Weighted Sobolev-Lieb-Thirring inequalities

Kazuya Tachizawa

Abstract. We give a weighted version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions. In the proof of our result we use φ-transform of Frazier-Jawerth.

AMS Subject Classification. 26D15, 42B25, 42C15.
Keywords. Sobolev-Lieb-Thirring inequalities, φ-transform, A_p-weights.

1 Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring inequality.

Theorem 1.1 ([2]). *Let $n \in \mathbb{N}, s > 0$ and p with*

$$\max\left(1, \frac{n}{2s}\right) < p \leq 1 + \frac{n}{2s}.$$

Then there exists a positive constant $c = c(p, n, s)$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^s(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$\left\{\int_{\mathbb{R}^n} \rho(x)^{p/(p-1)}\, dx\right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \|(-\Delta)^{s/2}\phi_i\|^2 \tag{1}$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

In this theorem $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order s and $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^n)$.

In [8] Lieb and Thirring proved this theorem for $s = 1$ and applied it to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved (1) for $s \in \mathbb{N}$ under the suborthonormal condtion on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$\sum_{i,j=1}^{N} \xi_i \overline{\xi_j} (\phi_i, \phi_j) \leq \sum_{i=1}^{N} |\xi_i|^2$$

holds for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$ ([4]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations(c.f.[13]). In this paper we shall give a weighted version of (1) under suborthonormal condition on $\{\phi_i\}$. In the proof of our theorem we shall use Frazier-Jawerth's φ-transform([3]).

For the statement of our result we need to recall the definition of A_p-weights(c.f. [5], [10]). By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let w be a non-negative, locally integrable function on $\mathbb{R}^n$. We say that w is an A_p-weight for $1 < p < \infty$ if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(x)\, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_p-constant of w. For example, $w(x) = |x|^\alpha$ is an A_p-weight when $-n < \alpha < n(p-1)$.

We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(y)\, dy \leq C w(x) \qquad a.e.\ x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_1-constant of w. Let A_p be the class of A_p-weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

For a nonnegative, locally integrable function w on $\mathbb{R}^n$ we define

$$L^p(w) = \left\{ f:\ \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx < \infty \right\}.$$

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ the cube Q defined by

$$Q = Q_{\nu k} = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1,\ i = 1, \ldots, n\}$$

is called a dyadic cube in $\mathbb{R}^n$. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$. For any $Q \in \mathcal{Q}$ there exists a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is double of that of Q. We call Q' the parent of Q.

For $s > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ we define via inverse Fourier transform

$$(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).$$

Let $w \in A_2$ and $\mathcal{H}^s(w)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x)\, dx + \|f\|^2 \right\}^{1/2}.$$

We remark that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x)\, dx < \infty$$

because

$$|(-\Delta)^{s/2} f(x)| \le \frac{c}{(1+|x|)^n} \quad (x \in \mathbb{R}^n)$$

and

$$\int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{2n}}\, dx < \infty$$

(c.f. [10, p.209]).

Let $f \in \mathcal{H}^s(w)$ and $\{f_i\}_{i=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\|f - f_i\|_{\mathcal{H}^s(w)} \to 0$ $(i \to \infty)$. This means that there exist $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(w)$ such that $\|g_1 - f_i\| \to 0$ and

$$\int_{\mathbb{R}^n} |g_2(x) - (-\Delta)^{s/2} f_i(x)|^2 w(x)\, dx \to 0$$

as $i \to \infty$. We denote $(-\Delta)^{s/2} f = g_2$. We remark that $g_1 \equiv 0$ means $g_2 \equiv 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} g_2 \overline{\varphi}\, dx = \lim_{i\to\infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \overline{\varphi}\, dx = \lim_{i\to\infty} \int_{\mathbb{R}^n} f_i \overline{(-\Delta)^{s/2}\varphi}\, dx = 0.$$

Hence we have $g_2 \equiv 0$. This means that we can identify $\mathcal{H}^s(w)$ as a subspace of $L^2(\mathbb{R}^n)$.

The following is the main result of this paper.

Theorem 1.2. *Let $n \in \mathbb{N}, s > 0$, and*

$$\max(1, \frac{n}{2s}) < p \le 1 + \frac{n}{2s}.$$

Let $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \ge n$, then we assume that $w^{-n/(2s)} \in A_p$ and

$$\int_{Q'} w\, dx \le 2^{2s} \int_Q w\, dx \tag{2}$$

for all dyadic cubes $Q \in \mathcal{Q}$ and its parent Q'.

