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Weighted Sobolev-Lieb-Thirring inequalities

Kazuya Tachizawa

Abstract. We give a weighted version of the Sobolev-Lieb-Thirring inequality for
suborthonormal functions. In the proof of our result we use $\varphi$-transform of Frazier-
Jawerth.

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Keywords. Sobolev-Lieb-Thirring inequalities, $\varphi$-transform, $A_p$-weights.

1 Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring in-
equality.

Theorem 1.1 ([2]). Let $n \in \mathbb{N}, s > 0$ and $p$ with
\[
\max \left( 1, \frac{n}{2s} \right) < p \leq 1 + \frac{n}{2s}.
\]
Then there exists a positive constant $c = c(p, n, s)$ such that for every family \{\phi_i\}_{i=1}^N in
$H^s(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \|(-\Delta)^{s/2} \phi_i\|^2
\]
where
\[
\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.
\]

In this theorem $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order $s$ and $\| \cdot \|$ is the norm
of $L^2(\mathbb{R}^n)$.

In [8] Lieb and Thirring proved this theorem for $s = 1$ and applied it to the problem
of the stability of matter. Ghidaglia, Marion, and Temam proved (1) for $s \in \mathbb{N}$ under
the suborthonormal condition on \( \{ \phi_i \} \), where \( \{ \phi_i \}_{i=1}^N \) in \( L^2(\mathbb{R}^n) \) is called suborthonormal if the inequality
\[
\sum_{i,j=1}^N \xi_i \overline{\xi_j} (\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2
\]
holds for all \( \xi_i \in \mathbb{C}, i = 1, \ldots, N \) ([4]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations (c.f. [13]). In this paper we shall give a weighted version of (1) under suborthonormal condition on \( \{ \phi_i \} \). In the proof of our theorem we shall use Frazier-Jawerth’s \( \varphi \)-transform ([3]).

For the statement of our result we need to recall the definition of \( A_p \)-weights (c.f. [5], [10]). By a cube in \( \mathbb{R}^n \) we mean a cube which sides are parallel to coordinate axes. Let \( w \) be a non-negative, locally integrable function on \( \mathbb{R}^n \). We say that \( w \) is an \( A_p \)-weight for \( 1 < p < \infty \) if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \quad \text{for all cubes } Q \subset \mathbb{R}^n.
\]
For example, \( w(x) = |x|^\alpha \) is an \( A_p \)-weight when \( -n < \alpha < n(p-1) \).

We say that \( w \) is an \( A_1 \)-weight if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(y) \, dy \leq C w(x) \quad \text{a.e. } x \in Q
\]
for all cubes \( Q \subset \mathbb{R}^n \). The infimum of the constant \( C \) is called the \( A_1 \)-constant of \( w \).

Let \( A_p \) be the class of \( A_p \)-weights. The inclusion \( A_p \subset A_q \) holds for \( p < q \).

For a nonnegative, locally integrable function \( w \) on \( \mathbb{R}^n \) we define
\[
L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.
\]

For \( \nu \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) the cube \( Q \) defined by
\[
Q = Q_{\nu k} = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1, \ i = 1, \ldots, n\}
\]
is called a dyadic cube in \( \mathbb{R}^n \). Let \( Q \) be the set of all dyadic cubes in \( \mathbb{R}^n \). For any \( Q \in Q \) there exists a unique \( Q' \in Q \) such that \( Q \subset Q' \) and the side-length of \( Q' \) is double of that of \( Q \). We call \( Q' \) the parent of \( Q \).

For \( s > 0 \) and \( f \in C_0^\infty(\mathbb{R}^n) \) we define via inverse Fourier transform
\[
(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).
\]
Let \( w \in A_2 \) and \( H^s(w) \) be the completion of \( C_0^\infty(\mathbb{R}^n) \) with respect to the norm
\[
\|f\|_{H^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx + \|f\|^2 \right\}^{1/2}.
\]
We remark that for $f \in C_0^\infty(\mathbb{R}^n)$ we have
\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx < \infty
\]
because
\[
|(-\Delta)^{s/2} f(x)| \leq \frac{c}{(1 + |x|)^n} \quad (x \in \mathbb{R}^n)
\]
and
\[
\int_{\mathbb{R}^n} \frac{w(x)}{(1 + |x|)^{2n}} \, dx < \infty
\]
(c.f. [10, p.209]).

