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# ON VISCOUS CONSERVATION LAWS WITH GROWING INITIAL DATA

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**Abstract.** A local unique solvability is established for viscous conservation laws when the initial data may grow at the space infinity with a natural order. It is also shown that such a classical solution can be extended to a global-in-time solution proved that the growth order of the initial data is less than the critical order. <sup>1</sup>

## 1 INTRODUCTION

We consider a viscous conservation laws of the form

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div} \mathbf{G}(u) = 0 & \text{in } \mathbf{R}^n \times (0, T), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n, \end{cases} \quad (1.1)$$

where  $\partial_t = \partial/\partial t$ . It is well-known that if  $u_0$  is bounded, (1.1) admits a unique global solution (cf. [7]). In this paper we consider the case that  $u_0$  is not bounded at the space infinity. In [4] we show that (1.1) has a unique local-in-time classical solution when  $\mathbf{G}'(r) \sim |r|$  for large  $r$  and  $u_0(x) \sim |x|$  for large  $x$  provided that  $u_0$  is global Lipschitz continuous. A typical example is the viscous Burgers equation:  $\partial_t u - \Delta u + u \partial_x u = 0$  where  $\partial_x = \partial/\partial x$ . Our goal is to solve the initial value problem for more general growth of nonlinear term and initial data  $u_0$  without assuming global Lipschitz continuous of  $u_0$ . Specifically we assume that  $\mathbf{G}'(r) \sim |r|^\beta$  for large  $r$  and  $u_0 \sim |x|^\alpha$  with  $\alpha\beta \leq 1$ . The global existence is not expected in general even for  $n = 1$  since  $u(x, t) = -x/(1-t)$  is a solution of the viscous Burgers equation with  $u_0(x) = -x$ . We also obtain an optimal estimate of the existence time.

It is natural to consider growth order  $\alpha\beta \leq 1$  for (1.1). Let us give formal argument to say  $\alpha\beta \leq 1$  is necessary and sufficient condition for local solvability. We postulate that  $\mathbf{G}(r) = r^{\beta+1}$ , and that  $u(x, t) = x^\alpha f(t)$  is solution of (1.1). By (1.1)  $u$  must satisfy

$$x^\alpha f'(t) = \alpha(\alpha - 1)x^{\alpha-2}f(t) + (\alpha\beta + \alpha)x^{\alpha\beta+\alpha-1}f(t)^{\beta+1}.$$

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We observe that the growth of the left hand side is  $x^\alpha$ , right hand side is  $x^{\alpha\beta+\alpha-1}$ . Hence  $\alpha, \beta$  must satisfy  $\alpha \geq \alpha\beta + \alpha - 1$  so that  $\alpha\beta \leq 1$ .

To state our main result precisely we define function spaces.

$$L_\alpha^\infty = L_\alpha^\infty(\mathbf{R}^n) = \left\{ f \in L_{loc}^\infty(\mathbf{R}^n) \mid \|f\|_\alpha := \left\| \frac{f(x)}{\langle x \rangle^\alpha} \right\|_{L^\infty} < \infty \right\},$$

where  $\langle x \rangle := (1 + |x|^2)^{1/2}$ . We introduce equivalent norm with parameter  $a$ :

$$\|f\|_{\alpha,a} := \left\| \frac{f(x)}{\langle x \rangle_a^\alpha} \right\|_{L^\infty},$$

where  $\langle x \rangle_a = (a + |x|^2)^{1/2}$ .

We define  $BC_\alpha := BC_\alpha(\mathbf{R}^n) = L_\alpha^\infty(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ .

Moreover we assume the following bounds for  $\mathbf{G} = (G_1, \dots, G_n) \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$  with some  $\theta \in (0, 1)$ :

$$\sup_{r \in \mathbf{R}} \frac{|\mathbf{G}'(r)|}{1 + |r|^\beta} < \infty. \quad (1.2)$$

**Definition 1.1.** *By a classical solution  $u$  of (1.1) we mean that  $u \in C(\mathbf{R}^n \times [0, T))$  is  $C^2$  in space and  $C^1$  in time, and it solves (1.1) in  $\mathbf{R}^n \times (0, T)$ .*

**Theorem 1.2 (Existence and uniqueness of solution of (1.1)).** *Assume that  $\alpha, \beta > 0$ . Assume that  $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$  satisfies bounds (1.2). Assume that  $u_0 \in BC_\alpha$ .*

- (1) *If  $\alpha\beta = 1$  there exist unique local classical solution  $u \in L_{loc}^\infty([0, T); L_\alpha^\infty)$  that satisfies (1.1) in  $\mathbf{R}^n \times (0, T)$  with  $u|_{t=0} = u_0$ . Moreover existence time  $T$  is estimated from below by  $T \geq T_0 := 1/C_0^\beta C_{\mathbf{G}'}$ , where  $C_0 = \limsup_{|x| \rightarrow \infty} |u_0(x)|/|x|^\alpha$ ,  $C_{\mathbf{G}'} = \limsup_{|r| \rightarrow \infty} |\mathbf{G}'(r)|/|r|^\beta$ .*
- (2) *If  $\alpha\beta < 1$  there exist unique global classical solution  $u \in L_{loc}^\infty([0, \infty); L_\alpha^\infty)$  that satisfies (1.1) in  $\mathbf{R}^n \times (0, \infty)$  with  $u|_{t=0} = u_0$ .*

**Remark 1.3.** *The existence time is  $T \geq T_0$  is optimal in the sense that a classical solution may not exist in  $\mathbf{R}^n \times (0, T)$ . Because  $u(x, t) = -x/(1 - t)$  is a solution of viscous Burgers equation with  $u_0(x) = -x$ .*

