cN^{2sp/n-1} \varepsilon^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 \right| w \, dx \\
\leq cN^{2sp/n-1} \varepsilon^{2sp/n},
\]

where we used (12).

Therefore

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \right\}^{2s(p-1)/n} \\
\leq \left\{ \int_{\mathbb{R}^n} \left( 2 \sum_{i=1}^{N} |\phi_i - \chi_i|^2 + 2 \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \right\}^{2s(p-1)/n} \\
\leq 2^{2sp/n} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \right\}^{(p-1)/p} 2sp/n \\
\leq 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \right\}^{2s(p-1)/n} \\
\leq 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \right\}^{2s(p-1)/n} \\
\leq c2^{sp/n-1} N^{2sp/n-1} \varepsilon^{2sp/n} + c2^{sp/n-2} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 \right| w \, dx, 
\]
where we used (13) and (3) for $\chi_i$. Since
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 w \, dx
\]
\[
\leq 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i - (-\Delta)^{s/2} \phi_i|^2 w \, dx + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx
\]
\[
\leq 2 \varepsilon + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx,
\]
we have by (14)
\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n}
\leq c 2^{4sp/n-1} K 2^{sp/n-1} \varepsilon 2^{sp/n-1} + c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \varepsilon
\]
\[
+ c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx.
\]
Since we can take $\varepsilon$ arbitrary small, we conclude
\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n}
\leq c 2^{6sp/n-1} K 2^{sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx.
\]
Hence we get (3).

4 Proof of Lemma 3.3

The arguments of the proof is similar to those in [11] and [12]. First we consider the case $n > 2s$. For $\lambda > 0$ we set
\[
(I) = \{ Q \in \mathcal{Q} : \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx - \beta |Q|^{-1} \int_{Q} v \, dx < -\lambda \}.
\]
Then we have for $Q \in I$\$
\alpha |Q|^{-2s/n-1} \int_{Q} w \, dx < |Q|^{-1} \int_{Q} (\beta v - \lambda)_{+} \, dx,
\$
where
\[
(\beta v - \lambda)_{+}(x) = \max\{0, \beta v(x) - \lambda\}.
\]
Since \( p = n/(2s) + \gamma, \gamma \in (0, 1] \), and
\[
\beta^{-p}\gamma \int_0^\infty \int_{\beta v > \lambda} \frac{(\beta v - \lambda)^{n/(2s)}}{w^{-n/(2s)}} dx \lambda \gamma^{-1} d\lambda \leq \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx < \infty,
\]
we have
\[
\int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} dx < \infty
\]
for all \( \lambda > 0 \). By the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) and (ii) of Proposition 2.1, there exists a \( \kappa' \in (1, n/(2s)) \) such that \( w^{-n/(2s)} \in A_{n/(2s\kappa')} \). We set
\[
v^*_\lambda(x) = M((\beta v - \lambda)^{\kappa'})(x)^{1/\kappa'}.
\]
Then
\[
\int_{\mathbb{R}^n} (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx \leq c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} dx < \infty
\]
and \( v^*_\lambda \in A_1 \) by (iii) of Proposition 2.1, where \( c_1 \) depends only on \( n, s \) and \( A_{n/(2s)} \)-constant of \( w^{-n/(2s)} \).