Let $f \in \mathcal{H}^s(w)$ and $\{f_i\}_{i=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\|f - f_i\|_{\mathcal{H}^s(w)} \to 0$ ($i \to \infty$). This means that there exist $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(w)$ such that $\|g_1 - f_i\| \to 0$ and
\[
\int_{\mathbb{R}^n} |g_2(x) - (-\Delta)^{s/2} f_i(x)|^2 w(x) \, dx \to 0
\]
as $i \to \infty$. We denote $(-\Delta)^{s/2} f = g_2$. We remark that $g_1 \equiv 0$ means $g_2 \equiv 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have
\[
\int_{\mathbb{R}^n} g_2 \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} f_i (-\Delta)^{s/2} \varphi \, dx = 0.
\]
Hence we have $g_2 \equiv 0$. This means that we can identify $\mathcal{H}^s(w)$ as a subspace of $L^2(\mathbb{R}^n)$.

The following is the main result of this paper.

**Theorem 1.2.** Let $n \in \mathbb{N}, s > 0$, and
\[
\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}.
\]
Let $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \geq n$, then we assume that $w^{-n/(2s)} \in A_p$ and
\[
(2) \quad \int_{Q'} w \, dx \leq 2^{2s} \int_{Q} w \, dx
\]
for all dyadic cubes $Q \in \mathcal{Q}$ and its parent $Q'$.

Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $\mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$, we have
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx,
\]
where
\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]
and \( c \) depends only on \( n, s, p, A_2 \)-constant of \( w \), and \( A_{n/(2s)} \) or \( A_p \)-constant of \( w^{-n/(2s)} \).

When \( 2s < n \), an example of weight function \( w \) is given by \( w(x) = |x|^\alpha \) for \( -n+2s < \alpha < 2s \). When \( 2s > n \), an example of weight function \( w \) is given by \( w(x) = |x|^\alpha \) for \( 0 \leq \alpha < \min\{2s-n, n\} \) (c.f. [12, Section 4]). When \( 2s = n \), the condition (2) means \( w \) is equivalent to a constant almost everywhere (c.f. [12, Proposition 4.1]).

2 Preliminaries

Let \( \psi \) be a function which satisfies the following conditions.

(A1) \( \psi \in S(\mathbb{R}^n) \).

(A2) \( \text{supp} \hat{\psi} \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \} \)

(A3) \( |\hat{\psi}(\xi)| \geq c > 0 \) if \( \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \).

(A4) \( \sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 \) = 1 for all \( \xi \neq 0 \).

For \( \nu \in \mathbb{Z}, k \in \mathbb{Z}^n \) and \( Q = Q_{\nu k} \), we set
\[ \psi_Q(x) = 2^{\nu n/2} \psi(2^\nu x - k) \quad (x \in \mathbb{R}^n). \]

Let \( M \) be the Hardy-Littlewood maximal operator, that is,
\[ M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \]
where \( f \) is a locally integrable function on \( \mathbb{R}^n \) and the supremum is taken over all cubes \( Q \) which contain \( x \).

**Proposition 2.1.**

(i) Let \( 1 < p < \infty \) and \( w \) be a non-negative locally integrable function on \( \mathbb{R}^n \). Then there exists a positive constant \( c \) such that
\[ \int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx \]
for all \( f \in L^p(w) \) if and only if \( w \in A_p \). The constant \( c \) depends only on \( n, p \) and \( A_p \)-constant of \( w \).
(ii) Let \(1 < p < \infty\) and \(w \in A_p\). Then there exists a \(q \in (1, p)\) such that \(w \in A_q\).

(iii) Let \(0 < \tau < 1\) and \(f\) be a locally integrable function on \(\mathbb{R}^n\) such that \(M(f)(x) < \infty\) a.e. Then \((M(f))^{\tau} \in A_1\) and the \(A_1\)-constant of \((M(f))^{\tau}\) depends only on \(n\) and \(\tau\).

(iv) Let \(1 \leq p < \infty\) and \(w \in A_p\). Then there exists a positive constant \(c\) such that

\[
\int_{2Q} w \, dx \leq c \int_Q w \, dx
\]

for all cubes \(Q \in \mathbb{R}^n\), where \(2Q\) denotes the double of \(Q\) and \(c\) depend only on \(n\) and \(A_p\)-constant of \(w\).