Let us give the idea of the proof of main theorem. If initial data  $u_0$  is bounded, the equation (1.1) admits a global solution. So we solve;

$$\begin{cases} \partial_t u_k - \Delta u_k + \operatorname{div} \mathbf{G}(u_k) = 0 & \text{in } \mathbf{R}^n \times (0, T), \\ u_k|_{t=0} = u_k(0) := \varphi_k u_0 & \text{in } \mathbf{R}^n, \end{cases}$$

by cutting off initial data, where  $\varphi_k(x) = \varphi(x/k)$ ,  $\varphi \in C_0^\infty$  satisfies

$$\varphi(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2). \end{cases}$$

By using the maximum principle and the Gronwall inequality we see that  $\|u_k(t)\|_\alpha$  is uniformly bounded for  $k$  and  $t \leq T$  for some  $T$ . Estimating the integral equation for (1.1), we observe that  $\{u_k\}$  is local equicontinuous. By using Ascoli-Arzela theorem there exists  $u$  satisfying  $u_{k_l} \rightarrow u$  as  $l \rightarrow \infty$ . Moreover, we observe that  $u$  is solution of (1.1).

The idea of global existence when  $\alpha\beta < 1$  and estimate of existence time when  $\alpha\beta = 1$  is to introduce a weight with a parameter  $a$  for  $\|\cdot\|_{\alpha,a}$ . By using the maximum principle and the Gronwall inequality we see that  $\|u_k(t)\|_{\alpha,a}$  is uniformly bounded for  $k$  and locally bounded for  $t < T(a)$ , where  $T(a)$  is the blow up time of our uniformly estimate. Blow up time depends only on parameter  $a$ . If  $a_1 \leq a_2$ , then  $T(a_1) \leq T(a_2)$ . Moreover,  $\lim_{a \rightarrow \infty} T(a) = T_0 = 1/C_0^\beta C_{\mathbf{G}}$ . Then we can say that existence time  $T \geq T_0$ .

The idea of uniqueness is to introduce the energy:

$$E(f) := \int_{\mathbf{R}^n} e^{-|x|^2} |f(x)|^2 dx.$$

We set  $u, v$  is solution of (1.1), and we set  $w = u - v$ . We calculate

$$\int_0^t \int_{\mathbf{R}^n} e^{-|x|^2} w(\partial_t w - \Delta w + \operatorname{div}(\mathbf{G}(u) - \mathbf{G}(v))) dx ds.$$

Then we conclude that

$$E(w(t)) \leq e^{-|R|^2/2} + M_4 \langle R \rangle^2 \int_0^t E(w(s)) ds.$$

for all sufficiently large  $R$  where  $M_4$  depends only on  $T_0, C_1$ . By using the Gronwall inequality we conclude that

$$E(w(t)) \leq e^{-|R|^2/2 + M_4 t \langle R \rangle^2},$$

sending  $R \rightarrow \infty$ . We conclude that  $E(w(t)) = 0$ .

A classical result of Tychonov [8] states that the Cauchy problem for heat equation has a unique classical solution if solution of the heat equation  $u$  satisfies

$$|u(x, t)| \leq C e^{a|x|^2}$$

for some positive constants  $C, a$ . As remarked in [4] our results for Burgers equation follows from this classical results through the Hopf-Cole transform.

D. G. Aronson [1] generalized the result of Tychonov for a linear parabolic equation of variables coefficients. K. Ishige and M. Murata [6] generalized his results for linear equations when coefficient grows at space infinity. There are some results for nonlinear equations when initial data is not bounded (see e.g. [2], [3], [5]).

For example a paper [3] of A. Gladkov and M. Guedda and R. Kersner studied the unique solvability of

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \lambda \left| \frac{\partial v}{\partial x} \right|^q \text{ in } \mathbf{R} \times (0, T]$$

with  $\lambda > 0, q > 1$ , when initial data  $v_0$  is not necessary bounded. In fact, they proved that if  $v_0(x) \leq M_0(\alpha_0 + x^2)^{q/[2(q-1)]-\gamma}$  with some positive constant  $M_0, \alpha$ , some non negative constant  $\gamma$ . Then there exists a unique local solution on  $\mathbf{R} \times (0, T)$ .  $T$  is estimated explicitly from below only by  $M_0, \lambda, q$  when  $\gamma = 0$ . If  $\gamma$  is positive then the solution can be extended globally in time.

## 2 ESTIMATES FOR THE HEAT SEMIGROUP IN WEIGHTED SPACE

We recall several elementary properties of the heat kernel

$$G_t(x) = \frac{1}{\sqrt{4\pi t}^n} e^{-\frac{|x|^2}{4t}}.$$

For a multi-index  $a = (a_1, \dots, a_n) \in \mathbf{N}^n$  by  $\partial^a$  we mean  $\partial^a = \partial^{a_1} \dots \partial^{a_n}$ ,  $\partial^i = \partial/\partial x_i$ , where  $\mathbf{N}$  is the set of nonnegative integers. We set  $e^{t\Delta} f = G_t * f$ .

In [4] the estimate of heat kernel is proved only  $f \in L_m^\infty$  where  $m \in \mathbf{N}$ . But this estimate is still valid for  $f \in L_\alpha^\infty$  where  $\alpha \geq 0$ . In this paper we use this form of the estimate of the heat kernel. The proof is essentially, the same for  $m \in \mathbf{N}$ .

**Lemma 2.1.** (1) For all  $f \in L_\alpha^\infty, a \in \mathbf{N}^n, t > 0$  we have

$$\|\partial^a e^{t\Delta} f\|_\alpha \leq C t^{-\frac{|a|}{2}} (1+t)^{\frac{\alpha}{2}} \|f\|_\alpha,$$

where  $C$  depends only on  $n, \alpha, |a|$ , and  $|a| = \sum_{i=1}^n a_i$ .