Then there exists a positive constant c such that for every family $\{\phi_i\}_{i=1}^N$ in $\mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$, we have

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))}\, dx \right\}^{2s(p-1)/n} \le c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x)\, dx,$$

where

$$\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2$$

and c *depends only on* n, s, p, A_2*-constant of* w*, and* $A_{n/(2s)}$ *or* A_p*-constant of* $w^{-n/(2s)}$*.*

When $2s < n$, an example of weight function w is given by $w(x) = |x|^\alpha$ for $-n+2s < \alpha < 2s$. When $2s > n$, an example of weight function w is given by $w(x) = |x|^\alpha$ for $0 \le \alpha < \min\{2s - n, n\}$(c.f.[12, Section 4]). When $2s = n$, the condition (2) means w is equivalent to a constant almost everywhere(c.f.[12, Proposition 4.1]).

2 Preliminaries

Let ψ be a function which satisfies the following conditions.

(A1) $\psi \in \mathcal{S}(\mathbb{R}^n)$.

(A2) $\operatorname{supp} \hat{\psi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$

(A3) $|\hat{\psi}(\xi)| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$.

(A4) $\sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 = 1$ for all $\xi \ne 0$.

For $\nu \in \mathbb{Z}, k \in \mathbb{Z}^n$ and $Q = Q_{\nu k}$, we set

$$\psi_Q(x) = 2^{\nu n/2} \psi(2^\nu x - k) \quad (x \in \mathbb{R}^n).$$

Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where f is a locally integrable function on $\mathbb{R}^n$ and the supremum is taken over all cubes Q which contain x.

Proposition 2.1.

(*i*) *Let* $1 < p < \infty$ *and* w *be a non-negative locally integrable function on* $\mathbb{R}^n$*. Then there exists a positive constant* c *such that*

$$\int_{\mathbb{R}^n} M(f)^p w \, dx \le c \int_{\mathbb{R}^n} |f|^p w \, dx$$

for all $f \in L^p(w)$ *if and only if* $w \in A_p$*. The constant* c *depends only on* n, p *and* A_p*-constant of* w*.*

(ii) *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.*

(iii) *Let $0 < \tau < 1$ and f be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $(M(f))^\tau \in A_1$ and the A_1-constant of $(M(f))^\tau$ depends only on n and τ.*

(iv) *Let $1 \le p < \infty$ and $w \in A_p$. Then there exists a positive constant c such that*

$$\int_{2Q} w\,dx \le c \int_Q w\,dx$$

for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of Q and c depend only on n and A_p-constant of w.

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3 Proof of Theorem 1.2

The suborthonormal condition on $\{\phi_i\}$ is equivalent to the inequality

$$\sum_{i=1}^N |(\phi_i, f)|^2 \le \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$ (c.f.[1, p57]).

We shall prove the inequality

$$\left\{\int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n} \le cK^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i(x)|^2 w(x)\,dx \tag{3}$$

under the assumption

$$\sum_{i=1}^N |(\phi_i, f)|^2 \le K\|f\|^2 \tag{4}$$

for all $f \in L^2(\mathbb{R}^n)$ where K is a positive constant. This is equivalent to the statement of Theorem 1.2. We remark that K may depend on $\{\phi_i\}$. For example, the inequality (3) says that

$$\left\{\int_{\mathbb{R}^n} |\phi|^{2p/(p-1)} w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n} \le c\|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi|^2 w\,dx \tag{5}$$

holds for all $\phi \in \mathcal{H}^s(w)$ under suitable condition on s, p, n and w because

$$|(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C_0^\infty(\mathbb{R}^n), i = 1, \ldots, N$. Let

$$V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2s(p-1))}$$

where the value of the constant $\delta_1 > 0$ will be given later. Since $\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$ is a bounded function with compact support and $w^{n/(2s(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have

$$\int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx < \infty.$$

We may also assume that

$$0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx.$$

By (ii) of Proposition 2.1 there exists a constant κ such that $1 < \kappa < p$ and $w^{-n/(2s)} \in A_{p/\kappa}$. We set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^p w^{-n/(2s)}\, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)}\, dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx < \infty. \tag{6}$$

Furthermore v is an A_1-weight by (iii) of Proposition 2.1.

We have the following lemmas.

Lemma 3.1. *For $s > 0$ and $w \in A_2$ there exists a positive constant α such that*

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w\, dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where α is given by

$$\alpha^{-1} = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and c is a constant depending only on n, s and A_2-constant of w.

