Weighted Sobolev-Lieb-Thirring inequalities

Kazuya Tachizawa

Abstract. We give a weighted version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions. In the proof of our result we use $\varphi$-transform of Frazier-Jawerth.

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1 Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring inequality.

Theorem 1.1 ([2]). Let $n \in \mathbb{N}, s > 0$ and $p$ with
\[
\max \left( 1, \frac{n}{2s} \right) < p \leq 1 + \frac{n}{2s}.
\]
Then there exists a positive constant $c = c(p, n, s)$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^s(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have
\[
\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} \, dx \right\}^{2s/(p-1)/n} \leq c \sum_{i=1}^N \| (-\Delta)^{s/2} \phi_i \|_2^2
\]
where
\[
\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.
\]

In this theorem $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order $s$ and $\| \cdot \|$ is the norm of $L^2(\mathbb{R}^n)$.

In [8] Lieb and Thirring proved this theorem for $s = 1$ and applied it to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved (1) for $s \in \mathbb{N}$ under
the suborthonormal condition on \{\phi_i\}, where \{\phi_i\}_{i=1}^{N} in \(L^2(\mathbb{R}^n)\) is called suborthonormal if the inequality
\[
\sum_{i,j=1}^{N} \xi_i \overline{\xi_j} (\phi_i, \phi_j) \leq \sum_{i=1}^{N} |\xi_i|^2
\]
holds for all \(\xi_i \in \mathbb{C}, i = 1, \ldots, N\) ([4]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations (c.f. [13]). In this paper we shall give a weighted version of (1) under suborthonormal condition on \{\phi_i\}. In the proof of our theorem we shall use Frazier-Jawerth’s \(\varphi\)-transform ([3]).

For the statement of our result we need to recall the definition of \(A_p\)-weights (c.f. [5], [10]). By a cube in \(\mathbb{R}^n\) we mean a cube which sides are parallel to coordinate axes. Let \(w\) be a non-negative, locally integrable function on \(\mathbb{R}^n\). We say that \(w\) is an \(A_p\)-weight for \(1 < p < \infty\) if there exists a positive constant \(C\) such that
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1}
\]
for all cubes \(Q \subset \mathbb{R}^n\). The infimum of the constant \(C\) is called the \(A_p\)-constant of \(w\).

We say that \(w\) is an \(A_1\)-weight if there exists a positive constant \(C\) such that
\[
\frac{1}{|Q|} \int_Q w(y) \, dy \leq C w(x) \quad \text{a.e. } x \in Q
\]
for all cubes \(Q \subset \mathbb{R}^n\). The infimum of the constant \(C\) is called the \(A_1\)-constant of \(w\). Let \(A_p\) be the class of \(A_p\)-weights. The inclusion \(A_p \subset A_q\) holds for \(p < q\).

For a nonnegative, locally integrable function \(w\) on \(\mathbb{R}^n\) we define
\[
L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.
\]

For \(\nu \in \mathbb{Z}\) and \(k \in \mathbb{Z}^n\) the cube \(Q\) defined by
\[
Q = Q_{\nu k} = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1, \; i = 1, \ldots, n\}
\]
is called a dyadic cube in \(\mathbb{R}^n\). Let \(Q\) be the set of all dyadic cubes in \(\mathbb{R}^n\). For any \(Q \in Q\) there exists a unique \(Q' \in Q\) such that \(Q \subset Q'\) and the side-length of \(Q'\) is double of that of \(Q\). We call \(Q'\) the parent of \(Q\).

For \(s > 0\) and \(f \in C_0^\infty(\mathbb{R}^n)\) we define via inverse Fourier transform
\[
(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).
\]
Let \(w \in A_2\) and \(H^s(w)\) be the completion of \(C_0^\infty(\mathbb{R}^n)\) with respect to the norm
\[
\|f\|_{H^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx + \|f\|^2 \right\}^{1/2}.
\]
We remark that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx < \infty$$

because

$$|(-\Delta)^{s/2} f(x)| \leq \frac{c}{(1 + |x|)^n} \quad (x \in \mathbb{R}^n)$$

and

$$\int_{\mathbb{R}^n} \frac{w(x)}{(1 + |x|)^{2n}} \, dx < \infty$$

(c.f. [10, p.209]).