(2) For all  $f \in L_\alpha^\infty, a \in \mathbf{N}^n, 0 < \theta \leq 1, 0 < s \leq t$  we have

$$\|\partial^a e^{t\Delta} f - \partial^a e^{s\Delta} f\|_\alpha \leq C s^{-|a|/2} (t-s)^\theta (s^{-\theta} + s^{(\alpha-2\theta)/2} + t^{(\alpha-2\theta)/2}) \|f\|_\alpha,$$

where  $C$  is depend on  $n, \alpha, |a|, \theta$ .

In this paper we skip the proof of this Lemma.

**Lemma 2.2 (Hölder continuity for heat semigroup).** We set  $\alpha' = \max\{\alpha - \theta, 0\}$ . There is a constant  $C = C(n, \alpha, |a|, \theta)$  such that

$$|(\partial^a e^{t\Delta} f)(x) - (\partial^a e^{t\Delta} f)(y)| \leq C \langle x \rangle^\alpha (1 + |x - y|^\alpha) t^{-|a|/2} (t^{\alpha'} + t^{-\theta/2}) |x - y|^\theta \|f\|_\alpha$$

holds for all  $f \in L_\alpha^\infty(\mathbf{R}^n)$ ,  $0 < s \leq t$ ,  $0 < \theta \leq 1$ .

In this paper we prove this Lemma only for  $a = (0, \dots, 0)$ , see for [4] for general  $a$ .

*Proof.* It is enough to prove that  $n = 1$ . We set  $x \leq y$ ,  $\delta = y - x$ .

$$\begin{aligned} & \left| \int_{\mathbf{R}} (G_t(x - z) - G_t(y - z)) f(z) dz \right| \\ & \leq \|f\|_\alpha \int_{\mathbf{R}} |G_t(x - z) - G_t(y - z)| \langle z \rangle^\alpha dz \\ & \leq C \langle x \rangle^\alpha \|f\|_\alpha \int_{\mathbf{R}} |G_t(x - z) - G_t(y - z)| (1 + |x - z|^\alpha) dz \\ & = C \langle x \rangle^\alpha \|f\|_\alpha \int_{\mathbf{R}} |G_t(z - x) - G_t(z - y)| (1 + |z - x|^\alpha) dz \\ & = C \langle x \rangle^\alpha \|f\|_\alpha \int_{\mathbf{R}} |G_t(z) - G_t(z - \delta)| (1 + |z|^\alpha) dz \\ & = C \langle x \rangle^\alpha \|f\|_\alpha \left( \int_{[-4\delta, 4\delta]} |G_t(z) - G_t(z - \delta)| (1 + |z|^\alpha) dz \right. \\ & \quad \left. + \int_{[-4\delta, 4\delta]^c} |G_t(z) - G_t(z - \delta)| (1 + |z|^\alpha) dz \right). \end{aligned}$$

The first term is estimated as follows:

$$\begin{aligned} & \int_{[-4\delta, 4\delta]} |G_t(z) - G_t(z - \delta)| (1 + |z|^\alpha) dz \\ & \leq (1 + (4\delta)^\alpha) \int_{\mathbf{R}} |G_t(z) - G_t(z - \delta)| dz \\ & \leq C (1 + \delta^\alpha) t^{-\theta/2} \delta^\theta. \end{aligned}$$

The second term is estimated as follows:

$$\begin{aligned}
& \int_{[-4\delta, 4\delta]^c} |G_t(z) - G_t(z - \delta)| (1 + |z|^\alpha) dz \\
& \leq C \int_{[-4\delta, 4\delta]^c} |G_t(z) - G_t(z - \delta)| \langle z \rangle^\alpha dz \\
& = C \left( \int_{-\infty}^{-4\delta} - \int_{4\delta}^{\infty} \right) (G_t(z) - G_t(z - \delta)) \langle z \rangle^\alpha dz \\
& = C \left( - \int_{-\infty}^{-4\delta} + \int_{4\delta}^{\infty} \right) G_t(z - \delta) \langle z \rangle^\alpha dz \\
& = C \left( - \int_{-\infty}^{-5\delta} + \int_{3\delta}^{\infty} \right) G_t(z) \langle z + \delta \rangle^\alpha dz \\
& = C \int_{3\delta}^{5\delta} G_t(z) \langle z + \delta \rangle^\alpha dz + C \int_{5\delta}^{\infty} G_t(z) (\langle z + \delta \rangle^\alpha - \langle z - \delta \rangle^\alpha) dz \\
& \leq C(1 + \delta^\alpha) t^{-\theta/2} \delta^\theta + C\delta^\theta \int_{5\delta}^{\infty} G_t(z) (\langle z + \delta \rangle^{\alpha-\theta} + \langle z - \delta \rangle^{\alpha-\theta}) dz \\
& \leq C(1 + \delta^\alpha) t^{-\theta/2} \delta^\theta + C\delta^\theta \int_0^{\infty} G_t(z) (1 + z^{\alpha'} + \delta^{\alpha'}) dz \\
& \leq C(1 + \delta^\alpha) t^{-\theta/2} \delta^\theta + C\delta^\theta (1 + t^{\alpha'/2} + \delta^{\alpha'}) \\
& \leq C(1 + \delta^\alpha) (t^{\alpha'} + t^{-\theta/2}) \delta^\theta
\end{aligned}$$

The proof is now complete . □

### 3 EXISTENCE OF SOLUTION

In this section we prove existence of solution of (1.1). This is proved by using Ascoli-Arzela theorem.

It is well-known that if initial data  $u_0$  is bounded and if  $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$  for some  $\theta \in (0, 1)$  then (1.1) has unique global classical solution  $u \in L^\infty((0, \infty); L^\infty)$ . So we set  $\varphi_k(x) = \varphi(x/k)$  where  $\varphi \in C_0^\infty(\mathbf{R}^n)$  satisfies

$$\varphi(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2), \end{cases}$$

and set  $\{u_k\} \subset L^\infty((0, \infty); L^\infty)$  is a sequence of classical solution of

$$\begin{cases} \partial_t u_k - \Delta u_k + \operatorname{div} \mathbf{G}(u_k) = 0 & \text{in } \mathbf{R}^n \times (0, \infty), \\ u_k|_{t=0} = u_k(0) := \varphi_k u_0 & \text{in } \mathbf{R}^n. \end{cases} \quad (3.1)$$

The first lemma is useful to estimate existence time.