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3 Proof of Theorem 1.2

The suborthonormal condition on \(\{\phi_i\}\) is equivalent to the inequality

\[
\sum_{i=1}^{N} |(\phi_i, f)|^2 \leq \|f\|^2
\]

for all \(f \in L^2(\mathbb{R}^n)\) (c.f.[1, p57]).

We shall prove the inequality

\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{\alpha_n/(p-1)} w(x)^{\alpha_n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq cK^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx
\]

under the assumption

\[
\sum_{i=1}^{N} |(\phi_i, f)|^2 \leq K\|f\|^2 \quad \quad (4)
\]

for all \(f \in L^2(\mathbb{R}^n)\) where \(K\) is a positive constant. This is equivalent to the statement of Theorem 1.2. We remark that \(K\) may depend on \(\{\phi_i\}\). For example, the inequality (3) says that

\[
\left\{ \int_{\mathbb{R}^n} |\phi|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c\|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi|^2 w \, dx
\]

(5)
holds for all $\phi \in H^s(w)$ under suitable condition on $s, p, n$ and $w$ because
\[ |(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2 \]
for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C^\infty_0(\mathbb{R}^n), i = 1, \ldots, N$. Let
\[ V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2(p-1))} \]
where the value of the constant $\delta_1 > 0$ will be given later. Since $\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$ is a bounded function with compact support and $w^{n/(2(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have
\[ \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty. \]
We may also assume that
\[ 0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx. \]

By (ii) of Proposition 2.1 there exists a constant $\kappa$ such that $1 < \kappa < p$ and $w^{-n/(2s)} \in A_{p/\kappa}$. We set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to
\[ \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty. \]
Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 2.1.

We have the following lemmas.

**Lemma 3.1.** For $s > 0$ and $w \in A_2$ there exists a positive constant $\alpha$ such that
\[ \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \]
for all $f \in C^\infty_0(\mathbb{R}^n)$, where $\alpha$ is given by
\[ \alpha^{-1} = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2 \]
and $c$ is a constant depending only on $n, s$ and $A_2$-constant of $w$.

**Lemma 3.2.** For $v \in A_2$ there exist positive constants $\beta$ and $\beta'$ such that
\[ \beta' \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \leq \int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \]
for all $f \in C^\infty_0(\mathbb{R}^n)$, where $\beta$ is given by
\[ \beta = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2 \]
and $c$ is a constant depending only on $n$ and $A_2$-constant of $v$. 

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The proof of Lemmas 3.1 and 3.2 are in [11, Proposition 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader’s convenience because the dependence of ψ in α and β is not explained in [11].

For \( f \in C_0^\infty(\mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}^n} |f|^2 V \, dx \leq \int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx,
\]

where we used Lemma 3.2. Hence by Lemma 3.1

\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 \, w \, dx - \int_{\mathbb{R}^n} V |f|^2 \, dx \geq \alpha \sum_{Q \in \mathcal{Q}} \frac{|Q|^{-2s/n}}{\alpha |Q| \left( \int_Q w \, dx - \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \right)}.
\]

Now we set

\[
\mathcal{I} = \left\{ Q \in \mathcal{Q} : \beta \int_Q v \, dx > \alpha |Q|^{-2s/n} \int_Q w \, dx \right\}.
\]

Let \( \{ \mu_k \}_{1 \leq k} \) be the non-decreasing rearrangement of

\[
\left\{ \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx \right\}_{Q \in \mathcal{I}}.
\]

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When

\[
\mu_k = \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx,
\]

we define \( \Psi_k = \psi_Q \). Then we have by (7)

\[
\sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 \, w \, dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V |\phi_i|^2 \, dx \geq \sum_{i=1}^N \sum_{Q \in \mathcal{Q}} |(\phi_i, \psi_Q)|^2 \left\{ \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx \right\} \geq \sum_{i=1}^N \sum_{k} \mu_k |(\phi_i, \Psi_k)|^2 = \sum_{k} \mu_k \sum_{i=1}^N |(\phi_i, \Psi_k)|^2 \geq -K \|\psi\|^2 \sum_{k} |\mu_k| \geq -K \|\psi\|^2 \left( \sum_{k} |\mu_k|^\gamma \right)^{1/\gamma},
\]

where \( \gamma = p - n/(2s) \in (0, 1] \) and we used (4).

Now the following lemma holds.

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Lemma 3.3.

\[ \sum_k |\mu_k|^{\gamma} \leq c \int_{\mathbb{R}^n} r^p w^{-n/(2s)} \, dx, \]

where \( c \) is given by

\[ c = c' \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_{\infty}^{n/s+2p} \]

and \( c' \) depends only on \( n, s, p \) and \( w \).

