ON VISCOUS CONSERVATION LAWS WITH GROWING INITIAL DATA

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Abstract. A local unique solvability is established for viscous conservation laws when the initial data may grow at the space infinity with a natural order. It is also shown that such a classical solution can be extended to a global-in-time solution proved that the growth order of the initial data is less than the critical order. [1]

1 INTRODUCTION

We consider a viscous conservation laws of the form

$$\begin{cases} \partial_t u - \Delta u + \mathrm{div}\mathbf{G}(u) = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$. It is well-known that if u_0 is bounded, (1.1) admits a unique global solution (cf. [7]). In this paper we consider the case that u_0 is not bounded at the space infinity. In [4] we show that (1.1) has a unique local-in-time classical solution when $\mathbf{G}'(r) \sim |r|$ for large r and $u_0(x) \sim |x|$ for large x provided that u_0 is global Lipschitz continuous. A typical example is the viscous Burgers equation: $\partial_t u - \Delta u + u\partial_x u = 0$ where $\partial_x = \partial/\partial x$. Our goal is to solve the initial value problem for more general growth of nonlinear term and initial data u_0 without assuming global Lipschitz continuous of u_0. Specifically we assume that $\mathbf{G}'(r) \sim |r|^\beta$ for large r and $u_0 \sim |x|^\alpha$ with $\alpha\beta \le 1$. The global existence is not expected in general even for $n = 1$ since $u(x,t) = -x/(1-t)$ is a solution of the viscous Burgers equation with $u_0(x) = -x$. We also obtain an optimal estimate of the existence time.

It is natural to consider growth order $\alpha\beta \le 1$ for (1.1). Let us give formal argument to say $\alpha\beta \le 1$ is necessary and sufficient condition for local solvability. We postulate that $\mathbf{G}(r) = r^{\beta+1}$, and that $u(x,t) = x^\alpha f(t)$ is solution of (1.1). By (1.1) u must satisfy

$$x^\alpha f'(t) = \alpha(\alpha-1)x^{\alpha-2}f(t) + (\alpha\beta+\alpha)x^{\alpha\beta+\alpha-1}f(t)^{\beta+1}.$$

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We observe that the growth of the left hand side is x^α, right hand side is $x^{\alpha\beta+\alpha-1}$. Hence α, β must satisfy $\alpha \geq \alpha\beta + \alpha - 1$ so that $\alpha\beta \leq 1$.

To state our main result precisely we define function spaces.

$$L^\infty_\alpha = L^\infty_\alpha(\mathbf{R}^n) = \left\{ f \in L^\infty_{loc}(\mathbf{R}^n) \;\middle|\; \|f\|_\alpha := \left\| \frac{f(x)}{\langle x \rangle^\alpha} \right\|_{L^\infty} < \infty \right\},$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$. We introduce equivalent norm with parameter a:

$$\|f\|_{\alpha,a} := \left\| \frac{f(x)}{\langle x \rangle_a^\alpha} \right\|_{L^\infty},$$

where $\langle x \rangle_a = (a + |x|^2)^{1/2}$.

We define $BC_\alpha := BC_\alpha(\mathbf{R}^n) = L^\infty_\alpha(\mathbf{R}^n) \cap C(\mathbf{R}^n)$.

Moreover we assume the following bounds for $\mathbf{G} = (G_1, \cdots, G_n) \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$ with some $\theta \in (0, 1)$:

$$\sup_{r \in \mathbf{R}} \frac{|\mathbf{G}'(r)|}{1 + |r|^\beta} < \infty. \tag{1.2}$$

Definition 1.1. *By a classical solution u of (1.1) we mean that $u \in C(\mathbf{R}^n \times [0, T))$ is C^2 in space and C^1 in time, and it solves (1.1) in $\mathbf{R}^n \times (0, T)$.*

Theorem 1.2 (Existence and uniqueness of solution of (1.1)). *Assume that $\alpha, \beta > 0$. Assume that $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$ satisfies bounds (1.2). Assume that $u_0 \in BC_\alpha$.*

(1) If $\alpha\beta = 1$ there exist unique local classical solution $u \in L^\infty_{loc}([0, T); L^\infty_\alpha)$ that satisfies (1.1) in $\mathbf{R}^n \times (0, T)$ with $u|_{t=0} = u_0$. Moreover existence time T is estimated from below by $T \geq T_0 := 1/C_0^\beta C_{\mathbf{G}'}$, where $C_0 = \limsup_{|x| \to \infty} |u_0(x)|/|x|^\alpha$, $C_{\mathbf{G}'} = \limsup_{|r| \to \infty} |\mathbf{G}'(r)|/|r|^\beta$.

(2) If $\alpha\beta < 1$ there exist unique global classical solution $u \in L^\infty_{loc}([0, \infty); L^\infty_\alpha)$ that satisfies (1.1) in $\mathbf{R}^n \times (0, \infty)$ with $u|_{t=0} = u_0$.

Remark 1.3. *The existence time is $T \geq T_0$ is optimal in the sense that a classical solution may not exist in $\mathbf{R}^n \times (0, T)$. Because $u(x, t) = -x/(1 - t)$ is a solution of viscous Burgers equation with $u_0(x) = -x$.*

Let us give the idea of the proof of main theorem. If initial data u_0 is bounded, the equation (1.1) admits a global solution. So we solve;

$$\begin{cases} \partial_t u_k - \Delta u_k + \mathrm{div}\mathbf{G}(u_k) = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u_k|_{t=0} = u_k(0) := \varphi_k u_0 & \text{in } \mathbf{R}^n, \end{cases}$$

by cutting off initial data, where $\varphi_k(x) = \varphi(x/k)$, $\varphi \in C_0^\infty$ satisfies

$$\varphi(x) = \begin{cases} 1 & (|x| \le 1), \\ 0 & (|x| \ge 2). \end{cases}$$

By using the maximum principle and the Gronwall inequality we see that $\|u_k(t)\|_\alpha$ is uniformly bounded for k and $t \le T$ for some T. Estimating the integral equation for (1.1), we observe that $\{u_k\}$ is local equicontinuous. By using Ascoli-Arzela theorem there exists u satisfying $u_{k_l} \to u$ as $l \to \infty$. Moreover, we observe that u is solution of (1.1).

