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Interpolation Of Weighted $\ell^q$ Sequences By $H^p$ Functions

by

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Abstract. Let \((z_n)\) be a sequence of points in the open unit disc \(D\) and \(\rho_n = \prod_{m \neq n} |(z_n - z_m)(1 - \overline{z_m}z_n)^{-1}| > 0\). Let \(a = (a_j)_{j=1}^{\infty}\) be a sequence of positive numbers and \(\ell^s(a) = \{(w_j); (a_jw_j) \in \ell^s\}\) where \(1 \leq s \leq \infty\). When \(1 \leq p \leq \infty\) and \(1/p + 1/q = 1\), we show that \(\{(f(z_n)); f \in H^p\} \supset \ell^s(a)\) if and only if there exists a finite positive constant \(\gamma\) such that \(\left\{\sum_{n=1}^{\infty} (a_n\rho_n)^{-t}(1 - |z_n|^2)^t|f(z_n)|^t\right\}^{1/t} \leq \gamma\|f\|_q\) (\(f \in H^q\)), where \(1/s + 1/t = 1\). As results, we show that \(\{(f(z_j)); f \in H^p\} \supset \ell^1(a)\) if and only if \(\sup(a_n\rho_n)^{-1}(1 - |z_n|^2)^{1/p} < \infty\), and \(\{(f(z_n)); f \in H^1\} \supset \ell^\infty(a)\) if and only if \(\sum_{n=1}^{\infty} (a_n\rho_n)^{-1}(1 - |z_n|^2)\delta_{z_n}\) is finite measure on \(D\). These are also proved in the case of weighted Hardy spaces.
§1. Introduction

$H^p$ ($0 < p \leq \infty$) denotes the usual Hardy space in the open unit disc $D$. In this paper, we assume that a sequence $(z_j)$ in $D$ satisfies that $\sum_{j=1}^{\infty} (1 - |z_j|) < \infty$, that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \frac{z - z_j}{|z_j|}.$$  

Let

$$\rho_{k,n} = \prod_{j=1}^{n} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|, \quad 1 \leq k \leq n,$$

$$\rho_k = \prod_{j=1}^{\infty} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|.$$  

Then $\rho_{k,n} \geq \rho_{k,n+1}$ and $\lim_{n \to \infty} \rho_{k,n} = \rho_k$ for $k \geq 1$. We assume that $\rho_k > 0$ for $k = 1, 2, \cdots$.

For a positive sequence $a = (a_j)$, $\ell^s(a)$ denotes $\{(w_j) ; w_j \in \mathcal{C}$ and $\sum_{j=1}^{\infty} (a_j |w_j|)^s < \infty\}$ and $\ell^\infty(a)$ denotes $\{(w_j) ; w_j \in \mathcal{C}$ and $\sup_{1 \leq j < \infty} a_j |w_j| < \infty\}$. In this paper, we study the following problem: Find a necessary and sufficient condition on $(z_j)$ so that $\{(f(z_j)) ; f \in H^p\} \supset \ell^s(a)$ where $1 \leq p \leq \infty$ and $1 \leq s \leq \infty$.

Suppose $a_j = 1$ for all $j \geq 1$. When $p = s = \infty$, this was solved by L. Carleson [1]. That is, $\{(f(z_j)) ; f \in H^\infty\} \supset \ell^\infty$ if and only if $\inf_{j} \rho_j > 0$. (z_j) is called a uniformly separated sequence when $\inf \rho_j > 0$. When $p = \infty$ and $1 \leq s < \infty$, A. K. Snyder [13] (cf. [7], [11]) proved that $\{(f(z_j)) ; f \in H^\infty\} \supset \ell^s$ if and only if $\inf \rho_j > 0$.

A. K. Snyder [13] and P. L. Duren and H. S. Shapiro [3] showed that there exists a sequence $(z_j)$ which is not uniformly separated, that is, $\inf \rho_j = 0$ and has the property: $\{(f(z_j)) ; f \in H^p\} \supset \ell^\infty$ when $p \neq \infty$. B. A. Taylor and D. L. Williams [14] showed that for $1 \leq p \leq \infty$ $\{(f(z_j)) ; f \in H^p\} \supset \ell^\infty$ and only if there exists a positive finite constant $\gamma$ such that $\sum_{j=1}^{\infty} \frac{1}{\rho_j} (1 - |z_j|^2)|g(z_j)| \leq \gamma \| g \|_q$ for all $g \in H^q$ and $1/p + 1/q = 1$.