We can show that \( \mathcal{I}_\lambda \) is a finite set as follows. Let \( Q \in \mathcal{I}_\lambda \). Then we have
\[
\alpha |Q|^{-2s/n} \int_Q w dx \leq \int_Q v^*_\lambda dx \leq \left\{ \int_Q (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx \right\}^{2s/n} \left\{ \int_Q w^{n/(n-2s)} dx \right\}^{(n-2s)/n}.
\]
Since \( w^{-n/(2s)} \in A_{n/(2s)} \), the last quantity is bounded by
\[
c_2 \left\{ \int_Q (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx \right\}^{2s/n} |Q| \left( \int_Q w^{-n/(2s)} dx \right)^{-2s/n} \leq c_2 \left\{ \int_Q (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx \right\}^{2s/n} |Q|^{-2s/n} \int_Q w dx,
\]
where we used the inequality
\[
1 \leq \frac{1}{|Q|} \int_Q w dx \left( \frac{1}{|Q|} \int_Q w^{-n/(2s)} dx \right)^{2s/n}.
\]
The above calculation says
\[
1 \leq c_3 \int_Q (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx,
\]
where \( c_3 = c' \alpha^{-n/(2s)} \) and \( c' \) is the \( A_{n/(2s)} \)-constant of \( w^{-n/(2s)} \).
First we assume that $I_\lambda$ includes infinite disjoint cubes $\{Q_i\}_{i=1}^\infty$. Then we have
\[
\infty = \sum_{i=1}^\infty 1 \leq \sum_{i=1}^\infty c_3 \int_{Q_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \leq c_3 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx < \infty.
\]
This is a contradiction. Hence $I_\lambda$ does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes $\{Q_i\}_{i=1}^\infty \subset I_\lambda$ such that $Q_i \neq Q_j$ ($i \neq j$) and $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$. Let $\tilde{Q}_i$ be a half size dyadic sub-cube of $Q_{i+1}$ such that $Q_i \cap \tilde{Q}_i = \emptyset$. Since $Q_{i+1} \in I_\lambda$, we have
\[
\alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \leq \int_{Q_{i+1}} v_\lambda^* \, dx.
\]
Now we get
\[
\int_{Q_{i+1}} v_\lambda^* \, dx \leq c_4 \int_{\tilde{Q}_i} v_\lambda^* \, dx,
\]
where we used the doubling property of $v_\lambda^*$. Since
\[
|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \geq 2^{-2s} |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w \, dx,
\]
we get
\[
c_5 |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w \, dx \leq \int_{\tilde{Q}_i} v_\lambda^* \, dx.
\]
The similar calculation as before leads to
\[
1 \leq c_6 \int_{\tilde{Q}_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx,
\]
where $c_6 = c'' \alpha^{-n/(2s)}$ and $c''$ depends only on $n, s,$ and $w$. Since $\{\tilde{Q}_i\}_{i=1}^\infty$ is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in $I_\lambda$ such that $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$ has a maximal element. Similarly we can show that any sequence in $I_\lambda$ such that $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in $I_\lambda$ with respect to the inclusion relation is finite. Hence $I_\lambda$ is a finite set. We remark that the non-decreasing rearrangement of $I$ in (8) is possible because $I_\lambda$ is a finite set for every $\lambda > 0$.

Let $N(\lambda) = |I_\lambda|$, that is, the number of elements of $I_\lambda$. Let $\hat{I}_\lambda$ be the set of all $Q \in I_\lambda$ which satisfy the following condition: there exists a half size dyadic sub-cube $\hat{Q} \subset Q$ such that $\hat{Q} \notin I_\lambda$ and $\hat{Q}$ does not contain any dyadic cube in $I_\lambda$. Then we have the following lemma.
Lemma 4.1. $\|\mathcal{I}_\lambda\| \leq 2\|\tilde{\mathcal{I}}_\lambda\|$.

Lemma 4.1 is proved in Rochberg and Taibleson’s paper([9, Lemma 1]).

Let $Q \in \mathcal{I}_\lambda$ and $\tilde{Q}$ be a dyadic cube which satisfies the condition in the definition of $\mathcal{I}_\lambda$. Then by similar calculations as before we get

$$1 \leq c_6 \int_Q (\nu_\lambda^n)^{n/(2s)} w^{-n/(2s)} dx.$$

For every $Q \in \mathcal{I}_\lambda$ we choose a $\tilde{Q}$ as above. Let $\{\tilde{Q}_j\}_{j \in J}$ be the set of all such cubes $\tilde{Q}$. Then the cubes in $\{\tilde{Q}_j\}_{j \in J}$ are mutually disjoint. Therefore we get

$$\|\mathcal{I}_\lambda\| \leq \sum_{j \in J} c_6 \int_{\tilde{Q}_j} (\nu_\lambda^n)^{n/(2s)} w^{-n/(2s)} dx \leq c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} dx,$$  

where we used (16). Hence we have

$$N(\lambda) \leq 2c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} dx.$$

Therefore we conclude

$$\sum_k |\mu_k|^\gamma = \int_0^\infty \gamma \lambda^{\gamma-1} N(\lambda) d\lambda \leq 2c_7 \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} dx \gamma \lambda^{\gamma-1} d\lambda \leq c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx,$$

where $c_8 = c'' \alpha^{-n/(2s)} \beta^p$ and $c''$ depends only on $n, s, p$ and $w$.