The idea of global existence when $\alpha\beta < 1$ and estimate of existence time when $\alpha\beta = 1$ is to introduce a weight with a parameter a for $\|\cdot\|_{\alpha,a}$. By using the maximum principle and the Gronwall inequality we see that $\|u_k(t)\|_{\alpha,a}$ is uniformly bounded for k and locally bounded for $t < T(a)$, where $T(a)$ is the blow up time of our uniformly estimate. Blow up time depends only on parameter a. If $a_1 \le a_2$, then $T(a_1) \le T(a_2)$. Moreover, $\lim_{a\to\infty} T(a) = T_0 = 1/C_0^\beta C_{\mathbf{G}}$. Then we can say that existence time $T \ge T_0$.

The idea of uniqueness is to introduce the energy:

$$E(f) := \int_{\mathbf{R}^n} e^{-|x|^2} |f(x)|^2 dx.$$

We set u, v is solution of (1.1), and we set $w = u - v$. We calculate

$$\int_0^t \int_{\mathbf{R}^n} e^{-|x|^2} w(\partial_t w - \Delta w + \mathrm{div}(\mathbf{G}(u) - \mathbf{G}(v)))dxds.$$

Then we conclude that

$$E(w(t)) \le e^{-|R|^2/2} + M_4 \langle R \rangle^2 \int_0^t E(w(s))ds.$$

for all sufficiently large R where M_4 depends only on T_0, C_1. By using the Gronwall inequality we conclude that

$$E(w(t)) \le e^{-|R|^2/2 + M_4 t \langle R \rangle^2},$$

sending $R \to \infty$. We conclude that $E(w(t)) = 0$.

A classical result of Tychonov [8] states that the Cauchy problem for heat equation has a unique classical solution if solution of the heat equation u satisfies

$$|u(x,t)| \le C e^{a|x|^2}$$

for some positive constants C, a. As remarked in [4] our results for Burgers equation follows from this classical results through the Hopf-Cole transform.

D. G. Aronson [1] generalized the result of Tychonov for a linear parabolic equation of variables coefficients. K. Ishige and M. Murata [6] generalized his results for linear equations when coefficient grows at space infinity. There are some results for nonlinear equations when initial data is not bounded (see e.g. [2], [3], [5]).

For example a paper [3] of A. Gladkov and M. Guedda and R. Kersner studied the unique solvability of

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \lambda \left| \frac{\partial v}{\partial x} \right|^q \text{ in } \mathbf{R} \times (0, T]$$

with $\lambda > 0$, $q > 1$, when initial data v_0 is not necessary bounded. In fact, they proved that if $v_0(x) \leq M_0(\alpha_0 + x^2)^{q/[2(q-1)]-\gamma}$ with some positive constant M_0, α, some non negative constant γ. Then there exists a unique local solution on $\mathbf{R} \times (0, T)$. T is estimated explicitly from below only by M_0, λ, q when $\gamma = 0$. If γ is positive then the solution can be extended globally in time.

2 Estimates for the heat semigroup in weighted space

We recall several elementary properties of the heat kernel

$$G_t(x) = \frac{1}{\sqrt{4\pi t}^n} e^{-\frac{|x|^2}{4t}}.$$

For a multi-index $a = (a_1, \cdots, a_n) \in \mathbf{N}^n$ by ∂^a we mean $\partial^a = \partial^{a_1} \cdots \partial^{a_n}$, $\partial^i = \partial/\partial x_i$, where $\mathbf{N}$ is the set of nonnegative integers. We set $e^{t\Delta} f = G_t * f$.

In [4] the estimate of heat kernel is proved only $f \in L^\infty_m$ where $m \in \mathbf{N}$. But this estimate is still valid for $f \in L^\infty_\alpha$ where $\alpha \geq 0$. In this paper we use this form of the estimate of the heat kernel. The proof is essentially, the same for $m \in \mathbf{N}$.

Lemma 2.1. *(1) For all $f \in L^\infty_\alpha$, $a \in \mathbf{N}^n$, $t > 0$ we have*

$$\|\partial^a e^{t\Delta} f\|_\alpha \leq C t^{-\frac{|a|}{2}} (1 + t)^{\frac{\alpha}{2}} \|f\|_\alpha,$$

where C depends only on n, α, $|a|$, and $|a| = \sum_{i=1}^n a_i$.

(2) For all $f \in L^\infty_\alpha$, $a \in \mathbf{N}^n$, $0 < \theta \leq 1$, $0 < s \leq t$ we have

$$\|\partial^a e^{t\Delta} f - \partial^a e^{s\Delta} f\|_\alpha \leq C s^{-|a|/2} (t - s)^\theta \left(s^{-\theta} + s^{(\alpha - 2\theta)/2} + t^{(\alpha-2\theta)/2}\right) \|f\|_\alpha,$$

where C is depend on n, α, $|a|$, θ.

In this paper we skip the proof of this Lemma.