Suppose $1 \leq p = s \leq \infty$. When $a_j = (1 - |z_j|^2)^{1/p}$ for all $j \geq 1$, this was solved by H. S. Shapiro and A. L. Shields [11]. That is, $\{(f(z_j)) ; f \in H^p\} \supset \ell^p(a)$ if and only if $\inf \rho_j > 0$. When $a_j = \rho_j^2$ for all $j \geq 1$, J. P. Earl [4] showed that $\{(f(z_j)) ; f \in H^\infty\}$ contains $\ell^\infty(a)$ always. This was pointed out by A. M. Gleason (see [6]). On the other hand, when $a_j = \rho_j$ for all $j \geq 1$, T. Nakazi [10] showed that $\{(f(z_j)) ; f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $(z_j)$ is the union of a finite number of uniformly separated sequences. For a general weight $a = (a_j)$, J. D. McPhail [9] gave a necessary and sufficient condition about
sequences. This is a generalization of a result in [10] for\( H^p \) spaces. If we put\( \sup_j (a_j \rho_j)^{-t} (1 - |z_j|^2)^t < \infty \)
where \( 1/s + 1/t = 1 \). Moreover, when \( 1 < p \leq \infty \) and \( a = (\rho_j^{-1})_j \), we show that \( \{ (f(z_j)) ; f \in H^p \} \supset \ell^p(a) \) if and only if \( (f(z_j)) ; f \in H^p \) is a finite sum of uniformly separated sequences. This is a generalization of a result in [10] for \( p = \infty \).

In §3, when \( 1 \leq p \leq \infty \), we show that \( \{ (f(z_j)) ; f \in H^p \} \supset \ell^1(a) \) if and only if \( \sup (a_j \rho_j)^{-1} (1 - |z_j|^2)^{1/p} < \infty \). As a result, a theorem of A. K. Snyder [13] follows, that is, \( \{ (f(z_j)) ; f \in H^\infty \} \supset \ell^s \) if and only if \( \inf_j \rho_j > 0 \).

In §4, we give a necessary and sufficient condition about \( (z_j) \) for that \( \{ (f(z_j)) ; f \in H^p \} \supset \ell^\infty(a) \). Put \( \mu = \sum_{j=1}^\infty \rho_j^{-1} \delta_{z_j} \). Then \( \{ (f(z_j)) ; f \in H^1 \} \supset \ell^\infty(a) \) if and only if \( \mu \) is a finite measure on \( D \), and \( \{ (f(z_j)) ; f \in H^\infty \} \supset \ell^\infty(a) \) if and only if \( \mu \) is a Carleson measure on \( D \).

In §5, we give a necessary and sufficient condition about \( (z_j) \) for that \( \{ (s(z_j)f(z_j)) ; f \in H^p(W) \} \supset \ell^p \), where \( H^p(W) \) is a weighted Hardy space and \( s(z_j) = \inf \left\{ \int |f|^p Wd\theta/2\pi ; f(z_j) = 1 \right\} \). We assume only that \( \log W \) is in \( L^1 \). J.D. McPhail [9] studied such a problem when \( W \) satisfies the \((A_p)\)-condition of Muckenhoupt.

Our interests in this paper are in the differences between interpolations for \( \ell^1(a) \) and \( \ell^\infty(a) \) and in the interpolation problems for weighted Hardy spaces. For example, it is very easy to prove that \( \{ (f(z_j)) ; f \in H^\infty \} \supset \ell^1 \) if and only if \( \{ z_j \} \) is uniformly separated.

§2. General results

In this section, we obtain a general result for interpolation problems for \( \ell^s(a) \) (\( 1 \leq s \leq \infty \)) by \( H^p \) (\( 1 \leq p \leq \infty \)). For \( 1 \leq j \leq n \), let

\[
B_n(z) = \prod_{j=1}^n \frac{z - z_j}{1 - \bar{z}_j z} \quad \text{and} \quad B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}.
\]

If we put \( b_{nj} = B_{nj}(z_j) \), then

\[
\rho_{j,n} = |b_{nj}| \quad (1 \leq j \leq n).
\]

Suppose for \( n = 1, 2, \cdots \)

\[
f_n(z) = \sum_{j=1}^n b_{nj}^{-1} w_j B_{nj}(z).
\]
Then $f_n$ is in $H^\infty$ and $f_n(z_j) = w_j$ ($1 \leq j \leq n$). Lemma 1 is essentially known.

