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Interpolation Of Weighted $\ell^q$ Sequences By $H^p$ Functions

by

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Abstract. Let \((z_n)\) be a sequence of points in the open unit disc \(D\) and \(\rho_n = \prod_{m \neq n} |(z_n - z_m)(1 - \bar{z}_m z_n)^{-1}| > 0\). Let \(a = (a_j)_{j=1}^{\infty}\) be a sequence of positive numbers and \(\ell^s(a) = \{(w_j) ; (a_j w_j) \in \ell^s\}\) where \(1 \leq s \leq \infty\). When \(1 \leq p \leq \infty\) and \(1/p + 1/q = 1\), we show that \(\{(f(z_n)) ; f \in H^p\} \supset \ell^s(a)\) if and only if there exists a finite positive constant \(\gamma\) such that \(\sum_{n=1}^{\infty} (a_n \rho_n)^{-t}(1 - |z_n|^2)^t |f(z_n)|^t \leq \gamma \|f\|_q (f \in H^q), \) where \(1/s + 1/t = 1\). As results, we show that \(\{(f(z_j)) ; f \in H^p\} \supset \ell^1(a)\) if and only if \(\sup (a_n \rho_n)^{-1}(1 - |z_n|^2)^{1/p} < \infty\), and \(\{(f(z_n)) ; f \in H^1\} \supset \ell^{\infty}(a)\) if and only if \(\sum (a_n \rho_n)^{-1}(1 - |z_n|^2)\delta_{z_n}\) is finite measure on \(D\). These are also proved in the case of weighted Hardy spaces.
§1. Introduction

$H^p$ ($0 < p \leq \infty$) denotes the usual Hardy space in the open unit disc $D$. In this paper, we assume that a sequence $(z_j)$ in $D$ satisfies that $\sum_{j=1}^{\infty} (1 - |z_j|) < \infty$, that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \frac{z_j - z}{1 - \overline{z_j}z}.$$  

Let

$$\rho_{k,n} = \prod_{j=1}^{n} \left| \frac{z_k - z_j}{1 - \overline{z_j}z_k} \right|, \quad 1 \leq k \leq n,$$

$$\rho_k = \prod_{j=1}^{\infty} \left| \frac{z_k - z_j}{1 - \overline{z_j}z_k} \right|.$$  

Then $\rho_{k,n} \geq \rho_{k,n+1}$ and $\lim_{n \to \infty} \rho_{k,n} = \rho_k$ for $k \geq 1$. We assume that $\rho_k > 0$ for $k = 1, 2, \ldots$.

For a positive sequence $a = (a_j)$, $\ell^s(a)$ denotes $\{(w_j) ; w_j \in \mathcal{C} \text{ and } \sum_{j=1}^{\infty} (a_j|w_j|)^s < \infty\}$ and $\ell^\infty(a)$ denotes $\{(w_j) ; w_j \in \mathcal{C} \text{ and } \sup_{1 \leq j < \infty} a_j|w_j| < \infty\}$. In this paper, we study the following problem: Find a necessary and sufficient condition on $(z_j)$ so that $\{(f(z_j)) ; f \in H^p\} \supset \ell^s(a)$ where $1 \leq p \leq \infty$ and $1 \leq s \leq \infty$.

Suppose $a_j = 1$ for all $j \geq 1$. When $p = s = \infty$, this was solved by L. Carleson [1]. That is, $\{(f(z_j)) ; f \in H^\infty\} \supset \ell^\infty$ if and only if inf $\rho_j > 0$. $(z_j)$ is called a uniformly separated sequence when inf $\rho_j > 0$. When $p = \infty$ and $1 \leq s < \infty$, A. K. Snyder [13] (cf. [7],[11]) proved that $\{(f(z_j)) ; f \in H^\infty\} \supset \ell^s$ if and only if inf $\rho_j > 0$. A. K. Snyder [13] and P. L. Duren and H. S. Shapiro [3] showed that there exists a sequence $(z_j)$ which is not uniformly separated, that is, inf $\rho_j = 0$ and has the property: $\{(f(z_j)) ; f \in H^p\} \supset \ell^\infty$ when $p \neq \infty$. B. A. Taylor and D. L. Williams [14] showed that for $1 \leq p \leq \infty$ $\{(f(z_j)) ; f \in H^p\} \supset \ell^\infty$ if and only if there exists a positive finite constant $\gamma$ such that $\sum_{j=1}^{\infty} \frac{1}{\rho_j} (1 - |z_j|^2)|g(z_j)| \leq \gamma\|g\|_q$ for all $g$ in $H^q$ and $1/p + 1/q = 1$.