**Lemma 3.1.** *Let  $f \in L_{loc}^\infty(\mathbf{R}^n)$ . Then*

$$\lim_{a \rightarrow \infty} \|f\|_{\alpha, a} = \limsup_{|x| \rightarrow \infty} \frac{|f(x)|}{|x|^\alpha}.$$

*Proof.* We observe that

$$\begin{aligned} \|f\|_{\alpha, a} &= \sup_{x \in \mathbf{R}^n} \frac{|f(x)|}{\langle x \rangle_a^\alpha} \\ &\leq \sup_{|x| \leq R} \frac{|f(x)|}{\langle x \rangle_a^\alpha} + \sup_{|x| \geq R} \frac{|f(x)|}{\langle x \rangle_a^\alpha} \\ &\leq \sup_{|x| \leq R} \frac{|f(x)|}{\langle x \rangle_a^\alpha} + \sup_{|x| \geq R} \frac{|f(x)|}{|x|^\alpha}. \end{aligned}$$

Let  $a$  tend to  $\infty$ . Then we have

$$\lim_{a \rightarrow \infty} \|f\|_{\alpha, a} \leq \sup_{|x| \geq R} \frac{|f(x)|}{|x|^\alpha}.$$

And let  $R$  tend to  $\infty$ , then we have

$$\lim_{a \rightarrow \infty} \|f\|_{\alpha, a} \leq \limsup_{|x| \rightarrow \infty} \frac{|f(x)|}{|x|^\alpha}.$$

We set

$$M = \limsup_{|x| \rightarrow \infty} \frac{|f(x)|}{|x|^\alpha},$$

then for all  $\varepsilon > 0$ , there exists  $r > 0$ ,

$$\begin{aligned} M - \varepsilon &< \sup_{|x| \geq R} \frac{|f(x)| \langle x \rangle_a^\alpha}{\langle x \rangle_a^\alpha |x|^\alpha} \\ &\leq \frac{\langle R \rangle_a^\alpha}{|R|^\alpha} \sup_{|x| \geq R} \frac{|f(x)|}{\langle x \rangle_a^\alpha} \\ &\leq \frac{\langle R \rangle_a^\alpha}{|R|^\alpha} \|f\|_{\alpha, a}, \end{aligned}$$

for all  $R > r$ . Let  $R$  tend to  $\infty$ . Then we have

$$M - \varepsilon \leq \|f\|_{\alpha, a}.$$

Letting  $\varepsilon$  tend to 0, we have

$$M \leq \|f\|_{\alpha, a}.$$

Letting  $a$  tend to  $\infty$ , we have

$$\limsup_{|x| \rightarrow \infty} \frac{|f(x)|}{|x|^\alpha} \leq \lim_{a \rightarrow \infty} \|f\|_{\alpha, a}.$$

The proof is now complete . □



Before stating next Lemma, we define following quantity:

$$T_0 = \begin{cases} 1/C_0^\beta C_{\mathbf{G}'} & (\alpha\beta = 1), \\ \infty & (\alpha\beta < 1), \end{cases}$$

where  $C_0 = \limsup_{|x| \rightarrow \infty} |u_0(x)|/|x|^\alpha$ ,  $C_{\mathbf{G}'} = \limsup_{|r| \rightarrow \infty} |\mathbf{G}'(r)|/|r|^\beta$ .

**Lemma 3.2 (Uniform boundedness).** *Assume that  $u_0 \in BC_\alpha$ ,  $\alpha\beta \leq 1$ , and  $\{u_k\}_{k \geq 1} \subset L^\infty((0, \infty); L^\infty)$  is a classical solution of (3.1). Then  $\{u_k\}_{k \geq 1} \subset L^\infty((0, T_0); L^\infty)$  is uniformly bounded for  $k$ .*

*Proof.* We set  $v_k = \frac{u_k}{\langle x \rangle_a^\alpha}$  and observe that  $\nabla \langle x \rangle_a^\alpha$  and  $\Delta \langle x \rangle_a^\alpha$  satisfy

$$\begin{aligned} \nabla \langle x \rangle_a^\alpha &= \alpha \langle x \rangle_a^{\alpha-2} x, \\ \Delta \langle x \rangle_a^\alpha &= \alpha n \langle x \rangle_a^{\alpha-2} + \alpha(\alpha-2) \langle x \rangle_a^{\alpha-4} |x|^2. \end{aligned}$$

By these identities we see that

$$\partial_t v_k - \Delta v_k + \nabla v_k \cdot \mathbf{H} + K = 0,$$

where

$$\begin{aligned} \mathbf{H} &= \mathbf{G}'(u_k) - \alpha \langle x \rangle_a^{-2} x, \\ K &= \left( \alpha n \langle x \rangle_a^{-2} + \alpha(\alpha-2) \langle x \rangle_a^{-4} |x|^2 + \alpha \frac{x}{\langle x \rangle_a} \cdot \frac{\mathbf{G}'(v_k \langle x \rangle_a^\alpha)}{\langle x \rangle_a} \right) v_k. \end{aligned}$$