Let $f \in \mathcal{H}^s(w)$ and \{${f_i}$\}$_{i=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\|f - f_i\|_{\mathcal{H}^s(w)} \to 0$ ($i \to \infty$). This means that there exist $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(w)$ such that $\|g_1 - f_i\| \to 0$ and

$$\int_{\mathbb{R}^n} |g_2(x) - (-\Delta)^{s/2} f_i(x)|^2 w(x) \, dx \to 0$$

as $i \to \infty$. We denote $(-\Delta)^{s/2} f = g_2$. We remark that $g_1 \equiv 0$ means $g_2 \equiv 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} g_2 \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \varphi \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} f_i (-\Delta)^{s/2} \varphi \, dx = 0.$$

Hence we have $g_2 \equiv 0$. This means that we can identify $\mathcal{H}^s(w)$ as a subspace of $L^2(\mathbb{R}^n)$.

The following is the main result of this paper.

**Theorem 1.2.** Let $n \in \mathbb{N}, s > 0$, and

$$\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s},$$

Let $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \geq n$, then we assume that $w^{-n/(2s)} \in A_p$ and

$$\int_{Q'} w \, dx \leq 2^{2s} \int_Q w \, dx$$

for all dyadic cubes $Q \in \Omega$ and its parent $Q'$.

Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $\mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$, we have

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx,$$
where
\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]
and \( c \) depends only on \( n, s, p, A_2 \)-constant of \( w \), and \( A_{n/(2s)} \) or \( A_p \)-constant of \( w^{-n/(2s)} \).

When \( 2s < n \), an example of weight function \( w \) is given by \( w(x) = |x|^\alpha \) for \(-n+2s < \alpha < 2s\). When \( 2s > n \), an example of weight function \( w \) is given by \( w(x) = |x|^\alpha \) for \( 0 \leq \alpha < \min\{2s - n, n\}\). When \( 2s = n \), the condition (2) means \( w \) is equivalent to a constant almost everywhere (c.f. [12, Proposition 4.1]).

2 Preliminaries

Let \( \psi \) be a function which satisfies the following conditions.

(A1) \( \psi \in S(\mathbb{R}^n) \).

(A2) \( \text{supp } \hat{\psi} \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \} \)

(A3) \( |\hat{\psi}(\xi)| \geq c > 0 \) if \( \frac{2}{5} \leq |\xi| \leq \frac{5}{3} \).

(A4) \( \sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 = 1 \) for all \( \xi \neq 0 \).

For \( \nu \in \mathbb{Z}, k \in \mathbb{Z}^n \) and \( Q = Q_{\nu k} \), we set
\[ \psi_Q(x) = 2^{n\nu/2} \psi(2^{\nu} x - k) \quad (x \in \mathbb{R}^n) \]

Let \( M \) be the Hardy-Littlewood maximal operator, that is,
\[ M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \]
where \( f \) is a locally integrable function on \( \mathbb{R}^n \) and the supremum is taken over all cubes \( Q \) which contain \( x \).

Proposition 2.1.

(i) Let \( 1 < p < \infty \) and \( w \) be a non-negative locally integrable function on \( \mathbb{R}^n \). Then there exists a positive constant \( c \) such that
\[ \int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx \]
for all \( f \in L^p(w) \) if and only if \( w \in A_p \). The constant \( c \) depends only on \( n, p \) and \( A_p \)-constant of \( w \).
(ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.

(iii) Let $0 < \tau < 1$ and $f$ be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $(M(f))^\tau \in A_1$ and the $A_1$-constant of $(M(f))^\tau$ depends only on $n$ and $\tau$.

(iv) Let $1 \leq p < \infty$ and $w \in A_p$. Then there exists a positive constant $c$ such that
\[ \int_{2Q} w \, dx \leq c \int_{Q} w \, dx \]
for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of $Q$ and $c$ depend only on $n$ and $A_p$-constant of $w$.

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3 Proof of Theorem 1.2

The suborthonormal condition on $\{\phi_i\}$ is equivalent to the inequality
\[ \sum_{i=1}^{N} |(\phi_i, f)|^2 \leq \|f\|^2 \]
for all $f \in L^2(\mathbb{R}^n)$ (c.f.[1, p57]).

