Interpolation Of Weighted $\ell^q$ Sequences By $H^p$ Functions

by

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**Abstract**. Let $(z_n)$ be a sequence of points in the open unit disc $D$ and $\rho_n = \prod_{m\neq n} |(z_n - z_m)(1-\bar{z}_m z_n)^{-1}| > 0$. Let $a = (a_j)_{j=1}^{\infty}$ be a sequence of positive numbers and $\ell^s(a) = \{(w_j)\ ;\ (a_j w_j) \in \ell^s\}$ where $1 \leq s \leq \infty$. When $1 \leq p \leq \infty$ and $1/p + 1/q = 1$, we show that $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^s(a)$ if and only if there exists a finite positive constant $\gamma$ such that $\left\{\sum_{n=1}^{\infty} (a_n\rho_n)^{-t}(1-|z_n|^2)^t |f(z_n)|^t\right\}^{1/t} \leq \gamma \|f\|_q$ $(f \in H^q)$, where $1/s + 1/t = 1$. As results, we show that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^1(a)$ if and only if $\sup_n (a_n\rho_n)^{-1}(1-|z_n|^2)^{1/p} < \infty$, and $\{(f(z_n))\ ;\ f \in H^1\} \supset \ell^{\infty}(a)$ if and only if $\sum_n (a_n\rho_n)^{-1}(1-|z_n|^2)\delta_{z_n}$ is finite measure on $D$. These are also proved in the case of weighted Hardy spaces.

## §1. Introduction

$H^p$ $(0 < p \le \infty)$ denotes the usual Hardy space in the open unit disc $D$. In this paper, we assume that a sequence $(z_j)$ in $D$ satisfies that $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$, that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^{\infty} -\frac{\bar{z}_j}{|z_j|}\,\frac{z-z_j}{1-\bar{z}_j z}.$$

Let

$$\begin{aligned}
\rho_{k,n} &= \prod_{\substack{j=1\\ j\neq k}}^{n} \left|\frac{z_k-z_j}{1-\bar{z}_j z_k}\right|, \ 1 \le k \le n, \\
\rho_k &= \prod_{\substack{j=1\\ j\neq k}}^{\infty} \left|\frac{z_k-z_j}{1-\bar{z}_j z_k}\right|.
\end{aligned}$$

Then $\rho_{k,n} \ge \rho_{k,n+1}$ and $\lim_{n\to\infty}\rho_{k,n} = \rho_k$ for $k \ge 1$. We assume that $\rho_k > 0$ for $k = 1, 2, \cdots$.

For a positive sequence $a = (a_j)$, $\ell^s(a)$ denotes $\{(w_j)\ ;\ w_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty}(a_j|w_j|)^s < \infty\}$ and $\ell^\infty(a)$ denotes $\{(w_j)\ ;\ w_j \in \mathbb{C}$ and $\sup_{1\le j<\infty} a_j|w_j| < \infty\}$. In this paper, we study the following problem : Find a necessary and sufficient condition on $(z_j)$ so that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^s(a)$ where $1 \le p \le \infty$ and $1 \le s \le \infty$.

Suppose $a_j = 1$ for all $j \ge 1$. When $p = s = \infty$, this was solved by L. Carleson [1]. That is, $\{(f(z_j))\ ;\ f \in H^\infty\} \supset \ell^\infty$ if and only if $\inf_j \rho_j > 0$. $(z_j)$ is called a uniformly separated sequence when $\inf_j \rho_j > 0$. When $p = \infty$ and $1 \le s < \infty$, A. K. Snyder [13] (cf. [7],[11]) proved that $\{(f(z_j))\ ;\ f \in H^\infty\} \supset \ell^s$ if and only if $\inf_j \rho_j > 0$. A. K. Snyder [13] and P. L. Duren and H. S. Shapiro [3] showed that there exists a sequence $(z_j)$ which is not uniformly separated, that is, $\inf_j \rho_j = 0$ and has the property : $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^\infty$ when $p \ne \infty$. B. A. Taylor and D. L. Williams [14] showed that for $1 \le p \le \infty$ $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^\infty$ if and only if there exists a positive finite constant $\gamma$ such that $\sum_{j=1}^{\infty} \frac{1}{\rho_j}(1-|z_j|^2)|g(z_j)| \le \gamma \|g\|_q$ for all $g$ in $H^q$ and $1/p + 1/q = 1$.