Moreover,  $\mathbf{H} \in L_1^\infty$ ,  $K \in L^\infty$ . By the maximum principle [4] we see that  $\|v_k(t)\|_\infty$  satisfies

$$\|v_k(t)\|_\infty \leq \|v_k(0)\|_\infty + \int_0^t C_1(a) \|v_k(s)\|_\infty + C_2(a) \|v_k(s)\|_\infty^{1+\beta} ds,$$

where  $C_1(a) = \alpha(|\alpha-2| + n + a^{3/4})/a$ ,  $C_2(a) = \alpha a^{\alpha\beta-1} (\sup_{r \in \mathbf{R}} |\mathbf{G}'(r)|/(a^{1/4} + |r|^\beta))$ . By using the Gronwall inequality [4] we see that  $\|v_k(t)\|_\infty$  satisfies

$$\|v_k(t)\|_\infty \leq \left( \|v_k(0)\|_\infty^{-\beta} e^{-t\beta C_1(a)} - \frac{C_2(a)}{C_1(a)} (1 - e^{-t\beta C_1(a)}) \right)^{-1/\beta} \quad (3.2)$$

in  $(0, T(a))$ , where  $T(a)$  satisfies  $(C_1(a) + C_2(a) \|u_k(0)\|_{\alpha,a}^\beta) - C_2(a) \|u_k(0)\|_{\alpha,a}^\beta e^{T(a)\beta C_1(a)} = 0$ . The right hand side of (3.2) is monotone increasing for  $\|v_k(0)\|_\infty$ , and  $\|v_k(0)\|_\infty$  is monotone increasing for  $k$ . Then (3.2) becomes

$$\|v_k(t)\|_\infty \leq \left( \|v(0)\|_\infty^{-\beta} e^{-t\beta C_1(a)} - \frac{C_2(a)}{C_1(a)} (1 - e^{-t\beta C_1(a)}) \right)^{-1/\beta},$$

where  $v(0) = u_0/\langle x \rangle_a^\alpha$ .

Thus  $\|u_k(t)\|_{\alpha,a}$  satisfies

$$\|u_k(t)\|_{\alpha,a} \leq \left( \frac{C_1(a)\|u_0\|_{\alpha,a}^\beta e^{t\beta C_1(a)}}{\left(C_1(a) + C_2(a)\|u_0\|_{\alpha,a}^\beta\right) - C_2(a)\|u_0\|_{\alpha,a}^\beta e^{t\beta C_1(a)}} \right)^{1/\beta}.$$

By definition,  $\|u_k(t)\|_{\alpha,a}$  and  $\|u_k(t)\|_\alpha$  satisfy

$$\|u_k(t)\|_{\alpha,a} \leq \|u_k(t)\|_\alpha \leq a^{\alpha/2} \|u_k(t)\|_{\alpha,a}$$

for all  $a \geq 1$ . Thus we conclude that

$$\|u_k(t)\|_\alpha \leq \left( \frac{a^{\alpha/2\beta} C_1(a)\|u_0\|_{\alpha,a}^\beta e^{t\beta C_1(a)}}{\left(C_1(a) + C_2(a)\|u_0\|_{\alpha,a}^\beta\right) - C_2(a)\|u_0\|_{\alpha,a}^\beta e^{t\beta C_1(a)}} \right)^{1/\beta},$$

for all  $a \geq 1$ . In other words,  $\|u_k(t)\|_\alpha$  satisfies

$$\|u_k(t)\|_\alpha \leq \inf_{a \geq 1} F(a, t),$$

where

$$F(a, t) = \begin{cases} \left( \frac{a^{\alpha/2\beta} C_1(a)\|u_0\|_{\alpha,a}^\beta e^{t\beta C_1(a)}}{\left(C_1(a) + C_2(a)\|u_0\|_{\alpha,a}^\beta\right) - C_2(a)\|u_0\|_{\alpha,a}^\beta e^{t\beta C_1(a)}} \right)^{1/\beta} & (0 \leq t < T(a)), \\ \infty & (t \geq T(a)). \end{cases}$$

Moreover,  $T(a) = (\log(C_1(a) + C_2(a)\|u_0\|_{\alpha,a}^\beta) - \log(C_2(a)\|u_0\|_{\alpha,a}^\beta))/\beta C_1(a) \leq T_0$  converges to  $T_0$  by Lemma 3.1. Thus  $\{u_k\}_{k \geq 1} \subset L^\infty((0, T_0); L^\infty)$  is uniformly bounded for  $k$ .  $\square$

**Lemma 3.3 (equicontinuity of  $\{u_k\}$ ).** *Assume that  $u_0 \in BC_\alpha$ ,  $\alpha\beta \leq 1$ , and  $\{u_k\}_{k \geq 1} \subset L^\infty((0, \infty); L^\infty)$  is a classical solution of (3.1). Then  $\{u_k\}$  are locally equicontinuous functions in  $\mathbf{R}^n \times (0, T_0)$ .*

*Proof.* Since  $\{u_k\}$  satisfies

$$u_k(t) = e^{t\Delta} u_k(0) - \int_0^t \operatorname{div} e^{(t-\tau)\Delta} \mathbf{G}(u_k(\tau)) d\tau.$$

By Lemma 3.2,  $\{u_k\}$  are uniformly bounded in  $L^\infty((0, T); L^\infty)$  for all  $T < T_0$ . We set

$$\begin{aligned} M &:= \sup_{k \in \mathbf{N}} \sup_{0 \leq t \leq T} \|u_k(t)\|_\alpha \\ M_{\mathbf{G}} &:= \sup_{k \in \mathbf{N}} \sup_{0 \leq t \leq T} \|\mathbf{G}(u_k(t))\|_{\alpha+1}. \end{aligned}$$

Then  $\|u_k(t) - u_k(s)\|_{\alpha+1}$  follows inequality.

$$\begin{aligned} \|u_k(t) - u_k(s)\|_{\alpha+1} &\leq \|(e^{t\Delta} - e^{s\Delta})u_k(0)\|_{\alpha+1} + \int_0^s \|\operatorname{div}(e^{(t-\tau)\Delta} - e^{(s-\tau)\Delta})\mathbf{G}(u_k(\tau))\|_{\alpha+1} d\tau \\ &\quad + \int_s^t \|\operatorname{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))\|_{\alpha+1} d\tau. \end{aligned}$$