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (6) the last quantity in (10) is estimated from below by

\[ -cK \left( \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx \right)^{1/\gamma} \]

\[ = -cK \delta_1^{p/(p-n/(2s))} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{1/(p-n/(2s))}, \]

where

\[ c = c' \|\psi\|^2 \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_{\infty}^{(4ps+2n)/(2ps-n)} \]

and \( c' \) depends only on \( n, s, p \) and \( w \). We may take the infimum of the above constant with respect to possible \( \psi \) and replace \( c \) by this infimum.

Let

\[ \delta_1 = \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-n/(2s)-1)/n}, \]

where \( \delta_2 \) is a positive constant. Then we have by (9)

\[ \sum_{i=1}^N \int_{\mathbb{R}^n} \left| (-\Delta)^{s/2} \phi_i \right|^2 w \, dx \]

\[ \geq \delta_2 K^{-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ - cK \delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ = \{ \delta_2 - c \delta_2^{p/(p-n/(2s))} \} K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n}. \]

If we take \( \delta_2 \) small enough, then we get the inequality (3) because \( 1 < p/(p-n/(2s)). \)

Next we shall show (3) for \( \phi_i \in \mathcal{H}^s(w), i = 1, \ldots, N \). First we show

\[ \mathcal{H}^s(w) \subset L^{2p/(p-1)}(w^{n/(2s(p-1))}). \]
Let \( h \in \mathcal{H}^s(w) \). Then there exists a sequence \( \{h_m\}_{m=1}^{\infty} \subset C_0^\infty(\mathbb{R}^n) \) such that
\[
\|h - h_m\|_{\mathcal{H}^s(w)} \to 0 \quad (m \to \infty).
\]
Since we proved that (5) holds for \( h_m \in C_0^\infty(\mathbb{R}^n) \), we get
\[
\left\{ \int_{\mathbb{R}^n} |h_m|^{2p/(p-1)} u^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|h_m\|^{4sp/n - 2} \int_{\mathbb{R}^n} \|(-\Delta)^{s/2} h_m\|^2 w \, dx,
\]
where \( c \) does not depend on \( h_m \). Since \( 4sp/n - 2 > 0 \) and \( \{h_m\} \) is a Cauchy sequence in \( \mathcal{H}^s(w) \), the above inequality says that \( \{h_m\} \) is a Cauchy sequence in \( L^{2p/(p-1)}(u^{n/(2s(p-1))}) \). Let \( g \) be the limit of \( \{h_m\} \) in \( L^{2p/(p-1)}(u^{n/(2s(p-1))}) \). For any compact set \( E \) in \( \mathbb{R}^n \) we have
\[
\int_E |g - h_m| \, dx \leq \left( \int_E |g - h_m|^{2p/(p-1)} u^{n/(2s(p-1))} \, dx \right)^{(p-1)/(2p)} \times \left( \int_E w^{-n/(2s(p+1))} \, dx \right)^{(p+1)/(2p)}.
\]
Since \( w^{-n/(2s)} \) is locally integrable by the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) or \( w^{-n/(2s)} \in A_p \), we get \( h_m \to g \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( m \to \infty \). Hence we have \( g = h \) and (11). Furthermore we have
\[
\left\{ \int_{\mathbb{R}^n} |h|^{2p/(p-1)} u^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|h\|^{4sp/n - 2} \int_{\mathbb{R}^n} \|(-\Delta)^{s/2} h\|^2 w \, dx.
\]
We fix a positive number \( \varepsilon \). Let \( \chi_1, \ldots, \chi_N \) be functions in \( C_0^\infty(\mathbb{R}^n) \) such that
\[
\sum_{i=1}^N \|\phi_i - \chi_i\|_{\mathcal{H}^s(w)}^2 < \varepsilon.
\]
Now the inequalities
\[
\sum_{i=1}^N \langle \chi_i, f \rangle^2 \leq 2 \sum_{i=1}^N \langle \chi_i - \phi_i, f \rangle^2 + 2 \sum_{i=1}^N \langle \phi_i, f \rangle^2
\]
\[
\leq 2 \sum_{i=1}^N \|\chi_i - \phi_i\|^2 \|f\|^2 + 2K\|f\|^2 \leq 2(K + \varepsilon)\|f\|^2
\]
hold for all \( f \in L^2(\mathbb{R}^n) \). On the other hand

\[
\begin{align*}
&\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
&\leq \left\{ \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{(p-1)/p} \right\}^{2sp/n} \\
&\leq N^{2sp/n-1} \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \\
&\leq cN^{2sp/n-1} \sum_{i=1}^{N} \|\phi_i - \chi_i\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w \, dx \\
&\leq cN^{2sp/n-1} \varepsilon^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w \, dx \\
&\leq cN^{2sp/n-1} \varepsilon^{2sp/n},
\end{align*}
\]

where we used (12).