We shall prove the inequality
\[ \left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c K^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx \]
under the assumption
\[ \sum_{i=1}^{N} |(\phi_i, f)|^2 \leq K \|f\|^2 \]
for all $f \in L^2(\mathbb{R}^n)$ where $K$ is a positive constant. This is equivalent to the statement of Theorem 1.2. We remark that $K$ may depend on $\{\phi_i\}$. For example, the inequality (3) says that
\[ \left\{ \int_{\mathbb{R}^n} |\phi|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi|^2 w \, dx \]
holds for all $\phi \in H^s(w)$ under suitable condition on $s, p, n$ and $w$ because

$$|(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C_0^\infty(\mathbb{R}^n), i = 1, \ldots, N$. Let

$$V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2s(p-1))}$$

where the value of the constant $\delta_1 > 0$ will be given later. Since $\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$ is a bounded function with compact support and $w^{n/(2s(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have

$$\int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty.$$ 

We may also assume that

$$0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx.$$

By (ii) of Proposition 2.1 there exists a constant $\kappa$ such that $1 < \kappa < p$ and $w^{-n/(2s)} \in A_p/\kappa$. We set $v(x) = M(V^\kappa)(x)_{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)} \, dx < \infty.$$ 

Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 2.1.

We have the following lemmas.

**Lemma 3.1.** For $s > 0$ and $w \in A_2$ there exists a positive constant $\alpha$ such that

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where $\alpha$ is given by

$$\alpha^{-1} = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and $c$ is a constant depending only on $n, s$ and $A_2$-constant of $w$.

**Lemma 3.2.** For $v \in A_2$ there exist positive constants $\beta$ and $\beta'$ such that

$$\beta' \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \leq \int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where $\beta$ is given by

$$\beta = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and $c$ is a constant depending only on $n$ and $A_2$-constant of $v$. 

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The proof of Lemmas 3.1 and 3.2 are in [11, Proposition 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader’s convenience because the dependence of \( \psi \) in \( \alpha \) and \( \beta \) is not explained in [11].

For \( f \in C_0^\infty (\mathbb{R}^n) \) we have
\[
\int_{\mathbb{R}^n} |f|^2 V \, dx \leq \int_{\mathbb{R}^n} |f|^2 \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} v \, dx,
\]
where we used Lemma 3.2. Hence by Lemma 3.1
\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx - \int_{\mathbb{R}^n} V |f|^2 \, dx \geq \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_{Q} v \, dx.
\]

Now we set
\[
I = \left\{ Q \in \mathcal{Q} : \beta \int_{Q} v \, dx > \alpha |Q|^{-2s/n} \int_{Q} w \, dx \right\}.
\]

Let \( \{\mu_k\}_{1 \leq k} \) be the non-decreasing rearrangement of
\[
\left\{ \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx - \beta |Q|^{-1} \int_{Q} v \, dx \right\}_{Q \in I}.
\]

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When
\[
\mu_k = \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx - \beta |Q|^{-1} \int_{Q} v \, dx,
\]
we define \( \Psi_k = \psi_Q \). Then we have by (7)
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx - \sum_{i=1}^{N} \int_{\mathbb{R}^n} V |\phi_i|^2 \, dx \geq \sum_{i=1}^{N} \sum_{Q \in \mathcal{Q}} |(\phi_i, \psi_Q)|^2 \left\{ \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx - \beta |Q|^{-1} \int_{Q} v \, dx \right\}
\]
\[
\geq \sum_{i=1}^{N} \sum_{k \in \mathbb{K}} \mu_k (|\phi_i, \Psi_k|)^2 = \sum_{k \in \mathbb{K}} \mu_k \sum_{i=1}^{N} (|\phi_i, \Psi_k|)^2
\]
\[
\geq -K||\psi||^2 \sum_{k \in \mathbb{K}} |\mu_k| \geq -K||\psi||^2 \left( \sum_{k \in \mathbb{K}} |\mu_k|^\gamma \right)^{1/\gamma},
\]
where \( \gamma = p - n/(2s) \in (0, 1] \) and we used (4).

Now the following lemma holds.
Lemma 3.3.

\[ \sum_k |\mu_k|^\gamma \leq c \int_{\mathbb{R}^n} \nu^p w^{-n/(2s)} \, dx, \]

where \( c \) is given by

\[ c = c' \max_{|\sigma| \leq n} \| \partial^\sigma \tilde{\psi} \|^{n/s + 2p}_{\infty} \]

and \( c' \) depends only on \( n, s, p \) and \( w \).