Suppose $1 \le p = s \le \infty$. When $a_j = (1-|z_j|^2)^{1/p}$ for all $j \ge 1$, this was solved by H. S. Shapiro and A. L. Shields [11]. That is, $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^p(a)$ if and only if $\inf_j \rho_j > 0$. When $a_j = \rho_j^2$ for all $j \ge 1$, J. P. Earl [4] showed that $\{(f(z_j))\ ;\ f \in H^\infty\}$ contains $\ell^\infty(a)$ always. This was pointed out by A. M. Gleason (see [6]). On the other hand, when $a_j = \rho_j$ for all $j \ge 1$, T. Nakazi [10] showed that $\{(f(z_j))\ ;\ f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $(z_j)$ is the union of a finite number of uniformly separated sequences. For a general weight $a = (a_j)$, J. D. McPhail [9] gave a necessary and sufficient condition about

$(z_j)$ that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^p(a)$. In fact, he studied such a problem in weighted Hardy spaces.

In §2, we give a necessary and sufficient condition about $(z_j)$ for that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^s(a)$ where $1 \le p \le \infty,\ 1 \le s \le \infty$ and $a = (a_j)$ is arbitrary weight . As a result, we show that $\{(f(z_j))\ ;\ f \in H^1\} \supset \ell^s(a)$ if and only if $\sum_{j=1}^{\infty}(a_j\rho_j)^{-t}(1-|z_j|^2)^t < \infty$ where $1/s + 1/t = 1$. Moreover, when $1 < p \le \infty$ and $a = (\rho_j^{-1})$, we show that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^p(a)$ if and only if $(z_j)$ is a finite sum of uniformly separated sequences. This is a generalization of a result in [10] for $p = \infty$.

In §3, when $1 \le p \le \infty$, we show that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^1(a)$ if and only if $\sup_j (a_j\rho_j)^{-1}(1-|z_j|^2)^{1/p} < \infty$. As a result, a theorem of A. K. Snyder [13] follows, that is, $\{(f(z_j))\ ;\ f \in H^\infty\} \supset \ell^s$ if and only if $\inf_j \rho_j > 0$.

In §4, we give a necessary and sufficient condition about $(z_j)$ for that $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^\infty(a)$. Put $\mu = \sum_{j=1}^{\infty}(a_j\rho_j)^{-1}(1-|z_j|^2)\delta_{z_j}$. Then $\{(f(z_j))\ ;\ f \in H^1\} \supset \ell^\infty(a)$ if and only if $\mu$ is a finite measure on $D$, and $\{(f(z_j))\ ;\ f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $\mu$ is a Carleson measure on $D$.

In §5, we give a necessary and sufficient condition about $(z_j)$ for that $\{(s(z_j)f(z_j))\ ;\ f \in H^p(W)\} \supset \ell^p$, where $H^p(W)$ is a weighted Hardy space and $s(z_j) = \inf\{\int |f|^p W d\theta/2\pi\ ;\ f(z_j) = 1\}$. We assume only that $\log W$ is in $L^1$. J.D.McPhail [9] studied such a problem when $W$ satisfies the $(A_p)$-condition of Muckenhoupt.

Our interests in this paper are in the differences between interpolations for $\ell^1(a)$ and $\ell^\infty(a)$ and in the interpolation problems for weighted Hardy spaces. For example, it is very easy to prove that $\{(f(z_j))\ ;\ f \in H^\infty\} \supset \ell^1$ if and only if $\{z_j\}$ is uniformly separated.