By Lemma 2.1 the first term of right hand side has following inequality for all  $0 < \theta \leq 1$ :

$$\|(e^{t\Delta} - e^{s\Delta})u_k(0)\|_{\alpha+1} \leq CM(t-s)^\theta (s^{-\theta} + s^{(\alpha+2-2\theta)/2} + t^{(\alpha+2-2\theta)/2}),$$

where  $C$  depends only on  $\alpha, \theta, n$ . By Lemma 2.1 the second term of the right hand side satisfies

$$\begin{aligned} &\int_0^s \|\operatorname{div}(e^{(t-\tau)\Delta} - e^{(s-\tau)\Delta})\mathbf{G}(u_k(\tau))\|_{\alpha+1} d\tau \\ &\leq CM_{\mathbf{G}}(t-s)^\theta \int_0^s (s-\tau)^{(-2\theta-1)/2} + (s-\tau)^{(\alpha-2\theta-1)/2} + (s-\tau)^{-1/2}(t-\tau)^{(\alpha-2\theta)/2} d\tau \\ &\leq CM_{\mathbf{G}}(t-s)^\theta, \end{aligned}$$

for all  $0 < \theta < 1/2$ , where  $C$  depends only on  $\alpha, \theta, n, T$ . By Lemma 2.1 the third term of the right hand side enjoys

$$\begin{aligned} &\int_s^t \|\operatorname{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))\|_{\alpha+1} d\tau \\ &\leq CM_{\mathbf{G}} \int_s^t (t-\tau)^{-1/2} + (t-\tau)^{(\alpha-1)/2} d\tau \\ &\leq CM_{\mathbf{G}} ((t-s)^{1/2} + (t-s)^{(\alpha+1)/2}), \end{aligned}$$

where  $C$  depends only on  $\alpha, n$ . We thus obtain

$$\|u_k(t) - u_k(s)\|_{\alpha+1} \leq C(t-s)^\theta s^{-\theta}$$

for all  $0 < \theta \leq 1/2$ , where  $C$  is depending only  $\alpha, \theta, n, T, M, M_{\mathbf{G}}$ .

Moreover,  $|u_k(x, t) - u_k(y, t)|$  satisfies

$$\begin{aligned} &|u_k(x, t) - u_k(y, t)| \\ &\leq |(e^{t\Delta}u_k(0))(x) - (e^{t\Delta}u_k(0))(y)| \\ &\quad + \int_0^t |(\operatorname{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau)))(x) - (\operatorname{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau)))(y)| d\tau. \end{aligned}$$

By Lemma 2.2 the first term of the right hand side fulfills the following inequality for all  $0 < \theta \leq 1$ :

$$\begin{aligned} &|(e^{t\Delta}u_k(0))(x) - (e^{t\Delta}u_k(0))(y)| \\ &\leq CM\langle x \rangle^\alpha (1 + |x-y|^\alpha) (t^{\alpha'} + t^{-\theta/2}) |x-y|^\theta \end{aligned}$$

where  $C$  depends only on  $\alpha, \theta, n$ . By Lemma 2.2 the second term of the right hand fulfills for all  $0 < \theta \leq 1$ :

$$\begin{aligned} & \int_0^t |(\operatorname{div} e^{(t-\tau)\Delta} \mathbf{G}(u_k(\tau)))(x) - (\operatorname{div} e^{(t-\tau)\Delta} \mathbf{G}(u_k(\tau)))(y)| d\tau \\ & \leq CM_{\mathbf{G}} \langle x \rangle^{\alpha+1} (1 + |x - y|^{\alpha+1}) |x - y|^\theta \int_0^t (t - \tau)^{\alpha+1-\theta} + (t - \tau)^{-\theta/2} d\tau \\ & \leq CM_{\mathbf{G}} \langle x \rangle^{\alpha+1} (1 + |x - y|^{\alpha+1}) |x - y|^\theta (t^{\alpha+2-\theta} + t^{(2-\theta)/2}) \end{aligned}$$

for all  $0 < \theta \leq 1$ , where  $C$  depends only on  $\alpha, \theta, n$ . We thus conclude that

$$|u_k(x, t) - u_k(y, t)| \leq Ct^{-\theta/2} |x - y|^\theta,$$

for all  $0 \leq \theta \leq 1$ ,  $x, y \in B_R$ , where  $C$  depends only on  $\alpha, \theta, n, T, M, M_{\mathbf{G}}, R$ .

The proof is now complete.  $\square$

**Theorem 3.4 (Existence of classical solution of (1.1)).** *Assume that  $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$  satisfies bounds (1.2). Assume that  $u_0 \in BC_\alpha$ . Then there exists a unique classical solution  $u \in L_{loc}^\infty([0, T_0]; L_\alpha^\infty)$  of (1.1).*

*Proof.* By using the Ascoli-Arzelà theorem and Lemmas 3.2, 3.3, there exists  $\tilde{u} \in L_{loc}^\infty([0, T_0]; L_\alpha^\infty) \cap C(\mathbf{R}^n \times [0, T_0])$  and  $\{u_{k_j}\} \subset \{u_k\}$  such that

$$\lim_{j \rightarrow \infty} \|u_{k_j} - \tilde{u}\|_{L^\infty(B_R \times (\varepsilon, T - \varepsilon))} = 0$$

for all  $R > 0$  and  $\varepsilon > 0$ . Clearly

$$\lim_{j \rightarrow \infty} \|e^{\cdot\Delta} u_{0, k_j} - e^{\cdot\Delta} u_0\|_{L^\infty(B_R \times (\varepsilon, T - \varepsilon))} = 0.$$