**Lemma 1.** Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Suppose $w_j$ is a complex number for $j = 1, 2, \cdots$. There exists a function $f$ in $H^p$ such that $f(z_j) = w_j$ for $j = 1, 2, \cdots$ if and only if there exists a positive finite constant $\gamma$ such that for any $n \geq 1$ and for all $g$ in $H^q$,

$$\left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma \| g \|_q.$$  

Proof. Put for $n \geq 1$

$$m_{p,n}(w) = \inf \{ \| f_n + B_n h \|_p ; h \in H^p \}.$$  

Then by [2, p142],

$$m_{p,n}(w) = \sup \left\{ \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) ; g \in H^q \text{ and } \| g \|_q \leq 1 \right\}.$$  

There exists a function $f$ in $H^p$ such that $f(z_j) = w_j$ for $j = 1, 2, \cdots$ if and only if $\sup_{n} m_{p,n}(w) < \infty$ because the unit ball of $H^p$ is compact in the weak topology or the weak * topology. This implies the lemma.

**Theorem 1.** Let $1 \leq p \leq \infty$ and $1 \leq s \leq \infty$. \{(f(z_n)) ; f \in H^p \} \supset \ell^s(a)$ if and only if there exists a finite positive constant $\gamma$ such that

$$\left\{ \sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2) \left| f(z_n) \right|^{t} \right\}^{1/t} \leq \gamma \| f \|_q$$

for $f$ in $H^q$, where $1/p + 1/q = 1$ and $1/s + 1/t = 1$.

Proof. For the ‘only if’ part, since \{(f(z_j)) ; f \in H^p \} \supset \ell^s(a)$, by Lemma 1 there exists a positive finite constant $\gamma$ such that for any $n \geq 1$

$$\sup_{w \in \ell^s(a)} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma \| g \|_q \text{ (} g \in H^q \text{)}.$$  

where $w = (w_j)$ and $\| w \| = \left( \sum_{j=1}^{\infty} |w_j a_j|^s \right)^{1/s}$. Hence for any $n \geq 1$

$$\left\{ \sum_{j=1}^{\infty} (a_j \rho_{nj})^{-t} (1 - |z_j|^2) \left| g(z_j) \right|^{t} \right\}^{1/t} \leq \gamma \| g \|_q \text{ (} g \in H^q \text{)}.$$

Assuming $\| g \|_q = 1$,

$$\sum_{j=1}^{\infty} (a_j \rho_{nj})^{-t} (1 - |z_j|^2) \left| g(z_j) \right|^{t} \leq \gamma^{t}.$$  

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For any $\varepsilon > 0$, there exists a positive integer $n_j$ for each $j$ such that for all $n \geq n_j$
\[(a_j \rho_{j,n})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \frac{\varepsilon}{2^j} \leq (a_j \rho_{j,n})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t\]
because $\rho_{j,n} \geq \rho_{j,n+1}$ and $\lim_{n \to \infty} \rho_{j,n} = \rho_j$. Thus, $\{(f(z_j)) ; f \in H^p \supset \ell^s(a)\}$ if and only if
\[\sum_{j=1}^{n} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \varepsilon \leq \sum_{j=1}^{n} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \leq \gamma^t\]
This implies the ‘only if’ part.

For the ‘if’ part, by Lemma 1 it is sufficient to show that there exists a finite positive constant $\gamma$ such that for all $n \geq 1$
\[
\sup_{\|w\| \leq 1} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma < \infty.
\]
In fact, for all $n \geq 1$
\[
\sup_{\|w\| \leq 1} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \\
\leq \sup_{\|g\|_q \leq 1} \left\{ \sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} \\
\leq \sup_{\|g\|_q \leq 1} \left\{ \sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} < \infty
\]

**Corollary 1.** Let $1 \leq s \leq \infty$. $\{(f(z_n)) ; f \in H^1 \supset \ell^s(a)\}$ if and only if
\[\sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2)^t < \infty\]
where $1/s + 1/t = 1$. Hence, when $a = (a_n) = (\rho_n^{-1})$ it is always true that $\{(f(z_n)) ; f \in H^1 \supset \ell^s(a)\}$.

**Proof.** The first part is clear by Theorem 1. When $a = (\rho_n^{-1})$, $\{(f(z_n)) ; f \in H^1 \supset \ell^s(a)\}$ if and only if $\sum_{n=1}^{\infty} (1 - |z_n|^2)^t < \infty$. This implies the second part.