## §2. General results

In this section, we obtain a general result for interpolation problems for $\ell^s(a)$ $(1 \le s \le \infty)$ by $H^p$ $(1 \le p \le \infty)$. For $1 \le j \le n$, let

$$B_n(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z} \quad \text{and} \quad B_{nj}(z) = B_n(z)\frac{1 - \bar{z}_j z}{z - z_j}.$$

If we put $b_{nj} = B_{nj}(z_j)$, then

$$\rho_{j,n} = |b_{nj}| \quad (1 \le j \le n).$$

Suppose for $n = 1, 2, \cdots$

$$f_n(z) = \sum_{j=1}^{n} b_{nj}^{-1} w_j B_{nj}(z).$$

Then $f_n$ is in $H^\infty$ and $f_n(z_j) = w_j$ $(1 \le j \le n)$. Lemma 1 is essentially known.

**Lemma 1**. *Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Suppose $w_j$ is a complex number for $j = 1, 2, \cdots$. There exists a function $f$ in $H^p$ such that $f(z_j) = w_j$ for $j = 1, 2, \cdots$ if and only if there exists a positive finite constant $\gamma$ such that for any $n \ge 1$ and for all $g$ in $H^q$,*

$$\left|\sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1 - |z_j|^2)g(z_j)\right| \le \gamma \|g\|_q.$$

Proof. Put for $n \ge 1$

$$m_{p,n}(w) = \inf\{\|f_n + B_n h\|_p \ ; \ h \in H^p\}.$$

Then by [2, p142],

$$m_{p,n}(w) = \sup\left\{\left|\sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1 - |z_j|^2)g(z_j)\right| \ ; \ g \in H^q \ \text{ and } \ \|g\|_q \le 1\right\}.$$

There exists a function $f$ in $H^p$ such that $f(z_j) = w_j$ for $j = 1, 2, \cdots$ if and only if $\sup\limits_n m_{p,n}(w) < \infty$ because the unit ball of $H^p$ is compact in the weak topology or the weak $*$ topology. This implies the lemma.

**Theorem 1**. *Let $1 \le p \le \infty$ and $1 \le s \le \infty$. $\{(f(z_n)) \ ; \ f \in H^p\} \supset \ell^s(a)$ if and only if there exists a finite positive constant $\gamma$ such that*

$$\left\{\sum_{n=1}^{\infty}(a_n\rho_n)^{-t}(1 - |z_n|^2)^t|f(z_n)|^t\right\}^{1/t} \le \gamma\|f\|_q$$

*for $f$ in $H^q$, where $1/p + 1/q = 1$ and $1/s + 1/t = 1$.*

Proof. For the 'only if' part, since $\{(f(z_j)) \ ; \ f \in H^p\} \supset \ell^s(a)$, by Lemma 1 there exists a positive finite constant $\gamma$ such that for any $n \ge 1$

$$\sup_{\substack{w \in \ell^s(a) \\ \|w\| \le 1}} \left|\sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1 - |z_j|^2)g(z_j)\right| \le \gamma\|g\|_q \quad (g \in H^q)$$

where $w = (w_j)$ and $\|w\| = \left(\sum_{j=1}^{\infty}|w_j a_j|^s\right)^{1/s}$. Hence for any $n \ge 1$

$$\left\{\sum_{j=1}^{\infty}(a_j\rho_{n,j})^{-t}(1 - |z_j|^2)^t|g(z_j)|^t\right\}^{1/t} \le \gamma\|g\|_q \quad (g \in H^q).$$

Assuming $\|g\|_q = 1$,

$$\sum_{j=1}^{\infty}(a_j\rho_{n,j})^{-t}(1 - |z_j|^2)^t|g(z_j)|^t \le \gamma^t.$$

For any $\varepsilon > 0$, there exists a positive integer $n_j$ for each $j$ such that for all $n \geq n_j$

$$(a_j\rho_{n,j})^{-t}(1-|z_j|^2)^t|g(z_j)|^t - \frac{\varepsilon}{2^j} \leq (a_j\rho_{n,j})^{-t}(1-|z_j|^2)^t|g(z_j)|^t$$

because $\rho_{j,n} \geq \rho_{j,n+1}$ and $\lim_{n\to\infty}\rho_{j,n} = \rho_j$. Thus, $\{(f(z_j))\ ;\ f \in H^p\} \supset \ell^s(a)$ if and only if for any $\varepsilon > 0$ and any $n \geq \max(n_1, \cdots, n_n)$

$$\sum_{j=1}^{n}(a_j\rho_j)^{-t}(1-|z_j|^2)^t|g(z_j)|^t - \varepsilon \leq \sum_{j=1}^{n}(a_j\rho_j)^{-t}(1-|z_j|^2)^t|g(z_j)|^t \leq \gamma^t$$

This implies the 'only if' part.