Lemma 2.2 (Hölder continuity for heat semigroup). *We set* $\alpha' = \max\{\alpha-\theta, 0\}$. *There is a constant* $C = C(n, \alpha, |a|, \theta)$ *such that*

$$\left|\left(\partial^a e^{t\Delta} f\right)(x) - \left(\partial^a e^{t\Delta} f\right)(y)\right| \le C\langle x\rangle^\alpha \left(1 + |x-y|^\alpha\right) t^{-|a|/2}\left(t^{\alpha'} + t^{-\theta/2}\right)|x-y|^\theta \|f\|_\alpha$$

holds for all $f \in L^\infty_\alpha(\mathbf{R}^n)$, $0 < s \le t$, $0 < \theta \le 1$.

In this paper we prove this Lemma only for $a = (0, \cdots, 0)$, see for [4] for general a.

Proof. It is enough to prove that $n = 1$. We set $x \le y$, $\delta = y - x$.

$$\begin{aligned}
&\left|\int_{\mathbf{R}} (G_t(x-z) - G_t(y-z)) f(z) dz\right| \\
\le\ & \|f\|_\alpha \int_{\mathbf{R}} |G_t(x-z) - G_t(y-z)| \langle z\rangle^\alpha dz \\
\le\ & C\langle x\rangle^\alpha \|f\|_\alpha \int_{\mathbf{R}} |G_t(x-z) - G_t(y-z)| \left(1 + |x-z|^\alpha\right) dz \\
=\ & C\langle x\rangle^\alpha \|f\|_\alpha \int_{\mathbf{R}} |G_t(z-x) - G_t(z-y)| \left(1 + |z-x|^\alpha\right) dz \\
=\ & C\langle x\rangle^\alpha \|f\|_\alpha \int_{\mathbf{R}} |G_t(z) - G_t(z-\delta)| \left(1 + |z|^\alpha\right) dz \\
=\ & C\langle x\rangle^\alpha \|f\|_\alpha \left(\int_{[-4\delta, 4\delta]} |G_t(z) - G_t(z-\delta)| \left(1 + |z|^\alpha\right) dz\right. \\
& \qquad \left. + \int_{[-4\delta, 4\delta]^c} |G_t(z) - G_t(z-\delta)| \left(1 + |z|^\alpha\right) dz\right).
\end{aligned}$$

The first term is estimated as follows:

$$\begin{aligned}
&\int_{[-4\delta, 4\delta]} |G_t(z) - G_t(z-\delta)| \left(1 + |z|^\alpha\right) dz \\
\le\ & \left(1 + (4\delta)^\alpha\right) \int_{\mathbf{R}} |G_t(z) - G_t(z-\delta)| dz \\
\le\ & C\left(1 + \delta^\alpha\right) t^{-\theta/2} \delta^\theta.
\end{aligned}$$

The second term is estimated as follows:

$$\begin{aligned}
&\int_{[-4\delta,4\delta]^c} |G_t(z) - G_t(z-\delta)|\,(1+|z|^\alpha)\,dz \\
\le\;& C\int_{[-4\delta,4\delta]^c} |G_t(z) - G_t(z-\delta)|\langle z\rangle^\alpha dz \\
=\;& C\left(\int_{-\infty}^{-4\delta} - \int_{4\delta}^{\infty}\right)(G_t(z) - G_t(z-\delta))\langle z\rangle^\alpha dz \\
=\;& C\left(-\int_{-\infty}^{-4\delta} + \int_{4\delta}^{\infty}\right) G_t(z-\delta)\langle z\rangle^\alpha dz \\
=\;& C\left(-\int_{-\infty}^{-5\delta} + \int_{3\delta}^{\infty}\right) G_t(z)\langle z+\delta\rangle^\alpha dz \\
=\;& C\int_{3\delta}^{5\delta} G_t(z)\langle z+\delta\rangle^\alpha dz + C\int_{5\delta}^{\infty} G_t(z)\left(\langle z+\delta\rangle^\alpha - \langle z-\delta\rangle^\alpha\right)dz \\
\le\;& C\,(1+\delta^\alpha)\,t^{-\theta/2}\delta^\theta + C\delta^\theta\int_{5\delta}^{\infty} G_t(z)\left(\langle z+\delta\rangle^{\alpha-\theta} + \langle z-\delta\rangle^{\alpha-\theta}\right)dz \\
\le\;& C\,(1+\delta^\alpha)\,t^{-\theta/2}\delta^\theta + C\delta^\theta\int_{0}^{\infty} G_t(z)\left(1 + z^{\alpha'} + \delta^{\alpha'}\right)dz \\
\le\;& C\,(1+\delta^\alpha)\,t^{-\theta/2}\delta^\theta + C\delta^\theta\left(1 + t^{\alpha'/2} + \delta^{\alpha'}\right) \\
\le\;& C\,(1+\delta^\alpha)\left(t^{\alpha'} + t^{-\theta/2}\right)\delta^\theta
\end{aligned}$$

The proof is now complete . □

3 Existence of solution

In this section we prove existence of solution of (1.1). This is proved by using Ascoli-Arzela theorem.

It is well-known that if initial data u_0 is bounded and if $\mathbf{G} \in C^{1+\theta}(\mathbf{R};\mathbf{R}^n)$ for some $\theta \in (0,1)$ then (1.1) has unique global classical solution $u \in L^\infty((0,\infty);L^\infty)$. So we set $\varphi_k(x) = \varphi(x/k)$ where $\varphi \in C_0^\infty(\mathbf{R}^n)$ satisfies

$$\varphi(x) = \begin{cases} 1 & (|x| \le 1), \\ 0 & (|x| \ge 2), \end{cases}$$

and set $\{u_k\} \subset L^\infty((0,\infty);L^\infty)$ is a sequence of classical solution of

$$\begin{cases} \partial_t u_k - \Delta u_k + \mathrm{div}\mathbf{G}(u_k) = 0 & \text{in } \mathbf{R}^n \times (0,\infty), \\ u_k|_{t=0} = u_k(0) := \varphi_k u_0 & \text{in } \mathbf{R}^n. \end{cases} \tag{3.1}$$

The first lemma is useful to estimate existence time.