Lemma 3.2. *For $v \in A_2$ there exist positive constants β and β' such that*

$$\beta' \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v\, dx \leq \int_{\mathbb{R}^n} |f|^2 v\, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where β is given by

$$\beta = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and c is a constant depending only on n and A_2-constant of v.

The proof of Lemmas 3.1 and 3.2 are in [11, Proposition 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader's convenience because the dependence of ψ in α and β is not explained in [11].

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f|^2 V\, dx \le \int_{\mathbb{R}^n} |f|^2 v\, dx \le \beta \sum_{Q\in\mathcal{Q}} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q v\, dx,$$

where we used Lemma 3.2. Hence by Lemma 3.1

$$\begin{aligned} &\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w\, dx - \int_{\mathbb{R}^n} V|f|^2\, dx \\ \tag{7} &\ge \alpha \sum_{Q\in\mathcal{Q}} |Q|^{-2s/n} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q w\, dx - \beta \sum_{Q\in\mathcal{Q}} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q v\, dx. \end{aligned}$$

Now we set

$$\mathcal{I} = \left\{ Q \in \mathcal{Q} \,:\, \beta \int_Q v\, dx > \alpha |Q|^{-2s/n} \int_Q w\, dx \right\}. \tag{8}$$

Let $\{\mu_k\}_{1\le k}$ be the non-decreasing rearrangement of

$$\left\{ \alpha |Q|^{-2s/n-1} \int_Q w\, dx - \beta |Q|^{-1} \int_Q v\, dx \right\}_{Q\in\mathcal{I}}.$$

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When

$$\mu_k = \alpha |Q|^{-2s/n-1} \int_Q w\, dx - \beta |Q|^{-1} \int_Q v\, dx,$$

we define $\Psi_k = \psi_Q$. Then we have by (7)

$$\begin{aligned} \tag{9} &\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\, dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V|\phi_i|^2\, dx \\ &\ge \sum_{i=1}^N \sum_{Q\in\mathcal{Q}} |(\phi_i,\psi_Q)|^2 \left\{ \alpha |Q|^{-2s/n-1} \int_Q w\, dx - \beta |Q|^{-1} \int_Q v\, dx \right\} \\ &\ge \sum_{i=1}^N \sum_k \mu_k |(\phi_i,\Psi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\phi_i,\Psi_k)|^2 \\ \tag{10} &\ge -K\|\psi\|^2 \sum_k |\mu_k| \ \ge -K\|\psi\|^2 \left(\sum_k |\mu_k|^\gamma \right)^{1/\gamma}, \end{aligned}$$

where $\gamma = p - n/(2s) \in (0,1]$ and we used (4).

Now the following lemma holds.

Lemma 3.3.

$$\sum_k |\mu_k|^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\, dx,$$

where c is given by

$$c = c' \max_{|\sigma|\leq n} \|\partial^\sigma \hat{\psi}\|_\infty^{n/s+2p}$$

and c' depends only on n, s, p and w.

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (6) the last quantity in (10) is estimated from below by

$$\begin{aligned}
&-cK \left(\int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx \right)^{1/\gamma} \\
&= -cK\delta_1^{p/(p-n/(2s))} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{1/(p-n/(2s))},
\end{aligned}$$

where

$$c = c'\|\psi\|^2 \max_{|\sigma|\leq n} \|\partial^\sigma \hat{\psi}\|_\infty^{(4ps+2n)/(2ps-n)}$$

and c' depends only on n, s, p and w. We may take the infimum of the above constant with respect to possible ψ and replace c by this infimum.

Let

$$\delta_1 = \delta_2 K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-n/(2s)-1)/n},$$

where δ_2 is a positive constant. Then we have by (9)

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\, dx \\
&\geq \delta_2 K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-1)/n} \\
&\quad - cK\delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-1)/n} \\
&= \{\delta_2 - c\delta_2^{p/(p-n/(2s))}\} K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-1)/n}.
\end{aligned}$$

If we take δ_2 small enough, then we get the inequality (3) because $1 < p/(p-n/(2s))$.