Next we consider the case $n \leq 2s$. We remark that $v(x) > 0$ for all $x \in \mathbb{R}^n$. In fact if $v(x_0) = 0$ at some point $x_0$, then by the definition of the maximal operator we have $V \equiv 0$, that is, $\phi_i \equiv 0, i = 1, \ldots, N$.

We also remark that $\mathcal{I}$ in (8) is not empty. In fact if $\mathcal{I}$ is empty, then we have

$$\beta \int_Q v dx \leq \alpha |Q|^{-2s/n} \int_Q w dx$$

for all $Q \in \mathcal{Q}$. Let $Q_0 \in \mathcal{Q}$ and $Q_0 \subset Q_1 \subset Q_2 \subset \cdots$ be the infinite sequence of dyadic cubes such that $Q_{i+1}$ is the parent of $Q_i$ for all $i = 0, 1, 2, \ldots$. By (2) we have

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \leq |Q_i|^{-2s/n} \int_{Q_i} w dx$$

for all $i$. Hence we have
for all $i$. On the other hand, since $v \in A_1$, there exists a constant $d > 1$ such that
\[ d \int_{Q_i} v \, dx \leq \int_{Q_{i+1}} v \, dx \]
for all $i$ (c.f. [5, p141]). Hence we have
\[ d^i \int_{Q_0} v \, dx \leq \int_{Q_i} v \, dx \]
and
\[ \lim_{i \to \infty} \int_{Q_i} v \, dx = \infty, \]
which contradicts to (17). Therefore $\mathcal{I}$ is not empty.

Let $Q \in \mathcal{I}$ and $Q'$ be the parent of $Q$. Then we have
\[ \alpha |Q'|^{-2s/n} \int_{Q'} w \, dx \leq \alpha |Q|^{-2s/n} \int_{Q} w \, dx < \beta \int_{Q} v \, dx \leq \beta \int_{Q'} v \, dx, \]
where we used the assumption (2). Hence we have $Q' \in \mathcal{I}$, which means that $\mathcal{I}$ is an infinite set.

**Lemma 4.2.** There exists a $c > 0$ such that
\[ \sum_{Q \in \mathcal{I}} \left( \frac{1}{|Q|} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx, \]
where $c = c' \beta^{n/(2s)} \beta^p$ and $c'$ depends only on $n, p, s$ and $w$.

This lemma is proved in [12, Lemma 3.3]. Let $\mathcal{I}_\lambda$ be the set defined by (15).

**Lemma 4.3.** For each $\lambda > 0$, $\mathcal{I}_\lambda$ is a finite set.

Lemma 4.3 is easily proved by Lemma 4.2 (c.f. [12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of $\mathcal{I}$ is possible.

By Lemma 4.2 we conclude
\[ \sum_{k=1}^\infty |\mu_k|^\gamma = c \sum_{Q \in \mathcal{I}} \left( \beta |Q|^{-1} \int_{Q} v \, dx - \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx \right)^\gamma \]
\[ \leq c \sum_{Q \in \mathcal{I}} \left( \beta |Q|^{-1} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx, \]
where $c = c'' \alpha^{-n/(2s)} \beta^p$ and $c''$ depends only on $n, p, s$ and $w$. This ends the proof of Lemma 3.3.
5 Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

Lemma 5.1. Let \( w \in A_2 \) and \( m \in C^n(\mathbb{R}^n \setminus \{0\}) \). Suppose that

\[
B = \max_{|\sigma| \leq n} \sup_{0 < r < \infty} r^{2|\sigma| - n} \int_{r \leq |\xi| \leq 2r} \left| \left( \frac{\partial}{\partial \xi} \right)^{\sigma} m(\xi) \right|^2 d\xi < \infty.
\]

Then the operator \( T \) defined by \( \hat{T} f(\xi) = m(\xi) \hat{f}(\xi) \) is bounded from \( L^2(w) \) to \( L^2(w) \) and the operator norm \( \|T\| \) is bounded by \( CB^{1/2} \) where \( C \) is a constant which depends only on \( n \) and \( w \).

