Proceedings of Sapporo Guest House Minisymposium on NONLINEAR WAVE EQUATIONS

Edited by T. Ozawa

Sapporo, November 1999

Partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, Science, Sports, and Culture (Monbusho).
文部省科学研究費補助金（基盤研究 A (1) 課題番号 09304012)

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The place of minisymposium:

Sapporo Guest House, 8 minutes walk from Sumikawa(Subway station)
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Energy Scattering for Nonlinear Klein-Gordon and Schrödinger Equations in Spatial Dimensions 1 and 2

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1. Introduction

In this talk, we consider asymptotic behavior of solutions to the nonlinear Klein-Gordon equations (NLKG) and the nonlinear Schrödinger equations (NLS):

$$\ddot{u} - \Delta u + u + |u|^p u = 0, \quad \text{(NLKG)}, \tag{1}$$

$$i\dot{u} - \Delta u + |u|^p u = 0, \quad \text{(NLS)}, \tag{2}$$

where $u = u(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$ and $p > 1$. For any function $u(t,x)$, denote by $\mathrm{eq}(u)$ the left hand side of the equation, denote by $\mathrm{eq}_L(u) := \mathrm{eq}(u) - |u|^p u$ the linear (free) part, and

$$\mathbf{u} := \begin{cases} (u, \sqrt{1-\Delta}^{-1}\dot{u}), & \text{(for NLKG)} \\ u, & \text{(for NLS)} \end{cases} \tag{3}$$

We have the following conservation law of the energy:

$$E(u;t) := \int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 + |\mathbf{u}|^2 + \frac{2|u|^{p+2}}{p+2} = E(u;0). \tag{4}$$

Let $u_\pm(t,x)$ and $v(t,x)$ satisfy $\mathrm{eq}(u_\pm) = 0 = \mathrm{eq}_L(v)$ and

$$\lim_{t\to\pm\infty} \|\mathbf{v}(t) - \mathbf{u}_\pm(t)\|_{H^1} = 0. \tag{5}$$

Then the wave operators are given by the correspondences $W_\pm : \mathbf{v}(0) \mapsto \mathbf{u}_\pm(0)$. Our main result is the asymptotic completeness of the wave operators, namely that they are well-defined as homeomorphisms from H^1 into itself.

In the following, $C(\cdot, \dots)$ denotes any positive continuous function, whose explicit form we can write but we will not for the sake of simplicity.

2. Known results

First we mention the known results on the asymptotic completeness.

1. [6, 7, 11, 12] Let $n \geq 3$ and $4/n < p < 4/(n-2)$. Then we have the asymptotic completeness for (NLKG) and (NLS) in the energy class H^1.
2. [18] Let $n \geq 3$ and $p = 4/(n-2)$. Then we have the asymptotic completeness for (NLKG) in the energy class.
3. [10, 25, 8] Let $n \in \mathbb{N}$, $p \geq 8/(\sqrt{(n+2)^2 + 8n} + n - 2)$ and $(n-2)p < 4$. Then we have the asymptotic completeness for (NLS) in $\Sigma := \{\varphi \in H^1 \mid x\varphi \in L^2\}$.

3. Main result

Theorem 1. *Let $n \in \mathbb{N}$ and $(n-2)p < 4 < np$. Then we have the asymptotic completeness for* (NLKG) *and* (NLS) *in the energy class.*

The large data scattering of NLKG for $n \leq 2$ was one of the major open problems in [23, pp. 247]. Here we consider a single power for simplicity, but our proof can be applied to more general nonlinearity $f(u)$ satisfying

$$\exists F : \mathbb{R} \to \mathbb{R} \text{ s.t. } 2f(u) = F'(|u|)\frac{u}{|u|}, \quad F(0) = f(0) = 0, \tag{6}$$

$$|f(u) - f(v)| \leq C|u - v|(|u|^{p_1} + |v|^{p_1} + |u|^{p_2} + |v|^{p_2}), \tag{7}$$

for some $p_1 \leq p_2$ satisfying $(n-2)p_2 < 4 < np_1$, and

$$G(u) := f(u)\overline{u} - F(|u|) \geq 0. \tag{8}$$

In the preceding works [6, 7, 11, 12, 4], it was needed that

$$G(u) \geq C \min(|u|^2, |u|^{p_3}), \quad \exists p_3 > 2. \tag{9}$$

If we write $F(|u|) = V(|u|)|u|^2$, (8) is equivalent to $V' \geq 0$. We remark that if $V(r) < 0 = V(0)$ for some $r > 0$, there exist standing wave solutions to (NLKG) and (NLS) (see [2]), so that the asymptotic completeness does not hold.

4. Difficulties in low spatial dimensions

There were two difficulties in proving the asymptotic completeness for $n < 3$. The first problem was that we can not prove the Morawetz estimate, which has been essentially the only a priori estimate for global space-time integral used to prove the asymptotic completeness. The second problem is on the decay order $t^{-n/2}$ of the free evolution. For $n \geq 3$, it is integrable on $(1, \infty)$, so that we can deduce only from the boundedness of the solution that the nonlinear interaction in the distant future or past has little influence on the behavior of the solution . Such an argument was essentially used in the preceding works, but it becomes invalid if $n < 3$, because the decay order is no longer integrable. We overcome the first difficulty by a new Morawetz type estimate which holds in any spatial dimension and independent of the nonlinearity, so that we can also improve the generality of the nonlinearity (regarding (8)). To avoid the second difficulty, we employ a new idea inspired by Bourgain [5]. Separating localized energy into rapidly decreasing free solutions, we can reduce the problem to that for small energy data.

5. New Morawetz type estimates

In this section, we deal with the equation with the general form of nonlinearity

$$\mathrm{eq}(u) + f(u) = 0, \tag{10}$$

to show how we can replace the assumption (9) with (8). Assume (6) and (7). The Morawetz estimate is

$$\iint_{\mathbb{R}^{1+n}} \frac{G(u)}{|x|} dxdt \leq CE(u), \tag{11}$$

where u is any solution to (10) and $n \geq 3$. Our estimate is the following, which has two advantages. First, our estimate holds in any spatial dimension. Secondly, our estimate does not depend on the nonlinearity, as long as (8) is satisfied.

Lemma 2. *Let $n \in \mathbb{N}$ and assume (6), (7) and (8). Then, for any finite enrgy solution u and p satisfying $(n-2)p \leq 4 \leq np$, we have*

$$\iint_K \frac{t^2|u|^{p+2}}{|(t,x)|^3} dx\, dt \leq C(p, E(u)), \tag{12}$$

where $K = \{(t,x) \mid |x| < |t|\}$ for (NLKG) *and $K = \mathbb{R}^{1+n}$ for* (NLS).

Any Morawetz-type estimate or conservation law is based on some integral identities derived by variations of the Lagrangian. We mention a general formula for such identities. First we have to prepare some notation.

$$\langle a, b\rangle := \Re(a\bar{b}), \quad \partial = (\partial_t, \nabla_x), \quad \mathcal{D} = \begin{cases} (-\partial_t, \nabla_x), & \text{(for NLKG)} \\ (-i/2, \nabla_x), & \text{(for NLS)} \end{cases} \tag{13}$$

$$2\ell(u) = \begin{cases} -|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + F(u), & \text{(for NLKG)} \\ \langle i\dot{u}, u\rangle + |\nabla u|^2 + F(u), & \text{(for NLS)} \end{cases} \tag{14}$$

$\ell(u)$ is the Lagrangian density associated to the equation $\mathrm{eq}(u) = 0$. The operator $\mathcal{D}$ naturally appears from the variation of ℓ:

$$\delta_v \ell(u) := \lim_{\varepsilon \to 0} \frac{\ell(u + \varepsilon v) - \ell(u)}{\varepsilon} = \langle \mathrm{eq}(u), v\rangle + \partial \cdot \langle \mathcal{D}u, v\rangle. \tag{15}$$

Using this identity, we can easily obtain the following formula, where $h : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ and $q : \mathbb{R}^{1+n} \to \mathbb{R}$ are sufficiently smooth.

$$\begin{aligned} \langle \mathrm{eq}(u), h \cdot \mathcal{D}u + qu\rangle &= -\partial \cdot \langle \mathcal{D}u, h \cdot \mathcal{D}u + qu\rangle + \mathcal{D} \cdot \left(h\ell(u) + \frac{|u|^2}{2}\partial q \right) \\ &\quad + \langle \mathcal{D}u, (\partial h)\mathcal{D}u\rangle - \frac{|u|^2}{2}\mathcal{D} \cdot \partial q + (2q - \mathcal{D} \cdot h)\ell(u) + G(u)q. \end{aligned} \tag{16}$$

Now let

$$h := \frac{(t,x)}{|(t,x)|}, \quad q := \Re \frac{\mathcal{D} \cdot h}{2}, \tag{17}$$

and integrate the real part of (16) in

$$K_1 := \begin{cases} \{(t,x) \mid |t|^2 > |x|^2 + 1\}, & \text{(for NLKG)} \\ \{(t,x) \mid |t| > 1\}, & \text{(for NLS)} \end{cases}. \tag{18}$$

Then we have

$$\iint_{K_1} \langle \mathcal{D}u, (\partial h)\mathcal{D}u\rangle - \frac{|u|^2}{2}\Re\mathcal{D} \cdot \partial q + G(u)q\, dxdt \leq CE(u). \tag{19}$$

Since $q \geq 0$ and $|\Re\mathcal{D} \cdot \partial q| \leq C/t^3$, we obtain

$$\iint_{K_1} \frac{|t\nabla u - x\mathcal{D}_0 u|^2}{|(t,x)|^3} dxdt \leq CE(u), \tag{20}$$

which comes from the first term in (19). Such an estimate was first derived in [17, Proposition 4.4] for (NLKG) with $n \geq 3$. Now we use the following Sobolev type inequality.

Lemma 3. *Let $n \in \mathbb{N}$. Let $\chi(x)$ and $\lambda(x)$ be real-valued functions. Let $p > 0$ and $q := np/2$. Then for any complex-valued $u(x) \in H^1(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \chi^2 |u|^{p+2} dx \leq C(p) \|u\|_{L^q}^p \int_{\mathbb{R}^n} \chi^2 |\nabla u + i\lambda u|^2 + |u \nabla \chi|^2 dx. \tag{21}$$

We can apply this inequality directly to (20) in the NLS case. In the NLKG case, we apply it to the function $v(\tau, x) := u(\sqrt{\tau^2 + |x|^2}, x)$ and use the boundedness of the energy on the hyperboloids. Then we obtain the estimate on K_1. Indeed, the estimate on $K \setminus K_1$ is trivial from the Hardy inequality: $\||x|^{-\theta} u\|_{L^2} \leq C \|u\|_{H^1}$, where $0 \leq \theta \leq 1$ and $\theta < n/2$.