Moreover,

$$\lim_{j \rightarrow \infty} \left\| \int_0^{\cdot} \operatorname{div} e^{(\cdot-s)\Delta} \mathbf{G}(u_{k_j}(s)) ds - \int_0^{\cdot} \operatorname{div} e^{(\cdot-s)\Delta} \mathbf{G}(\tilde{u}(s)) ds \right\|_{L^\infty(B_R \times (\varepsilon, T - \varepsilon))} = 0.$$

Thus we conclude that  $\tilde{u}$  satisfies

$$\tilde{u}(t) = e^{t\Delta} u_0 - \int_0^t \operatorname{div} e^{(t-s)\Delta} \mathbf{G}(\tilde{u}(s)) ds$$

in  $\mathbf{R}^n \times (0, T_0)$ . We define

$$u(x, t) = \begin{cases} \tilde{u}(x, t) & (t > 0), \\ u_0(x) & (t = 0), \end{cases}$$

then  $u \in L_{loc}^\infty([0, T_0]; BC_\alpha)$  satisfies

$$u(t) = e^{t\Delta}u_0 - \int_0^t \operatorname{div}e^{(t-s)\Delta}\mathbf{G}(u(s))ds$$

in  $\mathbf{R}^n \times [0, T_0)$ , and  $u \in C(\mathbf{R}^n \times [0, T_0))$ , since  $\{u_k\}$  satisfies the integral equation. We thus conclude  $u$  is solution of (1.1).  $\square$

## 4 UNIQUENESS OF SOLUTION

In this section we prove the uniqueness of the solution of (1.1). Before proving the uniqueness, we define several energies.

**Definition 4.1.** Let  $\psi(x) = e^{-|x|^2}$ . We define  $E_R^I$ ,  $E_R^O$ ,  $E$ :

$$E_R^I(f) := \int_{|x| \leq R} \psi(x)|f(x)|^2 dx,$$

$$E_R^O(f) := \int_{|x| \geq R} \psi(x)|f(x)|^2 dx,$$

$$E(f) := \int_{\mathbf{R}^n} \psi(x)|f(x)|^2 dx.$$

**Lemma 4.2 (Derivative estimate).** Assume that  $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$  satisfies bounds (1.2). Assume that  $u \in L^\infty((0, T); L_\alpha^\infty)$  is solution of (1.1) for some  $\alpha > 0$ . Then  $E(|\nabla u(t)|) \in L_{loc}^1(0, T)$ .

*Proof.* The function  $u$  is solution of (1.1) so that  $u$  satisfies integral equation:

$$u(t) = e^{t\Delta}u_0 - \int_0^t \operatorname{div}e^{(t-s)\Delta}\mathbf{G}(u(s))ds.$$

By the proof of Lemma 3.3  $u$  is Hölder continuous function and  $\nabla u$  satisfies

$$\begin{aligned} \nabla u(x, t) &= \int_{\mathbf{R}^n} (\nabla G_t)(x - y)u_0(y)dy \\ &\quad + \int_0^t \int_{\mathbf{R}^n} (\nabla \operatorname{div}G_{t-s})(x - y)(\mathbf{G}(u(x, s)) - \mathbf{G}(u(y, s)))dyds. \end{aligned} \tag{4.1}$$

By Lemma 2.1 the first term of the right hand of (4.1) fulfills following inequality:

$$\left\| \int_{\mathbf{R}^n} (\nabla G_t)(x - y)u_0(y)dy \right\|_\alpha \leq Ct^{-1/2}\|u_0\|_\alpha,$$

where  $C$  depends only on  $n, \alpha$ .

By assumption of  $\mathbf{G}$  and the proof of the Lemma 3.3,  $|\mathbf{G}(u(x, s)) - \mathbf{G}(u(y, s))|$  satisfies

$$|\mathbf{G}(u(x, s)) - \mathbf{G}(u(y, s))| \leq Cs^{-\delta} \langle |x| + |y| \rangle^{\alpha\beta + \alpha + 1} |x - y|^\delta,$$

for all  $0 < \delta \leq 1/2$  where  $C$  depends only on  $\alpha, \beta, \delta, n, \sup_{0 \leq t \leq T} \|u(t)\|_\alpha, \|\mathbf{G}\|_{\beta+1}$ . Thus the second term of the right hand side of (4.1) is dominated by

$$\begin{aligned} & \left| \int_0^t \int_{\mathbf{R}^n} (\nabla \operatorname{div} G_{t-s})(x-y) (\mathbf{G}(u(x, s)) - \mathbf{G}(u(y, s))) dy ds \right| \\ & \leq C \int_0^t \int_{\mathbf{R}^n} \frac{|x-y|^2 + (t-s)}{(t-s)^2} G_{t-s}(x-y) s^{-\delta} \langle |x| + |y| \rangle^{\alpha\beta + \alpha + 1} |x-y|^\delta dy ds \\ & \leq C \int_0^t \int_{\mathbf{R}^n} \frac{|x-y|^2 + (t-s)}{(t-s)^2} G_{t-s}(x-y) s^{-\delta} \langle |x| + |y| \rangle^{\alpha\beta + \alpha + 1} |x-y|^\delta dy ds \\ & = C \int_0^t \int_{\mathbf{R}^n} \frac{1}{(t-s)^{1-\delta}} e^{-|z|^2} s^{-\delta} \langle |x| + |x - 2(t-s)^{1/2}z| \rangle^{\alpha\beta + \alpha + 1} dz ds \\ & \leq C \langle x \rangle^{\alpha\beta + \alpha + 1} \int_0^t \frac{s^{-\delta}}{(t-s)^{1-\delta}} ds \end{aligned}$$

for all  $0 < \delta \leq 1/2$ , where  $z = (x-y)/[2(t-s)^{1/2}]$ ,  $C$  depends only on  $\alpha, \beta, \delta, n, T, \sup_{0 \leq t \leq T} \|u(t)\|_\alpha, \|\mathbf{G}\|_{\beta+1}$ . Combining these results, we have

$$|\nabla u(x, t)| \leq Ct^{-1/2} \langle x \rangle^{\alpha\beta + \alpha + 1},$$

where  $C$  depends only on  $\alpha, \beta, \delta, n, T, \sup_{0 \leq t \leq T} \|u(t)\|_\alpha, \|\mathbf{G}\|_{\beta+1}$ . Thus we conclude that  $E(|\nabla u(t)|) \in L^1_{loc}(0, T)$ .  $\square$

