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Interpolation Of Weighted $\ell^q$ Sequences By $H^p$ Functions

by

Takahiko Nakazi*

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Abstract. Let \((z_n)\) be a sequence of points in the open unit disc \(D\) and \(\rho_n = \prod_{m \neq n} |(z_n - z_m)(1 - \bar{z}_m z_n)^{-1}| > 0\). Let \(a = (a_j)_{j=1}^\infty\) be a sequence of positive numbers and \(\ell^s(a) = \{(w_j) ; (a_j w_j) \in \ell^s\}\) where \(1 \leq s \leq \infty\). When \(1 \leq p \leq \infty\) and \(1/p + 1/q = 1\), we show that \(\{(f(z_n)) ; f \in H^p\} \supset \ell^s(a)\) if and only if there exists a finite positive constant \(\gamma\) such that
\[
\left\{ \sum_{n=1}^\infty (a_n \rho_n)^{-1}(1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q \quad (f \in H^q),\]
where \(1/s + 1/t = 1\). As results, we show that \(\{(f(z_j)) ; f \in H^p\} \supset \ell^1(a)\) if and only if \(\sup (a_n \rho_n)^{-1}(1 - |z_n|^2)^{1/p} < \infty\), and \(\{(f(z_n)) ; f \in H^1\} \supset \ell^\infty(a)\) if and only if \(\sum (a_n \rho_n)^{-1}(1 - |z_n|^2)^{\delta} < \infty\) is finite measure on \(D\). These are also proved in the case of weighted Hardy spaces.

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§1. Introduction

$H^p (0 < p \leq \infty)$ denotes the usual Hardy space in the open unit disc $D$. In this paper, we assume that a sequence $(z_j)$ in $D$ satisfies that $\sum_{j=1}^{\infty}(1 - |z_j|) < \infty$, that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \frac{z - \bar{z}_j}{|z_j| - 1 - \bar{z}_j z}.$$ 

Let

$$\rho_{k,n} = \prod_{j=1}^{n} \left| \frac{z_k - \bar{z}_j}{1 - \bar{z}_j z_k} \right|, \quad 1 \leq k \leq n,$$

$$\rho_k = \prod_{j=1}^{\infty} \left| \frac{z_k - \bar{z}_j}{1 - \bar{z}_j z_k} \right|.$$ 

Then $\rho_{k,n} \geq \rho_{k,n+1}$ and $\lim_{n \to \infty} \rho_{k,n} = \rho_k$ for $k \geq 1$. We assume that $\rho_k > 0$ for $k = 1, 2, \ldots$.

For a positive sequence $a = (a_j)$, $\ell^s(a)$ denotes $\{(w_j) ; w_j \in \mathcal{C} \text{ and } \sum_{j=1}^{\infty}(a_j|w_j|)^s < \infty\}$ and $\ell^\infty(a)$ denotes $\{(w_j) ; w_j \in \mathcal{C} \text{ and } \sup_{1 \leq j < \infty} a_j|w_j| < \infty\}$. In this paper, we study the following problem: Find a necessary and sufficient condition on $(z_j)$ so that $\{(f(z_j)) ; f \in H^p \supset \ell^s(a) \text{ where } 1 \leq p \leq \infty \text{ and } 1 \leq s \leq \infty\}$.

Suppose $a_j = 1$ for all $j \geq 1$. When $p = s = \infty$, this was solved by L. Carleson [1]. That is, $\{(f(z_j)) ; f \in H^\infty \supset \ell^s(a) \text{ if and only if } \inf_j \rho_j > 0\}$. (z_j) is called a uniformly separated sequence when $\inf \rho_j > 0$. When $p = \infty$ and $1 \leq s < \infty$, A. K. Snyder [13] (cf. [7],[11]) proved that $\{(f(z_j)) ; f \in H^\infty \supset \ell^s(a) \text{ if and only if } \inf_j \rho_j > 0\}$.

A. K. Snyder [13] and P. L. Duren and H. S. Shapiro [3] showed that there exists a sequence $(z_j)$ which is not uniformly separated, that is, $\inf \rho_j = 0$ and has the property $\{(f(z_j)) ; f \in H^p \supset \ell^\infty \text{ when } p \neq \infty\}$. B. A. Taylor and D. L. Williams [14] showed that for $1 \leq p \leq \infty \{(f(z_j)) ; f \in H^p \supset \ell^\infty \text{ if and only if } \exists \text{ a positive finite constant } \gamma \text{ such that } \sum_{j=1}^{\infty} \frac{1}{\rho_j} (1 - |z_j|^2)|g(z_j)| \leq \gamma \|g\|_q \text{ for all } g \in H^q \text{ and } 1/p + 1/q = 1\}$. 