6. Global space-time estimates and energy concentration

For simplicity, in this section we consider (NLKG) for $n < 3$ and (NLS) for any n. For (NLKG) with $n \geq 3$, we have to change the exponents of the space-time norms below, though the arguments are essentially the same. The asymptotic completeness means that at time infinity any free solution can be approximated by a nonlinear solution and any nonlinear solution can be approximated by a free solution. That is possible because the nonlinear interaction term loses its effect as $|t|$ tends to infinity, since $|u|$ decays by the dispersion of wave. However, we can not expect any uniform decay estimate for the solutions like

$$\|u(t)\| \leq C(E(u)) t^{-\alpha}, \tag{22}$$

because our setting is invariant under space-time translations and time inversion. Since the decay property of the solutions comes from the finiteness of the energy, it is natural that the decay property is also described in (space-time) integral forms. We introduce two space-time norms:

$$\|u\|_{(K;I)} := \|u\|_{L^\rho(I; B^\sigma_{\rho,2}(\mathbb{R}^n))}, \quad \|u\|_{(X;I)} := \|u\|_{L^q(I \times \mathbb{R}^n)}, \tag{23}$$

where $\rho := 2 + 4/n$, $q := p(n+2)/2$, $\sigma = 1/2$ for (NLKG), $\sigma = 1$ for (NLS) and $B^\sigma_{\rho,2}$ is the inhomogeneous Besov space (cf. [3]). We know from the Strichartz estimates

$$\|v\|_{(K;\mathbb{R})} + \|v\|_{(X;\mathbb{R})} \leq C \|\mathbf{v}(0)\|_{H^1}, \tag{24}$$

for any linear solution v. Moreover, let w be the solution for the linear inhomogeneous equation $\mathrm{eq}_L(w) = -|v|^p v$ and $\mathbf{w}(0) = 0$. We have also by the Strichartz estimate and well-known power estimates,

$$\|\mathbf{w}\|_{L^\infty_t((S,T);H^1)} + \|w\|_{(K;(S,T))} + \|w\|_{(X;(S,T))} \leq \|v\|^p_{(X;(S,T))} \|v\|_{(K;(S,T))}. \tag{25}$$

Since $\|v\|_{(K;(S,T))}$ vanishes as $S \to \infty$ by (24), (25) means that the nonlinear interaction loses its effect for linear solutions. It is easy to construct the wave operators by a fixed point argument using such estimates as (25). Thus, the asymptotic completeness will immediately follow if we can prove that global space-time norms such as (24) are finite also for the nonlinear solutions:

$$\|u\|_{(K;\mathbb{R})} + \|u\|_{(X;\mathbb{R})} \leq C(E(u)). \tag{26}$$

This can be derived by a standard argument from the following weaker estimate:

$$\|u\|_{(X;\mathbb{R})} \le C(E(u)). \tag{27}$$

Our objective is hereafter (27). Indeed, it is the hardest step to prove the global estimate for the nonlinear solutions in the proof of asymptotic completeness, for we can not approximate the solution by one free solution as in the construction of local solutions and wave operators or as in the small data analysis. We can divide the time axis into many intervals such that we can approximate the solution on each interval by a free solution respectively. But how can we get any asymptotic imformation from those many free solutions? Bourgain [5] considered instead the space-time distribution of the energy density of u on each time interval. More precisely, we have the following lemma essentially due to Bourgain (here the situation is simpler because we are considering the subcritical case).

Lemma 4. *Let* $\mathrm{eq}(u) = 0$, $E(u) = E < \infty$ *and* $\|u\|_{(X;I)} = \eta$ *for some interval* I. *There exists a constant* η_0 *such that if* $\eta \in (0, \eta_0]$ *then we have a subinterval* $J \subset I$, $X \in \mathbb{R}^n$ *and* $R > 0$ *such that for any* $t \in J$ *and* $s \ge 1$ *we have*

$$\int_{|x-X|<R} |u|^s dx > C(E, \eta, s), \tag{28}$$

$|J| > C(E, \eta)$ *and* $R < C(E, \eta)$.

Outline of proof. Let v be the free solution with the same data at the top of I. Then, by (25) we have

$$\|u\|_{(K;I)} \le \|v\|_{(K;I)} + C\|u\|^p_{(X;I)}\|u\|_{(K;I)} \le C(E) + C\eta^p\|u\|_{(K;I)}. \tag{29}$$

From this, we have $\|u\|_{(K;I)} \le C(E)$ if η_0 is sufficiently small. By the interpolation inequality and the Sobolev embedding, we have

$$\eta = \|u\|_{(X)} \le C\|u\|^{1-\rho/q}_{(B)}\|u\|^{\rho/q}_{(K)} \le C(E)\|u\|^{1-\rho/q}_{(B)}, \tag{30}$$

where we denote $(B) := L^\infty(B^{1-n/2-\varepsilon}_{\infty,\infty})$ with a constant $\varepsilon > 0$ small enough for the above interpolation to hold. Thus we obtain $\|u\|_{(B)} > C(E, \eta)$, which means by the definition of (B),

$$2^{N(1-n/2-\varepsilon)}|(\varphi_N * u)(T, X)| > C(E, \eta), \tag{31}$$

for some $-1 \le N \in \mathbb{Z}$, $T \in I$ and $X \in \mathbb{R}^n$, where $\{\varphi_j\}_{j=-1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ is a Paley-Littlewood partition of $\delta(x)$ satisfing $\varphi_j(x) = 2^{jn}\varphi_0(2^j x)$ for $j > 0$. On the other hand by the Sobolev embedding we have

$$2^{N(1-n/2)}\|\varphi_N * u(t)\|_{L^\infty} \le C\|u(t)\|_{H^1} \le C(E), \tag{32}$$

so that $N < C(E, \eta)$. Moreover we have by the equation,

$$\|\varphi_N * (u(t) - u(T))\|_{L^\infty} \le 2^{(1+n/2)N}\|u(t) - u(T)\|_{H^{-1}} \le C(E, \eta)|t - T|, \tag{33}$$

so that the estimate (31) remains valid for $T \in J$ with some interval J of length $> C(E, \eta)$. Since $|\varphi_N(x)|$ is sufficiently small for $|x| > C2^{-N}$, we obtain the desired result from (31). □

Now let I be a finite interval and let us estimate $\|u\|_{(X;I)}$. First we divide I into subintervals $\{I_j\}_{j=1}^N$ such that $\|u\|_{(X;I_j)} = \eta_0$ on each subinterval. Applying the above lemma on each subinterval, we obtain $|J_j| > C(E)$, $X_j \in \mathbb{R}^n$ and $R < C(E)$ such that for any $t \in J_j \subset I_j$ and any $2 \le s \le q$ we have

$$\int_{|x-X_j|<R} |u|^s dx > C(E) =: \nu. \tag{34}$$

Let $T_j := \inf J_j$, $B_j := \{(T_j, x) \mid |x - X_j| < R\}$ and $K_j := \{(t,x) \mid t \ge T_j, |x - X_j| < M|t - T_j| + 3R\}$, where $M = 1$ for (NLKG) and we take $M = C(E)$ sufficiently large for (NLS) such that the loss of the L^2 norm inside K_j is at most $\nu/2$. We can choose $P \subset \{1, \dots, N\} =: U$ such that

(i). $k, j \in P$, $k \ne j \Longrightarrow B_j \not\subset K_k$.
(ii). $\forall j \in U$, $\exists k \in P$, $B_j \subset K_k$.

From (i) and the energy propagation estimate, we have $E \ge \#P\nu/2$, so that $\#P < C(E)$. Using (ii) and the Morawetz type estimate, we have

$$\begin{aligned} C(E) &\ge \sum_{k\in P} \int_{K_k} \frac{|u|^q}{M|t - T_k| + R} dx dt \\ &\ge \sum_{j\in U} \frac{\nu |J_j|}{C(E)(|T_j - T_1| + 1)} \ge \frac{C(E)\log N}{\max_j |I_j| + 1}, \end{aligned} \tag{35}$$

so that $\max_j |I_j| + 1 \ge C(E)\log N \ge C(E)\log\|u\|_{(X;I)}$. Now assume that $\|u\|_{(X;I)}$ is very large. Then, there exists a very long I_j with $\|u\|_{(X;I_j)} = \eta_0$ fixed, which means that the mean density in I_j is very low. Nevertheless, we have $(T_j, X_j) \in I_j \times \mathbb{R}^n$, where exists a certain amount of energy $\nu > C(E)$ in a fixed radius $R < C(E)$. Now we want to extract a subinterval where the space-time norms are very small, without losing the localized energy ν. To this end, we divide $I_j =: (S, T)$ into further subintervals as follows. We suppose that $T_j \le (S + T)/2$. Otherwise the time direction should be reversed in the following argument. Let $\Lambda > 1$, $S_k := T_j - R + (M\Lambda)^k R$, $H_k := (S_k, S_{k+1})$. Let $A \in \mathbb{N}$ and assume that for $k \le 3A$ we have $S_k \in I_j$. Then, there exists some $k < 3A$ such that $\|u\|_{(X;H_k)} \le \eta_0/A^{1/q}$ and $\|\mathbf{u}\|_{(K;H_k)} \le C(E)/A^{1/\rho}$. By the energy propagation estimate we have

$$\int_{|x-X_j|<R'} e(u; S_k) dx \ge \nu/2, \tag{36}$$

where $e(u;t)$ denotes the energy density and $R' := R + M(S_k - T_j) \le M(M\Lambda)^k R \le |H_k|/(\Lambda - 1)$. If $|I_j|$ is very large, we can take Λ and A also large. Thus, for any $\varepsilon > 0$, there exists $\mathcal{N} \le C(E, \varepsilon)$ such that if $\|u\|_{(X;I)} > \mathcal{N}$ then we have a subinterval $J = (S, T) \subset I$, $X \in \mathbb{R}^n$ and $R > 1$ such that $\|u\|_{(X;J)} + \|\mathbf{u}\|_{(K;J)} < \varepsilon$, $R < \varepsilon|J|$ and $\int_{|x-X|<R} e(u;S)dx \ge \nu/2$. Then, we can separate the energy around (S, X) by a free solution v such that

$$E(v;S) \le C\nu, \quad E(u - v; S) \le E - \nu/3, \quad \operatorname{diam}\operatorname{supp}\mathbf{v}(S) \le CR. \tag{37}$$

Using the support property of $\mathbf{v}(S)$ and the decay estimate for the free evolution, we have for $t > T$,

$$\|v(t)\|_{B^{-n-3}_{\infty,2}} \le C|J|^{-n/2}\|\mathbf{v}(S)\|_{B^{-1}_{1,2}} \le C(R/|J|)^{n/2}\|\mathbf{v}(S)\|_{H^1} \le C\varepsilon^{n/2}\sqrt{\nu}. \tag{38}$$

Interpolating with the Strichartz estimate, we obtain $\|v\|_{(X;(T,\infty))} \leq C(E)\varepsilon^C$. By the energy identity, if $\varepsilon < \sqrt{\nu}$ we have $E(u-v;T) \leq E(u-v;S) + C(E)\nu^{1+p/2}$. We may assume that ν is so small that $C(E)\nu^{1+p/2} < \nu/12$, and then we obtain $E(u-v;T) < E - \nu/4$. Now we can reduce the energy level by the following perturbation lemma essentially due to Bourgain.

Lemma 5. *Let* $\mathrm{eq}(u) = \mathrm{eq}(w) = \mathrm{eq}_L(v) = 0$ *and* $\mathbf{u}(0) = \mathbf{v}(0) + \mathbf{w}(0)$. *Let* $E(u), E(w) \leq E$ *and* $\|w\|_{(X;(0,\infty))} < M$. *Then there exists* $\varepsilon = \varepsilon(E,M) > 0$ *such that if* $\|v\|_{(X;(0,\infty))} < \varepsilon$, *we have* $\|u\|_{(X;(0,\infty))} < C(E,M)$.