Suppose $1 \leq p = s \leq \infty$. When $a_j = (1 - |z_j|^2)^{1/p}$ for all $j \geq 1$, this was solved by H. S. Shapiro and A. L. Shields [11]. That is, $\{(f(z_j)) ; f \in H^p\} \supset \ell^p(a)$ if and only if inf $\rho_j > 0$. When $a_j = \rho_j^2$ for all $j \geq 1$, J. P. Earl [4] showed that $\{(f(z_j)) ; f \in H^\infty\}$ contains $\ell^\infty(a)$ always. This was pointed out by A. M. Gleason (see [6]). On the other hand, when $a_j = \rho_j$ for all $j \geq 1$, T. Nakazi [10] showed that $\{(f(z_j)) ; f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $(z_j)$ is the union of a finite number of uniformly separated sequences. For a general weight $a = (a_j)$, J. D. McPhail [9] gave a necessary and sufficient condition about
we show that \( \{ f(\{z_j\}) \mid f \in H^p \} \supset \ell^p(\{a_j\}) \) where \( 1 \leq p \leq \infty \) and \( \{a_j\} \) is arbitrary weight. As a result, we show that \( \{ f(\{z_j\}) \mid f \in H^1 \} \supset \ell^s(\{a_j\}) \) if and only if \( \sum_{j=1}^{\infty} (a_j \rho_j)^{-\ell}(1 - |z_j|^2)^t < \infty \) where \( 1/s + 1/t = 1 \). Moreover, when \( 1 < p \leq \infty \) and \( \{a_j\} = (\rho_j^{-1}) \), we show that \( \{ f(\{z_j\}) \mid f \in H^p \} \supset \ell^p(\{a_j\}) \) if and only if \( \{z_j\} \) is a finite sum of uniformly separated sequences. This is a generalization of a result in [10] for \( p = \infty \).

In §3, when \( 1 \leq p \leq \infty \), we show that \( \{ f(\{z_j\}) \mid f \in H^p \} \supset \ell^1(\{a_j\}) \) if and only if \( \sup(a_j \rho_j)^{-1}(1 - |z_j|^2)^{1/p} < \infty \). As a result, a theorem of A. K. Snyder [13] follows, that is, \( \{ f(\{z_j\}) \mid f \in H^\infty \} \supset \ell^s \) if and only if \( \inf \rho_j > 0 \).

In §4, we give a necessary and sufficient condition about \( \{z_j\} \) for that \( \{ f(\{z_j\}) \mid f \in H^p \} \supset \ell^\infty(\{a_j\}) \). Put \( \mu = \sum_{j=1}^{\infty} (a_j \rho_j)^{-1}(1 - |z_j|^2)\delta_{z_j} \). Then \( \{ f(\{z_j\}) \mid f \in H^1 \} \supset \ell^\infty(\{a_j\}) \) if and only if \( \mu \) is a finite measure on \( D \), and \( \{ f(\{z_j\}) \mid f \in H^\infty \} \supset \ell^\infty(\{a_j\}) \) if and only if \( \mu \) is a Carleson measure on \( D \).

In §5, we give a necessary and sufficient condition about \( \{z_j\} \) for that \( \{ (s(z_j)f(\{z_j\})) \mid f \in H^p(W) \} \supset \ell^p \), where \( H^p(W) \) is a weighted Hardy space and \( s(z_j) = \inf \{ \int |f|^p W d\theta/2\pi \mid f(z_j) = 1 \} \). We assume only that \( \log W \) is in \( L^1 \). J.D. McPhail [9] studied such a problem when \( W \) satisfies the \( (A_p) \)-condition of Muckenhoupt.