Therefore

\[
\begin{align*}
&\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
&\leq \left\{ \int_{\mathbb{R}^n} \left( 2 \sum_{i=1}^{N} |\phi_i - \chi_i|^2 + 2 \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
&\leq 2^{2sp/n} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{(p-1)/p} \\
&\quad + \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{(p-1)/p} \\
&\leq 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
&\quad + 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
&\leq c2^{sp/n-1} N^{2sp/n-1} \varepsilon^{2sp/n} + c2^{6sp/n-2} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 w \, dx,
\end{align*}
\]

(14)
where we used (13) and (3) for $\chi_i$. Since

$$\sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\chi_i|^2 w \, dx$$

$$\leq 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\chi_i - (-\Delta)^{s/2}\phi_i|^2 w \, dx + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w \, dx$$

$$\leq 2\varepsilon + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w \, dx,$$

we have by (14)

$$\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c 2^{4sp/n-1} x^{2sp/n-1} \varepsilon^{2sp/n} + c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \varepsilon$$

$$+ c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w \, dx.$$ 

Since we can take $\varepsilon$ arbitrary small, we conclude

$$\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c 2^{4sp/n-1} K^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w \, dx.$$ 

Hence we get (3).

### 4 Proof of Lemma 3.3

The arguments of the proof is similar to those in [11] and [12]. First we consider the case $n > 2s$. For $\lambda > 0$ we set

$$(15) \quad \mathcal{I}_\lambda = \{ Q \in \mathcal{Q} : \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx - \beta |Q|^{-1} \int_{Q} v \, dx < -\lambda \}.$$ 

Then we have for $Q \in \mathcal{I}_\lambda$

$$\alpha |Q|^{-2s/n-1} \int_{Q} w \, dx < |Q|^{-1} \int_{Q} (\beta v - \lambda)_+ \, dx,$$

where

$$(\beta v - \lambda)_+(x) = \max\{0, \beta v(x) - \lambda\}.$$
Since \( p = n/(2s) + \gamma, \gamma \in (0, 1] \), and
\[
\beta^{-p} \gamma \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx \lambda^{\gamma - 1} \, d\lambda \leq \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx < \infty,
\]
we have
\[
\int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx < \infty
\]
for all \( \lambda > 0 \). By the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) and (ii) of Proposition 2.1, there exists a \( \kappa' \in (1, n/(2s)) \) such that \( w^{-n/(2s)} \in A_{n/(2s^\kappa')} \). We set
\[
v_\lambda^*(x) = M((\beta v - \lambda)^{\kappa'})(x)^{1/\kappa'}.
\]
Then
\[
\int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx < \infty
\]
and \( v_\lambda^* \in A_1 \) by (iii) of Proposition 2.1, where \( c_1 \) depends only on \( n, s \) and \( A_{n/(2s^\kappa')} \)-constant of \( w^{-n/(2s)} \).

We can show that \( \mathcal{I}_\lambda \) is a finite set as follows. Let \( Q \in \mathcal{I}_\lambda \). Then we have
\[
\alpha |Q|^{-2s/n} \int_Q w \, dx \leq \int_Q v_\lambda^* \, dx
\]
\[
\leq \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} \left\{ \int_Q w^{n/(n-2s)} \, dx \right\}^{(n-2s)/n}.
\]
Since \( w^{-n/(2s)} \in A_{n/(2s)} \), the last quantity is bounded by
\[
c_2 \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} |Q|^{-2s/n} \left( \int_Q w^{-n/(2s)} \, dx \right)^{-2s/n}
\]
\[
\leq c_2 \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} |Q|^{-2s/n} \int_Q w \, dx,
\]
where we used the inequality
\[
1 \leq \frac{1}{|Q|} \int_Q w \, dx \left( \frac{1}{|Q|} \int_Q w^{-n/(2s)} \, dx \right)^{2s/n}.
\]
The above calculation says
\[
1 \leq c_3 \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx,
\]
where \( c_3 = c' \alpha^{-n/(2s)} \) and \( c' \) is the \( A_{n/(2s)} \)-constant of \( w^{-n/(2s)} \).
First we assume that \( \mathcal{I}_\lambda \) includes infinite disjoint cubes \( \{Q_i\}_{i=1}^{\infty} \). Then we have

\[
\infty = \sum_{i=1}^{\infty} 1 \leq \sum_{i=1}^{\infty} c_3 \int_{Q_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx \leq c_3 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx < \infty.
\]