Next we shall show (3) for $\phi_i \in \mathcal{H}^s(w),\ i = 1, \ldots, N$. First we show

$$\mathcal{H}^s(w) \subset L^{2p/(p-1)}(w^{n/(2s(p-1))}). \tag{11}$$

Let $h \in \mathcal{H}^s(w)$. Then there exists a sequence $\{h_m\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ such that $\|h - h_m\|_{\mathcal{H}^s(w)} \to 0$ $(m \to \infty)$. Since we proved that (5) holds for $h_m \in C_0^\infty(\mathbb{R}^n)$, we get

$$\left\{\int_{\mathbb{R}^n} |h_m|^{2p/(p-1)} w^{n/(2s(p-1))}\, dx\right\}^{2s(p-1)/n} \leq c\|h_m\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h_m|^2 w\, dx,$$

where c does not depend on h_m. Since $4sp/n-2>0$ and $\{h_m\}$ is a Cauchy sequence in $\mathcal{H}^s(w)$, the above inequality says that $\{h_m\}$ is a Cauchy sequence in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. Let g be the limit of $\{h_m\}$ in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. For any compact set E in $\mathbb{R}^n$ we have

$$\begin{aligned}\int_E |g-h_m|\, dx \leq &\left(\int_E |g-h_m|^{2p/(p-1)} w^{n/(2s(p-1))}\, dx\right)^{(p-1)/(2p)} \\ &\times \left(\int_E w^{-n/(2s(p+1))}\, dx\right)^{(p+1)/(2p)}.\end{aligned}$$

Since $w^{-n/(2s)}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ or $w^{-n/(2s)} \in A_p$, we get $h_m \to g$ in $L^1_{loc}(\mathbb{R}^n)$ as $m \to \infty$. Hence we have $g = h$ and (11). Furthermore we have

$$\left\{\int_{\mathbb{R}^n} |h|^{2p/(p-1)} w^{n/(2s(p-1))}\, dx\right\}^{2s(p-1)/n} \leq c\|h\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h|^2 w\, dx. \tag{12}$$

We fix a positive number ε. Let $\chi_1, \dots, \chi_N$ be functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{i=1}^N \|\phi_i - \chi_i\|^2_{\mathcal{H}^s(w)} < \varepsilon.$$

Now the inequalities

$$\begin{aligned}\sum_{i=1}^N |(\chi_i, f)|^2 &\leq 2\sum_{i=1}^N |(\chi_i - \phi_i, f)|^2 + 2\sum_{i=1}^N |(\phi_i, f)|^2 \\ &\leq 2\sum_{i=1}^N \|\chi_i - \phi_i\|^2 \|f\|^2 + 2K\|f\|^2 \leq 2(K+\varepsilon)\|f\|^2 \end{aligned} \tag{13}$$

hold for all $f \in L^2(\mathbb{R}^n)$. On the other hand

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\phi_i-\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\
&\le\left\{\sum_{i=1}^N\left(\int_{\mathbb{R}^n}|\phi_i-\chi_i|^{2p/(p-1)}w^{n/(2s(p-1))}\,dx\right)^{(p-1)/p}\right\}^{2sp/n}\\
&\le N^{2sp/n-1}\sum_{i=1}^N\left(\int_{\mathbb{R}^n}|\phi_i-\chi_i|^{2p/(p-1)}w^{n/(2s(p-1))}\,dx\right)^{2s(p-1)/n}\\
&\le cN^{2sp/n-1}\sum_{i=1}^N\|\phi_i-\chi_i\|^{4sp/n-2}\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\phi_i-(-\Delta)^{s/2}\chi_i|^2w\,dx\\
&\le cN^{2sp/n-1}\varepsilon^{2sp/n-1}\sum_{i=1}^N\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\phi_i-(-\Delta)^{s/2}\chi_i|^2w\,dx\\
&\le cN^{2sp/n-1}\varepsilon^{2sp/n},
\end{aligned}$$

where we used (12).

Therefore

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\phi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\
&\le\left\{\int_{\mathbb{R}^n}\left(2\sum_{i=1}^N|\phi_i-\chi_i|^2+2\sum_{i=1}^N|\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\
&\le 2^{2sp/n}\left[\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\phi_i-\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{(p-1)/p}\right.\\
&\quad\left.+\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{(p-1)/p}\right]^{2sp/n}\\
&\le 2^{4sp/n-1}\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\phi_i-\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\
&\quad+2^{4sp/n-1}\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\
&\le c2^{4sp/n-1}N^{2sp/n-1}\varepsilon^{2sp/n}+c2^{6sp/n-2}(K+\varepsilon)^{2sp/n-1}\sum_{i=1}^N\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\chi_i|^2w\,dx, \qquad (14)
\end{aligned}$$

where we used (13) and (3) for χ_i. Since

$$\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\chi_i|^2 w\,dx$$
$$\le 2\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\chi_i - (-\Delta)^{s/2}\phi_i|^2 w\,dx + 2\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\,dx$$
$$\le 2\varepsilon + 2\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\,dx,$$

we have by (14)

$$\left\{\int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i|^2\right)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}$$
$$\le c2^{4sp/n-1}N^{2sp/n-1}\varepsilon^{2sp/n} + c2^{6sp/n-1}(K+\varepsilon)^{2sp/n-1}\varepsilon$$
$$+c2^{6sp/n-1}(K+\varepsilon)^{2sp/n-1}\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\,dx.$$