Suppose $1 \leq p = s \leq \infty$. When $a_j = (1 - |z_j|^2)^{1/p}$ for all $j \geq 1$, this was solved by H. S. Shapiro and A. L. Shields [11]. That is, $\{(f(z_j)) ; f \in H^p \supset \ell^p(a) \text{ if and only if } \inf_j \rho_j > 0\}$. When $a_j = \rho_j^2$ for all $j \geq 1$, J. P. Earl [4] showed that $\{(f(z_j)) ; f \in H^\infty\}$ contains $\ell^\infty(a)$ always. This was pointed out by A. M. Gleason (see [6]). On the other hand, when $a_j = \rho_j$ for all $j \geq 1$, T. Nakazi [10] showed that $\{(f(z_j)) ; f \in H^\infty \supset \ell^\infty(a) \text{ if and only if } (z_j) \text{ is the union of a finite number of uniformly separated sequences. For a general weight } a = (a_j), \text{ J. D. McPhail [9] gave a necessary and sufficient condition about}$. 

(z_j) that \{ (f(z_j)) \ ; \ f \in H^p \} \supset \ell^p(a). In fact, he studied such a problem in weighted Hardy spaces.

In §2, we give a necessary and sufficient condition about (z_j) for that \{ (f(z_j)) \ ; \ f \in H^p \} \supset \ell^p(a) where 1 \leq p \leq \infty, 1 \leq s \leq \infty and a = (a_j) is arbitrary weight. As a result, we show that \{ (f(z_j)) \ ; \ f \in H^1 \} \supset \ell^s(a) if and only if \[ \sum_{j=1}^{\infty} (a_j \rho_j)^{-\ell}(1-|z_j|^2)^t < \infty \]
where 1/s + 1/t = 1. Moreover, when 1 < p \leq \infty and a = (\rho_j^{-1}), we show that \{ (f(z_j)) \ ; \ f \in H^p \} \supset \ell^p(a) if and only if (z_j) is a finite sum of uniformly separated sequences. This is a generalization of a result in [10] for p = \infty.

In §3, when 1 \leq p \leq \infty, we show that \{ (f(z_j)) \ ; \ f \in H^p \} \supset \ell^1(a) if and only if \sup(a_j \rho_j)^{-1}(1-|z_j|^2)^{1/p} < \infty. As a result, a theorem of A. K. Snyder [13] follows, that is, \{ (f(z_j)) \ ; \ f \in H^\infty \} \supset \ell^s if and only if \inf \rho_j > 0.

In §4, we give a necessary and sufficient condition about (z_j) for that \{ (f(z_j)) \ ; \ f \in H^p \} \supset \ell^\infty(a). Put \mu = \sum_{j=1}^{\infty} (a_j \rho_j)^{-1}(1-|z_j|^2)\delta_{z_j}. Then \{ (f(z_j)) \ ; \ f \in H^1 \} \supset \ell^\infty(a) if and only if \mu is a finite measure on D, and \{ (f(z_j)) \ ; \ f \in H^\infty \} \supset \ell^\infty(a) if and only if \mu is a Carleson measure on D.

In §5, we give a necessary and sufficient condition about (z_j) for that \{ (s(z_j)f(z_j)) \ ; \ f \in H^p(W) \} \supset \ell^p, where H^p(W) is a weighted Hardy space and \( s(z_j) = \inf \left\{ \int |f|^p W d\theta/2\pi \ ; \ f(z_j) = 1 \right\}. We assume only that \log W is in L^1. J.D. McPhail [9] studied such a problem when W satisfies the \( A_p \)-condition of Muckenhoupt.

Our interests in this paper are in the differences between interpolations for \ell^1(a) and \ell^\infty(a) and in the interpolation problems for weighted Hardy spaces. For example, it is very easy to prove that \{ (f(z_j)) \ ; \ f \in H^\infty \} \supset \ell^1 if and only if \{ z_j \} is uniformly separated.