Our interests in this paper are in the differences between interpolations for \( \ell^1(\{a_j\}) \) and \( \ell^\infty(\{a_j\}) \) and in the interpolation problems for weighted Hardy spaces. For example, it is very easy to prove that \( \{ f(\{z_j\}) \mid f \in H^\infty \} \supset \ell^1 \) if and only if \( \{z_j\} \) is uniformly separated.

§2. General results

In this section, we obtain a general result for interpolation problems for \( \ell^s(\{a_j\}) \) \( (1 \leq s \leq \infty) \) by \( H^p \) \( (1 \leq p \leq \infty) \). For \( 1 \leq j \leq n \), let

\[
B_n(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z} \quad \text{and} \quad B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}.
\]

If we put \( b_{nj} = B_{nj}(z_j) \), then

\[
\rho_{j,n} = |b_{nj}| \quad (1 \leq j \leq n).
\]

Suppose for \( n = 1, 2, \ldots \)

\[
f_n(z) = \sum_{j=1}^{n} b_{nj}^{-1} w_j B_{nj}(z).
\]
Then \( f_n \) is in \( H^\infty \) and \( f_n(z_j) = w_j \) (1 \( \leq j \leq n \)). Lemma 1 is essentially known.

**Lemma 1.** Let 1 \( \leq p \leq \infty \) and \( 1/p + 1/q = 1 \). Suppose \( w_j \) is a complex number for \( j = 1, 2, \ldots \). There exists a function \( f \) in \( H^p \) such that \( f(z_j) = w_j \) for \( j = 1, 2, \ldots \) if and only if there exists a positive finite constant \( \gamma \) such that for any \( n \geq 1 \) and for all \( g \) in \( H^q \),

\[
\left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1 - |z_j|^2)g(z_j) \right| \leq \gamma \|g\|_q.
\]

**Proof.** Put for \( n \geq 1 \)

\[
m_{p,n}(w) = \inf\{\|f_n + B_nh\|_p \ ; \ h \in H^p\}.
\]

Then by [2, p142],

\[
m_{p,n}(w) = \sup \left\{ \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1 - |z_j|^2)g(z_j) \right| \ ; \ g \in H^q \text{ and } \|g\|_q \leq 1 \right\}.
\]

There exists a function \( f \) in \( H^p \) such that \( f(z_j) = w_j \) for \( j = 1, 2, \ldots \) if and only if \( \sup_n m_{p,n}(w) < \infty \) because the unit ball of \( H^p \) is compact in the weak topology or the weak * topology. This implies the lemma.

**Theorem 1.** Let 1 \( \leq p \leq \infty \) and 1 \( \leq s \leq \infty \). \( \{f(z_n) \ ; \ f \in H^p\} \supset \ell^s(a) \) if and only if there exists a finite positive constant \( \gamma \) such that

\[
\left\{ \sum_{n=1}^{\infty} (a_n \rho_n)^{-t}(1 - |z_n|^2)^t|f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q
\]

for \( f \) in \( H^q \), where \( 1/p + 1/q = 1 \) and \( 1/s + 1/t = 1 \).

**Proof.** For the ‘only if’ part, since \( \{f(z_j) \ ; \ f \in H^p\} \supset \ell^s(a) \), by Lemma 1 there exists a positive finite constant \( \gamma \) such that for any \( n \geq 1 \)

\[
\sup_{w \in \ell^s(a)} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1 - |z_j|^2)g(z_j) \right| \leq \gamma \|g\|_q \ (g \in H^q)
\]

where \( w = (w_j) \) and \( \|w\| = \left( \sum_{j=1}^{\infty} |w_j a_j|^s \right)^{1/s} \). Hence for any \( n \geq 1 \)

\[
\left\{ \sum_{j=1}^{\infty} (a_j \rho_{nj})^{-t}(1 - |z_j|^2)^t|g(z_j)|^t \right\}^{1/t} \leq \gamma \|g\|_q \ (g \in H^q).
\]