Lemma 3.1. *Let $f \in L^\infty_{loc}(\mathbf{R}^n)$. Then*

$$\lim_{a\to\infty} \|f\|_{\alpha,a} = \limsup_{|x|\to\infty} \frac{|f(x)|}{|x|^\alpha}.$$

Proof. We observe that

$$\begin{aligned}
\|f\|_{\alpha,a} &= \sup_{x\in\mathbf{R}^n} \frac{|f(x)|}{\langle x\rangle_a^\alpha} \\
&\le \sup_{|x|\le R} \frac{|f(x)|}{\langle x\rangle_a^\alpha} + \sup_{|x|\ge R} \frac{|f(x)|}{\langle x\rangle_a^\alpha} \\
&\le \sup_{|x|\le R} \frac{|f(x)|}{\langle x\rangle_a^\alpha} + \sup_{|x|\ge R} \frac{|f(x)|}{|x|^\alpha}.
\end{aligned}$$

Let a tend to ∞. Then we have

$$\lim_{a\to\infty} \|f\|_{\alpha,a} \le \sup_{|x|\ge R} \frac{|f(x)|}{|x|^\alpha}.$$

And let R tend to ∞, then we have

$$\lim_{a\to\infty} \|f\|_{\alpha,a} \le \limsup_{|x|\to\infty} \frac{|f(x)|}{|x|^\alpha}.$$

We set

$$M = \limsup_{|x|\to\infty} \frac{|f(x)|}{|x|^\alpha},$$

then for all $\varepsilon > 0$, there exists $r > 0$,

$$\begin{aligned}
M - \varepsilon &< \sup_{|x|\ge R} \frac{|f(x)|}{\langle x\rangle_a^\alpha} \frac{\langle x\rangle_a^\alpha}{|x|^\alpha} \\
&\le \frac{\langle R\rangle_a^\alpha}{|R|^\alpha} \sup_{|x|\ge R} \frac{|f(x)|}{\langle x\rangle_a^\alpha} \\
&\le \frac{\langle R\rangle_a^\alpha}{|R|^\alpha} \|f\|_{\alpha,a},
\end{aligned}$$

for all $R > r$. Let R tend to ∞. Then we have

$$M - \varepsilon \le \|f\|_{\alpha,a}.$$

Letting ε tend to 0, we have

$$M \le \|f\|_{\alpha,a}.$$

Letting a tend to ∞, we have

$$\limsup_{|x|\to\infty} \frac{|f(x)|}{|x|^\alpha} \le \lim_{a\to\infty} \|f\|_{\alpha,a}.$$

The proof is now complete . □

Before stating next Lemma, we define following quantity:

$$T_0 = \begin{cases} 1/C_0^\beta C_{\mathbf{G}'} & (\alpha\beta = 1), \\ \infty & (\alpha\beta < 1), \end{cases}$$

where $C_0 = \limsup_{|x|\to\infty} |u_0(x)|/|x|^\alpha$, $C_{\mathbf{G}'} = \limsup_{|r|\to\infty} |\mathbf{G}'(r)|/|r|^\beta$.

Lemma 3.2 (Uniform boundedness). *Assume that $u_0 \in BC_\alpha$, $\alpha\beta \le 1$, and $\{u_k\}_{k\ge1} \subset L^\infty((0,\infty); L^\infty)$ is a classical solution of (3.1). Then $\{u_k\}_{k\ge1} \subset L^\infty((0,T_0); L^\infty_\alpha)$ is uniformly bounded for k.*

Proof. We set $v_k = \dfrac{u_k}{\langle x\rangle_a^\alpha}$ and observe that $\nabla\langle x\rangle_a^\alpha$ and $\Delta\langle x\rangle_a^\alpha$ satisfy

$$\begin{aligned} \nabla\langle x\rangle_a^\alpha &= \alpha\langle x\rangle_a^{\alpha-2}x, \\ \Delta\langle x\rangle_a^\alpha &= \alpha n\langle x\rangle_a^{\alpha-2} + \alpha(\alpha-2)\langle x\rangle_a^{\alpha-4}|x|^2. \end{aligned}$$

By these identities we see that

$$\partial_t v_k - \Delta v_k + \nabla v_k\cdot\mathbf{H} + K = 0,$$

where

$$\mathbf{H} = \mathbf{G}'(u_k) - \alpha\langle x\rangle_a^{-2}x,$$

$$K = \left(\alpha n\langle x\rangle_a^{-2} + \alpha(\alpha-2)\langle x\rangle_a^{-4}|x|^2 + \alpha\frac{x}{\langle x\rangle_a}\cdot\frac{\mathbf{G}'(v_k\langle x\rangle_a^\alpha)}{\langle x\rangle_a}\right)v_k.$$