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (6) the last quantity in (10) is estimated from below by

\[ -cK \left( \int_{\mathbb{R}^n} \nu^p w^{-n/(2s)} \, dx \right)^{1/\gamma} \]

where

\[ c = c' \| \psi \|^2 \max_{|\sigma| \leq n} \| \partial^\sigma \tilde{\psi} \|^{(4ps+2n)/(2ps-n)}_{\infty} \]

and \( c' \) depends only on \( n, s, p \) and \( w \). We may take the infimum of the above constant with respect to possible \( \psi \) and replace \( c \) by this infimum.

Let

\[ \delta_1 = \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-n/(2s)-1)/n}, \]

where \( \delta_2 \) is a positive constant. Then we have by (9)

\[ \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx \]

\[ \geq \delta_2 K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ -cK \delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \]

\[ = \{ \delta_2 - c\delta_2^{p/(p-n/(2s))} \} K^{1-2sp/n} \left( \int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n}. \]

If we take \( \delta_2 \) small enough, then we get the inequality (3) because \( 1 < p/(p - n/(2s)) \).

Next we shall show (3) for \( \phi_i \in \mathcal{H}^s(w), \ i = 1, \ldots , N \). First we show

\[ (11) \quad \mathcal{H}^s(w) \subset L^{2p/(p-1)}(w^{n/(2s(p-1))}). \]
Let \( h \in \mathcal{H}^s(w) \). Then there exists a sequence \( \{h_m\}_{m=1}^{\infty} \subset C_0^\infty(\mathbb{R}^n) \) such that \( \|h - h_m\|_{\mathcal{H}^s(w)} \to 0 \) (\( m \to \infty \)). Since we proved that (5) holds for \( h_m \in C_0^\infty(\mathbb{R}^n) \), we get

\[
\left\{ \int_{\mathbb{R}^n} |h_m|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|h_m\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h_m|^2 w \, dx,
\]

where \( c \) does not depend on \( h_m \). Since \( 4sp/n - 2 > 0 \) and \( \{h_m\} \) is a Cauchy sequence in \( \mathcal{H}^s(w) \), the above inequality says that \( \{h_m\} \) is a Cauchy sequence in \( L^{2p/(p-1)}(w^{n/(2s(p-1))}) \).

Let \( g \) be the limit of \( \{h_m\} \) in \( L^{2p/(p-1)}(w^{n/(2s(p-1))}) \). For any compact set \( E \) in \( \mathbb{R}^n \) we have

\[
\int_E |g - h_m| \, dx \leq \left( \int_E |g - h_m|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{(p-1)/(2p)} \times \left( \int_E w^{-n/(2s(p+1))} \, dx \right)^{(p+1)/(2p)}.
\]

Since \( w^{-n/(2s)} \) is locally integrable by the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) or \( w^{-n/(2s)} \in A_p \), we get \( h_m \to g \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( m \to \infty \). Hence we have \( g = h \) and (11). Furthermore we have

\[
\left\{ \int_{\mathbb{R}^n} |h|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|h\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h|^2 w \, dx.
\]

We fix a positive number \( \varepsilon \). Let \( \chi_1, \ldots, \chi_N \) be functions in \( C_0^\infty(\mathbb{R}^n) \) such that

\[
\sum_{i=1}^N \|\phi_i - \chi_i\|_{\mathcal{H}^s(w)}^2 < \varepsilon.
\]

Now the inequalities

\[
\sum_{i=1}^N |(\chi_i, f)|^2 \leq 2 \sum_{i=1}^N |(\chi_i - \phi_i, f)|^2 + 2 \sum_{i=1}^N |(\phi_i, f)|^2 \leq 2(2K\|f\|^2 + 2K\|f\|^2) \leq 2(K + \varepsilon)\|f\|^2
\]

(13)
hold for all \( f \in L^2(\mathbb{R}^n) \). On the other hand

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq \left\{ \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{(p-1)/p} \right\}^{2sp/n} \\
\leq N^{2sp/n-1} \sum_{i=1}^{N} \left( \int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right)^{2s(p-1)/n} \\
\leq cN^{2sp/n-1} \sum_{i=1}^{N} \|\phi_i - \chi_i\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w \, dx \\
\leq cN^{2sp/n-1} \varepsilon^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w \, dx \\
\leq cN^{2sp/n-1} \varepsilon^{2sp/n},
\]

where we used (12).