Assuming \( \|g\|_q = 1 \),

\[
\sum_{j=1}^{\infty} (a_j \rho_{nj})^{-t}(1 - |z_j|^2)^t|g(z_j)|^t \leq \gamma^t.
\]
For any $\varepsilon > 0$, there exists a positive integer $n_j$ for each $j$ such that for all $n \geq n_j$

\[
(a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \frac{\varepsilon}{2j} \leq (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t
\]

because $\rho_{j,n} \geq \rho_{j,n+1}$ and $\lim_{n \to \infty} \rho_{j,n} = \rho_j$. Thus, \{$(f(z_j)) ; f \in H^p \} \supset \ell^s(a)$ if and only if for any $\varepsilon > 0$ and any $n \geq \max(n_1, \cdots, n_n)$

\[
\sum_{j=1}^{n} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \varepsilon \leq \sum_{j=1}^{n} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \leq \gamma^t
\]

This implies the 'only if' part.

For the 'if' part, by Lemma 1 it is sufficient to show that there exists a finite positive constant $\gamma$ such that for all $n \geq 1$

\[
\sup_{\|w\| \leq 1} \sup_{\|g\| \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma < \infty.
\]

In fact, for all $n \geq 1$

\[
\sup_{\|w\| \leq 1} \sup_{\|g\| \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \\
\leq \sup_{\|g\| \leq 1} \left\{ \sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} \\\n\leq \sup_{\|g\| \leq 1} \left\{ \sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} < \infty
\]

**Corollary 1.** Let $1 \leq s \leq \infty$. $\{ (f(z_n)) ; f \in H^1 \} \supset \ell^s(a)$ if and only if

\[
\sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2)^t < \infty
\]

where $1/s + 1/t = 1$. Hence, when $a = (a_n) = (\rho_n^{-1})$ it is always true that $\{ (f(z_n)) ; f \in H^1 \} \supset \ell^s(a)$.

Proof. The first part is clear by Theorem 1. When $a = (\rho_n^{-1})$, $\{ (f(z_n)) ; f \in H^1 \} \supset \ell^s(a)$ if and only if $\sum_{n=1}^{\infty} (1 - |z_n|^2)^t < \infty$. This implies the second part.

**Corollary 2.** Let $1 \leq p \leq \infty$, $1 \leq s \leq \infty$ and $a = (\rho_n^{-1})$. $\{ (f(z_n)) ; f \in H^p \} \supset \ell^s(a)$ if and only if there exists a finite positive constant $\gamma$ such that

\[
\left\{ \sum_{n=1}^{\infty} (1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q
\]
for $f$ in $H^q$, where $1/p + 1/q = 1$ and $1/s + 1/t = 1$. When $1 < p \leq \infty$, $\{(f(z_n)) ; f \in H^p\} \supset \ell^p(a)$ if and only if $(z_n)$ is a finite sum of uniformly separated sequences.

Proof. The first part is clear by Theorem 1. The second part follows from the first one and [8].

In Corollary 2, when $1 < p \leq \infty$ and $1 < s \leq \infty$ and $s > p$, if $\{(f(z_n)) ; f \in H^p\} \supset \ell^s(a)$ then $(z_n)$ is a finite sum of uniformly separated sequences but the converse is not true. When $s < p$, if $(z_n)$ is a finite sum of uniformly separated sequences then $\{(f(z_n)) ; f \in H^p\} \supset \ell^s(a)$ but the converse is not true.

§3. Interpolations for $\ell^1(a)$

$\ell^1(a)$ is the smallest sequence space among $\ell^p(a)$ ($1 \leq p \leq \infty$) for the same $a = \{a_j\}$. Then the interpolations for $\ell^1(a)$ are very special as the following shows.