For the 'if' part, by Lemma 1 it is sufficient to show that there exists a finite positive constant $\gamma$ such that for all $n \geq 1$

$$\sup_{\substack{w\in\ell^s(a)\\ \|w\|\leq 1}} \sup_{\|g\|_q\leq 1} \left|\sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1-|z_j|^2)g(z_j)\right| \leq \gamma < \infty.$$

In fact, for all $n \geq 1$

$$\begin{aligned}
&\sup_{\substack{w\in\ell^s(a)\\ \|w\|\leq 1}} \sup_{\|g\|_q\leq 1} \left|\sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1-|z_j|^2)g(z_j)\right| \\
&\leq \sup_{\|g\|_q\leq 1} \left\{\sum_{j=1}^{n}(a_j\rho_{j,n})^{-t}(1-|z_j|^2)^t|g(z_j)|^t\right\}^{1/t} \\
&\leq \sup_{\|g\|_q\leq 1} \left\{\sum_{j=1}^{\infty}(a_j\rho_j)^{-t}(1-|z_j|^2)^t|g(z_j)|^t\right\}^{1/t} < \infty
\end{aligned}$$

**Corollary 1**. *Let $1 \leq s \leq \infty$. $\{(f(z_n))\ ;\ f \in H^1\} \supset \ell^s(a)$ if and only if*

$$\sum_{n=1}^{\infty}(a_n\rho_n)^{-t}(1-|z_n|^2)^t < \infty$$

*where $1/s + 1/t = 1$. Hence, when $a = (a_n) = (\rho_n^{-1})$ it is always true that $\{(f(z_n))\ ;\ f \in H^1\} \supset \ell^s(a)$.*

Proof. The first part is clear by Theorem 1. When $a = (\rho_n^{-1})$, $\{(f(z_n))\ ;\ f \in H^1\} \supset \ell^s(a)$ if and only if $\sum_{n=1}^{\infty}(1-|z_n|^2)^t < \infty$. This implies the second part.

**Corollary 2**. *Let $1 \leq p \leq \infty,\ 1 \leq s \leq \infty$ and $a = (\rho_n^{-1})$. $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^s(a)$ if and only if there exists a finite positive constant $\gamma$ such that*

$$\left\{\sum_{n=1}^{\infty}(1-|z_n|^2)^t|f(z_n)|^t\right\}^{1/t} \leq \gamma\|f\|_q$$

*for $f$ in $H^q$, where $1/p+1/q=1$ and $1/s+1/t=1$. When $1<p\le\infty$, $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^p(a)$ if and only if $(z_n)$ is a finite sum of uniformly separated sequences.*

Proof. The first part is clear by Theorem 1. The second part follows from the first one and [8].

In Corollary 2, when $1<p\le\infty$ and $1<s\le\infty$ and $s>p$, if $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^s(a)$ then $(z_n)$ is a finite sum of uniformly separated sequences but the converse is not true. When $s<p$, if $(z_n)$ is a finite sum of uniformly separated sequences then $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^s(a)$ but the converse is not true.

## §3. Interpolations for $\ell^1(a)$

$\ell^1(a)$ is the smallest sequence space among $\ell^p(a)$ $(1\le p\le\infty)$ for the same $a=\{a_j\}$. Then the inlerpolations for $\ell^1(a)$ are very special as the following shows.

The case of $p=\infty$ in Corollary 3 was proved by A.Snyder [13] (see [7], [11]). Corollary 4 is due to O. Hatori [7].