This is a contradiction. Hence \( \mathcal{I}_\lambda \) does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes \( \{Q_i\}_{i=1}^{\infty} \subset \mathcal{I}_\lambda \) such that \( Q_i \neq Q_j \) (\( i \neq j \)) and \( Q_1 \subset Q_2 \subset Q_3 \subset \cdots \). Let \( \tilde{Q}_i \) be a half size dyadic sub-cube of \( Q_{i+1} \) such that \( Q_i \cap \tilde{Q}_i = \emptyset \). Since \( Q_{i+1} \in \mathcal{I}_\lambda \), we have

\[
\alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \leq \int_{Q_{i+1}} v_\lambda^* \, dx.
\]

Now we get

\[
\int_{Q_{i+1}} v_\lambda^* \, dx \leq \int_{3\tilde{Q}_i} v_\lambda^* \, dx \leq c_4 \int_{\tilde{Q}_i} v_\lambda^* \, dx,
\]

where we used the doubling property of \( v_\lambda^* \). Since

\[
|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \geq 2^{-2s/|\tilde{Q}_i|} |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w \, dx,
\]

we get

\[
c_5 |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w \, dx \leq \int_{\tilde{Q}_i} v_\lambda^* \, dx.
\]

The similar calculation as before leads to

\[
1 \leq c_6 \int_{\tilde{Q}_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} \, dx,
\]

where \( c_6 = c'^n \alpha^{-n/(2s)} \) and \( c' \) depends only on \( n, s, \) and \( w \). Since \( \{\tilde{Q}_i\}_{i=1}^{\infty} \) is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in \( \mathcal{I}_\lambda \) such that \( Q_1 \subset Q_2 \subset Q_3 \subset \cdots \) has a maximal element. Similarly we can show that any sequence in \( \mathcal{I}_\lambda \) such that \( Q_1 \supset Q_2 \supset Q_3 \supset \cdots \) has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in \( \mathcal{I}_\lambda \) with respect to the inclusion relation is finite. Hence \( \mathcal{I}_\lambda \) is a finite set. We remark that the non-decreasing rearrangement of \( \mathcal{I} \) in (8) is possible because \( \mathcal{I}_\lambda \) is a finite set for every \( \lambda > 0 \).

Let \( N(\lambda) = |\mathcal{I}_\lambda| \), that is, the number of elements of \( \mathcal{I}_\lambda \). Let \( \tilde{\mathcal{I}}_\lambda \) be the set of all \( Q \in \mathcal{I}_\lambda \) which satisfy the following condition: there exists a half size dyadic sub-cube \( \tilde{Q} \subset Q \) such that \( \tilde{Q} \notin \mathcal{I}_\lambda \) and \( \tilde{Q} \) does not contain any dyadic cube in \( \mathcal{I}_\lambda \). Then we have the following lemma.
Lemma 4.1. \( \|I_\lambda \| \leq 2\|I_\lambda \| \).

Lemma 4.1 is proved in Rochberg and Taibleson's paper([9, Lemma 1]).

Let \( Q \in \tilde{I}_\lambda \) and \( \tilde{Q} \) be a dyadic cube which satisfies the condition in the definition of \( \tilde{I}_\lambda \). Then by similar calculations as before we get

\[
1 \leq c_6 \int_Q (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} dx.
\]

For every \( Q \in \tilde{I}_\lambda \) we choose a \( \tilde{Q} \) as above. Let \( \{\tilde{Q}_j\}_{j \in J} \) be the set of all such cubes \( \tilde{Q} \). Then the cubes in \( \{\tilde{Q}_j\}_{j \in J} \) are mutually disjoint. Therefore we get

\[
\|\tilde{I}_\lambda \| \leq \sum_{j \in J} c_6 \int_{\tilde{Q}_j} (v_\lambda^n)^{n/(2s)} w^{-n/(2s)} dx \leq c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx,
\]

where we used (16). Hence we have

\[
N(\lambda) \leq 2c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx.
\]

Therefore we conclude

\[
\sum_k |\mu_k| \gamma = \int_0^\infty \gamma \lambda^{\gamma-1} N(\lambda) d\lambda \leq c_8 \int_{\mathbb{R}^n} \beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \gamma \lambda^{\gamma-1} d\lambda \leq c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx,
\]

where \( c_8 = c'' \alpha^{-n/(2s)} \beta^p \) and \( c'' \) depends only on \( n, s, p \) and \( w \).