Moreover, $\mathbf{H} \in L^\infty_1$, $K \in L^\infty$. By the maximum principle [4] we see that $\|v_k(t)\|_\infty$ satisfies

$$\|v_k(t)\|_\infty \le \|v_k(0)\|_\infty + \int_0^t C_1(a)\|v_k(s)\|_\infty + C_2(a)\|v_k(s)\|_\infty^{1+\beta}ds,$$

where $C_1(a) = \alpha(|\alpha-2| + n + a^{3/4})/a$, $C_2(a) = \alpha a^{\alpha\beta-1}\left(\sup_{r\in\mathbf{R}}|\mathbf{G}'(r)|/(a^{1/4}+|r|^\beta)\right)$. By using the Gronwall inequality [4] we see that $\|v_k(t)\|_\infty$ satisfies

$$\|v_k(t)\|_\infty \le \left(\|v_k(0)\|_\infty^{-\beta}e^{-t\beta C_1(a)} - \frac{C_2(a)}{C_1(a)}(1-e^{-t\beta C_1(a)})\right)^{-1/\beta} \tag{3.2}$$

in $(0,T(a))$, where $T(a)$ satisfies $\left(C_1(a) + C_2(a)\|u_k(0)\|_{\alpha,a}^\beta\right) - C_2(a)\|u_k(0)\|_{\alpha,a}^\beta e^{T(a)\beta C_1(a)} = 0$. The right hand side of (3.2) is monotone increasing for $\|v_k(0)\|_\infty$, and $\|v_k(0)\|_\infty$ is monotone increasing for k. Then (3.2) becomes

$$\|v_k(t)\|_\infty \le \left(\|v(0)\|_\infty^{-\beta}e^{-t\beta C_1(a)} - \frac{C_2(a)}{C_1(a)}(1-e^{-t\beta C_1(a)})\right)^{-1/\beta},$$

where $v(0) = u_0/\langle x\rangle_a^\alpha$.

Thus $\|u_k(t)\|_{\alpha,a}$ satisfies

$$\|u_k(t)\|_{\alpha,a} \leq \left(\frac{C_1(a)\|u_0\|_{\alpha,a}^{\beta} e^{t\beta C_1(a)}}{\left(C_1(a)+C_2(a)\|u_0\|_{\alpha,a}^{\beta}\right) - C_2(a)\|u_0\|_{\alpha,a}^{\beta} e^{t\beta C_1(a)}} \right)^{1/\beta}.$$

By definition, $\|u_k(t)\|_{\alpha,a}$ and $\|u_k(t)\|_{\alpha}$ satisfy

$$\|u_k(t)\|_{\alpha,a} \leq \|u_k(t)\|_{\alpha} \leq a^{\alpha/2}\|u_k(t)\|_{\alpha,a}$$

for all $a \geq 1$. Thus we conclude that

$$\|u_k(t)\|_{\alpha} \leq \left(\frac{a^{\alpha/2\beta} C_1(a)\|u_0\|_{\alpha,a}^{\beta} e^{t\beta C_1(a)}}{\left(C_1(a)+C_2(a)\|u_0\|_{\alpha,a}^{\beta}\right) - C_2(a)\|u_0\|_{\alpha,a}^{\beta} e^{t\beta C_1(a)}} \right)^{1/\beta},$$

for all $a \geq 1$. In other words, $\|u_k(t)\|_{\alpha}$ satisfies

$$\|u_k(t)\|_{\alpha} \leq \inf_{a\geq 1} F(a,t),$$

where

$$F(a,t) = \begin{cases} \left(\dfrac{a^{\alpha/2\beta} C_1(a)\|u_0\|_{\alpha,a}^{\beta} e^{t\beta C_1(a)}}{\left(C_1(a)+C_2(a)\|u_0\|_{\alpha,a}^{\beta}\right) - C_2(a)\|u_0\|_{\alpha,a}^{\beta} e^{t\beta C_1(a)}} \right)^{1/\beta} & (0 \leq t < T(a)), \\ \infty & (t \geq T(a)). \end{cases}$$

Moreover, $T(a) = (\log(C_1(a) + C_2(a)\|u_0\|_{\alpha,a}^{\beta}) - \log(C_2(a)\|u_0\|_{\alpha,a}^{\beta}))/\beta C_1(a) \leq T_0$ converges to T_0 by Lemma 3.1. Thus $\{u_k\}_{k\geq 1} \subset L^{\infty}((0,T_0); L^{\infty}_{\alpha})$ is uniformly bounded for k. $\square$

Lemma 3.3 (equicontinuity of $\{u_k\}$). *Assume that $u_0 \in BC_{\alpha}$, $\alpha\beta \leq 1$, and $\{u_k\}_{k\geq 1} \subset L^{\infty}((0,\infty); L^{\infty})$ is a classical solution of (3.1). Then $\{u_k\}$ are locally equicontinuous functions in $\mathbf{R}^n \times (0, T_0)$.*

Proof. Since $\{u_k\}$ satisfies

$$u_k(t) = e^{t\Delta}u_k(0) - \int_0^t \operatorname{div} e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))d\tau.$$

By Lemma 3.2, $\{u_k\}$ are uniformly bounded in $L^{\infty}((0,T); L^{\infty}_{\alpha})$ for all $T < T_0$. We set

$$\begin{aligned} M &:= \sup_{k\in\mathbf{N}} \sup_{0\leq t\leq T} \|u_k(t)\|_{\alpha} \\ M_{\mathbf{G}} &:= \sup_{k\in\mathbf{N}} \sup_{0\leq t\leq T} \|\mathbf{G}(u_k(t))\|_{\alpha+1}. \end{aligned}$$

Then $\|u_k(t)-u_k(s)\|_{\alpha+1}$ follows inequality.

$$\begin{aligned}\|u_k(t)-u_k(s)\|_{\alpha+1} &\leq \|(e^{t\Delta}-e^{s\Delta})u_k(0)\|_{\alpha+1}+\int_0^s \|\mathrm{div}(e^{(t-\tau)\Delta}-e^{(s-\tau)\Delta})\mathbf{G}(u_k(\tau))\|_{\alpha+1}d\tau \\ &\quad + \int_s^t \|\mathrm{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))\|_{\alpha+1}d\tau.\end{aligned}$$