The proof of Lemma 5.1 is in [6] or [7].

For \( \nu \in \mathbb{Z} \) we define \( \psi_\nu(x) = 2^{n\nu} \psi(2^\nu x) \). Let \( w \in A_2 \) and \( s \geq 0 \). Frazier and Jawerth proved that there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w(x) dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} 2^{2\nu s} |f * \psi_\nu(x)|^2 \right\} w(x) dx,
\]

\[
\leq c_2 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w(x) dx
\]

for all \( f \in C^\infty_0(\mathbb{R}^n) \) where \( c_1 \) and \( c_2 \) depend only on \( n \) and \( w \) ([3, Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let \( \{r_\nu(t)\} \) be the Rademacher functions on \([0,1]\) indexed by \( \nu \in \mathbb{Z} \) and

\[
T_t f(x) = \sum_{\nu \in \mathbb{Z}} r_\nu(t) f * \psi_\nu(x).
\]

Then \( T_t \) satisfies the condition of Lemma 5.1. Hence

\[
\int_{\mathbb{R}^n} |T_t f(x)|^2 w(x) dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx,
\]

for all \( f \in C^\infty_0(\mathbb{R}^n) \) where \( M = \max_{|\sigma| \leq n} \|\partial^\sigma \psi\|_\infty \) and \( C \) is a positive constant depending only on \( n \) and \( w \). By integrating from 0 to 1 with respect to \( t \), we get

\[
\int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx.
\]

By the duality argument and the fact \( w^{-1} \in A_2 \) we obtain

\[
\int_{\mathbb{R}^n} |f(x)|^2 w(x) dx \leq CM \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$. Hence we have
\[ c_3 M^{-1} \int_{\mathbb{R}^n} |f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f \ast \psi_\nu|^2 \right\} w \, dx \leq c_4 M \int_{\mathbb{R}^n} |f|^2 w \, dx, \]
where $c_3$ and $c_4$ are constants depending only on $n$ and $w$.

Therefore we get
\[ c_3 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |(-\Delta)^{s/2} f \ast \psi_\nu|^2 \right\} w \, dx \leq c_4 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \]
for all $f \in C_0^\infty(\mathbb{R}^n)$ (c.f. [11]).

Let $\Phi \in S(\mathbb{R}^n)$ satisfy supp $\Phi \subset \{ \xi : 1/4 \leq |\xi| \leq 4 \}$ and $\Phi(\xi) = 1$ for $1/2 \leq |\xi| \leq 2$. For $\nu \in \mathbb{Z}$ the multiplier $m_\nu(\xi) = 2^{-s\nu} |\xi|^s \Phi(\xi/2^\nu)$ satisfies the condition of Lemma 5.1.

Hence we have
\[ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f \ast \psi_\nu(x)|^2 w(x) \, dx \leq c_5 \int_{\mathbb{R}^n} 2^{2s\nu}|f \ast \psi_\nu(x)|^2 w(x) \, dx, \]
where $c_5 = c_6 \inf_{\Phi} \| \partial^s \Phi \|_{\infty}^2$ and $c_6$ is a positive constant depending only on $n, s$ and $w$ and the infimum is taken over all possible $\Phi$.

Similarly there exists a positive constant $c_7$ depending only on $n, s$ and $w$ such that
\[ \int_{\mathbb{R}^n} 2^{2s\nu}|f \ast \psi_\nu(x)|^2 w(x) \, dx \leq c_7 \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f \ast \psi_\nu(x)|^2 w(x) \, dx. \]

Hence we get
\[ c_8 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \leq \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \leq c_9 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \]
for all $f \in C_0^\infty(\mathbb{R}^n)$, where $c_8$ and $c_9$ are positive constant depending only on $n, s$ and $w$. This ends the proof of Lemmas 3.1 and 3.2.

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