§2. General results

In this section, we obtain a general result for interpolation problems for \ell^s(a) (1 \leq s \leq \infty) by H^p (1 \leq p \leq \infty). For 1 \leq j \leq n, let

\[ B_n(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z} \] and \[ B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}. \]

If we put \( b_{nj} = B_{nj}(z_j) \), then

\[ \rho_{j,n} = |b_{nj}| \quad (1 \leq j \leq n). \]

Suppose for \( n = 1, 2, \cdots \)

\[ f_n(z) = \sum_{j=1}^{n} b_{nj}^{-1} w_j B_{nj}(z). \]
Then \( f_n \) is in \( H^\infty \) and \( f_n(z_j) = w_j \) \((1 \leq j \leq n)\). Lemma 1 is essentially known.

**Lemma 1.** Let \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \). Suppose \( w_j \) is a complex number for \( j = 1, 2, \cdots \). There exists a function \( f \) in \( H^p \) such that \( f(z_j) = w_j \) for \( j = 1, 2, \cdots \) if and only if there exists a positive finite constant \( \gamma \) such that for any \( n \geq 1 \) and for all \( g \) in \( H^q \),

\[
\left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma \|g\|_q.
\]

Proof. Put for \( n \geq 1 \)

\[
m_{p,n}(w) = \inf \{ \|f_n + B_n h\|_p \ ; \ h \in H^p \}.
\]

Then by [2, p142],

\[
m_{p,n}(w) = \sup \left\{ \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \ ; \ g \in H^p \text{ and } \|g\|_q \leq 1 \right\}.
\]

There exists a function \( f \) in \( H^p \) such that \( f(z_j) = w_j \) for \( j = 1, 2, \cdots \) if and only if \( \sup m_{p,n}(w) < \infty \) because the unit ball of \( H^p \) is compact in the weak topology or the weak * topology. This implies the lemma.

**Theorem 1.** Let \( 1 \leq p \leq \infty \) and \( 1 \leq s \leq \infty \). \{\((f(z)) \ ; \ f \in H^p \) \supset \ell^s(a)\} if and only if there exists a finite positive constant \( \gamma \) such that

\[
\left\{ \sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q
\]

for \( f \) in \( H^q \), where \( 1/p + 1/q = 1 \) and \( 1/s + 1/t = 1 \).

Proof. For the 'only if' part, since \{\((f(z)) \ ; \ f \in H^p \) \supset \ell^s(a)\}, by Lemma 1 there exists a positive finite constant \( \gamma \) such that for any \( n \geq 1 \)

\[
\sup_{w \in \ell^s(a), \|w\| \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma \|g\|_q \ (g \in H^q)
\]

where \( w = (w_j) \) and \( \|w\| = \left( \sum_{j=1}^{\infty} |w_j a_j|^s \right)^{1/s} \). Hence for any \( n \geq 1 \)

\[
\left\{ \sum_{j=1}^{\infty} (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} \leq \gamma \|g\|_q \ (g \in H^q).
\]

Assuming \( \|g\|_q = 1 \),

\[
\sum_{j=1}^{\infty} (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \leq \gamma^t.
\]
For any \( \varepsilon > 0 \), there exists a positive integer \( n_j \) for each \( j \) such that for all \( n \geq n_j \):

\[
(a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \frac{\varepsilon}{2} \leq (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t
\]

because \( \rho_{j,n} \geq \rho_{j,n+1} \) and \( \lim_{n \to \infty} \rho_{j,n} = \rho_j \). Thus, \( \{(f(z_j)) ; f \in H^p \} \supset \ell^s(a) \) if and only if for any \( \varepsilon > 0 \) and any \( n \geq \max(n_1, \ldots, n_n) \):

\[
\sum_{j=1}^{n}(a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \varepsilon \leq \sum_{j=1}^{n}(a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \leq \gamma^t
\]

This implies the ‘only if’ part.

For the ‘if’ part, by Lemma 1 it is sufficient to show that there exists a finite positive constant \( \gamma \) such that for all \( n \geq 1 \):

\[
\sup_{w \in \ell^s(a)} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma < \infty.
\]

In fact, for all \( n \geq 1 \):

\[
\sup_{w \in \ell^s(a)} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \\
\leq \sup_{\|g\|_q \leq 1} \left\{ \sum_{j=1}^{n} (a_j \rho_{j,n})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} \\
\leq \sup_{\|g\|_q \leq 1} \left\{ \sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} < \infty
\]

**Corollary 1.** Let \( 1 \leq s \leq \infty \). \( \{(f(z_n)) ; f \in H^1 \} \supset \ell^s(a) \) if and only if

\[
\sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2)^t < \infty
\]

where \( 1/s + 1/t = 1 \). Hence, when \( a = (a_n) = (\rho_n^{-1}) \) it is always true that \( \{(f(z_n)) ; f \in H^1 \} \supset \ell^s(a) \).