Next we consider the case \( n \leq 2s \). We remark that \( v(x) > 0 \) for all \( x \in \mathbb{R}^n \). In fact if \( v(x_0) = 0 \) at some point \( x_0 \), then by the definition of the maximal operator we have \( V \equiv 0 \), that is, \( \phi_i \equiv 0 \), \( i = 1, \ldots, N \).

We also remark that \( I \) in (8) is not empty. In fact if \( I \) is empty, then we have

\[
\beta \int_Q v dx \leq \alpha |Q|^{-2s/n} \int_Q w dx
\]

for all \( Q \in \mathcal{Q} \). Let \( Q_0 \in \mathcal{Q} \) and \( Q_0 \subset Q_1 \subset Q_2 \subset \cdots \) be the infinite sequence of dyadic cubes such that \( Q_{i+1} \) is the parent of \( Q_i \) for all \( i = 0, 1, 2, \ldots \). By (2) we have

\[
|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \leq |Q_i|^{-2s/n} \int_{Q_i} w dx
\]

for all \( i \). Hence we have
\begin{align}
& \beta \int_{Q_i} v \, dx \leq \alpha |Q_0|^{-2s/n} \int_{Q_0} w \, dx \\
& \text{for all } i. \quad \text{On the other hand, since } v \in A_1, \text{ there exists a constant } d > 1 \text{ such that } \\
& d \int_{Q_i} v \, dx \leq \int_{Q_{i+1}} v \, dx \\
& \text{for all } i \text{ (c.f.}[5, \text{p141}]). \quad \text{Hence we have } \\
& d^i \int_{Q_0} v \, dx \leq \int_{Q_i} v \, dx \\
& \text{and } \\
& \lim_{i \to \infty} \int_{Q_i} v \, dx = \infty,
\end{align}
which contradicts to (17). Therefore $I$ is not empty.

Let $Q \in I$ and $Q'$ be the parent of $Q$. Then we have
\begin{align}
& \alpha |Q'|^{-2s/n} \int_{Q'} w \, dx \leq \alpha |Q|^{-2s/n} \int_{Q} w \, dx < \beta \int_{Q} v \, dx \leq \beta \int_{Q'} v \, dx,
\end{align}
where we used the assumption (2). Hence we have $Q' \in I$, which means that $I$ is an infinite set.

\textbf{Lemma 4.2.} There exists a $c > 0$ such that
\begin{align}
& \sum_{Q \in I} \left( \frac{1}{|Q|} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,
\end{align}
where $c = c' \alpha^{-n/(2s)} \beta^{n/(2s)}$ and $c'$ depends only on $n, p, s$ and $w$.

This lemma is proved in [12, Lemma 3.3]. Let $I_\lambda$ be the set defined by (15).

\textbf{Lemma 4.3.} For each $\lambda > 0$, $I_\lambda$ is a finite set.

Lemma 4.3 is easily proved by Lemma 4.2 (c.f.[12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of $I$ is possible.

By Lemma 4.2 we conclude
\begin{align}
& \sum_{k=1}^\infty |\mu_k|^\gamma = c \sum_{Q \in I} \left( \beta |Q|^{-1} \int_{Q} v \, dx - \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx \right)^\gamma \\
& \leq c \sum_{Q \in I} \left( \beta |Q|^{-1} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,
\end{align}
where $c = c'' \alpha^{-n/(2s)} \beta^p$ and $c''$ depends only on $n, p, s$ and $w$. This ends the proof of Lemma 3.3.
5 Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

Lemma 5.1. Let \( w \in A_2 \) and \( m \in C^n(\mathbb{R}^n \setminus \{0\}) \). Suppose that

\[
B = \max_{|\sigma| \leq n} \sup_{0 < r} \frac{1}{r} \left| \int_{r \leq |\xi| \leq 2r} \left( \frac{\partial}{\partial \xi} \right)^\sigma m(\xi) \right|^2 \, d\xi < \infty.
\]

Then the operator \( T \) defined by \( \hat{T} f(\xi) = m(\xi) \hat{f}(\xi) \) is bounded from \( L^2(w) \) to \( L^2(w) \) and the operator norm \( \|T\| \) is bounded by \( CB^{1/2} \) where \( C \) is a constant which depends only on \( n \) and \( w \).