Therefore

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq \left\{ \int_{\mathbb{R}^n} \left( 2 \sum_{i=1}^{N} |\phi_i - \chi_i|^2 + 2 \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq 2^{2sp/n} \left[ \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{(p-1)/p} \right]^{2sp/n} \\
+ \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{(p-1)/p} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
+ 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
\leq c^{2sp/n-1} N^{2sp/n-1} \varepsilon^{2sp/n} + c^{6sp/n-2} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 w \, dx,
\]
where we used (13) and (3) for $\chi_i$. Since

$$\sum_{i=1}^{N} \int_{\mathbb{R}^n} \left| (-\Delta)^{s/2} \chi_i \right|^2 w \, dx \leq 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left| (-\Delta)^{s/2} \phi_i \right|^2 w \, dx + 2 \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left| (-\Delta)^{s/2} \phi_i \right|^2 w \, dx$$

we have by (14)

$$\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} \left| \phi_i \right|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c 2^{4sp/n-1} K^{2sp/n-1} \varepsilon^{2sp/n} + c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \varepsilon$$

$$+ c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left| (-\Delta)^{s/2} \phi_i \right|^2 w \, dx.$$ 

Since we can take $\varepsilon$ arbitrary small, we conclude

$$\left\{ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} \left| \phi_i \right|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c 2^{6sp/n-1} K^{2sp/n-1} \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left| (-\Delta)^{s/2} \phi_i \right|^2 w \, dx.$$ 

Hence we get (3).

4 Proof of Lemma 3.3

The arguments of the proof is similar to those in [11] and [12]. First we consider the case $n > 2s$. For $\lambda > 0$ we set

$$\mathcal{I}_{\lambda} = \{ Q \in \mathcal{Q} : \alpha |Q|^{-2s/n-1} \int_{Q} w \, dx - \beta |Q|^{-1} \int_{Q} v \, dx < -\lambda \}.$$ 

Then we have for $Q \in \mathcal{I}_{\lambda}$

$$\alpha |Q|^{-2s/n-1} \int_{Q} w \, dx < |Q|^{-1} \int_{Q} (\beta v - \lambda)_+ \, dx,$$

where

$$(\beta v - \lambda)_+(x) = \max\{0, \beta v(x) - \lambda\}.$$
Since \( p = n/(2s) + \gamma, \gamma \in (0,1], \) and
\[
\beta^{-p} \gamma \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx \, d\lambda \, \gamma^{-1} \, d\lambda \leq \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx < \infty,
\]
we have
\[
\int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx < \infty
\]
for all \( \lambda > 0. \) By the assumption \( w^{-n/(2s)} \in A_{n/(2s)} \) and (ii) of Proposition 2.1, there exists a \( \kappa' \in (1, n/(2s)) \) such that \( w^{-n/(2s)} \in A_{n/(2s\kappa')} \). We set
\[
v^*_{\lambda}(x) = M((\beta v - \lambda)^{\kappa'})(x)^{1/\kappa'}.
\]
Then
\[
\int_{\mathbb{R}^n} (v^*_{\lambda})^{n/(2s)} w^{-n/(2s)} \, dx \leq c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx < \infty
\]
and \( v^*_{\lambda} \in A_1 \) by (iii) of Proposition 2.1, where \( c_1 \) depends only on \( n, s \) and \( A_{n/(2s)} \) constant of \( w^{-n/(2s)} \).