Proof. The first part is clear by Theorem 1. When \( a = (\rho_n^{-1}) \), \( \{(f(z_n)) ; f \in H^1 \} \supset \ell^s(a) \) if and only if \( \sum_{n=1}^{\infty} (1 - |z_n|^2)^t < \infty \). This implies the second part.

**Corollary 2.** Let \( 1 \leq p \leq \infty \), \( 1 \leq s \leq \infty \) and \( a = (\rho_n^{-1}) \). \( \{(f(z_n)) ; f \in H^p \} \supset \ell^s(a) \) if and only if there exists a finite positive constant \( \gamma \) such that

\[
\left\{ \sum_{n=1}^{\infty} (1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q
\]
for \( f \) in \( H^q \), where \( 1/p + 1/q = 1 \) and \( 1/s + 1/t = 1 \). When \( 1 < p \leq \infty \), \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^p(a) \) if and only if \((z_n)\) is a finite sum of uniformly separated sequences.

Proof. The first part is clear by Theorem 1. The second part follows from the first one and [8].

In Corollary 2, when \( 1 < p \leq \infty \) and \( 1 < s \leq \infty \) and \( s > p \), if \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^s(a) \) then \((z_n)\) is a finite sum of uniformly separated sequences but the converse is not true. When \( s < p \), if \((z_n)\) is a finite sum of uniformly separated sequences then \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^s(a) \) but the converse is not true.

\[\text{§3. Interpolations for } \ell^1(a)\]

\( \ell^1(a) \) is the smallest sequence space among \( \ell^p(a) \) (\( 1 \leq p \leq \infty \)) for the same \( a = \{a_j\} \). Then the interpolations for \( \ell^1(a) \) are very special as the following shows.

The case of \( p = \infty \) in Corollary 3 was proved by A.Snyder [13] (see [7], [11]). Corollary 4 is due to O. Hatori [7].

**Theorem 2.** Let \( 1 \leq p \leq \infty \). \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^1(a) \) if and only if

\[
\sup_n (a_n \rho_n)^{-1}(1 - |z_n|^2)^{1/p} < \infty.
\]

Proof. By Theorem 1, \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^1(a) \) if and only if there exists a finite positive constant \( \gamma \) such that

\[
\sup_n (a_n \rho_n)^{-1}(1 - |z_n|^2)|f(z_n)| \leq \gamma \|f\|_q
\]

for all \( f \) in \( H^q \). For each \( n \), \( \sup_{\|f\|_q=1} |f(z_n)| = (1 - |z_n|^2)^{-1/q} \) by [2, p144] and so the theorem follows.

**Corollary 3.** Let \( 1 \leq p \leq \infty \). \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^1 \) if and only if \( \sup_n (1 - |z_n|^2)^{1/p} < \infty \). Hence if \( p = \infty \), \( \{(f(z_n)) \mid f \in H^\infty\} \supset \ell^1 \) if and only if \( \inf_n \rho_n > 0 \).

**Corollary 4.** Let \( 1 \leq p \leq \infty \). \( \{(1 - |z_n|^2)^{1/p} f(z_n) \mid f \in H^p\} \supset \ell^1 \) if and only if \( \inf_n \rho_n > 0 \).

Proof. Note that \( \{(1 - |z_n|^2)^{1/p} f(z_n) \mid f \in H^p\} \supset \ell^1 \) if and only if \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^1(a) \) and \( a = (1 - |z_n|^2)^{1/p} \).

**Corollary 5.** Let \( 1 \leq p \leq \infty \). For any \((z_n)\), \( \{(f(z_n)) \mid f \in H^p\} \supset \ell^1(a) \) where \( a = (\rho_n^{-1}) \).
Let \((b_j)\) be a uniformly separated sequence in \(D\) such that \(0 < \text{Re}b_j \neq 1\) and \(\text{Im} b_j \searrow 0\). For \(j \geq 1\), put \(z_{2j-1} = b_j\) and \(z_{2j} = b_j\). Let \(B\) be the Blaschke product associated with \(\{z_n\}\). Then for each \(j\)

\[
B = \frac{z - b_j}{1 - b_j z} \frac{z - \bar{b}_j}{1 - \bar{b}_j z} B_{1j} B_{2j}
\]