Now we prove the global estimate (27). It is well-known for the solutions with sufficiently small energy. We use induction on the energy E. Suppose that for any solution u with $E(u) \leq E - \nu(E)/4$ we have $\|u\|_{(X;\mathbb{R})} < M$. Let u be a solution with $E(u) \leq E$. Take $\tau < \tau'$ such that $\|u\|_{(X;(-\infty,\tau))} = \|u\|_{(X;(\tau,\tau'))} = \|u\|_{(X;(\tau',\infty))}$. By the above argument, there exists $\mathcal{N} = \mathcal{N}(E,M)$ such that if $\|u\|_{(X;(\tau,\tau'))} > \mathcal{N}$ then we have some $T \in (\tau,\tau')$ and a free solution v satisfying $E(u-v;T) < E - \nu/4$ and $\|v\|_{(X;(T,\infty))} < \varepsilon(E,M)$ or $\|v\|_{(X;(-\infty,T))} < \varepsilon$ (in the case where T_j is in the later half of I_j). Then, by the above lemma, we obtain $C(E,M) > \|u\|_{(X;(T,\infty))} > \|u\|_{(X;(\tau',\infty))}$ or $C(E,M) > \|u\|_{((X;(-\infty,\tau))}$. Thus we obtain $\|u\|_{(X;\mathbb{R})} \leq 3\max(\mathcal{N}(E,M), C(E,M))$ for any solution u with $E(u) \leq E$. Since it is obvious that we can take $\nu(E)$ depending continuously on E, by induction we obtain the desired estimate (27).

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CRITICAL SOBOLEV INEQUALITIES AND UNIQUENESS PROBLEM IN THE FLUID MECHANICS

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1. Introduction to Uniqueness Criterion

In this note, we consider a uniqueness problem for the Navier-Stokes equation

$$
(1.1) \qquad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \Delta u + f, & t > 0, x \in \mathbb{R}^n, \\ \operatorname{div} u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases}
$$

For simplicity, we assume that the external force $f \equiv 0$. It is well known that energy class weak solutions (so called Leray-Hopf's weak solution) with large initial data exist in $L^\infty(0,T; L^2_\sigma(\mathbb{R}^n)) \cap L^2(0,T; \dot{H}^1(\mathbb{R}^n))$. The regularity of this weak solution is based on the following energy inequality,

$$
\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(t)\|_2^2 d\tau \le \|u_0\|_2^2.
$$

and naturally we obtain interpolated regularity such as

$$
(1.2) \qquad \begin{aligned} & u \in L^\theta(0,T; L^p(\mathbb{R}^n)), \quad \frac{2}{\theta} + \frac{n}{p} = \frac{n}{2}, \\ & \qquad\qquad\qquad\qquad 2 \le n \le \frac{2n}{n-2}. \end{aligned}
$$

On the other hand, it is also known that there is a sufficient condition for the uniqueness and regularity for the weak solutions. Namely the Leray-Hopf weak solution is unique under the assumption;

$$
(1.3) \qquad u \in L^\theta(0,T; L^p(\mathbb{R}^n)), \quad \frac{2}{\theta} + \frac{n}{p} = 1, \quad n < p \le \infty
$$

Abstract for the workshop in Sapporo on Nov.22-24 1999,

† this work is a collaboration with Hideo Kozono and Yasushi Taniuchi.

See Ohyama [19], Serrin [24], Giga [12]. The conditions (1.3) is closely related to the estimate for the tri-linear form $(u \cdot \nabla v, w)$ induced from the nonlinear term.

The problem we would suggest here is to consider the corresponding condition for vorticity $\omega(t) = \text{rot } u(t)$.

By the Sobolev embedding theorem, the corresponding condition to $|\nabla|^s u$ is

$$|\nabla|^r u \in L^\theta(0,T;L^p(\mathbb{R}^n)), \quad \frac{2}{\theta} + \frac{n}{p} = 1 + r. \tag{1.4}$$

Hence the corresponding condition to ∇u is in the class $L^\theta(0,T;L^p(\mathbb{R}^n))$ with $\frac{2}{\theta} + \frac{n}{p} = 2$. Since rot ω = rot (rot u) $= -\Delta u + \nabla(\text{div } u) = -\Delta u$, the Bio-Savaurt law gives,

$$\nabla_i u = \nabla_i (-\Delta)^{-1} \text{rot } \omega \tag{1.5}$$

which involves the singular integral operator. Therefore if $p < \infty$, the condition to the vorticity ω immediately follows from the condition to ∇u . However, the limiting case $p = \infty$ (and hence $\theta = 1$) is not the case.

On the other hand, from the observation of the break down condition to the Euler equation (1.6),

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, & t > 0, x \in \mathbb{R}^n, \\ \text{div } u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{1.6}$$

it is desirable to control the situation by term of the vorticity of fluid, rot $u(t)$. In the well-known result due to Beale-Kato-Majda [2], the solution of the 3-dimensional Euler equation is shown to be regular over $[0,T]$ under the condition rot $u(t) \in L^1(0,T;L^\infty)$. This result is extended into a slightly larger class of solution by Kozono-Taniuichi [16]. (They also find the related uniqueness condition to the Navier-Stokes equations in terms of velocity u (c.f. [16]).)

Those results are based on the Sobolev inequalities of logarithmic type. For example, Beale-Kato-Majda [2] used the following type of inequality; for $f = (f_1, f_2, f_3) \in W^{s,p}$ $s > 1 + n/p$ with div $f = 0$,

$$\|\nabla f\|_\infty \le C\big(1 + \|\nabla f\|_2 + \|\text{rot} f\|_\infty (1 + \log^+ \|f\|_{W^{s,p}})\big)\big). \tag{1.7}$$

To handle the singular integral operator, it is needed to introduce some sort of semi-norm which allows those operator bounded. Kozono-Taniuchi derived a related inequality of BMO function ([16]); for $f \in W^{s,p}(\mathbb{R}^n)$ with div $f = 0$,

$$\|f\|_\infty \le C\big(1 + \|f\|_{BMO}(1 + \log^+ \|f\|_{W^{s,q}})\big)\big), \quad s > n/p + 1. \tag{1.8}$$

Here BMO is a set of $L^1_{loc}(\mathbb{R}^n)$ functions such that

$$\sup_{x,R} \frac{1}{|B_R|} \int_{B_R(x)} |u(x) - \bar{u}_{B_R(x)}| dx < \infty$$

where $\bar{u}_{B_R(x)}$ is the average of u over $B_R(x)$.

Our first aim is to extend those type of the Sobolev inequalities in terms of homogeneous Besov spaces where the singular integral operators are bounded.

The uniqueness condition for vorticity still have another difficulty. More specifically, in the case of regularity problem of the Euler equation, the solution is assumed to be regular until $t < T$. Then the question is if the solution is regular when $t \geq T$. Thanks to the logarithmic Sobolev inequalities ([2]), it is proved that the Euler equation can be continued to be regular after $t = T$ (see also Ponce [23] for the condition on the deformation tensor).

While the uniqueness problem to the Navier-Stokes equation is in slightly different situation. If u and $\tilde{u}$ are both weak solutions to the Navier-Stokes equation, we would assume some extra condition on one of solutions but not to to ont the both usually. Then there is a difficulty to handle with the term appearing inside of the logarithm function of the Sobolev inequalities (2.5) and (2.6). One possibility to avoid this ruck of regularity is that we may involve the term stems from the viscosity in the energy inequality. Then it is positively shown that the uniqueness criterion holds if we replace to the time regularity to $L \log L$ instead of L^1 but keeping the space regularity as the limiting case to the vorticity $\omega = \text{rot } u$ in BMO.

2. Besov Spaces and Critical Sobolev Inequality

Before presenting our result, we recall some notations and definition of the Besov spaces (c.f., [26]). Let ϕ_j $j = 0, \pm 1, \pm 2, \pm 3, \cdots$ be the Littlewood-Payley dyadic decomposition satisfying $\hat{\phi}_j(\xi) = \phi(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty} \phi_j(\xi) = 1$ except $\xi = 0$. We put a smooth cut off to fill the origin $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{\psi}(\xi) \in C_0^\infty(B_1)$ such that $\hat{\psi} + \sum_{j=0}^{\infty} \hat{\phi}_j(\xi) = 1$.

Definition. The homogeneous Besov space $\dot{B}^s_{p,\rho} = \{f \in \mathcal{S}'; \|f\|_{\dot{B}^s_{p,\rho}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{B}^s_{p,\rho}} = (\sum_{j=-\infty}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho'}$$

for $s \in \mathbb{R}$, $1 \le p, \rho \le \infty$ and the inhomogeneous Besov space $B^s_{p,\rho} = \{f \in \mathcal{S}'; \|f\|_{B^s_{p,\rho}} < \infty\}$ similarly defined by

$$\|f\|_{B^s_{p,\rho}} = (\|\psi * f\|_p^\rho + \sum_{j=0}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho'}.$$

We use the non-negative logarithmic function $\log^+ r$ which is

$$\log^+ r = \begin{cases} \log\ r, & e < r, \\ 1, & 0 \le r \le e. \end{cases}$$

Here we give some generalization of the logarithmic Sobolev inequality originally due to Brezis-Gallouet [4], Brezis-Wainger[5] and Beale-Kato-Majda [2] (see for some generalization [10], [25], [22], [16], [15]).

Theorem 2.1 ([15], [21]). *Let $1 \le \nu < \rho \le \infty$ and $\kappa > 0$. Then for $q \in [1, \infty]$, $\nu < \sigma$, there exists a constant C which is only depending on n q and σ such that for $f \in \dot{B}^{n/q+\kappa}_{\infty,\sigma} \cap \dot{B}^{n/q-\kappa}_{\infty,\sigma}$, we have*

$$\|f\|_{\dot{B}^0_{\infty,\nu}} \le C\|f\|_{\dot{B}^0_{\infty,\rho}}\left(1 + \left(\frac{1}{\kappa}\log^+ \frac{\|f\|_{\dot{B}^{n/q+\kappa}_{q,\sigma}} + \|f\|_{\dot{B}^{n/q-\kappa}_{q,\sigma}}}{\|f\|_{\dot{B}^0_{\infty,\rho}}}\right)^{1/\nu-1/\rho}\right). \tag{2.1}$$

<u>*Remark 1.*</u> By the embedding estimate $\|f\|_{\dot{B}^0_{\infty,\rho}} \le \|f\|_{\dot{B}^{n/q}_{n/q,\rho}}$, the term in the right hand side can be changed into $\|f\|_{\dot{B}^{n/q}_{q,\rho}}$ for $1 \le q \le \infty$. Furthermore, it can be generalized the right hand side into the general Besov exponent such as $\|f\|_{\dot{B}^0_{p,\nu}}$.

The above inequality is a sort of the interpolation inequality for functions in the Besov space. In fact the embedding

$$\dot{B}^0_{\infty,\nu} \subset \dot{B}^{n/q+\kappa}_{q,\sigma} \cap \dot{B}^{n/q-\kappa}_{q,\sigma}$$

is well known. The advantage of the above inequality is at the logarithmic order from the higher order norms. If $\rho < \nu$, then the inequality

$$\|f\|_{\dot{B}^0_{\infty,\nu}} \le C\|f\|_{\dot{B}^0_{\infty,\rho}}$$

always holds since $l^\rho \subset l^\nu$. To compensate the deficiency for the second summability exponent ν to ρ, we need a higher regularity which is shown by the logarithmic term. The extra regularity $f \in \dot{B}^{s_1}_{r_1,\sigma_1}$ is devoted for the regularity of f around the low frequency and $f \in \dot{B}^{s_1}_{r_1,\sigma_1}$ for high frequency. In fact it holds that

$$\|f\|_{\dot{B}^0_{\infty,\nu}} \le C\|f\|_{\dot{B}^0_{\infty,\rho}}\left(1 + \left(\frac{1}{\kappa}\log^+ \frac{\|f_+\|_{\dot{B}^{n/q+\kappa}_{q,\sigma}} + \|f_-\|_{\dot{B}^{n/q-\kappa}_{q,\sigma}}}{\|f\|_{\dot{B}^0_{\infty,\rho}}}\right)^{1/\nu-1/\rho}\right), \tag{2.2}$$

where $f_+ = \sum_{j>0} \phi_j * f$ and $f_- = f - f_+$.