The case of $p = \infty$ in Corollary 3 was proved by A.Snyder [13] (see [7], [11]). Corollary 4 is due to O. Hatori [7].

**Theorem 2.** Let $1 \leq p \leq \infty$. $\{(f(z_n)) ; f \in H^p\} \supset \ell^1(a)$ if and only if

$$\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2)^{1/p} < \infty.$$

Proof. By Theorem 1, $\{(f(z_n)) ; f \in H^p\} \supset \ell^1(a)$ if and only if there exists a finite positive constant $\gamma$ such that

$$\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2) |f(z_n)| \leq \gamma \|f\|_q$$

for all $f$ in $H^q$. For each $n$, $\sup_{\|f\|_q=1} |f(z_n)| = (1 - |z_n|^2)^{-1/q}$ by [2, p144] and so the theorem follows.

**Corollary 3.** Let $1 \leq p \leq \infty$. $\{(f(z_n)) ; f \in H^p\} \supset \ell^1$ if and only if $\sup_n (1 - |z_n|^2)^{1/p} < \infty$. Hence if $p = \infty$, $\{(f(z_n)) ; f \in H^\infty\} \supset \ell^1$ if and only if $\inf_n \rho_n > 0$.

**Corollary 4.** Let $1 \leq p \leq \infty$. $\{(1 - |z_n|^2)^{1/p} f(z_n) ; f \in H^p\} \supset \ell^1$ if and only if $\inf_n \rho_n > 0$.

Proof. Note that $\{(1 - |z_n|^2)^{1/p} f(z_n) ; f \in H^p\} \supset \ell^1(a)$ if and only if $\{(f(z_n)) ; f \in H^p\} \supset \ell^1(a)$ and $a = ((1 - |z_n|^2)^{1/p})$.

**Corollary 5.** Let $1 \leq p \leq \infty$. For any $(z_n)$, $\{(f(z_n)) ; f \in H^p\} \supset \ell^1(a)$ where $a = (\rho_n^{-1})$. 7
Hence \( \rho \) with 
\[
\sum_{j=0}^{\infty} \alpha_j = 1 \quad \text{and only if there exists a finite positive constant} \quad \rho \quad \text{of Corollary 6 is known in} \ [10].
\]

Let \( \{b_j\} \) be a uniformly separated sequence in \( D \) such that \( 0 < \text{Re} b_j < 1 \) and \( \text{Im} b_j \searrow 0 \). For \( j \geq 1 \), put \( z_{j-1} = b_j \) and \( z_j = b_j \). Let \( B \) be the Blaschke product associated with \( \{z_n\} \). Then for each \( j \)
\[
B = \frac{z - b_j}{1 - b_j \bar{z}} \frac{z - \bar{b}_j}{1 - \bar{b}_j \bar{z}} B_{1j} B_{2j}
\]
where \( B_{1j} \) (or \( B_{2j} \)) is a Blaschke product with zeros \( \{b_{\ell} \}_{\ell \neq j} \) (or \( \{\bar{b}_{\ell} \}_{\ell \neq j} \)). Then
\[
\rho_{2j-1} = \left| b_j - \bar{b}_j \right| \prod_{\ell \neq j} \left| b_j - b_{\ell} \right| \prod_{\ell \neq j} \left| \frac{b_j - \bar{b}_\ell}{1 - b_j b_\ell} \right|
\]
and
\[
\rho_{2j} = \left| \bar{b}_j - b_j \right| \prod_{\ell \neq j} \left| \frac{\bar{b}_j - b_{\ell}}{1 - b_j b_\ell} \right| \prod_{\ell \neq j} \left| \frac{\bar{b}_j - \bar{b}_\ell}{1 - b_j b_\ell} \right|
\]
Hence \( \rho_{2j-1} = \rho_{2j} \) for \( j \geq 1 \) and
\[
\delta^2 \left| \frac{\bar{b}_j - b_j}{1 - |b_j|^2} \right|^2 \leq \rho_{2j} = \rho_{2j-1} \leq \left| \frac{\bar{b}_j - b_j}{1 - |b_j|^2} \right|^2 \quad (j \geq 1)
\]
where
\[
0 < \delta = \min \left\{ \inf_j \left| \prod_{\ell \neq j} \frac{b_j - b_{\ell}}{1 - b_j b_{\ell}} \right|, \inf_j \left| \prod_{\ell \neq j} \frac{b_j - \bar{b}_{\ell}}{1 - b_j b_{\ell}} \right| \right\}
\]
Hence
\[
\frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|} \leq \frac{\rho_n}{\rho} \leq \delta^2 \frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|}.
\]
Thus \( \{(f(z_n)) \ ; \ f \in H^p \} \supset \ell^1 \) if and only if \( \sup_n (1 - |z_n|^2)^{1+1/p}/|z_n - \bar{z}_n| < \infty \).