Since we can take ε arbitrary small, we conclude

$$\left\{\int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i|^2\right)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}$$
$$\le c2^{6sp/n-1}K^{2sp/n-1}\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\,dx.$$

Hence we get (3).

4 Proof of Lemma 3.3

The arguments of the proof is similar to those in [11] and [12]. First we consider the case $n > 2s$. For $\lambda > 0$ we set

$$\mathcal{I}_\lambda = \{Q \in \mathcal{Q} \,:\, \alpha|Q|^{-2s/n-1}\int_Q w\,dx - \beta|Q|^{-1}\int_Q v\,dx < -\lambda\}. \tag{15}$$

Then we have for $Q \in \mathcal{I}_\lambda$

$$\alpha|Q|^{-2s/n-1}\int_Q w\,dx < |Q|^{-1}\int_Q (\beta v - \lambda)_+\,dx,$$

where

$$(\beta v - \lambda)_+(x) = \max\{0, \beta v(x) - \lambda\}.$$

Since $p = n/(2s) + \gamma, \gamma \in (0, 1]$, and

$$\beta^{-p}\gamma \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)}\, dx \lambda^{\gamma-1} d\lambda \le \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\, dx < \infty,$$

we have

$$\int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx < \infty$$

for all $\lambda > 0$. By the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ and (ii) of Proposition 2.1, there exists a $\kappa' \in (1, n/(2s))$ such that $w^{-n/(2s)} \in A_{n/(2s\kappa')}$. We set

$$v_\lambda^*(x) = M((\beta v - \lambda)_+^{\kappa'})(x)^{1/\kappa'}.$$

Then

$$\int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \le c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx < \infty \tag{16}$$

and $v_\lambda^* \in A_1$ by (iii) of Proposition 2.1, where c_1 depends only on n, s and $A_{n/(2s)}$-constant of $w^{-n/(2s)}$.

We can show that $\mathcal{I}_\lambda$ is a finite set as follows. Let $Q \in \mathcal{I}_\lambda$. Then we have

$$\begin{aligned} \alpha|Q|^{-2s/n} \int_Q w\, dx &\le \int_Q v_\lambda^*\, dx \\ \le \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \right\}^{2s/n} &\left\{ \int_Q w^{n/(n-2s)}\, dx \right\}^{(n-2s)/n}. \end{aligned}$$

Since $w^{-n/(2s)} \in A_{n/(2s)}$, the last quantity is bounded by

$$\begin{aligned} c_2 &\left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \right\}^{2s/n} |Q| \left(\int_Q w^{-n/(2s)}\, dx \right)^{-2s/n} \\ &\le c_2 \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \right\}^{2s/n} |Q|^{-2s/n} \int_Q w\, dx, \end{aligned}$$

where we used the inequality

$$1 \le \frac{1}{|Q|} \int_Q w\, dx \left(\frac{1}{|Q|} \int_Q w^{-n/(2s)}\, dx \right)^{2s/n}.$$

The above calculation says

$$1 \le c_3 \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx,$$

where $c_3 = c'\alpha^{-n/(2s)}$ and c' is the $A_{n/(2s)}$-constant of $w^{-n/(2s)}$.

First we assume that $\mathcal{I}_\lambda$ includes infinite disjoint cubes $\{Q_i\}_{i=1}^\infty$. Then we have

$$\infty = \sum_{i=1}^\infty 1 \le \sum_{i=1}^\infty c_3 \int_{Q_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \le c_3 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx < \infty.$$

This is a contradiction. Hence $\mathcal{I}_\lambda$ does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes $\{Q_i\}_{i=1}^\infty \subset \mathcal{I}_\lambda$ such that $Q_i \ne Q_j$ $(i \ne j)$ and $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$. Let $\tilde{Q}_i$ be a half size dyadic sub-cube of Q_{i+1} such that $Q_i \cap \tilde{Q}_i = \emptyset$. Since $Q_{i+1} \in \mathcal{I}_\lambda$, we have

$$\alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w\, dx \le \int_{Q_{i+1}} v_\lambda^*\, dx.$$