**Corollary 2.** Let $1 \leq p \leq \infty$, $1 \leq s \leq \infty$ and $a = (\rho_n^{-1})$. $\{(f(z_n)) ; f \in H^p \supset \ell^s(a)\}$ if and only if there exists a finite positive constant $\gamma$ such that
\[
\left\{ \sum_{n=1}^{\infty} (1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q
\]
for \( f \) in \( H^q \), where \( 1/p + 1/q = 1 \) and \( 1/s + 1/t = 1 \). When \( 1 < p \leq \infty \), \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^p(a) \) if and only if \( (z_n) \) is a finite sum of uniformly separated sequences.

Proof. The first part is clear by Theorem 1. The second part follows from the first one and [8].

In Corollary 2, when \( 1 < p \leq \infty \) and \( 1 < s \leq \infty \) and \( s > p \), if \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^s(a) \) then \( (z_n) \) is a finite sum of uniformly separated sequences but the converse is not true. When \( s < p \), if \( (z_n) \) is a finite sum of uniformly separated sequences then \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^s(a) \) but the converse is not true.

\[ \text{§3. Interpolations for } \ell^1(a) \]

\( \ell^1(a) \) is the smallest sequence space among \( \ell^p(a) \) \( (1 \leq p \leq \infty) \) for the same \( a = \{a_j\} \). Then the interpolations for \( \ell^1(a) \) are very special as the following shows.

The case of \( p = \infty \) in Corollary 3 was proved by A.Snyder [13] (see [7], [11]). Corollary 4 is due to O. Hatari [7].

**Theorem 2.** Let \( 1 \leq p \leq \infty \). \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^1(a) \) if and only if

\[
\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2)^{1/p} < \infty.
\]

Proof. By Theorem 1, \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^1(a) \) if and only if there exists a finite positive constant \( \gamma \) such that

\[
\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2) |f(z_n)| \leq \gamma \|f\|_q
\]

for all \( f \) in \( H^q \). For each \( n \), \( \sup_{\|f\|_q = 1} |f(z_n)| = (1 - |z_n|^2)^{-1/q} \) by [2, p144] and so the theorem follows.

**Corollary 3.** Let \( 1 \leq p \leq \infty \). \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^1 \) if and only if \( \sup_n (1 - |z_n|^2)^{1/p} < \infty \). Hence if \( p = \infty \), \( \{(f(z_n)) ; \ f \in H^\infty\} \supset \ell^1 \) if and only if \( \inf_n \rho_n > 0 \).

**Corollary 4.** Let \( 1 \leq p \leq \infty \). \( \{(1 - |z_n|^2)^{1/p} f(z_n) ; \ f \in H^p\} \supset \ell^1 \) if and only if \( \inf_n \rho_n > 0 \).

Proof. Note that \( \{(1 - |z_n|^2)^{1/p} f(z_n) ; \ f \in H^p\} \supset \ell^1 \) if and only if \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^1(a) \) and \( a = ((1 - |z_n|^2)^{1/p}) \).

**Corollary 5.** Let \( 1 \leq p \leq \infty \). For any \( (z_n) \), \( \{(f(z_n)) ; \ f \in H^p\} \supset \ell^1(a) \) where

\( a = (\rho_n^{-1}) \).
Let \((b_j)\) be a uniformly separated sequence in \(D\) such that \(0 < \text{Re}b_j \neq 1\) and \(\text{Im} b_j \neq 0\). For \(j \geq 1\), put \(z_{2j-1} = b_j\) and \(z_{2j} = b_j\). Let \(B\) be the Blaschke product associated with \(\{z_n\}\). Then for each \(j\)

\[
B = \frac{z - b_j}{1 - b_jz} \frac{z - \bar{b}_j}{1 - b_jz} B_{1j} B_{2j}
\]

where \(B_{1j}\) (or \(B_{2j}\)) is a Blaschke product with zeros \(\{b_{\ell}\}_{\ell \neq j}\) (or \(\{\bar{b}_{\ell}\}_{\ell \neq j}\)). Then

\[
\rho_{2j-1} = \left| b_j - \bar{b}_j \right| \prod_{\ell \neq j} \left| b_j - b_{\ell} \right| \prod_{\ell \neq j} \left| b_{\ell} - \bar{b}_j \right| \prod_{\ell \neq j} \left| b_{\ell} - b_{\ell} \right|
\]

and

\[
\rho_{2j} = \left| \bar{b}_j - b_j \right| \prod_{\ell \neq j} \left| \bar{b}_j - b_{\ell} \right| \prod_{\ell \neq j} \left| b_{\ell} - \bar{b}_j \right| \prod_{\ell \neq j} \left| b_{\ell} - b_{\ell} \right|
\]