Theorem 2.1 is a generalization to the known logarithmic Sobolev inequalities. Brezis-Gallouet [4] firstly presented this type of inequality. That is for $f \in H^1(\mathbb{R}^2)$,

$$\|f\|_\infty \le C\|\nabla f\|_2\big(1 + (\log^+ \|f\|_{W^{2,2}})\big)^{1/2} \tag{2.3}$$

and more general version is discussed by Brezis-Wainger [5]: for $f \in W^{s,n}(\mathbb{R}^n)$, $s > n/p+1$

$$\|f\|_\infty \le C\|\nabla f\|_n\big(1 + (\log^+ \|f\|_{W^{s,n}})\big)^{1-1/n} \tag{2.4}$$

Some more generalization was done by Ozawa [22] (see also Englar [10]). On the other hand, for the divergence free vector field $f = (f_1, f_2, f_3)$ div $f = 0$, the logarithmic Sobolev type inequalities is observed by Beale-Kato-Majda [2], that is for $f \in W^{1,\infty}(\mathbb{R}^3)$

$$\|\nabla f\|_\infty \le C\big(1 + \|\nabla f\|_2 + \|\mathrm{rot} f\|_\infty(1 + \log^+ \|f\|_{W^{s,p}})\big) \tag{2.5}$$

or Kozono-Taniuchi inequality in term of BMO function ([16]);

$$\|f\|_\infty \le C\big(1 + \|f\|_{BMO}(1 + \log^+ \|f\|_{W^{s,q}})\big), \quad f \in (W^{s,p}(\mathbb{R}^n))^3. \tag{2.6}$$

Theorem 2.1 shows a contrast to those previously obtained inequalities (2.3), (2.4) and (2.5), since the power of the logarithmic term is determined by the second exponent of the Besov semi norm but not the normal L^p exponents. We should emphasize that (2.6) is also included in (2.1) by $BMO \subset \dot{B}^0_{\infty,\infty}$.

3. Uniqueness condition by vorticity

Definition. Let X denote a normed space. For $\alpha > 0$, a class of function $u(t)$ is in $L(\log L)^\alpha(I; X)$ for an interval I if

$$\int_I \|u(t)\|_X (\log^+ \|u(t)\|_X)^\alpha dt < \infty.$$

Especially $L \log L(I; X)$ stands for a function $u(t)$ with

$$\int_I \|u(t)\|_X \log^+ \|u(t)\|_X dt < \infty.$$

Theorem 3.1 (Uniqueness[21]). *Let u and $\tilde{u}$ be the Leray-Hopf weak solutions for the Navier-Stokes system with the same initial data u_0. For $1 \le \rho \le \infty$, we suppose that the vorticity ω for one of the solution satisfies* rot $u = \omega \in L(\log L)^{1/\rho'}([0,T]; \dot{B}^0_{\infty,\rho})$ *with $1/\rho + 1/\rho' = 1$ and the other solution $\tilde{u}$ satisfies the energy inequality*

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla \tilde{u}(\tau)\|_2^2 d\tau \le \|u_0\|_2.$$

Then $u = \tilde{u}$.

Note that for $s = n/p$, it holds $\|f\|_{\dot{B}^0_{\infty,p}} \leq C\|f\|_{\dot{B}^s_{p,p}}$. Besides, since we are considering a weak solution, functions are restricted into the subspaces of the Besov spaces. In this case the following inclusion is holds: $\dot{B}^0_{\infty,1}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$. Therefore we have the following as a corollary.

Corollary 3.2 (limiting vorticity condition). *Let u and $\tilde{u}$ be the Leray-Hopf weak solutions for the Navier-Stokes system with the same initial data u_0. Suppose that the vorticity of the one of the solution u satisfies $\omega \equiv \text{rot } u \in L\log L([0,T]; BMO)$ and the other solution $\tilde{u}$ satisfies the energy inequality*

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla\tilde{u}(\tau)\|_2^2 d\tau \leq \|u_0\|_2.$$

Then $u = \tilde{u}$.

As is stated in the introduction, Beale-Kato-Majda [2] showed that the solution of Euler equation is regular if rot $u \in L^1([0,T]; L^\infty)$. In this case, the vorticity rot $u = \omega$ can dominate $\|\nabla u\|_\infty$ via the Bio-Savaurt law with aid of extra regularity assumption. (see also Ponce [23] and Kozono-Taniuchi [16] and Vishik [27]). In our case, however the regularity can be covered by the viscosity of the equation.

Proof of Theorem 3.1 . Set $w = u - \tilde{u}$. We note that $w \in L^\infty([0,T]; L^2_\sigma \cap \dot{H}^1_\sigma) \cap L^2([0,T]; \dot{H}^1_\sigma)$. Since w satisfies

$$\begin{cases} \partial_t w + \Delta w - w\cdot\nabla w + w\cdot\nabla u + u\cdot\nabla w + \nabla(p-q) = 0, & t>0, x\in\mathbb{R}^n, \\ \text{div } w = 0, \quad t>0, x\in\mathbb{R}^n, \\ w(t,0) = 0, \end{cases} \tag{3.1}$$

in the sense of distribution, we have the following weak form

$$\frac{d}{dt}\|w(t)\|_2^2 + 2\|\nabla w(t)\|_2^2 = (w(t)\cdot\nabla u, w(t)). \tag{3.2}$$

integrating over $[0,t]$, we have the starting inequality,

$$\|w(t)\|_2^2 + 2\int_0^t \|\nabla w(\tau)\|_2^2 d\tau = 2\int_0^t |(w(\tau)\cdot\nabla u(\tau), w(\tau))| d\tau. \tag{3.3}$$

This process can be justified by the following argument.

Under the assumption $\omega = \text{rot } u \in L\log L([0,T]; \dot{B}^0_{\infty,\rho})$, it is possible to show that u belongs to $C^1((0,T]; H^s)$ for any $s > 0$, i,e, u is smooth except $t = 0$, and hence satisfies the energy equality:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \|u_0\|_2^2, \tag{3.4}$$

(see Kozono-Taniuchi [16] and Kozono-Ogawa-Taniuchi [15]). We note that the energy equality guarantees strong continuity of $u(t)$ for t in L^2 on $[0,T]$. On the other hand, by assumption, $\tilde{u}$ satisfies the energy inequality:

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla\tilde{u}(\tau)\|_2^2 d\tau \le \|u_0\|_2^2, \tag{3.5}$$

Combining (3.4) and (3.5) we have (3.3).

Now we decompose the smoother solution u into the three parts in the phase variables such as

$$\begin{aligned} u(x) &= \sum_{j<-N} \phi_j * u(x) + \sum_{|j|\le N} \phi_j * u(x) + \sum_{j>N} \phi_j * u(x) \\ &= u_l(x) + u_m(x) + u_h(x) \end{aligned} \tag{3.6}$$

Then by the Hausdorff-Young inequality, the low frequency part is estimated as

$$\begin{aligned} |(w\cdot\nabla u_l, w)| =& |(w\cdot\nabla w, u_l)| \\ \le& \|\psi_{-N} * \nabla(w\otimes w)\|_2 \|u\|_2 \\ \le& C\|\nabla\psi_{-N}\|_2 \|w\|_2^2 \|u\|_2 \\ \le& C2^{-(n+2)N/2}\|w\|_2^2\|u\|_2 \end{aligned} \tag{3.7}$$

The second term giving a core part of the solutions, can be bound by the logarithmic Sobolev inequality that for small $\varepsilon > 0$,

$$\begin{aligned} |(w\cdot\nabla u_m, w)| &\le \|w\|_2^2 \|\nabla(\sum_{|j|\le N} \phi_j * u)\|_\infty \\ &\le C\|w\|_2^2 \|\nabla u_m\|_{\dot{B}^s_{p,\rho}} \Big\{1 + \Big(\frac{1}{\varepsilon}\log^+ \frac{\|\nabla u_m^+\|_{\dot{B}^\varepsilon_{\infty,\rho}} + \|\nabla u_m^-\|_{\dot{B}^{-\varepsilon}_{\infty,\rho}}}{\|\nabla u\|_{\dot{B}^0_{\infty,\rho}}}\Big)^{1/\rho'}\Big\} \\ &\le C\|w\|_2^2 \|\nabla u\|_{\dot{B}^0_{\infty,\rho}} \Big\{1 + \Big(\frac{1}{\varepsilon}\log^+ \frac{2^{\varepsilon N}\|\nabla u_m^+\|_{\dot{B}^0_{\infty,\rho}} + 2^{\varepsilon N}\|\nabla u_m^-\|_{\dot{B}^0_{\infty,\rho}}}{\|\nabla u\|_{\dot{B}^0_{\infty,\rho}}}\Big)^{1/\rho'}\Big\} \\ &\le CN^{1/\rho'}\|w\|_2^2 \|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}}, \end{aligned} \tag{3.8}$$

where we decompose $u_m = u_m^+ + u_m^- = \sum_{j\geq 0} \phi_j * u_m + \sum_{j<0} \phi_j * u_m$. While the last term is simply estimated by the Hausdorff-Young inequality that

$$
\begin{aligned}
|(w\cdot\nabla u_h, w)| =& |(w\cdot\nabla w, u_h)| \\
\leq& \|w\|_2\|\nabla w\|_2\|(\sum_{j>N}\mathcal{F}^{-1}(1-\hat{\psi}_N)*\phi_j*u)\|_\infty \\
\leq& \|w\|_2\|\nabla w\|_2\|(-\Delta)^{-1}\mathrm{rot}\ \mathcal{F}^{-1}(1-\hat{\psi}_N)*\sum_{j>N}\phi_j*\mathrm{rot}\ u)\|_\infty \\
\leq& \|w\|_2\|\nabla w\|_2\|(-\Delta)^{-1}\nabla\mathcal{F}^{-1}(1-\hat{\psi}_N)\|_{\dot{B}^0_{1,\rho'}}\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}} \\
\leq& C2^{-N}\|w\|_2\|\nabla w\|_2\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}}.
\end{aligned}
\tag{3.9}
$$

Gathering the estimates (3.7)-(3.9) with (3.6) we have

$$
\begin{aligned}
|(w\cdot\nabla u, w)| \leq& C2^{-N/2}\|w\|_2^2\|u\|_2 \\
&+ CN^{1/\rho'}\|w\|_2^2\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}} + C2^{-N}\|w\|_2\|\nabla w\|_2\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}}
\end{aligned}
\tag{3.10}
$$

Then choosing N properly large satisfying $2^{-N/2}\|u\|_2 \leq 1$,$2^{-N}\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}} \simeq 1$, we see that

$$
|(w\cdot\nabla u, w)| \leq C\|w\|_2^2(1+\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}}(1+(\log^+\|\mathrm{rot}\ u\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'}) + \|\nabla w\|_2^2. \tag{3.11}
$$

Hence we obtain from (3.3) and (3.11) that

(3.12)

$$
\begin{aligned}
\|w(t)\|_2^2 + 2\int_0^t \|\nabla w\|_2^2 \leq& C\int_0^t \{\|w(\tau)\|_2^2(1+\|\mathrm{rot}\ u(\tau)\|_{\dot{B}^0_{\infty,\rho}}(1+(\log^+\|\mathrm{rot}\ u(\tau)\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'}) \\
&+ \|\nabla w(\tau)\|_2^2\}d\tau
\end{aligned}
$$

or

$$
\|w(t)\|_2^2 \leq C\int_0^t \{\|w(\tau)\|_2^2(1+\|\mathrm{rot}\ u(\tau)\|_{\dot{B}^0_{\infty,\rho}}(1+(\log^+\|\mathrm{rot}\ u(\tau)\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'})\}d\tau \tag{3.13}
$$

Now the Gronwall argument gives

$$
\|w(t)\|_2^2 \leq C\|w(0)\|_2^2\exp\Big(\int_0^t \{(\|\mathrm{rot}\ u(\tau)\|_{\dot{B}^0_{\infty,\rho}}(\log^+\|\mathrm{rot}\ u(\tau)\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'})\}d\tau\Big). \tag{3.14}
$$

The right hand side is 0 under the condition rot $u \in L(\log L)^{1/\rho'}(0,T;\dot{B}^0_{\infty,\rho})$.