Now we get

$$\int_{Q_{i+1}} v_\lambda^*\, dx \le \int_{3\tilde{Q}_i} v_\lambda^*\, dx \le c_4 \int_{\tilde{Q}_i} v_\lambda^*\, dx,$$

where we used the doubling property of v_λ^*. Since

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w\, dx \ge 2^{-2s} |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w\, dx,$$

we get

$$c_5 |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w\, dx \le \int_{\tilde{Q}_i} v_\lambda^*\, dx.$$

The similar calculation as before leads to

$$1 \le c_6 \int_{\tilde{Q}_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx,$$

where $c_6 = c'' \alpha^{-n/(2s)}$ and c'' depends only on n, s, and w. Since $\{\tilde{Q}_i\}_{i=1}^\infty$ is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$ has a maximal element. Similarly we can show that any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in $\mathcal{I}_\lambda$ with respect to the inclusion relation is finite. Hence $\mathcal{I}_\lambda$ is a finite set. We remark that the non-decreasing rearrangement of $\mathcal{I}$ in (8) is possible because $\mathcal{I}_\lambda$ is a finite set for every $\lambda > 0$.

Let $N(\lambda) = \sharp \mathcal{I}_\lambda$, that is, the number of elements of $\mathcal{I}_\lambda$. Let $\tilde{\mathcal{I}}_\lambda$ be the set of all $Q \in \mathcal{I}_\lambda$ which satisfy the following condition: there exists a half size dyadic sub-cube $\tilde{Q} \subset Q$ such that $\tilde{Q} \notin \mathcal{I}_\lambda$ and $\tilde{Q}$ does not contain any dyadic cube in $\mathcal{I}_\lambda$. Then we have the following lemma.

Lemma 4.1. $\sharp\mathcal{I}_\lambda \le 2\sharp\tilde{\mathcal{I}}_\lambda$.

Lemma 4.1 is proved in Rochberg and Taibleson's paper([9, Lemma 1]).

Let $Q \in \tilde{\mathcal{I}}_\lambda$ and $\tilde{Q}$ be a dyadic cube which satisfies the condition in the definition of $\tilde{\mathcal{I}}_\lambda$. Then by similar calculations as before we get

$$1 \le c_6 \int_{\tilde{Q}} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx.$$

For every $Q \in \tilde{\mathcal{I}}_\lambda$ we choose a $\tilde{Q}$ as above. Let $\{\tilde{Q}_j\}_{j\in J}$ be the set of all such cubes $\tilde{Q}$. Then the cubes in $\{\tilde{Q}_j\}_{j\in J}$ are mutually disjoint. Therefore we get

$$\begin{aligned}\sharp\tilde{\mathcal{I}}_\lambda = \sharp J &\le \sum_{j\in J} c_6 \int_{\tilde{Q}_j} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \\ &\le c_6 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \le c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx,\end{aligned}$$

where we used (16). Hence we have

$$N(\lambda) \le 2c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx.$$

Therefore we conclude

$$\begin{aligned}\sum_k |\mu_k|^\gamma &= \int_0^\infty \gamma\lambda^{\gamma-1} N(\lambda)\, d\lambda \\ &\le 2c_7 \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx \gamma\lambda^{\gamma-1} d\lambda \\ &\le c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\, dx,\end{aligned}$$

where $c_8 = c'''\alpha^{-n/(2s)}\beta^p$ and c''' depends only on n, s, p and w.

Next we consider the case $n \le 2s$. We remark that $v(x) > 0$ for all $x \in \mathbb{R}^n$. In fact if $v(x_0) = 0$ at some point x_0, then by the definition of the maximal operator we have $V \equiv 0$, that is, $\phi_i \equiv 0, i = 1, \dots, N$.