Hence \(\rho_{2j-1} = \rho_{2j}\) for \(j \geq 1\) and

\[
\frac{\delta^2 \left| \bar{b}_j - b_j \right|}{1 - \left| b_j \right|^2} \leq \rho_{2j} = \rho_{2j-1} \leq \frac{\left| \bar{b}_j - b_j \right|}{1 - \left| b_j \right|^2} \quad (j \geq 1)
\]

where

\[
0 < \delta = \min \left\{ \inf_j \prod_{\ell \neq j} \left| \frac{b_j - b_{\ell}}{1 - b_{\ell}b_j} \right|, \inf_j \prod_{\ell \neq j} \left| \frac{b_{\ell} - \bar{b}_j}{1 - b_{\ell}b_j} \right| \right\}.
\]

Hence

\[
\frac{(1 - \left| z_n \right|^2)^{1+1/p}}{\left| z_n - \bar{z}_n \right|} \leq \frac{(1 - \left| z_n \right|^2)^{1+1/p}}{\rho_n} \leq \delta^2 \frac{(1 - \left| z_n \right|^2)^{1+1/p}}{\left| z_n - \bar{z}_n \right|}.
\]

Thus \(\{(f(z_n)) : f \in H^p\} \supset \ell^1\) if and only if \(\sup_n (1 - \left| z_n \right|^2)^{1+1/p}/|z_n - \bar{z}_n| < \infty\).

§4. Interpolations for \(\ell^\infty(a)\)

\(\ell^\infty(a)\) is the largest sequence space among \(\ell^p(a)\) \((1 \leq p \leq \infty)\) for the same \(a = (a_j)\). Then the interpolations for \(\ell^\infty(a)\) are special as the following shows. The case of \(p = \infty\) of Corollary 6 is known in [10].

**Theorem 3.** Let \(1 \leq p \leq \infty\) and \(1/p + 1/q = 1\), \(\{(f(z_n)) : f \in H^p\} \supset \ell^\infty(a)\) if and only if there exists a finite positive constant \(\gamma\) such that

\[
\sum_n (a_n \rho_n)^{-1}(1 - \left| z_n \right|^2)|f(z_n)| \leq \gamma \|f\|_q
\]

for all \(f\) in \(H^q\). When \(p = 1\), \(\{(f(z_n)) : f \in H^1\} \supset \ell^\infty(a)\) if and only if \(\mu = \sum_n (a_n \rho_n)^{-1}(1 - \left| z_n \right|^2)\delta_{z_n}\) is a finite measure on \(D\). When \(p = \infty\), \(\{(f(z_n)) : f \in H^\infty\} \supset \ell^\infty(a)\) if and only if \(\mu = \sum_n (a_n \rho_n)^{-1}(1 - \left| z_n \right|^2)\delta_{z_n}\) is a Carleson measure on \(D\).
Corollary 6. Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$ and $a = (p_n^{-1})$. \(\{ (f(z_n)) ; \ f \in H^p \} \supset \ell^\infty(a)\) if and only if there exists a finite positive constant $\gamma$ such that
\[
\sum_n (1 - |z_n|^2)|f(z_n)| \leq \gamma \| f \|_q
\]
for all $f$ in $H^q$.

(1) When $p = 1$, for any $(z_n)$, \(\{ (f(z_n)) ; \ f \in H^1 \} \supset \ell^\infty(a)\).

(2) When $p = \infty$, \(\{ (f(z_n)) ; \ f \in H^\infty \} \supset \ell^\infty(a)\) if and only if $(z_n)$ is a finite union of uniformly separated sequences.

(3) When $1 < p < \infty$, there exists a sequence $(z_n)$ in $D$ such that \(\{ (f(z_n)) ; \ f \in H^p \} \supset \ell^\infty(a)\) and $(z_n)$ is not a union of finitely many uniformly separated sequences. If $\sum_{n=1}^\infty (1 - |z_n|^2)^{1/p} < \infty$, then \(\{ (f(z_n)) ; \ f \in H^p \} \supset \ell^\infty\). This was proved by B. A. Taylor and D. L. Williams [14].