□

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Stability of Standing Waves for Nonlinear Schrödinger Equation with a Unbounded Potential

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This note is concerned with the nonlinear Schrödinger equation with a unbounded potential,

$$i\psi_t = -\frac{1}{2}\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi, \qquad t \geq 0, x \in R^N, \tag{1}$$

where $\psi = \psi(t,x)$ is a complex function of $(t,x) \in R^+ \times R^N$, the potential V is bounded below and satisfies $V(x) \to \infty$ as $|x| \to \infty$. $1 < p < \infty$ for $N = 1, 2$ and $1 \leq p < \frac{N+2}{(N-2)}$ for $N \geq 3$.

When $V(x) = |x|^2$, the model equation (1) describes the Bose-Einstein condensate with attractive interparticle interactions under magnetic trap (see Tsurumi and Wadati [10] as well as Dalfovo etc. [4]).

When $|D^\alpha V|$ is bounded for all $|\alpha| \geq 2$, in terms of the smoothness off the time 0 of Schrödinger kernel for potentials of quadratic growth provided by Fujiware [6], Oh [7] established the well-posedness of (1) in the corresponding energy space. Since Yajima [11] showed that for super-quadratic potentials, the Schrödinger kernel is nowhere C^1 , we see that quadratic potentials are the highest order potential for local well-posedness of (1) .

Let ω satisfy $\inf V + \omega > 0$ and u be a solution of the equation

$$-\frac{1}{2}\Delta u + V(x)u + \omega u - |u|^{p-1}u = 0, \qquad x \in R^N, \tag{2}$$

Supported by the JSPS Research Fellowship and the Monbusho's Grant No P-98029.

then , $\psi(t,x) = exp(i\omega t)u(x)$ is a standing wave of (1.1). Rabinowitz [9] showed the existence of the above standing waves (also see Ding and Ni [5]).

We study the orbital stability of the standing waves of (1) in terms of the argument of Cazenave and Lions [3](also see Cazenave and Estaban [2]).

First of all, we need refer to the global well-posedness of the Cauchy problem for (1).

Suppose that the initial data are that

$$\psi(0,x) = \psi_0(x), \qquad x \in R^N. \tag{3}$$

Let

$$H := \{\phi \in H^1(R^N), \quad \int_{R^N} V(x)|\phi|^2 dx < \infty\}.$$

H becomes a Hillbert space, continuously embedded in $H^1(R^N)$, when endowed with the inner product

$$< \phi, \varphi >_H = \int_{R^N} [\nabla\phi\nabla\bar{\varphi} + (V - \inf V)\phi\bar{\varphi} + \phi\bar{\varphi}]dx,$$

whose associated norm we denote $||\cdot||_H$.

In H, we define the energy functional

$$E(\phi) := \int_{R^N} [\frac{1}{4}|\nabla\phi|^2 + \frac{1}{2}V(x)|\phi|^2 - \frac{1}{p+1}|\phi|^{p+1}]dx, \qquad \phi \in H.$$

From the Sobolev's embedding theorem, we know that E is well defined. From the point of view of Hamiltonian systems, E is the generating Hamiltonian of (1) .

From Oh [7], we have

Lemma 1 *Let V satisfy that $\inf V > -\infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded , $1 \leq p < 1 + \frac{4}{N}$ and $\psi_0 \in H$. Then the Cauchy problem (1), (3) has a unique bounded solution $\psi(t,\cdot) \in C([0,\infty), H)$. Moreover $\psi(t,\cdot)$ satisfies the following conservation laws.*

$$\int_{R^N} |\psi(t,x)|^2 dx = \int_{R^N} |\psi_0(x)|^2 dx, \qquad t \in [0,\infty),$$

$$E(\psi(t,\cdot)) = E(\psi_0(\cdot)), \qquad t \in [0,\infty).$$

Then we state a compactness lemma (also see Omana and Willem [8]).

Lemma 2 *Let $V(x) \to \infty$ as $|x| \to \infty$,$1 \leq q < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 \leq q < \infty$ when $N = 1, 2$. Then the embedding $H \hookrightarrow L^{q+1}$ is compact.*

Now for $\mu > 0$, we define a variational problem as follows.

$$d_\mu := \inf_{\{u \in H, \int_{R^N} |u|^2 dx = \mu\}} E(u).$$

Theorem 1 *If $V(x) \to \infty$ as $|x| \to \infty$ and $1 < p < 1 + \frac{4}{N}$, then we have*

$$d_\mu = \min_{\{u \in H, \int_{R^N} |u|^2 dx = \mu\}} E(u).$$

For any $\mu > 0$, we denote the set of the minimizers of the above minimization problem by S_μ .Then for any $u \in S_\mu$, there exists a Lagrange multiplier $\Lambda \in R$ such that u is a solution of the elliptic equation

$$-\frac{1}{2}\Delta u + V(x)u + \Lambda u - u|u|^{p-1} = 0.$$

It follows that $\psi(t,x) = e^{i\Lambda t}u$ is a standing wave solution of (1). Thus $e^{i\Lambda t}u(\cdot)$ is the orbit of u. It is obvious that for any $t \geq 0$, if u is a solution of the above minimization problem , then $e^{i\Lambda t}u$ is also a solution of this minimization problem, that is $e^{i\Lambda t}u \in S_\mu$. Now in terms of Cazenave and Lions' argument, we have the following orbital stability theorem .

Theorem 2 *Assume that V satisfies that $\inf V > -\infty$, $V(x) \to \infty$ as $|x| \to \infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded .Let $1 < p < \frac{4}{N}$, $\mu > 0$. Then for arbitrary $\varepsilon > 0$, there exists $\sigma > 0$ such that for any $\psi_0 \in H$, if*

$$\inf_{u \in S_\mu} ||\psi_0 - u||_H < \sigma,$$

then the solution $\psi(t,x)$ of the Cauchy problem (1) - (3) satisfies

$$\inf_{u \in S_\mu} ||\psi(t,\cdot) - u(\cdot)||_H < \varepsilon, \quad for \quad all \quad t \geq 0.$$

In addition, when $1 + \frac{4}{N} \leq p < \infty$ to $N = 1, 2$ and $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)}$ to $N \geq 3$, for the important case $V(x) = \hbar^2|x|^2 \quad (\hbar > 0)$, we state the following instability result for the standing waves as a remark.

For $u \in H, \omega > 0$ and the above p, we define the following functionals .

$$I(u) := \int \frac{1}{4}|\nabla u|^2 + \frac{1}{2}\hbar^2|x|^2|u|^2 + \frac{1}{2}\omega|u|^2 - \frac{1}{p+1}|u|^{p+1}dx.$$

$$S(u) := \int \frac{1}{2}|\nabla u|^2 + \hbar^2|x|^2|u|^2 + \omega|u|^2 - |u|^{p+1}dx.$$

$$Q(u) := \int |\nabla u|^2 - 2\hbar^2|x|^2|u|^2 - \frac{p-1}{p+1}N|u|^{p+1}dx.$$

Since the Sobolev embedding theorem, the above functionals are well defined.

Then we define two constrained minimization problems.

$$d_\omega := \inf_{\{u\in H\setminus\{0\}, S(u)=0\}} I(u),$$

and

$$d_M := \inf_M I(u),$$

where M is a cross-region as follows

$$M := \{u \in H, S(u) < 0, Q(u) = 0\}.$$

It is known that d_ω is attained. It follows that there exists $u \in H\setminus\{0\}$ such that $I(u) = d^*$ and

$$-\frac{1}{2}\Delta u + \hbar^2|x|^2u + \omega u - u|u|^{p-1} = 0, \quad x \in R^N.$$

Thus $exp(i\omega t)u(x)$ is a standing wave. We have the following strong instability of the standing waves with some frequency ω.

Theorem 3 *For $1 + \frac{4}{N} \leq p < \infty$ to $N = 1, 2$ and $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)}$ to $N \geq 3$, let ω such that $d_M \geq d_\omega$. Then for the minimizer u of d_ω, and any $\varepsilon > 0$, there exists $\psi_0 \in H$ with $||\psi_0 - u||_H < \varepsilon$ such that the solution $\psi(t, x)$ of the Cauchy problem (1) - (3) blows up in a finite time.*

ACKNOWLEDGEMENT

The author wishes to thank Professor Yoshio Tsutsumi for his advice and proposition.

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Small solutions to nonlinear Schrödinger equations in the Sobolev spaces *

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1. Introduction

In this talk, we consider the Cauchy problem for nonlinear Schrödinger equations in the Sobolev space of fractional order. The problem is given by the form

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = f(u(t,x)), & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,\cdot) = \phi \in H^s(\mathbf{R}^n), \quad s \geq 0, \ n \geq 1, \end{cases}$$

where u and f are complex-valued functions, Δ is the Laplacian in $\mathbf{R}^n$.

We show the global solutions of (NLS) under the following assumptions.

(1) For $0 \leq s < n/2$, the nonlinearity f is given by $f(u) = c|u|^{p-1}u$ with $c \in \mathbf{C}$ and $1+4/n \leq p \leq 1+4/(n-2s)$. And $\|\phi; \dot{H}^{n/2-2/(p-1)}\|$ is sufficiently small.

(2) For $s = n/2$, $f(u)$ behaves as a conformal power $c|u|^{4/n}u$ near zero and has exponential growth rate such as $c\exp(\kappa|u|^2)$ with $\kappa > 0$ at infinity. And $\|u; L^2\|$ is relatively small with respect to $\|\phi; \dot{H}^{n/2}\|$.

(3) For $s > n/2$, $f(u)$ behaves as a conformal power $c|u|^{4/n}u$ near zero and has an arbitrary growth rate at infinity. And $\|u; L^2\|$ is relatively small with respect to $\|\phi; \dot{H}^{\sigma}\|$ for any fixed σ with $n/2 < \sigma \leq s$.

Our motivation is based on the scaling argument for (NLS) with $f(u) = c|u|^{p-1}u$ by which p is restricted as

$$p \leq 1 + 4/(n-2s) \tag{0.1}$$

for $0 \leq s < n/2$. Here it is natural that we have conjectures from the form of (0.1) that the growth rate of the nonlinearity is not restricted in the cases

* Joint work with Tohru Ozawa (Hokkaido University).
This work was partly supported by Japan Society for the Promotion of Science.