We also remark that $\mathcal{I}$ in (8) is not empty. In fact if $\mathcal{I}$ is empty, then we have

$$\beta \int_Q v\, dx \le \alpha |Q|^{-2s/n} \int_Q w\, dx$$

for all $Q \in \mathcal{Q}$. Let $Q_0 \in \mathcal{Q}$ and $Q_0 \subset Q_1 \subset Q_2 \subset \cdots$ be the infinite sequence of dyadic cubes such that Q_{i+1} is the parent of Q_i for all $i = 0, 1, 2, \dots$. By (2) we have

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w\, dx \le |Q_i|^{-2s/n} \int_{Q_i} w\, dx$$

for all i. Hence we have

$$\beta \int_{Q_i} v\,dx \le \alpha |Q_0|^{-2s/n} \int_{Q_0} w\,dx \tag{17}$$

for all i. On the other hand, since $v \in A_1$, there exists a constant $d > 1$ such that

$$d \int_{Q_i} v\,dx \le \int_{Q_{i+1}} v\,dx$$

for all i (c.f.[5, p141]). Hence we have

$$d^i \int_{Q_0} v\,dx \le \int_{Q_i} v\,dx$$

and

$$\lim_{i\to\infty} \int_{Q_i} v\,dx = \infty,$$

which contradicts to (17). Therefore $\mathcal{I}$ is not empty.

Let $Q \in \mathcal{I}$ and Q' be the parent of Q. Then we have

$$\alpha|Q'|^{-2s/n} \int_{Q'} w\,dx \le \alpha|Q|^{-2s/n} \int_Q w\,dx < \beta \int_Q v\,dx \le \beta \int_{Q'} v\,dx,$$

where we used the assumption (2). Hence we have $Q' \in \mathcal{I}$, which means that $\mathcal{I}$ is an infinite set.

Lemma 4.2. *There exists a $c > 0$ such that*

$$\sum_{Q\in\mathcal{I}} \left(\frac{1}{|Q|} \int_Q v\,dx \right)^{\gamma} \le c \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx,$$

where $c = c'\alpha^{-n/(2s)}\beta^{n/(2s)}$ and c' depends only on n, p, s and w.

This lemma is proved in [12, Lemma 3.3]. Let $\mathcal{I}_\lambda$ be the set defined by (15).

Lemma 4.3. *For each $\lambda > 0$, $\mathcal{I}_\lambda$ is a finite set.*

Lemma 4.3 is easily proved by Lemma 4.2 (c.f.[12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of $\mathcal{I}$ is possible.

By Lemma 4.2 we conclude

$$\begin{aligned}\sum_{k=1}^{\infty} |\mu_k|^{\gamma} &= c \sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1} \int_Q v\,dx - \alpha|Q|^{-2s/n-1} \int_Q w\,dx \right)^{\gamma} \\ &\le c \sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1} \int_Q v\,dx \right)^{\gamma} \le c \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx,\end{aligned}$$

where $c = c''\alpha^{-n/(2s)}\beta^p$ and c'' depends only on n, p, s and w. This ends the proof of Lemma 3.3.

5 Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

Lemma 5.1. *Let $w \in A_2$ and $m \in C^n(\mathbb{R}^n \setminus \{0\})$. Suppose that*

$$B = \max_{|\sigma| \le n} \sup_{0<r} r^{2|\sigma|-n} \int_{r \le |\xi| \le 2r} \left| \left(\frac{\partial}{\partial \xi} \right)^{\sigma} m(\xi) \right|^2 d\xi < \infty.$$

Then the operator T defined by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ is bounded from $L^2(w)$ to $L^2(w)$ and the operator norm $\|T\|$ is bounded by $CB^{1/2}$ where C is a constant which depends only on n and w.

The proof of Lemma 5.1 is in [6] or [7].

For $\nu \in \mathbb{Z}$ we define $\psi_\nu(x) = 2^{n\nu}\psi(2^\nu x)$. Let $w \in A_2$ and $s \ge 0$. Frazier and Jawerth proved that there exist positive constants c_1 and c_2 such that

$$\begin{aligned} c_1 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w\, dx &\le \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} 2^{2s\nu} |f * \psi_\nu(x)|^2 \right\} w(x)\, dx \\ &\le c_2 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w\, dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where c_1 and c_2 depend only on n, s and w ([3, Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let $\{r_\nu(t)\}$ be the Rademacher functions on $[0,1]$ indexed by $\nu \in \mathbb{Z}$ and

$$T_t f(x) = \sum_{\nu \in \mathbb{Z}} r_\nu(t) f * \psi_\nu(x).$$

Then T_t satisfies the condition of Lemma 5.1. Hence

$$\int_{\mathbb{R}^n} |T_t f(x)|^2 w(x)\, dx \le CM \int_{\mathbb{R}^n} |f(x)|^2 w(x)\, dx,$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where $M = \max_{|\sigma| \le n} \|\partial^\sigma \hat{\psi}\|_\infty^2$ and C is a positive constant depending only on n and w. By integrating from 0 to 1 with respect to t, we get

$$\int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x)\, dx \le CM \int_{\mathbb{R}^n} |f(x)|^2 w(x)\, dx.$$