By Lemma 2.1 the first term of right hand side has following inequality for all $0<\theta\leq 1$:

$$\|(e^{t\Delta}-e^{s\Delta})u_k(0)\|_{\alpha+1}\leq CM(t-s)^\theta\left(s^{-\theta}+s^{(\alpha+2-2\theta)/2}+t^{(\alpha+2-2\theta)/2)}\right),$$

where C depends only on α, θ, n. By Lemma 2.1 the second term of the right hand side satisfies

$$\begin{aligned}&\int_0^s \|\mathrm{div}(e^{(t-\tau)\Delta}-e^{(s-\tau)\Delta})\mathbf{G}(u_k(\tau))\|_{\alpha+1}d\tau \\ \leq\ & CM_{\mathbf{G}}(t-s)^\theta\int_0^s (s-\tau)^{(-2\theta-1)/2}+(s-\tau)^{(\alpha-2\theta-1)/2}+(s-\tau)^{-1/2}(t-\tau)^{(\alpha-2\theta)/2}d\tau \\ \leq\ & CM_{\mathbf{G}}(t-s)^\theta,\end{aligned}$$

for all $0<\theta<1/2$, where C depends only on α, θ, n, T. By Lemma 2.1 the third term of the right hand side enjoys

$$\begin{aligned}&\int_s^t \|\mathrm{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))\|_{\alpha+1}d\tau \\ \leq\ & CM_{\mathbf{G}}\int_s^t (t-\tau)^{-1/2}+(t-\tau)^{(\alpha-1)/2}d\tau \\ \leq\ & CM_{\mathbf{G}}\left((t-s)^{1/2}+(t-s)^{(\alpha+1)/2}\right),\end{aligned}$$

where C depends only on α, n. We thus obtain

$$\|u_k(t)-u_k(s)\|_{\alpha+1}\leq C(t-s)^\theta s^{-\theta}$$

for all $0<\theta\leq 1/2$, where C is depending only α, θ, n, T, M, $M_{\mathbf{G}}$.

Moreover, $|u_k(x,t)-u_k(y,t)|$ satisfies

$$\begin{aligned}&|u_k(x,t)-u_k(y,t)| \\ \leq\ & \left|\left(e^{t\Delta}u_k(0)\right)(x)-\left(e^{t\Delta}u_k(0)\right)(y)\right| \\ &+\int_0^t \left|\left(\mathrm{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))\right)(x)-\left(\mathrm{div}e^{(t-\tau)\Delta}\mathbf{G}(u_k(\tau))\right)(y)\right|d\tau.\end{aligned}$$

By Lemma 2.2 the first term of the right hand side fulfills the following inequality for all $0<\theta\leq 1$:

$$\begin{aligned}&\left|\left(e^{t\Delta}u_k(0)\right)(x)-\left(e^{t\Delta}u_k(0)\right)(y)\right| \\ \leq\ & CM\langle x\rangle^\alpha\left(1+|x-y|^\alpha\right)\left(t^{\alpha'}+t^{-\theta/2}\right)|x-y|^\theta\end{aligned}$$

where C depends only on α, θ, n. By Lemma 2.2 the second term of the right hand fulfills for all $0 < \theta \le 1$:

$$\begin{aligned}
&\int_0^t \left| \left(\mathrm{div} e^{(t-\tau)\Delta} \mathbf{G}(u_k(\tau))\right)(x) - \left(\mathrm{div} e^{(t-\tau)\Delta} \mathbf{G}(u_k(\tau))\right)(y) \right| d\tau \\
\le\ & C M_{\mathbf{G}} \langle x \rangle^{\alpha+1} \left(1 + |x-y|^{\alpha+1}\right) |x-y|^{\theta} \int_0^t (t-\tau)^{\alpha+1-\theta} + (t-\tau)^{-\theta/2} d\tau \\
\le\ & C M_{\mathbf{G}} \langle x \rangle^{\alpha+1} \left(1 + |x-y|^{\alpha+1}\right) |x-y|^{\theta} \left(t^{\alpha+2-\theta} + t^{(2-\theta)/2}\right)
\end{aligned}$$

for all $0 < \theta \le 1$, where C depends only on α, θ, n. We thus conclude that

$$|u_k(x,t) - u_k(y,t)| \le C t^{-\theta/2} |x-y|^{\theta},$$

for all $0 \le \theta \le 1$, $x, y \in B_R$, where C depends only on α, θ, n, T, M, $M_{\mathbf{G}}$, R.
The proof is now complete. □

Theorem 3.4 (Existence of classical solution of (1.1)). *Assume that* $\mathbf{G} \in C^{1+\theta}(\mathbf{R}; \mathbf{R}^n)$ *satisfies bounds (1.2). Assume that* $u_0 \in BC_\alpha$. *Then there exists a unique classical solution* $u \in L^\infty_{loc}([0, T_0); L^\infty_\alpha)$ *of (1.1).*

Proof. By using the Ascoli-Arzela theorem and Lemmas 3.2, 3.3, there exists $\tilde{u} \in L^\infty_{loc}([0,T_0); L^\infty_\alpha) \cap C(\mathbf{R}^n \times [0,T_0))$ and $\{u_{k_j}\} \subset \{u_k\}$ such that

$$\lim_{j\to\infty} \|u_{k_j} - \tilde{u}\|_{L^\infty(B_R \times (\varepsilon, T-\varepsilon))} = 0$$

for all $R > 0$ and $\varepsilon > 0$. Clearly

$$\lim_{j\to\infty} \left\|e^{\cdot\Delta} u_{0,k_j} - e^{\cdot\Delta} u_0\right\|_{L^\infty(B_R \times (\varepsilon, T-\varepsilon))} = 0.$$

Moreover,

$$\lim_{j\to\infty} \left\| \int_0^{\cdot} \mathrm{div} e^{(\cdot-s)\Delta} \mathbf{G}(u_{k_j}(s)) ds - \int_0^{\cdot} \mathrm{div} e^{(\cdot-s)\Delta} \mathbf{G}(\tilde{u}(s)) ds \right\|_{L^\infty(B_R \times (\varepsilon, T-\varepsilon))} = 0.$$