We can show that \( I_{\lambda} \) is a finite set as follows. Let \( Q \in I_{\lambda} \). Then we have
\[
\alpha |Q|^{-2s/n} \int_Q w \, dx \leq \int_Q v^*_{\lambda} \, dx \\
\leq \left\{ \int_Q (v^*_{\lambda})^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} \left\{ \int_Q w^{n/(n-2s)} \, dx \right\}^{(n-2s)/n}.
\]
Since \( w^{-n/(2s)} \in A_{n/(2s)} \), the last quantity is bounded by
\[
c_2 \left\{ \int_Q (v^*_{\lambda})^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} |Q| \left( \int_Q w^{-n/(2s)} \, dx \right)^{-2s/n} \\
\leq c_2 \left\{ \int_Q (v^*_{\lambda})^{n/(2s)} w^{-n/(2s)} \, dx \right\}^{2s/n} |Q|^{-2s/n} \int_Q w \, dx,
\]
where we used the inequality
\[
1 \leq \frac{1}{|Q|} \int_Q w \, dx \left( \frac{1}{|Q|} \int_Q w^{-n/(2s)} \, dx \right)^{2s/n}.
\]
The above calculation says
\[
1 \leq c_3 \int_Q (v^*_{\lambda})^{n/(2s)} w^{-n/(2s)} \, dx,
\]
where \( c_3 = c' \alpha^{-n/(2s)} \) and \( c' \) is the \( A_{n/(2s)} \) constant of \( w^{-n/(2s)} \).
First we assume that $\mathcal{I}_\lambda$ includes infinite disjoint cubes $\{Q_i\}_{i=1}^\infty$. Then we have
\[
\infty = \sum_{i=1}^\infty 1 \leq \sum_{i=1}^\infty c_3 \int_{Q_i} (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx \leq c_3 \int_{\mathbb{R}^n} (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx < \infty.
\]
This is a contradiction. Hence $\mathcal{I}_\lambda$ does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes $\{Q_i\}_{i=1}^\infty \subset \mathcal{I}_\lambda$ such that $Q_i \neq Q_j$ ($i \neq j$) and $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$. Let $\tilde{Q}_i$ be a half size dyadic sub-cube of $Q_{i+1}$ such that $Q_i \cap \tilde{Q}_i = \emptyset$. Since $Q_{i+1} \in \mathcal{I}_\lambda$, we have
\[
\alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \leq \int_{Q_{i+1}} v^*_\lambda dx.
\]
Now we get
\[
\int_{Q_{i+1}} v^*_\lambda dx \leq \int_{3\tilde{Q}_i} v^*_\lambda dx \leq c_4 \int_{\tilde{Q}_i} v^*_\lambda dx,
\]
where we used the doubling property of $v^*_\lambda$. Since
\[
|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \geq 2^{-2s} |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w dx,
\]
we get
\[
c_5 |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w dx \leq \int_{\tilde{Q}_i} v^*_\lambda dx.
\]
The similar calculation as before leads to
\[
1 \leq c_6 \int_{\tilde{Q}_i} (v^*_\lambda)^{n/(2s)} w^{-n/(2s)} dx,
\]
where $c_6 = c''\alpha^{-n/(2s)}$ and $c''$ depends only on $n, s$, and $w$. Since $\{\tilde{Q}_i\}_{i=1}^\infty$ is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$ has a maximal element. Similarly we can show that any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in $\mathcal{I}_\lambda$ with respect to the inclusion relation is finite. Hence $\mathcal{I}_\lambda$ is a finite set. We remark that the non-decreasing rearrangement of $\mathcal{I}$ in (8) is possible because $\mathcal{I}_\lambda$ is a finite set for every $\lambda > 0$.

Let $N(\lambda) = \sharp\mathcal{I}_\lambda$, that is, the number of elements of $\mathcal{I}_\lambda$. Let $\tilde{\mathcal{I}}_\lambda$ be the set of all $Q \in \mathcal{I}_\lambda$ which satisfy the following condition: there exists a half size dyadic sub-cube $\tilde{Q} \subset Q$ such that $\tilde{Q} \notin \mathcal{I}_\lambda$ and $\tilde{Q}$ does not contain any dyadic cube in $\mathcal{I}_\lambda$. Then we have the following lemma.
Lemma 4.1. \( \|I_\lambda \| \leq 2\|\hat{I}_\lambda \|. \)

Lemma 4.1 is proved in Rochberg and Taibleson’s paper([9, Lemma 1]).

Let \( Q \in \hat{I}_\lambda \) and \( \hat{Q} \) be a dyadic cube which satisfies the condition in the definition of \( \hat{I}_\lambda \). Then by similar calculations as before we get

\[
1 \leq c_6 \int_{\hat{Q}} (v_\lambda^s)^{n/(2s)} w^{-n/(2s)} \, dx.
\]

For every \( Q \in \hat{I}_\lambda \) we choose a \( \hat{Q} \) as above. Let \( \{\hat{Q}_j\}_{j \in J} \) be the set of all such cubes \( \hat{Q} \). Then the cubes in \( \{\hat{Q}_j\}_{j \in J} \) are mutually disjoint. Therefore we get

\[
\|\hat{I}_\lambda \| \leq \sum_{j \in J} c_6 \int_{\hat{Q}_j} (v_\lambda^s)^{n/(2s)} w^{-n/(2s)} \, dx
\]

where we used (16). Hence we have

\[
N(\lambda) \leq 2c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx.
\]

Therefore we conclude

\[
\sum_k |\mu_k|^\gamma = \int_0^\infty \gamma \lambda^{\gamma-1} N(\lambda) \, d\lambda
\]

\[
\leq 2c_7 \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} \, dx \gamma \lambda^{\gamma-1} \, d\lambda
\]

\[
\leq c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,
\]

where \( c_8 = c'''\alpha^{-n/(2s)}\beta^p \) and \( c''' \) depends only on \( n, s, p \) and \( w \).