§4. Interpolations for \( \ell^\infty(a) \)

\( \ell^\infty(a) \) is the largest sequence space among \( \ell^p(a) \) \( (1 \leq p \leq \infty) \) for the same \( a = (a_j) \). Then the interpolations for \( \ell^\infty(a) \) are special as the following shows. The case of \( p = \infty \) of Corollary 6 is known in [10].

**Theorem 3.** Let \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \), \( \{(f(z_n)) \ ; \ f \in H^p \} \supset \ell^\infty(a) \) if and only if there exists a finite positive constant \( \gamma \) such that
\[
\sum_n (a_n \rho_n)^{-1}(1 - |z_n|^2)|f(z_n)| \leq \gamma \|f\|_q
\]
for all \( f \) in \( H^q \). When \( p = 1 \), \( \{(f(z_n)) \ ; \ f \in H^1 \} \supset \ell^\infty(a) \) if and only if \( \mu = \sum_n (a_n \rho_n)^{-1}(1 - |z_n|^2)\delta_{z_n} \) is a finite measure on \( D \). When \( p = \infty \), \( \{(f(z_n)) \ ; \ f \in H^\infty \} \supset \ell^\infty(a) \) if and only if \( \mu = \sum_n (a_n \rho_n)^{-1}(1 - |z_n|^2)\delta_{z_n} \) is a Carleson measure on \( D \).
Corollary 6. Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$ and $a = (p_n^{-1})$. $\{(f(z_n)) ; f \in H^p \} \supset \ell^\infty(a)$ if and only if there exists a finite positive constant $\gamma$ such that

$$\sum_n (1 - |z_n|^2)|f(z_n)| \leq \gamma \|f\|_q$$

for all $f$ in $H^q$.

(1) When $p = 1$, for any $(z_n)$, $\{(f(z_n)) ; f \in H^1 \} \supset \ell^\infty(a)$.

(2) When $p = \infty$, $\{(f(z_n)) ; f \in H^\infty \} \supset \ell^\infty(a)$ if and only if $(z_n)$ is a finite union of uniformly separated sequences.

(3) When $1 < p < \infty$, there exists a sequence $(z_n)$ in $D$ such that $\{(f(z_n)) ; f \in H^p \} \supset \ell^\infty(a)$ and $(z_n)$ is not a union of finitely many uniformly separated sequences. If $\sum_{n=1}^\infty (1 - |z_n|^2)^{1/p} < \infty$, then $\{(f(z_n)) ; f \in H^p \} \supset \ell^\infty$. This was proved by B. A. Taylor and D. L. Williams [14].

§5. Weighted Hardy space

Let $W$ be a nonnegative function in $L^1$ with $\log W \in L^1$ and $1 \leq p < \infty$. $H^p(W)$ denotes the closure of the set of all analytic polynomials in $L^p(W) = L^p(Wd\theta/2\pi)$. $H^p(W)$ is called a weighted Hardy space. For $b \in D$, put

$$s(b) = s(b, p, W) = \inf \left\{ \int |f|^p Wd\theta/2\pi ; f(b) = 1 \right\}.$$ 

Let $h$ be an outer function in $H^p$ such that $|h|^p = W$.