The proof of Lemma 5.1 is in [6] or [7].

For \( \nu \in \mathbb{Z} \) we define \( \psi_\nu(x) = 2^{n\nu} \psi(2^\nu x) \). Let \( w \in A_2 \) and \( s \geq 0 \). Frazier and Jawerth proved that there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} \left| (f, \psi_Q) \right|^2 \frac{1}{|Q|} \int_Q w \, dx \leq \sum_{\nu \in \mathbb{Z}} 2^{2s\nu} \left| (f * \psi_\nu)(x) \right|^2 w(x) \, dx \leq c_2 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} \left| (f, \psi_Q) \right|^2 \frac{1}{|Q|} \int_Q w \, dx
\]

for all \( f \in C_0^\infty(\mathbb{R}^n) \) where \( c_1 \) and \( c_2 \) depend only on \( n \) and \( s \) (Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let \( \{r_\nu(t)\} \) be the Rademacher functions on \([0,1]\) indexed by \( \nu \in \mathbb{Z} \) and

\[ T_t f(x) = \sum_{\nu \in \mathbb{Z}} r_\nu(t) f * \psi_\nu(x). \]

Then \( T_t \) satisfies the condition of Lemma 5.1. Hence

\[
\int_{\mathbb{R}^n} |T_t f(x)|^2 w(x) \, dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx,
\]

for all \( f \in C_0^\infty(\mathbb{R}^n) \) where \( M = \max_{|\sigma| \leq n} \|\partial^\sigma \psi\|_{L_2} \) and \( C \) is a positive constant depending only on \( n \) and \( w \). By integrating from 0 to 1 with respect to \( t \), we get

\[
\int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) \, dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx.
\]

By the duality argument and the fact \( w^{-1} \in A_2 \) we obtain

\[
\int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx \leq CM \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) \, dx
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). Hence we have
\[
c_3M^{-1} \int_{\mathbb{R}^n} |f|^2w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu|^2 \right\} w \, dx \leq c_4M \int_{\mathbb{R}^n} |f|^2w \, dx,
\]
where \( c_3 \) and \( c_4 \) are constants depending only on \( n \) and \( w \).

Therefore we get
\[
c_3M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |(-\Delta)^{s/2} f * \psi_\nu|^2 \right\} w \, dx \leq c_4M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2w \, dx
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \) (c.f. [11]).

Let \( \Phi \in S(\mathbb{R}^n) \) satisfy \( \text{supp} \, \Phi \subset \{ \xi : 1/4 \leq |\xi| \leq 4 \} \) and \( \Phi(\xi) = 1 \) for \( 1/2 \leq |\xi| \leq 2 \). For \( \nu \in \mathbb{Z} \) the multiplier \( m_\nu(\xi) = 2^{-s\nu}|\xi|^s \Phi(\xi/2^\nu) \) satisfies the condition of Lemma 5.1.

Hence we have
\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2w(x) \, dx \leq c_5 \int_{\mathbb{R}^n} 2^{2s\nu}|f * \psi_\nu(x)|^2w(x) \, dx,
\]
where \( c_5 = c_6 \inf \max \| \partial^\sigma \Phi \|^2_\infty \) and \( c_6 \) is a positive constant depending only on \( n, s \) and \( w \) and the infimum is taken over all possible \( \Phi \).

Similarly there exists a positive constant \( c_7 \) depending only on \( n, s \) and \( w \) such that
\[
\int_{\mathbb{R}^n} 2^{2s\nu}|f * \psi_\nu(x)|^2w(x) \, dx \leq c_7 \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2w(x) \, dx.
\]

Hence we get
\[
c_8M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2w \, dx \leq \sum_{Q \in Q} |Q|^{-2s/n}|(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} w \, dx
\]
\[
\leq c_9M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2w \, dx
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \), where \( c_8 \) and \( c_9 \) are positive constant depending only on \( n, s \) and \( w \). This ends the proof of Lemmas 3.1 and 3.2.

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**References**