where \(B_{1j}\) (or \(B_{2j}\)) is a Blaschke product with zeros \(\{b_t\}_{t \neq j}\) (or \(\{\bar{b}_t\}_{t \neq j}\)). Then

\[
\rho_{2j-1} = \left| \frac{b_j - \bar{b}_j}{1 - b_j b_j} \prod_{\ell \neq j} \frac{b_j - b_\ell}{1 - b_j b_\ell} \prod_{\ell \neq j} \frac{b_j - \bar{b}_\ell}{1 - b_j \bar{b}_\ell} \right|
\]

and

\[
\rho_{2j} = \left| \frac{\bar{b}_j - b_j}{1 - \bar{b}_j b_j} \prod_{\ell \neq j} \frac{\bar{b}_j - b_\ell}{1 - \bar{b}_j b_\ell} \prod_{\ell \neq j} \frac{\bar{b}_j - \bar{b}_\ell}{1 - \bar{b}_j \bar{b}_\ell} \right|.
\]

Hence \(\rho_{2j-1} = \rho_{2j}\) for \(j \geq 1\) and

\[
\frac{\delta^2 |\bar{b}_j - b_j|}{1 - |b_j|^2} \leq \rho_{2j} = \rho_{2j-1} \leq \frac{|\bar{b}_j - b_j|}{1 - |b_j|^2} \quad (j \geq 1)
\]

where

\[
0 < \delta = \min \left\{ \inf_j \prod_{\ell \neq j} \left| \frac{b_j - b_\ell}{1 - b_j b_\ell} \right|, \inf_j \prod_{\ell \neq j} \left| \frac{b_j - \bar{b}_\ell}{1 - b_j \bar{b}_\ell} \right| \right\}.
\]

Hence

\[
\frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|} \leq \frac{(1 - |z_n|^2)^{1+1/p}}{\rho_n} \leq \delta^{-2} \frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|}.
\]

Thus \(\{(f(z_n)) : f \in H^p\} \supset \ell^1\) if and only if \(\sup_n (1 - |z_n|^2)^{1+1/p}/|z_n - \bar{z}_n| < \infty\).

### §4. Interpolations for \(\ell^\infty(a)\)

\(\ell^\infty(a)\) is the largest sequence space among \(\ell^p(a)\) \((1 \leq p \leq \infty)\) for the same \(a = (a_j)\). Then the interpolations for \(\ell^\infty(a)\) are special as the following shows. The case of \(p = \infty\) of Corollary 6 is known in [10].

**Theorem 3.** Let \(1 \leq p \leq \infty\) and \(1/p + 1/q = 1\), \(\{(f(z_n)) : f \in H^p\} \supset \ell^\infty(a)\) if and only if there exists a finite positive constant \(\gamma\) such that

\[
\sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2)|f(z_n)| \leq \gamma \|f\|_q
\]

for all \(f\) in \(H^q\). When \(p = 1\), \(\{(f(z_n)) : f \in H^1\} \supset \ell^\infty(a)\) if and only if \(\mu = \sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}\) is a finite measure on \(D\). When \(p = \infty\), \(\{(f(z_n)) : f \in H^\infty\} \supset \ell^\infty(a)\) if and only if \(\mu = \sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}\) is a Carleson measure on \(D\).
Corollary 6. Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$ and $a = (p_n^{-1})$. \(\{(f(z_n)) ; f \in H^p\} \supset \ell^\infty(a)\) if and only if there exists a finite positive constant $\gamma$ such that

$$\sum_n (1 - |z_n|^2)|f(z_n)| \leq \gamma \|f\|_q$$

for all $f$ in $H^q$.

1. When $p = 1$, for any $(z_n)$, \(\{(f(z_n)) ; f \in H^1\} \supset \ell^\infty(a)\).

2. When $p = \infty$, \(\{(f(z_n)) ; f \in H^\infty\} \supset \ell^\infty(a)\) if and only if $(z_n)$ is a finite union of uniformly separated sequences.

3. When $1 < p < \infty$, there exists a sequence $(z_n)$ in $D$ such that \(\{(f(z_n)) ; f \in H^p\} \supset \ell^\infty(a)\) and $(z_n)$ is not a union of finitely many uniformly separated sequences. If $\sum_{n=1}^{\infty} (1 - |z_n|^2)^{1/p} < \infty$, then \(\{(f(z_n)) ; f \in H^p\} \supset \ell^\infty\). This was proved by B. A. Taylor and D. L. Williams [14].