Next we consider the case \( n \leq 2s \). We remark that \( v(x) > 0 \) for all \( x \in \mathbb{R}^n \). In fact if \( v(x_0) = 0 \) at some point \( x_0 \), then by the definition of the maximal operator we have \( V \equiv 0 \), that is, \( \phi_i \equiv 0, i = 1, \ldots, N \).

We also remark that \( \mathcal{I} \) in (8) is not empty. In fact if \( \mathcal{I} \) is empty, then we have

\[
\beta \int_Q v \, dx \leq a|Q|^{-2s/n} \int_Q w \, dx
\]

for all \( Q \in \mathcal{Q} \). Let \( Q_0 \in \mathcal{Q} \) and \( Q_0 \subset Q_1 \subset Q_2 \subset \cdots \) be the infinite sequence of dyadic cubes such that \( Q_{i+1} \) is the parent of \( Q_i \) for all \( i = 0, 1, 2, \ldots \). By (2) we have

\[
|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w \, dx \leq |Q_i|^{-2s/n} \int_{Q_i} w \, dx
\]

for all \( i \). Hence we have
for all \( i \). On the other hand, since \( v \in A_1 \), there exists a constant \( d > 1 \) such that
\[
d \int_{Q_i} v \, dx \leq \int_{Q_{i+1}} v \, dx
\]
for all \( i \) (c.f.[5, p141]). Hence we have
\[
d^i \int_{Q_0} v \, dx \leq \int_{Q_i} v \, dx
\]
and
\[
\lim_{i \to \infty} \int_{Q_i} v \, dx = \infty,
\]
which contradicts to (17). Therefore \( \mathcal{I} \) is not empty.

Let \( Q \in \mathcal{I} \) and \( Q' \) be the parent of \( Q \). Then we have
\[
\alpha|Q'|^{-2s/n} \int_{Q'} w \, dx \leq \alpha|Q|^{-2s/n} \int_{Q} w \, dx < \beta \int_{Q} v \, dx \leq \beta \int_{Q'} v \, dx,
\]
where we used the assumption (2). Hence we have \( Q' \in \mathcal{I} \), which means that \( \mathcal{I} \) is an infinite set.

**Lemma 4.2.** There exists a \( c > 0 \) such that
\[
\sum_{Q \in \mathcal{I}} \left( \frac{1}{|Q|} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,
\]
where \( c = c' \alpha^{-n/(2s)} \beta^{n/(2s)} \) and \( c' \) depends only on \( n, p, s \) and \( w \).

This lemma is proved in [12, Lemma 3.3]. Let \( \mathcal{I}_\lambda \) be the set defined by (15).

**Lemma 4.3.** For each \( \lambda > 0 \), \( \mathcal{I}_\lambda \) is a finite set.

Lemma 4.3 is easily proved by Lemma 4.2 (c.f.[12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of \( \mathcal{I} \) is possible.

By Lemma 4.2 we conclude
\[
\sum_{k=1}^{\infty} |\mu_k|^\gamma = c \sum_{Q \in \mathcal{I}} \left( \beta|Q|^{-1} \int_{Q} v \, dx - \alpha|Q|^{-2s/n-1} \int_{Q} w \, dx \right)^\gamma \leq c \sum_{Q \in \mathcal{I}} \left( \beta|Q|^{-1} \int_{Q} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,
\]
where \( c = c'' \alpha^{-n/(2s)} \beta^p \) and \( c'' \) depends only on \( n, p, s \) and \( w \). This ends the proof of Lemma 3.3.
5 Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

Lemma 5.1. Let \( \omega \in A_2 \) and \( m \in C^n(\mathbb{R}^n \setminus \{0\}) \). Suppose that
\[
B = \max_{|\sigma| \leq n} \sup_{0 < r \leq 2r} \left( \frac{1}{r^{2|\sigma|}} \int_{|\xi| \leq 2r} \left| \frac{\partial}{\partial \xi} \right|^\sigma m(\xi) \right)^2 \, d\xi < \infty.
\]
Then the operator \( T \) defined by \( \hat{T}f(\xi) = m(\xi)\hat{f}(\xi) \) is bounded from \( L^2(\omega) \) to \( L^2(\omega) \) and the operator norm \( \|T\| \) is bounded by \( CB^{1/2} \) where \( C \) is a constant which depends only on \( n \) and \( \omega \).