$s \geq n/2$. Our results (2) and (3), above, relate to these conjectures and (2) seems to be optimal in terms of Trudinger's inequality.

Among a large literature on the problem (NLS), we must refer to the paper [3], in which T. Kato has already obtained the analogous results to ours. One of the main improvement compared to [3] is that on (2), in which we eased the restriction on the behavior of $f(u)$ at infinity to the level of exponential growth rate from the polynomial growth rate which is assumed in [3].

The key estimates in our proof are estimates for the nonlinearities in the scheme of Besov spaces, which gives us sharp estimates for nonlinearities of power type such as $|u|^{p-1}u$ by use of its property

$$|\partial^{[s]} f(z) - \partial^{[s]} f(w)| \leq C|z-w|^{p-[s]}, \quad z, w \in \mathbf{C}$$

for $s < p < [s]+1$ with the equivalent norm

$$\begin{aligned} &\|f(u); \dot{B}^s_{r,m}(\mathbf{R}^n)\| \\ \sim\ &\left\{\int_0^\infty (\tau^{[s]-s} \sup_{|y|<\tau} \|\partial^{[s]}(f(u(\cdot))) - \partial^{[s]}(f(u(\cdot+y)); L^r\|)^m d\tau/\tau\right\}^{1/m} \end{aligned}$$

for $0 < s \notin \mathbf{Z}$, $1 \leq r, m \leq \infty$, while the estimates are done in the scheme of Sobolev spaces in [3]. For simplicity, we write $\dot{B}^s_{r,2}(\mathbf{R}^n)$ as $\dot{B}^s_r$.

2. Estimates for nonlinear terms

The nonlinearities in our results are characterized by the following assumption $(N)_{s,p}$ with $0 \leq s < \infty$ and $1 \leq p < \infty$.

$(N)_{s,p}$ $f \in C^{[s]}(\mathbf{C}; \mathbf{C})$ and there exists a nonnegative, nondecreasing function M on $\mathbf{R}_+$ such that for all k with $0 \leq k \leq [s]$ $f^{(k)}$ satisfies the estimates

$$|f^{(k)}(z)| \leq |z|^{(p-k)_+} M(|z|),$$

$$\begin{aligned} &|f^{([s])}(z_1) - f^{([s])}(z_2)| \\ &\leq \begin{cases} |z_1 - z_2|^{p-[s]} M(|z_1| \vee |z_2|) & \text{if } s < p < [s]+1 \\ |z_1 - z_2|(|z_1| \vee |z_2|)^{(p-[s]-1)_+} M(|z_1| \vee |z_2|) & \text{otherwise} \end{cases} \end{aligned}$$

for all $z, z_1, z_2 \in \mathbf{C}$, where let $M(\cdot) \equiv 1$ for (1), $M(x) \equiv \exp(\kappa|x|^2)$ with $\kappa > 0$ for (2).

Here $f^{(k)}$ denotes any of the k-th order derivatives of f with respect to z and $\bar{z}$ and $|f^{(k)}|$ denotes the maximum of the moduli of those derivatives. For $a, b \in \mathbf{R}$ we denote by $a \vee b$ the maximum of a and b.

The main estimates are following. We use (0.2) for (1) and (3), (0.3) for (2).

Proposition 0. 1 *Let $0 < s < \infty$, $1 < p < \infty$. Let f satisfy $(N)_{s,p}$. Let $1 \le \tilde{r} < \infty$, $2 \le r^* \le \infty$, $2 \le r_0 < \infty$ satisfy*

$$1/\tilde{r} = (p-1)/r^* + 1/r_0.$$

Then

$$\|f(u); \dot{B}^s_{\tilde{r}}\| \le M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; L^{r^*} \cap \dot{B}^0_{r^*}\|^{p-1}\|u; \dot{B}^s_{r_0}\|, \tag{0.2}$$

$$\|f(u); \dot{B}^s_{\tilde{r}}\| \le C \sum_{k=1}^{[s]+1} \sum_{\ell=0}^{\infty} \frac{\kappa^\ell}{\ell!} \|u; L^{r_k(\ell)} \cap \dot{B}^0_{r_k(\ell)}\|^{p \vee k-1+2\ell} \|u; \dot{B}^s_{r_0}\|, \tag{0.3}$$

where $r_k(\ell)$ is given by $1/\tilde{r} = (p \vee k - 1 + 2\ell)/r_k(\ell) + 1/r_0$.

3. Sketch of the proof of (2)

Let $1/q_0 = 1/r_0 = n/2(n+2)$, by which (q_0, r_0) forms an admissible pair for the Strichartz estimates for Schrödinger equations. Let p_0 be a number with

$$1 \le p_0 \le (1 + 4/n) \vee p.$$

Let $\omega, \tilde{r}, \tilde{q}$ be numbers defined by

$$\omega \equiv 1 - (p_0 - 1)n/4, \quad 1/\tilde{r} \equiv p_0/r_0, \quad 1/\tilde{q} \equiv \omega + p_0/q_0.$$

Then putting $1/r_k(\ell) \equiv (p_0 - 1)/(p \vee k - 1 + 2\ell)r_0$, we have

$$1/\tilde{r} = (p \vee k - 1 + 2\ell)/r_k(\ell) + 1/r_0.$$

Let f satisfy $(N)_{s,p}$. Then by Proposition 0. 1 , we have

$$\|f(u); \dot{B}^s_{\tilde{r}}\| \le C \sum_{k=1}^{[s]+1} \sum_{\ell=0}^{\infty} \frac{\kappa^\ell}{\ell!} \|u; L^{r_k(\ell)} \cap \dot{B}^0_{r_k(\ell)}\|^{p \vee k-1+2\ell} \|u; \dot{B}^s_{r_0}\|. \tag{0.4}$$

Lemma 0. 1 [5, Lemma 2.2] *The following estimates holds.*

$$\|u; L^r\| \le C_0 r^{1/2+(r_0-2)/2r} \|u; \dot{H}^{n/2}\|^{1-r_0/r} \|u; L^{r_0}\|^{r_0/r},$$
$$\|u; \dot{B}^0_r\| \le C_0 r^{1/2+(r_0-2)/2r} \|u; \dot{H}^{n/2}\|^{1-r_0/r} \|u; \dot{B}^0_{r_0}\|^{r_0/r}$$

for any r_0, r with $0 < 1/r \le 1/r_0 \le 1$, where the constant C_0 is independent of r, but may be dependent on r_0.

Applying the above lemma to (0.4), we have

$$\begin{aligned} & \|f(u); \dot{B}^s_{\tilde{r}}\| \\ \leq \; & C \sum_{k=1}^{[s]+1} \sum_{\ell=0}^{\infty} a_k(\ell) \|u; \dot{H}^{n/2}\|^{p\vee k-1+2\ell-(p_0-1)} \|u; \dot{B}^0_{r_0}\|^{p_0-1} \|u; \dot{B}^s_{r_0}\|, \end{aligned}$$

where

$$a_k(\ell) \equiv \frac{\kappa^\ell}{\ell!} C_0^{p\vee k-1+2\ell} r_k(\ell)^{(p\vee k-1+2\ell)/2+(p_0-1)(r_0-2)/2r_0}.$$

Applying the Hölder inequality in time variable to the last inequality, we have

$$\begin{aligned} & \|f(u); L^{\tilde{q}}(I; \dot{B}^s_{\tilde{r}})\| \\ \leq \; & P(\|u; L^\infty(I; \dot{H}^{n/2})\|)|I|^\omega \|u; L^{q_0}(I; \dot{B}^0_{r_0})\|^{p_0-1} \|u; L^{q_0}(I; \dot{B}^s_{r_0})\| \quad (0.5) \end{aligned}$$

for any interval $I \subset \mathbf{R}$, where

$$P(x) \equiv C \sum_{k=1}^{[s]+1} \sum_{\ell=0}^{\infty} a_k(\ell) x^{p\vee k-1+2\ell-(p_0-1)}, \quad x > 0.$$

By (0.5), Strichartz estimates and the standard contraction argument, we can conclude that there exists P', which is P multiplied by a constant, such that if the initial data ϕ satisfies

$$P'(\|\phi; \dot{H}^{n/2}\|)|I|^\omega \|\phi; L^2\|^{p_0-1} \leq 1, \tag{0.6}$$

then there exists a solution of (NLS) in $C(I; H^s)$. In above, if p satisfies $p \geq 1 + 4/n$, then we can take $\omega = 0$. So that the solution is global with $\|\phi; L^2\|$ sufficiently small compared to $\|\phi; \dot{H}^{n/2}\|$. □

Remark. Corresponding to (0.6), the sufficient conditions on the initial data for the existence of the solutions of (NLS) are given by

$$C|I|^{1-(p-1)(n-2s_0)/4} \|\phi; \dot{H}^{s_0}\|^{p-1} \leq 1 \quad \text{and} \quad 0 \leq s_0 \leq s$$

for (1) with a constant $C > 0$, and

$$M(\|\phi; H^\sigma\|)|I|^\omega \|\phi; L^2\|^{p_0-1} \leq 1$$

for (3) with a nonnegative, nondecreasing function $M(\cdot)$.

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GAIN OF ANALYTICITY FOR SEMILINEAR SCHRÖDINGER EQUATIONS

HIROYUKI CHIHARA

This talk is concerned with gain of analyticity of solutions to the initial value problem for semilinear Schrödinger equations of the form

$$\partial_t u - i\Delta u = f(u, \partial u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^n, \tag{1}$$

$$u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{R}^n, \tag{2}$$

where u is a complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\partial = (\partial_1, \ldots, \partial_n)$, $\Delta = \partial_1^2 + \cdots + \partial_n^2$, n is the spatial dimension. Assume that the nonlinear term $f(u, v)$ is a cubic and smooth function on $\mathbb{R}^2 \times \mathbb{R}^{2n}$, has a holomorphic extension on $\mathbb{C}^2 \times \mathbb{C}^{2n}$, and satisfies the gauge invariance:

$$f(e^{i\psi}u, e^{i\psi}v) = e^{i\psi} f(u, v) \quad \text{for} \quad \psi \in \mathbb{R}, (u, v) \in \mathbb{C} \times \mathbb{C}^n.$$

We here recall Sobolev spaces. Let θ be a real number. H^θ is the set of all tempered distributions on $\mathbb{R}^n$ satisfying

$$\|u\|_\theta = \left(\int_{\mathbb{R}^n} |(1 - \Delta)^{\theta/2} u(x)|^2 dx \right)^{1/2} < +\infty.$$

$C(I; H^\theta)$ denotes the set of all H^θ-valued strongly continuous functions on the interval I

Recently, the author proved that if $u_0 = o(|x|^{-l})$ as $|x| \to \infty$ with some $l = 1, 2, 3, \ldots$, then the unique solution u to (1)-(2) gained extra smoothness of order l in x for $t \neq 0$. More precisely,

Proposition 1 ([1]). *Let θ be a real number greater than $n/2 + 3$, and let l be a nonnegative integer. Then for any u_0 satisfying $x^\alpha u_0 \in H^\theta$ for $|\alpha| \leqslant l$, there exist a positive time T depending only on $\|u_0\|_\theta$ and a unique solution u to (1)-(2) belonging to $C([-T, T]; H^\theta)$. Moreover u satisfies*

$$\langle x \rangle^{-|\alpha|} \partial^\alpha u \in C([-T, T] \setminus \{0\}; H^\theta)$$

for $|\alpha| \leqslant l$, where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\langle x \rangle = \sqrt{1 + |x|^2}$.