**Theorem 2**. *Let $1\le p\le\infty$. $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^1(a)$ if and only if*

$$\sup_n (a_n\rho_n)^{-1}(1-|z_n|^2)^{1/p}<\infty.$$

Proof. By Theorem 1, $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^1(a)$ if and only if there exists a finite positive constant $\gamma$ such that

$$\sup_n (a_n\rho_n)^{-1}(1-|z_n|^2)|f(z_n)|\le\gamma\|f\|_q$$

for all $f$ in $H^q$. For each $n$, $\sup_{\|f\|_q=1}|f(z_n)|=(1-|z_n|^2)^{-1/q}$ by [2, p144] and so the theorem follows.

**Corollary 3**. *Let $1\le p\le\infty$. $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^1$ if and only if $\sup_n \dfrac{1}{\rho_n}(1-|z_n|^2)^{1/p}<\infty$. Hence if $p=\infty$, $\{(f(z_n))\ ;\ f\in H^\infty\}\supset \ell^1$ if and only if $\inf_n \rho_n>0$.*

**Corollary 4**. *Let $1\le p\le\infty$. $\{((1-|z_n|^2)^{1/p}f(z_n))\ ;\ f\in H^p\}\supset \ell^1$ if and only if $\inf_n \rho_n>0$.*

Proof. Note that $\{((1-|z_n|^2)^{1/p}f(z_n))\ ;\ f\in H^p\}\supset \ell^1$ if and only if $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^1(a)$ and $a=((1-|z_n|^2)^{1/p})$.

**Corollary 5**. *Let $1\le p\le\infty$. For any $(z_n)$, $\{(f(z_n))\ ;\ f\in H^p\}\supset \ell^1(a)$ where $a=(\rho_n^{-1})$.*

Let $(b_j)$ be a uniformly separated sequence in $D$ such that $0 < \mathrm{Re} b_j \nearrow 1$ and $\mathrm{Im}\, b_j \searrow 0$. For $j \geq 1$, put $z_{2j-1} = b_j$ and $z_{2j} = \bar{b}_j$. Let $B$ be the Blaschke product associated with $\{z_n\}$. Then for each $j$

$$B = \frac{z - b_j}{1 - \bar{b}_j z} \frac{z - \bar{b}_j}{1 - b_j z} B_{1j} B_{2j}$$

where $B_{1j}$ (or $B_{2j}$) is a Blaschke product with zeros $\{b_\ell\}_{\ell \neq j}$ (or $\{\bar{b}_\ell\}_{\ell \neq j}$). Then

$$\rho_{2j-1} = \left| \frac{b_j - \bar{b}_j}{1 - \bar{b}_j b_j} \right| \prod_{\ell \neq j} \left| \frac{b_j - b_\ell}{1 - \bar{b}_\ell b_j} \right| \prod_{\ell \neq j} \left| \frac{b_j - \bar{b}_\ell}{1 - b_\ell b_j} \right|$$

and

$$\rho_{2j} = \left| \frac{\bar{b}_j - b_j}{1 - \bar{b}_j \bar{b}_j} \right| \prod_{\ell \neq j} \left| \frac{\bar{b}_j - b_\ell}{1 - \bar{b}_\ell \bar{b}_j} \right| \prod_{\ell \neq j} \left| \frac{\bar{b}_j - \bar{b}_\ell}{1 - b_\ell \bar{b}_j} \right|.$$

Hence $\rho_{2j-1} = \rho_{2j}$ for $j \geq 1$ and

$$\delta^2 \frac{|\bar{b}_j - b_j|}{1 - |b_j|^2} \leq \rho_{2j} = \rho_{2j-1} \leq \frac{|\bar{b}_j - b_j|}{1 - |b_j|^2} \quad (j \geq 1)$$

where

$$0 < \delta = \min \left\{ \inf_j \prod_{\ell \neq j} \left| \frac{b_j - b_\ell}{1 - \bar{b}_\ell b_j} \right|, \inf_j \prod_{\ell \neq j} \left| \frac{b_j - \bar{b}_\ell}{1 - b_\ell b_j} \right| \right\}.$$

Hence

$$\frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|} \leq \frac{(1 - |z_n|^2)^{1/p}}{\rho_n} \leq \delta^{-2} \frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|}.$$

Thus $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^1$ if and only if $\sup_n (1 - |z_n|^2)^{1+1/p} / |z_n - \bar{z}_n| < \infty$.