The proof of Lemma 5.1 is in [6] or [7].

For \( \nu \in \mathbb{Z} \) we define \( \psi_\nu(x) = 2^n \nu \psi(2^\nu x) \). Let \( \omega \in A_2 \) and \( s \geq 0 \). Frazier and Jawerth proved that there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} 2^{2\nu s} |f \ast \psi_\nu(x)|^2 \right\} w(x) \, dx
\]
\[
\leq c_2 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \) where \( c_1 \) and \( c_2 \) depend only on \( n \) and \( s \) and \( \omega \) ([3, Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let \( \{r_\nu(t)\} \) be the Rademacher functions on \([0,1]\) indexed by \( \nu \in \mathbb{Z} \) and
\[
T_t f(x) = \sum_{\nu \in \mathbb{Z}} r_\nu(t) f \ast \psi_\nu(x).
\]
Then \( T_t \) satisfies the condition of Lemma 5.1. Hence
\[
\int_{\mathbb{R}^n} |T_t f(x)|^2 w(x) \, dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx,
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \) where \( M = \max_{|\sigma| \leq n} \| \partial^\sigma \psi \|_{L^\infty} \) and \( C \) is a positive constant depending only on \( n \) and \( \omega \). By integrating from 0 to 1 with respect to \( t \), we get
\[
\int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f \ast \psi_\nu(x)|^2 \right\} w(x) \, dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx.
\]
By the duality argument and the fact \( \omega^{-1} \in A_2 \) we obtain
\[
\int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx \leq CM \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f \ast \psi_\nu(x)|^2 \right\} w(x) \, dx
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). Hence we have

\[
c_3 M^{-1} \int_{\mathbb{R}^n} |f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_{\nu}|^2 \right\} w \, dx \leq c_4 M \int_{\mathbb{R}^n} |f|^2 w \, dx,
\]

where \( c_3 \) and \( c_4 \) are constants depending only on \( n \) and \( w \).

Therefore we get

\[
c_3 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |(-\Delta)^{s/2} f * \psi_{\nu}|^2 \right\} w \, dx \leq c_4 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx
\]

for all \( f \in C_0^\infty(\mathbb{R}^n) \)(c.f.[11]).

Let \( \Phi \in S(\mathbb{R}^n) \) satisfy \( \text{supp} \, \Phi \subset \{ \xi : 1/4 \leq |\xi| \leq 4 \} \) and \( \Phi(\xi) = 1 \) for \( 1/2 \leq |\xi| \leq 2 \). For \( \nu \in \mathbb{Z} \) the multiplier \( m_\nu(\xi) = 2^{-s\nu} |\xi|^s \Phi(\xi/2^\nu) \) satisfies the condition of Lemma 5.1. Hence we have

\[
\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_{\nu}(x)|^2 w(x) \, dx \leq c_5 \int_{\mathbb{R}^n} 2^{2\nu s} |f * \psi_{\nu}(x)|^2 w(x) \, dx,
\]

where \( c_5 = c_6 \inf \max_{\Phi} \| \partial^\sigma \Phi \|^2_\infty \) and \( c_6 \) is a positive constant depending only on \( n, s \) and \( w \) and the infimum is taken over all possible \( \Phi \).

Similarly there exists a positive constant \( c_7 \) depending only on \( n, s \) and \( w \) such that

\[
\int_{\mathbb{R}^n} 2^{2\nu s} |f * \psi_{\nu}(x)|^2 w(x) \, dx \leq c_7 \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_{\nu}(x)|^2 w(x) \, dx.
\]

Hence we get

\[
c_8 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \leq \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \leq c_9 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx
\]

for all \( f \in C_0^\infty(\mathbb{R}^n) \), where \( c_8 \) and \( c_9 \) are positive constant depending only on \( n, s \) and \( w \). This ends the proof of Lemmas 3.1 and 3.2.

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References


Department of Mathematics
Faculty of Science, Hokkaido University
Sapporo 060-0810
JAPAN
tachizaw@math.sci.hokudai.ac.jp