More recently, Hayashi, Naumkin and Pipolo proved the infinite version of Proposition 1 for the one-dimensional equations. Roughly speaking, if $u_0 = o(e^{-\varepsilon|x|})$ as $|x| \to \infty$ with some $\varepsilon > 0$, then the unique solution u to (1)-(2) becomes real analytic in x for $t \neq 0$. See [4]. Reconsidering the method developed in [1], we here present the infinite version of Proposition 1.

1991 *Mathematics Subject Classification.* Primary 35B65; Secondary 35G25, 35Q55, 35S05.

Theorem 2. *Let θ, s and ε be real numbers satisfying $\theta > n/2+3$, $s \geqslant 1$ and $\varepsilon > 0$ respectively. Then for any u_0 satisfying $e^{\varepsilon\langle x\rangle^{1/s}}u_0 \in H^\theta$, there exist a positive time T depending only on $\|u_0\|_\theta$ and a unique solution u to (1)-(2) belonging to $C([-T,T];H^\theta)$. Moreover, there exist positive constants M, κ and ρ such that*

$$\|\langle x\rangle^{-2m-|\alpha|}\partial_t^m\partial^\alpha u(t)\|_\theta \leqslant M(\kappa\lambda(t))^{-2m}(\rho\lambda(t))^{-|\alpha|}m!^{2s}\alpha!^s,$$

for any $t \in [-T,T]\setminus\{0\}$, nonnegative integer m and multi-index α, where $\lambda(t) = 1/|t|^{1/2}+1/|t|$.

We would like to emphasize that the existence time T in Theorem 2 is independent of the weight $e^{\varepsilon\langle x\rangle^{1/s}}$. So, we can say that the solution to (1)-(2) gains Gevrey-s smoothness according to the exponential decay of the initial data.

Let r be a positive constant, and let $J = (J_1,\dots,J_n)$ be an operator defined by

$$J_k u = x_k u + 2it\partial_k u = e^{i|x|^2/4t}2it\partial_k(e^{-i|x|^2/4t}u).$$

To prove Theorem 2, we see (1) as a system for

$$w_l = \left[\frac{r^{|\alpha|}J^\alpha u}{\alpha!^s}, \frac{r^{|\alpha|}\overline{J^\alpha u}}{\alpha!^s}\right]_{|\alpha|\leqslant l},$$

and obtain the uniform energy estimates for $\{w_l\}_{l=1,2,3,\dots}$ step by step provided that r is sufficiently small. Doi-type operator discovered in [2] and block diagonalization of the system apply to getting the energy estimates. Note that the ellipticity of the principal part of the equation (1) is essential for the block diagonalization. About the initial value problem for semilinear Schrödinger equation with nonelliptic principal part, see [5]. Finally, we remark that from the view point of the Gevrey exponent s, our Gevrey estimate of solution is not optimal. Indeed, Hayashi and Kato studied the case $s = 1/2$ for the gauge invariant equation of the form

$$\partial_t u - i\Delta u = f(u),$$

and proved that the unique solution was real-analytic in $(t,x) \in ([-T,T]\setminus\{0\})\times\mathbb{R}^n$. See [3] for the detail.

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Scattering theory and self-similar solutions for the nonlinear Schrödinger equation

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We present some results in collaboration with F.B. Weissler concerning thesolutions of the model nonlinear Schrödinger equation

$$iu_t + \Delta u + \gamma |u|^\alpha u = 0, \tag{1}$$

where $u = u(t,x)$ is a function $(0,\infty) \times \mathbb{R}^N \to \mathbb{C}$, $\alpha > 0$ and $\gamma \in \mathbb{R}$.

It is well-known that if $\alpha < \dfrac{4}{N-2}$ ($\alpha < \infty$ if $N = 1$), then the Cauchy problem is locally well-posed $H^1(\mathbb{R}^N)$. Moreover, there is conservation of charge and energy. If, in addition, the initial value belongs to $L^2(\mathbb{R}^N, |x|^2 dx)$, then the solution stays in that space (see [6]). If $\gamma > 0$, or if $\alpha < 4/N$, or if the initial value has small H^1 norm, then the solution is global and uniformly bounded in H^1 as $t \to \infty$ (see [6, 9, 1]).

The asymptotic behavior of the solutions as $t \to \infty$ is often described by the scattering theory, at least if the initial value is small. More precisely, if $\alpha \geq 4/N$ and if the initial value has small H^1 norm, the there exists a unique $u^+ \in H^1(\mathbb{R}^N)$ such that $e^{-it\Delta}u(t) \to u^+$ in $H^1(\mathbb{R}^N)$ as $t \to \infty$, where $(e^{it\Delta})_{t\in\mathbb{R}}$ is the Schrödinger group (see [7] and [8]). The lower bound on α can be improved if one is willing to work in a smaller space. More precisely, if $\alpha > \dfrac{4}{N+2}$ ($\alpha > 2$ if $N = 1$), and if the initial value is small in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$, then there exists a unique $u^+ \in H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$ such that $e^{-it\Delta}u(t) \to u^+$ in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$ as $t \to \infty$ (see [2]). This last property implies easily that

$$\|u(t)\|_{L^{\alpha+2}} \approx t^{-\frac{N\alpha}{2(\alpha+2)}}, \tag{2}$$

as $t \to \infty$. (If $\gamma < 0$, some of these results hold for large data, see [7, 14, 2].)

We will study the asymptotic behavior of the solutions from a different point of view, and obtain different behaviors from (2). The main ingredients are only estimates for the

solutions of the linear equation with hmogeneous data and the elementary estimate

$$\|e^{it\Delta}\varphi\|_{L^{\rho+2}} \le t^{-\frac{N\rho}{2(\rho+2)}}\|\varphi\|_{L^{\frac{\rho+2}{\rho+1}}}, \tag{3}$$

which holds for all $0 \le \rho \le \infty$ and all $t > 0$. The estimate (3) is sharp in the sense that if $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, $\varphi \ne 0$, then $\liminf_{t\to\infty} t^{\frac{N\rho}{2(\rho+2)}}\|e^{it\Delta}\varphi\|_{L^{\rho+2}} > 0$ (see [10], p.228). In particular, if φ has sufficient decay $|x| \to \infty$, then $\|e^{it\Delta}\varphi\|_{L^{\rho+2}} \approx t^{-\frac{N\rho}{2(\rho+2)}}$ as $t \to \infty$,and for all $\rho \ge 0$.

It turns out that there are other decay rates. Indeed, consider $\psi(x) = |x|^{-p}$ with $0 < \mathrm{Re}p < N$, so that $\psi \in L^1_{loc}(\mathbb{R}^N) \cap \mathcal{S}'(\mathbb{R}^N)$. If $u(t,x) = [e^{it\Delta}\psi](x)$, then by uniqueness $u(t,x) \equiv \lambda^p u(\lambda^2 t, \lambda x)$ for all $\lambda > 0$; and so, letting $\lambda = t^{-\frac{1}{2}}$,

$$\|e^{it\Delta}\psi\|_{L^{\rho+2}} = t^{\frac{N}{2(\rho+2)}-\frac{\mathrm{Re}p}{2}}\|e^{i\Delta}\psi\|_{L^{\rho+2}}. \tag{4}$$

It turns out that $\|e^{i\Delta}\psi\|_{L^{\rho+2}} < \infty$ if

$$\rho + 2 > \min\Big\{\frac{N}{\mathrm{Re}p}, \frac{N}{N-\mathrm{Re}p}\Big\}. \tag{5}$$

The same property holds if, more generally, $\psi(x) = \omega(x)|x|^{-p}$ with ω homogeneous of degree 0 and sufficiently smooth (see [3, 4, 11, 13, 12]). Now, fix $0 \le \rho \le \infty$. Given $0 < \nu < \dfrac{N\rho}{2(\rho+2)}$, let $\psi(x) = |x|^{-2\nu-\frac{N}{\rho+2}}$. It follows from (4) and (5) that $\|e^{it\Delta}\psi\|_{L^{\rho+2}} = Ct^{-\nu}$. In particular, all possible decays up to the maximal one (3) are achieved.

The functions ψ considered above are not in any $L^q(\mathbb{R}^N)$. We now consider smooth functions φ that behave like ψ for $|x|$ large. For example, fix a function $\theta \in C_c^\infty(\mathbb{R}^N)$ with $\theta(x) = 0$ for $|x| \le 1$ and $\theta(x) = 1$ for $|x| \ge 2$, and let $\varphi(x) = |x|^{-2\nu-\frac{N}{\rho+2}}\theta(x)$ with ρ and ν as above. We have $\varphi \in C^\infty(\mathbb{R}^N)$, $\varphi \in H^\infty(\mathbb{R}^N)$ if $\nu > \dfrac{N\rho}{4(\rho+2)}$, and $\varphi - \psi \in L^{\frac{\rho+2}{\rho+1}}(\mathbb{R}^N)$ if $\nu < N/2$. Therefore, if $\dfrac{N\rho}{4(\rho+2)} < \nu < \dfrac{N\rho}{2(\rho+2)}$, then

$$t^\nu\big|\,\|e^{it\Delta}\varphi\|_{L^{\rho+2}} - \|e^{it\Delta}\psi\|_{L^{\rho+2}}\big| \le ct^{\nu-\frac{N\rho}{2(\rho+2)}} \underset{t\to\infty}{\longrightarrow} 0,$$

so that

$$\|e^{it\Delta}\varphi\|_{L^{\rho+2}} \approx t^{-\nu},$$

as $t \to \infty$. In particular, all the possible behaviors between $t^{-\frac{N\rho}{4(\rho+2)}}$ and the maximal one $t^{-\frac{N\rho}{2(\rho+2)}}$ are achieved with initial values $\varphi \in H^\infty(\mathbb{R}^N)$. Note that the lower bound on the decay rate is optimal by Strichartz' estimate if $\rho \le \dfrac{4}{N-2}$ ($\rho < \infty$ if $N = 1, 2$).