§5. Weighted Hardy space

Let $W$ be a nonnegative function in $L^1$ with $\log W \in L^1$ and $1 \leq p < \infty$. $H^p(W)$ denotes the closure of the set of all analytic polynomials in $L^p(W) = L^p(Wd\theta/2\pi)$. $H^p(W)$ is called a weighted Hardy space. For $b \in D$, put

$$s(b) = s(b, p, W) = \inf \left\{ \int |f|^p Wd\theta/2\pi ; f(b) = 1 \right\}.$$

Let $h$ be an outer function in $H^p$ such that $|h|^p = W$.

Lemma 2. For $1 \leq p < \infty$ and $b \in D$,

$$s(b, p, W) = (1 - |b|^2^2) \exp(\log W)\sim(b) = (1 - |b|^2) |h(b)|^p,$$

where $(\log W)\sim(b)$ denotes the Poisson integral of $\log W$ at $b$.

Proof. It is well known (cf. [5, p136]) that $s(0, p, W) = \exp \int_0^{2\pi} \log Wd\theta/2\pi$. It is easy to show the lemma using $f(b) = f \circ \phi_\theta(0)$, where $\phi_\theta(z) = (z + b)/(1 + \overline{b}z)$.

Lemma 3. Suppose $(z_j)$ is a sequence of points in $D$. For $1 \leq p < \infty$ and $1 \leq s < \infty$, \(\{(s(z_j, p, W))^{1/p} f(z_j)) ; f \in H^p(W)\} \supset \ell^s\) if and only if \(\{(F(z_j)) ; F \in H^p\} \supset \ell^s(a), \) where $a = (a_j)$ and $a_j = s(z_j)^{1/p}/h(z_j)$. 

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Proof. Since $H^p(W) = h^{-1}H^p$, $f \in H^p(W)$ if and only if $f = h^{-1}F$ and $F \in H^p$. For each $j$, $s(z_j)^{1/p}f(z_j) = w_j$ if and only if $F(z_j) = h(z_j)w_j/s(z_j)^{1/p}$ if and only if $F(z_j) = \zeta_j$, $w_j = a_j\zeta_j$. $(w_j) \in \ell^p$ if and only if $(\zeta_j) \in \ell^s(a)$. Now the lemma follows.

**Theorem 4.** Let $1 \leq p < \infty$, $1 \leq s \leq \infty$, and $1/p + 1/q = 1/s + 1/t = 1$. Then, 
\[
\{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^s \text{ if and only if }
\]
\[
\{s(z_n)^{t/q}\left|g(z_n)f(z_n)\right|^t\}^{1/t} \leq \gamma\|g\|_{H^q(W)}
\]
for $g$ in $H^q(W)$.

Proof. By Lemma 3, 
\[
\{(s(z_n)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^s \text{ if and only if }\{(F(z_n)) ; F \in H^p\} \supset \ell^s(a),\text{ where }a_n = s(z_n)^{1/p}/|h(z_n)|.
\]
By Theorem 1, this is equivalent to saying that there exists a finite positive constant $\gamma$ such that
\[
\{\sum_{n=1}^{\infty} \frac{1}{\rho_n^t} a_n^t (1 - |z_n|^2)^t |G(z_n)|^t\}^{1/t} \leq \gamma\|G\|_q
\]
for $G \in H^q$. Since $H^q(W) = h^{-p/q}H^q$, $g \in H^q(W)$ if and only if $g = h^{-p/q}G$ and $G \in H^q$. Hence $\|g\|_{H^q(W)} = \|G\|_{H^q}$ and for each $n \geq 1$
\[
a_n^{-t}(1 - |z_n|^2)^t |G(z_n)|^t
= s(z_n)^{-t(p/q)} |h(z_n)|^t (1 - |z_n|^2)^t |h(z_n)|^{p/q} |g(z_n)|^t
= s(z_n)^{-t(p/q)} s(z_n)^t |g(z_n)|^t
= s(z_n)^{t/q} |g(z_n)|^t.
\]
This implies the theorem.

**Corollary 7.** Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then 
\[
\{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^1 \text{ if and only if }\inf_n \rho_n > 0.
\]
Proof. By Theorem 4, 
\[
\{s(z_n)^{1/p}f(z_n) ; f \in H^p(W)\} \supset \ell^1 \text{ if and only if }\frac{1}{\rho_n} s(z_n,p,W)^{1/p} s(z_n,q,W)^{-1/q} < \infty.
\]
Now Lemma 2 implies the corollary.

References


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