## §4. Interpolations for $\ell^\infty(a)$

$\ell^\infty(a)$ is the largest sequence space among $\ell^p(a)$ $(1 \leq p \leq \infty)$ for the same $a = (a_j)$. Then the interpolations for $\ell^\infty(a)$ are special as the following shows. The case of $p = \infty$ of Corollary 6 is known in [10].

**Theorem 3**. *Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$, $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^\infty(a)$ if and only if there exists a finite positive constant $\gamma$ such that*

$$\sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) |f(z_n)| \leq \gamma \|f\|_q$$

*for all $f$ in $H^q$. When $p = 1$, $\{(f(z_n))\ ;\ f \in H^1\} \supset \ell^\infty(a)$ if and only if $\mu = \sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}$ is a finite measure on $D$. When $p = \infty$, $\{(f(z_n))\ ;\ f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $\mu = \sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}$ is a Carleson measure on $D$.*

**Corollary 6**. *Let $1 \le p \le \infty$ and $1/p + 1/q = 1$ and $a = (\rho_n^{-1})$. $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^\infty(a)$ if and only if there exists a finite positive constant $\gamma$ such that*

$$\sum_n (1 - |z_n|^2)|f(z_n)| \le \gamma \|f\|_q$$

*for all $f$ in $H^q$.*

*(1) When $p = 1$, for any $(z_n)$, $\{(f(z_n))\ ;\ f \in H^1\} \supset \ell^\infty(a)$.*

*(2) When $p = \infty$, $\{(f(z_n))\ ;\ f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $(z_n)$ is a finite union of uniformly separated sequences.*

*(3) When $1 < p < \infty$, there exists a sequence $(z_n)$ in $D$ such that $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^\infty(a)$ and $(z_n)$ is not a union of finitely many uniformly separated sequences. If $\sum_{n=1}^{\infty} (1 - |z_n|^2)^{1/p} < \infty$, then $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^\infty(a)$.*

Suppose that $(z_n)$ is the sequence in $D$ which was used in the end of Section 3, and $1 \le p < \infty$. If $0 < \gamma_1 \le \dfrac{(1 - |z_n|^2)^{1+1/p-\varepsilon}}{|z_n - \bar{z}_n|} \le \gamma_2 < \infty$ for some $0 < \varepsilon < 1/p$, then $\{(f(z_n))\ ;\ f \in H^p\} \supset \ell^\infty$. This was proved by B. A. Taylor and D. L. Williams [14].

## §5. Weighted Hardy space

Let $W$ be a nonnegative function in $L^1$ with $\log W \in L^1$ and $1 \le p < \infty$. $H^p(W)$ denotes the closure of the set of all analytic polynomials in $L^p(W) = L^p(W d\theta/2\pi)$. $H^p(W)$ is called a weighted Hardy space. For $b \in D$, put

$$s(b) = s(b, p, W) = \inf \left\{ \int |f|^p W d\theta/2\pi\ ;\ f(b) = 1 \right\}.$$

Let $h$ be an outer function in $H^p$ such that $|h|^p = W$.

**Lemma 2**. *For $1 \le p < \infty$ and $b \in D$,*

$$\begin{aligned} s(b, p, W) &= (1 - |b|^2)\ \exp(\log W)^{\sim}(b) \\ &= (1 - |b|^2)\ |h(b)|^p, \end{aligned}$$

where $(\log W)^{\sim}(b)$ denotes the Poisson integral of $\log W$ at $b$.