Thus we conclude that $\tilde{u}$ satisfies

$$\tilde{u}(t) = e^{t\Delta} u_0 - \int_0^t \mathrm{div} e^{(t-s)\Delta} \mathbf{G}(\tilde{u}(s)) ds$$

in $\mathbf{R}^n \times (0, T_0)$. We define

$$u(x,t) = \begin{cases} \tilde{u}(x,t) & (t > 0), \\ u_0(x) & (t = 0), \end{cases}$$

then $u \in L^\infty_{loc}([0,T_0); BC_\alpha)$ satisfies

$$u(t) = e^{t\Delta}u_0 - \int_0^t \mathrm{div} e^{(t-s)\Delta}\mathbf{G}(u(s))ds$$

in $\mathbf{R}^n \times [0,T_0)$, and $u \in C(\mathbf{R}^n \times [0,T_0))$, since $\{u_k\}$ satisfies the integral equation. We thus conclude u is solution of (1.1).

$\square$

4 Uniqueness of solution

In this section we prove the uniqueness of the solution of (1.1). Before proving the uniqueness, we define several energies.

Definition 4.1. *Let $\psi(x) = e^{-|x|^2}$. We define E^I_R, E^O_R, E:*

$$E^I_R(f) := \int_{|x|\le R} \psi(x)|f(x)|^2 dx,$$

$$E^O_R(f) := \int_{|x|\ge R} \psi(x)|f(x)|^2 dx,$$

$$E(f) := \int_{\mathbf{R}^n} \psi(x)|f(x)|^2 dx.$$

Lemma 4.2 (Derivative estimate). *Assume that $\mathbf{G} \in C^{1+\theta}(\mathbf{R};\mathbf{R}^n)$ satisfies bounds (1.2). Assume that $u \in L^\infty((0,T);L^\infty_\alpha)$ is solution of (1.1) for some $\alpha > 0$. Then $E(|\nabla u(t)|) \in L^1_{loc}(0,T)$.*

Proof. The function u is solution of (1.1) so that u satisfies integral equation:

$$u(t) = e^{t\Delta}u_0 - \int_0^t \mathrm{div} e^{(t-s)\Delta}\mathbf{G}(u(s))ds.$$

By the proof of Lemma 3.3 u is Hölder continuous function and ∇u satisfies

$$\begin{aligned}\nabla u(x,t) \quad &= \int_{\mathbf{R}^n} (\nabla G_t)(x-y)u_0(y)dy \\ &\quad + \int_0^t \int_{\mathbf{R}^n} (\nabla \mathrm{div} G_{t-s})(x-y)(\mathbf{G}(u(x,s)) - \mathbf{G}(u(y,s)))dyds.\end{aligned} \tag{4.1}$$

By Lemma 2.1 the first term of the right hand of (4.1) fulfills following inequality:

$$\left\| \int_{\mathbf{R}^n} (\nabla G_t)(x-y)u_0(y)dy \right\|_\alpha \le Ct^{-1/2}\|u_0\|_\alpha,$$

where C depends only on n, α.

By assumption of $\mathbf{G}$ and the proof of the Lemma 3.3, $|\mathbf{G}(u(x,s)) - \mathbf{G}(u(y,s))|$ satisfies

$$|\mathbf{G}(u(x,s)) - \mathbf{G}(u(y,s))| \leq Cs^{-\delta}\langle |x| + |y|\rangle^{\alpha\beta+\alpha+1}|x-y|^{\delta},$$

for all $0 < \delta \leq 1/2$ where C depends only on α, β, δ, n, $\sup_{0\leq t\leq T}\|u(t)\|_{\alpha}$, $\|\mathbf{G}\|_{\beta+1}$. Thus the second term of the right hand side of (4.1) is dominated by

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbf{R}^n} (\nabla \mathrm{div} G_{t-s})(x-y)(\mathbf{G}(u(x,s)) - \mathbf{G}(u(y,s)))dyds\right| \\
\leq\ & C\int_0^t \int_{\mathbf{R}^n} \frac{|x-y|^2 + (t-s)}{(t-s)^2} G_{t-s}(x-y)s^{-\delta}\langle |x| + |y|\rangle^{\alpha\beta+\alpha+1}|x-y|^{\delta}dyds \\
\leq\ & C\int_0^t \int_{\mathbf{R}^n} \frac{|x-y|^2 + (t-s)}{(t-s)^2} G_{t-s}(x-y)s^{-\delta}\langle |x| + |y|\rangle^{\alpha\beta+\alpha+1}|x-y|^{\delta}dyds \\
=\ & C\int_0^t \int_{\mathbf{R}^n} \frac{1}{(t-s)^{1-\delta}} e^{-|z|^2}s^{-\delta}\left\langle |x| + \left|x - 2(t-s)^{1/2}z\right|\right\rangle^{\alpha\beta+\alpha+1} dzds \\
\leq\ & C\langle x\rangle^{\alpha\beta+\alpha+1}\int_0^t \frac{s^{-\delta}}{(t-s)^{1-\delta}}ds
\end{aligned}$$

for all $0 < \delta \leq 1/2$, where $z = (x-y)/[2(t-s)^{1/2}]$, C depends only on α, β, δ, n, T, $\sup_{0\leq t\leq T}\|u(t)\|_{\alpha}$, $\|\mathbf{G}\|_{\beta+1}$. Combining these results, we have

$$|\nabla u(x,t)| \leq Ct^{-1/2}\langle x\rangle^{\alpha\beta+\alpha+1},$$

where C depends only on α, β, δ, n, T, $\sup_{0\leq t\leq T}\|u(t)\|_{\alpha}$, $\|\mathbf{G}\|_{\beta+1}$. Thus we conclude that $E(|\nabla u(t)|) \in L^1_{loc}(0,T)$. □

Theorem 4.3 (Uniqueness of classical solution of (1.1)). *Assume that* $\mathbf{G} \in C^{1+\theta}(\mathbf{R};\mathbf{R}^n)$ *satisfies bounds (1.2). Assume that* $u_0 \in BC_{\alpha}$. *Assume that* u, $v \in L^{\infty}_{loc}([0,T);L^{\infty}_{\alpha})$ *are classical solutions of (1.1). If* $\alpha\beta \leq 1$ *then* $u = v$ *in* $\mathbf{R}^n \times [0,T)$.