Lemma 2. For $1 \leq p < \infty$ and $b \in D$,

$$s(b, p, W) = (1 - |b|^2) \exp(\log W)\hat{\sim}(b)$$

$$= (1 - |b|^2) |h(b)|^p,$$

where $(\log W)\hat{\sim}(b)$ denotes the Poisson integral of $\log W$ at $b$.

Proof. It is well known (cf. [5, p136]) that $s(0, p, W) = \exp \int_0^{2\pi} \log Wd\theta/2\pi$. It is easy to show the lemma using $f(b) = f \circ \phi_b(0)$, where $\phi_b(z) = (z + b)/(1 + \bar{b}z)$.

Lemma 3. Suppose $(z_j)$ is a sequence of points in $D$. For $1 \leq p < \infty$ and $1 \leq s < \infty$, $\{(s(z_j, p, W)^{1/p}f(z_j)) ; f \in H^p(W) \} \supset \ell^s$ if and only if $\{(F(z_j)) ; F \in H^p \} \supset \ell^s(a)$, where $a = (a_j)$ and $a_j = s(z_j)^{1/p}/h(z_j)$. 

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Proof. Since $H^p(W) = h^{-1}H^p$, $f \in H^p(W)$ if and only if $f = h^{-1}F$ and $F \in H^p$. For each $j$, $s(z_j)^{1/p}f(z_j) = w_j$ if and only if $F(z_j) = h(z_j)w_j/s(z_j)^{1/p}$ if and only if $F(z_j) = \zeta_j$, $w_j = a_j\zeta_j$. $(w_j) \in \ell^p$ if and only if $(\zeta_j) \in \ell^s(a)$. Now the lemma follows.

**Theorem 4.** Let $1 \leq p < \infty$, $1 \leq s \leq \infty$, and $1/p + 1/q = 1/s + 1/t = 1$. Then, \{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^s$ if and only if

$$\{\sum_{n=1}^{\infty} \frac{1}{\rho_n} s(z_n)^{1/q} |g(z_n)|^t \}^{1/t} \leq \gamma \|g\|_{H^q(W)}$$

for $g$ in $H^q(W)$.

Proof. By Lemma 3, \{(s(z_n)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^s$ if and only if \{(F(z_n)) ; F \in H^p\} \supset \ell^s(a)$, where $a_n = s(z_n)^{1/p}/|h(z_n)|$. By Theorem 1, this is equivalent to saying that there exists a finite positive constant $\gamma$ such that

$$\{\sum_{n=1}^{\infty} \frac{1}{\rho_n} (1 - |z_n|^2)^t |G(z_n)|^t \}^{1/t} \leq \gamma \|G\|_q$$

for $G \in H^q$. Since $H^q(W) = h^{-p/q}H^q$, $g \in H^q(W)$ if and only if $g = h^{-p/q}G$ and $G \in H^q$. Hence $\|g\|_{H^q(W)} = \|G\|_{H^q}$ and for each $n \geq 1$

$$a_n^{-t}(1 - |z_n|^2)^t |G(z_n)|^t = s(z_n)^{-t/p} |h(z_n)|^t (1 - |z_n|^2)^t |h(z_n)|^{p/q} |g(z_n)|^t$$

$$= s(z_n)^{-t/p} (1 - |z_n|^2)^t |h(z_n)|^{t(q+p)/q} |g(z_n)|^t$$

$$= s(z_n)^{-t/p} s(z_n)^t |g(z_n)|^t$$

$$= s(z_n)^{t/q} |g(z_n)|^t.$$  

This implies the theorem.

**Corollary 7.** Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then \{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^1$ if and only if $\inf_{n} \rho_n > 0$.

Proof. By Theorem 4, \{(s(z_n,p,W)^{1/p}f(z_n)) ; f \in H^p(W)\} \supset \ell^1$ if and only if

$$\sup_n \frac{1}{\rho_n} s(z_n,p,W)^{1/p} s(z_n,q,W)^{-1/q} < \infty.$$  

Now Lemma 2 implies the corollary.

**References**


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