**Theorem 4.3 (Uniqueness of classical solution of (1.1)).** *Assume that  $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$  satisfies bounds (1.2). Assume that  $u_0 \in BC_\alpha$ . Assume that  $u, v \in L^\infty_{loc}([0, T]; L^\infty_\alpha)$  are classical solutions of (1.1). If  $\alpha\beta \leq 1$  then  $u = v$  in  $\mathbf{R}^n \times [0, T]$ .*

*Proof.* We set  $w = u - v$ . Since  $u, v \in L^\infty_{loc}([0, T]; L^\infty_\alpha) \cap C(\mathbf{R}^n \times [0, T])$  are solution of (1.1) following identity holds:

$$\int_\varepsilon^t \int_{\mathbf{R}^n} \psi w (\partial_t w - \Delta w + \operatorname{div}(\mathbf{G}(u) - \mathbf{G}(v))) dx ds = 0,$$

for all  $t \leq T$ . By Lemma 4.2,  $E(|\nabla u(t)|), E(|\nabla v(t)|) \in L^1_{loc}(0, T)$  so we have

$$\int_\varepsilon^t \int_{\mathbf{R}^n} \frac{1}{2} \partial_t(\psi w^2) + \nabla(\psi w) \cdot \nabla w - \nabla(\psi w) \cdot (\mathbf{G}(u) - \mathbf{G}(v)) dx ds = 0.$$

This yields

$$\begin{aligned} & \frac{1}{2}(E(w(t)) - E(w(\varepsilon))) + \int_{\varepsilon}^t \int_{\mathbf{R}^n} \psi |\nabla w|^2 dx ds \\ &= \int_{\varepsilon}^t \int_{\mathbf{R}^n} w \nabla w \cdot \nabla \psi + (w \nabla \psi + \psi \nabla w) \cdot (\mathbf{G}(u) - \mathbf{G}(v)) dx ds \end{aligned} \quad (4.2)$$

for all  $\varepsilon > 0$ .

By assumption  $\mathbf{G}$  satisfies

$$|\mathbf{G}(r_1) - \mathbf{G}(r_2)| \leq C_1 |r_1 - r_2| (1 + |r_1|^\beta + |r_2|^\beta),$$

where  $C_1 = C_1(\|\mathbf{G}'\|_\beta)$ . Since  $|u(x, t)|, |v(x, t)| \leq M_1 \langle x \rangle^\alpha$  for all  $(x, t) \in \mathbf{R}^n \times [0, T_0]$  where  $M_1$  depends only on  $T_0$ . Then we have

$$|\mathbf{G}(u) - \mathbf{G}(v)| \leq M_2 |w| \langle x \rangle,$$

where  $M_2$  depends only on  $T_0, C_1$ . Then (4.2) becomes

$$\begin{aligned} & \frac{1}{2}(E(w(t)) - E(w(\varepsilon))) + \int_{\varepsilon}^t \int_{\mathbf{R}^n} \psi |\nabla w|^2 dx ds \\ & \leq \int_{\varepsilon}^t \int_{\mathbf{R}^n} |w| |\nabla w| |\nabla \psi| + M_2 (|w| |\nabla \psi| + |\psi| |\nabla w|) w \langle x \rangle dx ds. \end{aligned} \quad (4.3)$$

The right hand side of (4.3) fulfills following inequalities:

$$\begin{aligned} & \int_{\varepsilon}^t \int_{\mathbf{R}^n} |w| |\nabla w| |\nabla \psi| + M_2 (|w| |\nabla \psi| + |\psi| |\nabla w|) w \langle x \rangle dx ds \\ & \leq \int_{\varepsilon}^t \int_{\mathbf{R}^n} 2 \langle x \rangle |w| |\nabla w| \psi + 2M_2 \langle x \rangle^2 |w|^2 \psi + M_2 |w| |\nabla w| \psi dx ds \\ & \leq \int_{\varepsilon}^t \int_{\mathbf{R}^n} \frac{1}{2} M_3 \langle x \rangle^2 |w|^2 \psi + |\nabla w|^2 \psi dx ds. \end{aligned}$$

where  $M_3$  depends only on  $T_0, C_1$ . The estimate (4.3) now becomes

$$E(w(t)) - E(w(\varepsilon)) \leq M_3 \int_{\varepsilon}^t E(\langle x \rangle w(s)) ds.$$

Let  $\varepsilon$  tend to 0, then we have

$$E(w(t)) \leq M_3 \int_0^t E(\langle x \rangle w(s)) ds. \quad (4.4)$$

On the other hand

$$\begin{aligned} E(\langle x \rangle w(s)) &= E_R^I(\langle x \rangle w(s)) + E_R^O(\langle x \rangle w(s)) \\ &\leq \langle R \rangle^2 E_R^I(w(s)) + e^{-|R|^2/2} \\ &\leq \langle R \rangle^2 E(w(s)) + e^{-|R|^2/2} \end{aligned}$$

for sufficiently large  $R$ . Thus (4.4) becomes

$$E(w(t)) \leq e^{-|R|^2/2} + M_4 \langle R \rangle^2 \int_0^t E(w(s)) ds.$$

By using Gronwall inequality

$$E(w(t)) \leq e^{-|R|^2/2 + M_4 t \langle R \rangle^2}.$$

Sending  $R \rightarrow \infty$  yields

$$E(w(t)) \rightarrow 0$$

for all  $0 \leq t < 1/(2M_4)$ .

Thus the proof is now complete. □

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