Proof. We set $w = u - v$. Since u, $v \in L^{\infty}_{loc}([0,T);L^{\infty}_{\alpha}) \cap C(\mathbf{R}^n \times [0,T))$ are solution of (1.1) following identity holds:

$$\int_{\varepsilon}^t \int_{\mathbf{R}^n} \psi w\left(\partial_t w - \Delta w + \mathrm{div}(\mathbf{G}(u) - \mathbf{G}(v)\right) dxds = 0,$$

for all $t \leq T$. By Lemma 4.2, $E(|\nabla u(t)|)$, $E(|\nabla v(t)|) \in L^1_{loc}(0,T)$ so we have

$$\int_{\varepsilon}^t \int_{\mathbf{R}^n} \frac{1}{2}\partial_t(\psi w^2) + \nabla(\psi w)\cdot\nabla w - \nabla(\psi w)\cdot(\mathbf{G}(u) - \mathbf{G}(v))dxds = 0.$$

This yields

$$
\begin{aligned}
&\frac{1}{2}(E(w(t)) - E(w(\varepsilon))) + \int_\varepsilon^t \int_{\mathbf{R}^n} \psi |\nabla w|^2 dxds \\
= &\int_\varepsilon^t \int_{\mathbf{R}^n} w \nabla w \cdot \nabla \psi + (w \nabla \psi + \psi \nabla w) \cdot (\mathbf{G}(u) - \mathbf{G}(v)) dxds
\end{aligned} \tag{4.2}
$$

for all $\varepsilon > 0$.

By assumption $\mathbf{G}$ satisfies

$$|\mathbf{G}(r_1) - \mathbf{G}(r_2)| \leq C_1 |r_1 - r_2|(1 + |r_1|^\beta + |r_2|^\beta),$$

where $C_1 = C_1(\|\mathbf{G}'\|_\beta)$. Since $|u(x,t)|$, $|v(x,t)| \leq M_1 \langle x \rangle^\alpha$ for all $(x,t) \in \mathbf{R}^n \times [0, T_0]$ where M_1 depends only on T_0. Then we have

$$|\mathbf{G}(u) - \mathbf{G}(v)| \leq M_2 |w| \langle x \rangle,$$

where M_2 depends only on T_0, C_1. Then (4.2) becomes

$$
\begin{aligned}
&\frac{1}{2}(E(w(t)) - E(w(\varepsilon))) + \int_\varepsilon^t \int_{\mathbf{R}^n} \psi |\nabla w|^2 dxds \\
\leq &\int_\varepsilon^t \int_{\mathbf{R}^n} |w||\nabla w||\nabla \psi| + M_2(|w||\nabla \psi| + |\psi||\nabla w|) w \langle x \rangle dxds.
\end{aligned} \tag{4.3}
$$

The right hand side of (4.3) fulfills following inequalities:

$$
\begin{aligned}
&\int_\varepsilon^t \int_{\mathbf{R}^n} |w||\nabla w||\nabla \psi| + M_2(|w||\nabla \psi| + |\psi||\nabla w|) w \langle x \rangle dxds \\
\leq &\int_\varepsilon^t \int_{\mathbf{R}^n} 2\langle x \rangle |w||\nabla w| \psi + 2M_2 \langle x \rangle^2 |w|^2 \psi + M_2 |w||\nabla w| \psi dxds \\
\leq &\int_\varepsilon^t \int_{\mathbf{R}^n} \frac{1}{2} M_3 \langle x \rangle^2 |w|^2 \psi + |\nabla w|^2 \psi dxds.
\end{aligned}
$$

where M_3 depends only on T_0, C_1. The estimate (4.3) now becomes

$$E(w(t)) - E(w(\varepsilon)) \leq M_3 \int_\varepsilon^t E(\langle x \rangle w(s)) ds.$$

Let ε tend to 0, then we have

$$E(w(t)) \leq M_3 \int_0^t E(\langle x \rangle w(s)) ds. \tag{4.4}$$

On the other hand

$$
\begin{aligned}
E(\langle x \rangle w(s)) = &\ E_R^I(\langle x \rangle w(s)) + E_R^O(\langle x \rangle w(s)) \\
\leq &\ \langle R \rangle^2 E_R^I(w(s)) + e^{-|R|^2/2} \\
\leq &\ \langle R \rangle^2 E(w(s)) + e^{-|R|^2/2}
\end{aligned}
$$

for sufficiently large R. Thus (4.4) becomes

$$E(w(t)) \leq e^{-|R|^2/2} + M_4\langle R\rangle^2 \int_0^t E(w(s))ds.$$

By using Gronwall inequality

$$E(w(t)) \leq e^{-|R|^2/2+M_4 t\langle R\rangle^2}.$$

Sending $R \to \infty$ yields

$$E(w(t)) \to 0$$

for all $0 \leq t < 1/(2M_4)$.

Thus the proof is now complete.

□

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