§5. Weighted Hardy space

Let $W$ be a nonnegative function in $L^1$ with $\log W \in L^1$ and $1 \leq p < \infty$. $H^p(W)$ denotes the closure of the set of all analytic polynomials in $L^p(W) = L^p(Wd\theta/2\pi)$. $H^p(W)$ is called a weighted Hardy space. For $b \in D$, put
\[
s(b) = s(b, p, W) = \inf \left\{ \int |f|^p Wd\theta/2\pi ; \ f(b) = 1 \right\}.
\]
Let $h$ be an outer function in $H^p$ such that $|h|^p = W$.

Lemma 2. For $1 \leq p < \infty$ and $b \in D$,
\[
s(b, p, W) = (1 - |b|^2) \exp(\log W)^\sim(b)
\]
where $(\log W)^\sim(b)$ denotes the Poisson integral of $\log W$ at $b$.

Proof. It is well known (cf. [5, p136]) that \(s(0,p,W) = \exp \int_0^{2\pi} \log W d\theta/2\pi\). It is easy to show the lemma using $f(b) = f \circ \phi_0(0)$, where $\phi_0(z) = (z + b)/(1 + \bar{b}z)$.

Lemma 3. Suppose $(z_j)$ is a sequence of points in $D$. For $1 \leq p < \infty$ and $1 \leq s < \infty$, \(\{ (s(z_j, p, W))^{1/p} f(z_j) ; \ f \in H^p(W) \} \supset \ell^s(a)\) if and only if \(\{ (F(z_j)) ; \ F \in H^p \} \supset \ell^s(a)\), where $a = (a_j)$ and $a_j = s(z_j)^{1/p}/h(z_j)$.
Proof. Since $H^p(W) = h^{-1}H^p$, $f \in H^p(W)$ if and only if $f = h^{-1}F$ and $F \in H^p$. For each $f$, $s(z_j)^{1/p}f(z_j) = w_j$ if and only if $F(z_j) = h(z_j)w_j/s(z_j)^{1/p}$ if and only if $F(z_j) = \zeta_j$, $w_j = a_j\zeta_j$. $(w_j) \in \ell^p$ if and only if $(\zeta_j) \in \ell^p(a)$. Now the lemma follows.

**Theorem 4.** Let $1 \leq p < \infty$, $1 \leq s \leq \infty$, and $1/p + 1/q = 1/s + 1/t = 1$. Then, \( \{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^s \) if and only if
\[
\left\{ \sum_{n=1}^{\infty} \frac{1}{\rho_n^t} s(z_n)^{t/q} |g(z_n)|^t \right\}^{1/t} \leq \gamma \|g\|_{H^q(W)}
\]
for $g$ in $H^q(W)$.

Proof. By Lemma 3, \( \{(s(z_n)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^s \) if and only if \( \{(F(z_n)) ; F \in H^p\} \supset \ell^s(a) \), where $a_n = s(z_n)^{1/p}/|h(z_n)|$. By Theorem 1, this is equivalent to saying that there exists a finite positive constant $\gamma$ such that
\[
\left\{ \sum_{n=1}^{\infty} \frac{1}{\rho_n^t} \frac{1}{a_n^t} (1 - |z_n|^2)^t |G(z_n)|^t \right\}^{1/t} \leq \gamma \|G\|_q
\]
for $G \in H^q$. Since $H^q(W) = h^{-p/q}H^q$, $g \in H^q(W)$ if and only if $g = h^{-p/q}G$ and $G \in H^q$. Hence $\|g\|_{H^q(W)} = \|G\|_{H^q}$ and for each $n \geq 1$
\[
a_n^{-t}(1 - |z_n|^2)^t |G(z_n)|^t = s(z_n)^{-t(p/q)} |h(z_n)|^t(1 - |z_n|^2)^t |h(z_n)|^{pt/q} |g(z_n)|^t
\]
\[
= s(z_n)^{-t(p/q)} (1 - |z_n|^2)^t |h(z_n)|^{t(q/p)} |g(z_n)|^t
\]
\[
= s(z_n)^{-t(p/q)} |g(z_n)|^t
\]
This implies the theorem.

**Corollary 7.** Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then \( \{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^1 \) if and only if $\inf_{n \rho_n > 0}$.

Proof. By Theorem 4, \( \{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^1 \) if and only if
\[
\left\{ \sup_{n} \frac{1}{\rho_n} s(z_n,p,W)^{1/p}s(z_n,q,W)^{-1/q} \right\} < \infty.
\]
Now Lemma 2 implies the corollary.

**References**


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