By the duality argument and the fact $w^{-1} \in A_2$ we obtain

$$\int_{\mathbb{R}^n} |f(x)|^2 w(x)\, dx \le CM \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x)\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Hence we have

$$c_3 M^{-1} \int_{\mathbb{R}^n} |f|^2 w\, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu|^2 \right\} w\, dx \leq c_4 M \int_{\mathbb{R}^n} |f|^2 w\, dx,$$

where c_3 and c_4 are constants depending only on n and w.

Therefore we get

$$\begin{aligned} c_3 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w\, dx &\leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |(-\Delta)^{s/2} f * \psi_\nu|^2 \right\} w\, dx \\ &\leq c_4 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w\, dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$(c.f.[11]).

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp} \Phi \subset \{\xi \,:\, 1/4 \leq |\xi| \leq 4\}$ and $\Phi(\xi) = 1$ for $1/2 \leq |\xi| \leq 2$. For $\nu \in \mathbb{Z}$ the multiplier $m_\nu(\xi) = 2^{-s\nu}|\xi|^s \Phi(\xi/2^\nu)$ satisfies the condition of Lemma 5.1. Hence we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2 w(x)\, dx \leq c_5 \int_{\mathbb{R}^n} 2^{2s\nu} |f * \psi_\nu(x)|^2 w(x)\, dx,$$

where $c_5 = c_6 \inf\limits_{\Phi} \max\limits_{|\sigma| \leq n} \|\partial^\sigma \Phi\|_\infty^2$ and c_6 is a positive constant depending only on n, s and w and the infimum is taken over all possible Φ.

Similarly there exists a positive constant c_7 depending only on n, s and w such that

$$\int_{\mathbb{R}^n} 2^{2s\nu} |f * \psi_\nu(x)|^2 w(x)\, dx \leq c_7 \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2 w(x)\, dx.$$

Hence we get

$$\begin{aligned} c_8 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w\, dx &\leq \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w\, dx \\ &\leq c_9 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w\, dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where c_8 and c_9 are positive constant depending only on n, s and w. This ends the proof of Lemmas 3.1 and 3.2.

Acknowledgment

The author was partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan.

References

[1] Daubechies, I., *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. SIAM, 1992.

[2] Edmunds, D.E. and Ilyin, A.A., On some multiplicative inequalities and approximation numbers, *Quart. J. Math. Oxford Ser. (2)* **45** (1994) 159–179.

[3] Frazier, M. and Jawerth, B., A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.* **93** (1990) 34–170.

[4] Ghidaglia, J.-M., Marion, M. and Temam, R., Generalization of the Sobolev-Lieb-Thirring inequalities and applications to the dimension of attractors, *Differential Integral Equations* **1** (1988) 1–21.

[5] García-Cuerva, J. and Rubio de Francia, J.L., *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, 116, North-Holland, 1985.

[6] Kurtz, D.S., Littlewood-Paley and multiplier theorems on weighted L^p spaces, *Trans. Amer. Math. Soc.* **259** (1980) 235–254.

[7] Kurtz, D.S. and Wheeden, R.L., Results on weighted norm inequalities for multipliers, *Trans. Amer. Math. Soc.* **255** (1979) 343–362.

[8] Lieb, E. and Thirring, W., Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities, *Studies in Mathematical Physics*, Princeton University Press, (1976), 269–303.

[9] Rochberg, R. and Taibleson, M., An averaging operator on a tree, *Harmonic analysis and partial differential equations (El Escorial)*, Lecture Notes in Mathematics, Springer-Verlag, **1384** (1987) 207–213.

[10] Stein, E.M., *Harmonic analysis : real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, **43**, Princeton University Press, 1993.

[11] Tachizawa, K., On the moments of negative eigenvalues of elliptic operators, *J. Fourier Anal. Appl.* **8** (2002) 233–244.

[12] Tachizawa, K., A generalization of the Lieb-Thirring inequalities in low dimensions, *Hokkaido Math. J.* **32** (2003) 383–399.

[13] Temam, R., *Infinite dimensional dynamical systems in mechanics and physics*, Springer, New York, 1988.

Department of Mathematics
Faculty of Science, Hokkaido University
Sapporo 060-0810
JAPAN
tachizaw@math.sci.hokudai.ac.jp