Proof. It is well known (cf. [5, p136]) that $s(0, p, W) = \exp \int_0^{2\pi} \log W d\theta/2\pi$. It is easy to show the lemma using $f(b) = f \circ \phi_b(0)$, where $\phi_b(z) = (z + b)/(1 + \bar{b}z)$.

**Lemma 3**. *Suppose $(z_j)$ is a sequence of points in $D$. For $1 \le p < \infty$ and $1 \le s < \infty$, $\{(s(z_j, p, W)^{1/p} f(z_j))\ ;\ f \in H^p(W)\} \supset \ell^s$ if and ony if $\{(F(z_j))\ ;\ F \in H^p\} \supset \ell^s(a)$, where $a = (a_j)$ and $a_j = s(z_j)^{1/p}/h(z_j)$.*

Proof. Since $H^p(W) = h^{-1}H^p,\ f \in H^p(W)$ if and only if $f = h^{-1}F$ and $F \in H^p$. For each $j,\ s(z_j)^{1/p} f(z_j) = w_j$ if and only if $F(z_j) = h(z_j)w_j/s(z_j)^{1/p}$ if and only if $F(z_j) = \zeta_j,\ w_j = a_j\zeta_j$. $(w_j) \in \ell^p$ if and only if $(\zeta_j) \in \ell^s(a)$. Now the lemma follows.

**Theorem 4**. *Let $1 \le p < \infty,\ 1 \le s \le \infty$, and $1/p + 1/q = 1/s + 1/t = 1$. Then, $\{(s(z_n, p, W)^{1/p} f(z_n))\ ;\ f \in H^p(W)\} \supset \ell^s$ if and only if*

$$\left\{\sum_{n=1}^{\infty} \frac{1}{\rho_n^t} s(z_n)^{t/q} |g(z_n)|^t\right\}^{1/t} \le \gamma \|g\|_{H^q(W)}$$

*for $g$ in $H^q(W)$.*

Proof. By Lemma 3, $\{(s(z_n)^{1/p} f(z_n))\ ;\ f \in H^p(W)\} \supset \ell^s$ if and only if $\{(F(z_n))\ ;\ F \in H^p\} \supset \ell^s(a)$, where $a_n = s(z_n)^{1/p}/|h(z_n)|$. By Theorem 1, this is equivalent to saying that there exists a finite positive constant $\gamma$ such that

$$\left\{\sum_{n=1}^{\infty} \frac{1}{\rho_n^t}\frac{1}{a_n^t}(1 - |z_n|^2)^t |G(z_n)|^t\right\}^{1/t} \le \gamma \|G\|_q$$

for $G \in H^q$. Since $H^q(W) = h^{-p/q}H^q,\ g \in H^q(W)$ if and only if $g = h^{-p/q}G$ and $G \in H^q$. Hence $\|g\|_{H^q(W)} = \|G\|_{H^q}$ and for each $n \ge 1$

$$\begin{aligned}
&a_n^{-t}(1 - |z_n|^2)^t |G(z_n)|^t \\
&\quad = s(z_n)^{-(t/p)} |h(z_n)|^t (1 - |z_n|^2)^t |h(z_n)|^{pt/q} |g(z_n)|^t \\
&\quad = s(z_n)^{-(t/p)} (1 - |z_n|^2)^t |h(z_n)|^{t(q+p)/q} |g(z_n)|^t \\
&\quad = s(z_n)^{-(t/p)} s(z_n)^t |g(z_n)|^t \\
&\quad = s(z_n)^{t/q} |g(z_n)|^t.
\end{aligned}$$

This implies the theorem.

**Corollary 7**. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then $\{(s(z_n, p, W)^{1/p} f(z_n))\ ;\ f \in H^p(W)\} \supset \ell^1$ if and only if $\inf_n \rho_n > 0$.*

Proof. By Theorem 4, $\{(s(z_n, p, W)^{1/p} f(z_n))\ ;\ f \in H^p(W)\} \supset \ell^1$ if and only if

$$\sup_n \frac{1}{\rho_n} s(z_n, p, W)^{1/p} s(z_n, q, W)^{-1/q} < \infty.$$

Now Lemma 2 implies the corollary.

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