We now turn to the nonlinear problem (1). Consider $p \in \mathbb{C}$, Re $p = 2/\alpha$. If u is a solution of (1), then $u_\lambda(t,x) = \lambda^p u(\lambda^2 t, \lambda x)$ is also a solution for every $\lambda > 0$. A solution u of (1) such that $u = u_\lambda$ for $\lambda > 0$ is called a *self-similar* solution. If u is a self-similar solution, then letting $\lambda = t^{-\frac{1}{2}}$, we see that $u(t,x) = t^{-\frac{p}{2}} u\Big(1, \dfrac{x}{\sqrt{t}}\Big)$. $f(x) = u(1,x)$ is called the *profile* of u. We see that $\|u(t)\|_{L^r} = t^{\frac{N}{2r} - \frac{1}{\alpha}} \|f\|_{L^r}$, for all $r \geq 1$. In particular, $\|u(t)\|_{L^2} = t^{\frac{N}{4} - \frac{1}{\alpha}} \|f\|_{L^2}$. Therefore, by conservation of charge, if u is a self-similar solution in the classical H^1 sense and if $\alpha \neq 4/N$, then $u \equiv 0$. Therefore, the energy space is not appropriate for the study of the self-similar solutions. It turns out that self-similar solutions can be studied in another space, which is also quite natural. Suppose the profile f belongs to $L^{\alpha+2}(\mathbb{R}^N)$. It follows that $\|u(t)\|_{L^{\alpha+2}} = ct^{-\beta}$ with

$$\beta = \frac{4 - (N-2)\alpha}{2\alpha(\alpha+2)}. \tag{6}$$

Therefore, a self-similar solution with profile in $L^{\alpha+2}(\mathbb{R}^N)$ must belong to the space

$$X_\alpha = \{u \in L^\infty_{loc}((0,\infty), L^{\alpha+2}(\mathbb{R}^N)); \|u\|_{X_\alpha} := \sup_{t>0} t^\beta \|u(t)\|_{L^{\alpha+2}} < \infty\},$$

with β given by (6). That property suggests to solve the Cauchy problem for the equation (1) in the space X_α, with initial values in the corresponding space

$$W_\alpha = \{\varphi \in \mathcal{S}'(\mathbb{R}^N); \|\varphi\|_{W_\alpha} := \sup_{t>0} t^\beta \|e^{it\Delta}\varphi\|_{L^{\alpha+2}} < \infty\}.$$

The consideration of these spaces introduces limitations on α. Indeed, if $(N-2)\alpha > 4$, then $\beta < 0$ so that $W_\alpha = \{0\}$. On the other hand, if $\alpha < \alpha_0$, where α_0 is the positive root of the polynomial $Nx^2 + (N-2)x - 4$, then $\beta > \dfrac{N\alpha}{2(\alpha+2)}$, so that again $W_\alpha = \{0\}$. Note that the exponent α_0 also appears in the scattering theory, see [14, 2]. Using the estimate (3), one shows by an elementary fixed point argument the following existence result (see Theorem 2.1 in [3]).

THEOREM 1. *Assume*

$$\alpha_0 < \alpha < \frac{4}{N-2}. \tag{7}$$

There exists $\delta > 0$ such that if $\varphi \in W_\alpha$ and $\|\varphi\|_{W_\alpha} \leq \delta$, then there exists a solution $u \in X_\alpha$ with $\|u\|_{X_\alpha} \leq 2\rho$ of the equation (1) with the initial condition $u(0) = \varphi$ in the sense that

$$u(t) = e^{it\Delta}\varphi + i\gamma \int_0^t e^{i(t-s)\Delta}|u(s)|^\alpha u(s)\, ds, \tag{8}$$

for all $t > 0$. u is unique in the class $\{\|u\|_{X_\alpha} \leq 2\rho\}$.

We now may apply Theorem 1 to homogeneous initial values (see Proposition 4.3 in [3]).

COROLLARY 2. *Assume (7). If $\psi(x) = c|x|^{-p}$ with Re $p = 2/\alpha$ and c sufficiently small, then the solution v of (8) with the initial value ψ given by Theorem 1 is self-similar.*

As observed above, the self-similar solutions constructed in Corollary 2 cannot in general be classical H^1 solutions. However, we can use them to describe the asymptotic behavior of certain finite energy solutions (see Propositions 4.7 and 4.8 in [3]).

THEOREM 3. *Assume*

$$\alpha_0 < \alpha < \frac{4}{N}.$$

Let $\psi(x) = c|x|^{-p}$ Re $p = 2/\alpha$. Let θ be a C^∞ cut-off function and $\varphi(x) = c\theta(x)|x|^{-p} \in H^\infty(\mathbb{R}^N)$. Let u be the classical H^1 solution of (1) with the initial value φ. If c is sufficiently small, then $\|u(t) - v(t)\|_{L^{\alpha+2}} = O(t^{-\frac{N\alpha}{2(\alpha+2)}+\varepsilon})$ for any $\varepsilon > 0$, where v is the (self-similar) solution of (8) with the initial value ψ as given by Corollary 2. In particular, $\|u(t)\|_{L^{\alpha+2}} \approx t^{-\beta}$ as $t \to \infty$.

One can interpret Theorem 3 as follows. If $\varphi(x) = c|x|^{-p}$ for x large, with c small, then $\|u(t)\|_{L^{\alpha+2}}$ behaves like $t^{-\beta}$. One may wonder what happens if $\varphi(x) = c|x|^{-p}$ for x large, with Re $p > 2/\alpha$. (We already know by the scattering theory that if Re p is large enough, then $\|u(t)\|_{L^{\alpha+2}}$ behaves like $t^{-\frac{N\alpha}{2(\alpha+2)}}$.) An answer to that question is given by the following result (see Proposition 7.7 in [5]).

THEOREM 4. *Assume* (7). *Let* $p \in \mathbb{C}$ *satisfy*

$$\max\left\{\frac{2}{\alpha}, \frac{N}{2}\right\} < Rep < N\frac{\alpha+1}{\alpha+2},$$

and set

$$\nu = \frac{Rep}{2} - \frac{N}{2(\alpha+2)} > \beta.$$

Let $\psi(x) = c|x|^{-p}$, *let* θ *be a* C^∞ *cut-off, and set* $\varphi(x) = c\theta(x)|x|^{-p} \in H^\infty(\mathbb{R}^N)$. *Let* u *be the classical* H^1 *solution of* (1) *with the initial value* φ. *If* c *is small enough, then* u *is global and there exists* $\varepsilon > 0$ *such that* $t^\nu \|u(t) - e^{it\Delta}\psi\|_{L^{\alpha+2}} = O(t^{-\varepsilon})$. *In particular,* $\|u(t)\|_{L^{\alpha+2}} \approx t^{-\nu}$ *as* $t \to \infty$.

Note that $u(t)$ behaves like $e^{it\Delta}\psi$, which is a self-similar solution of the linear Schrödinger equation. In view of the assumptions of Theorem 4, the range of decays that are achieved is given by

$$\beta < \nu < \frac{N\alpha}{2(\alpha+2)} \quad \text{if} \quad \alpha_0 < \alpha \le \frac{4}{N}, \tag{9}$$

and

$$\frac{N\alpha}{4(\alpha+2)} < \nu < \frac{N\alpha}{2(\alpha+2)} \quad \text{if} \quad \frac{4}{N} < \alpha < \frac{4}{N-2}. \tag{10}$$

The upper bound in (9) and (10) is optimal and it is achieved for small initial values in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2dx)$. The lower limit in (10) is never achieved because of Strichartz' estimate. The lower limit in (9) is probably not optimal.

Note also that in the scattering theory, the mapping $\varphi \to u^+$ is one to one. In Theorems 3 and 4 we see that, by taking different cut-off functions θ, many solutions of (1) are asymptotic as $t \to \infty$ to the same self-similar solution of (1), or of the linear Schrödinger equation in the case of Theorem 4. That may look surprising. However, it seems that the scattering theory cannot be applied to the initial values considered in Theorem 3. Indeed, the H^1 scattering does not apply because $\alpha < 4/N$ and the scattering in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2dx)$ does not apply because $\varphi \notin L^2(\mathbb{R}^N, |x|^2dx)$. On

the other hand, one may apply the H^1 scattering theory to the initial values considered in Theorem 4 in the case $\alpha > 4/N$. What happens is that, with respect to the scattering theory, Theorem 4 provides information of different nature (i.e. the fact that $\|u(t)\|_{L^{\alpha+2}}$ behaves like $t^{-\nu}$).

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GLOBAL EXISTENCE FOR A CLASS OF SYSTEMS OF NONLINEAR WAVE EQUATIONS

SOICHIRO KATAYAMA

1. Introduction

We consider the Cauchy problem for systems of nonlinear wave equations of the type

$$\begin{cases} \Box_i u_i = F_i(u, Du, D_x Du) & \text{in } (0,\infty)\times\mathbb{R}^n \ (i=1,\cdots,m), \\ u(0,x) = \varepsilon f(x),\ u_t(0,x) = \varepsilon g(x), \end{cases} \tag{1.1}$$

where $\Box_i = \partial_t^2 - c_i^2\Delta_x$ $(i = 1,\cdots,m)$ with $c_i > 0$, $u = (u_i)_{i=1,\cdots,m}$, $Du = (\partial_a u_i)_{\substack{i=1,\cdots,m\\ a=0,\cdots,n}}$ and $D_x Du = (\partial_j\partial_a u_i)_{\substack{i=1,\cdots,m\\ j=1,\cdots,n\\ a=0,\cdots,n}}$. Here we used the notation $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$ $(j = 1,\cdots,n)$. Without loss of generality, we may assume $c_1 \le c_2 \le \cdots \le c_m$.

We suppose that $F = (F_i)_{i=1,\cdots,m}$ is a smooth function around the origin, satisfying

$$\begin{aligned} F(u,v,w) &= F\left((u_i)_{i=1,\cdots,m},\ (v_{i,a})_{\substack{i=1,\cdots,m\\ a=0,\cdots,n}},\ (w_{i,ja})_{\substack{i=1,\cdots,m\\ j=1,\cdots,n\\ a=0,\cdots,n}}\right) \\ &= O\big(|u|^p + |v|^p + |w|^p\big) \end{aligned} \tag{1.2}$$

around the origin in $\mathbb{R}^m \times \mathbb{R}^{m(n+1)} \times \mathbb{R}^{mn(n+1)}$, with some integer $p(\ge 2)$. We also suppose that $f,\ g \in C_0^\infty(\mathbb{R}^n)$ and that ε is a small and positive parameter.

In order to ensure existence of local solutions to (1.1), we always assume

$$c_{ka}^{ij}(u,v,w) = c_{ka}^{ji}(u,v,w)\ \big(i,j\in\{1,\cdots,m\}, k\in\{1,\cdots,n\}, a\in\{0,\cdots,n\}\big) \tag{1.3}$$

for any $(u,v,w) \in \mathbb{R}^m \times \mathbb{R}^{m(n+1)} \times \mathbb{R}^{mn(n+1)}$, where

$$c_{ka}^{ij}(u,v,w) = \frac{\partial F_i}{\partial w_{j,ka}}(u,v,w). \tag{1.4}$$

Without loss of generality, we may also assume

$$c_{kl}^{ij} = c_{lk}^{ij} \quad \big(i,j\in\{1,\cdots,m\},\ k,l\in\{1,\cdots,n\}\big) \tag{1.5}$$

in addition to (1.3), because we only consider classical solutions.

In the following, we say that **(GE) holds** when for any $f,\ g \in C_0^\infty(\mathbb{R}^n)$, there exists a positive constant ε_0 such that (1.1) admits a unique global solution $u \in C^\infty\big([0,\infty)\times\mathbb{R}^n\big)$ for any $\varepsilon \in (0,\varepsilon_0]$.

We want to recall some known results briefly, restricting our attention to the cases $n = 2$ and $n = 3$.

First we assume that $c_1 = c_2 = \cdots = c_m (= c)$. In this case, it is known that (GE) holds if we assume $p \geq 3$ when $n = 3$, or $p \geq 4$ when $n = 2$, respectively. On the other hand, (GE) does not hold for general nonlinear terms F when $(n, p) = (3, 2)$ or $(2, 3)$. Hence we need some condition on F in order to get (GE) when $(n, p) = (3, 2)$ or $(2, 3)$.

Here we introduce some notations. For a given function $G = G(u, v, w)$ and a positive integer k, we define a function $G^{(k)}$ by

$$G^{(k)}(u, v, w) = \sum_{|\alpha|+|\beta|+|\gamma|=k} \partial_u^\alpha \partial_v^\beta \partial_w^\gamma G(0, 0, 0) \frac{u^\alpha v^\beta w^\gamma}{\alpha!\beta!\gamma!},$$

where $\partial_u^\alpha = \partial_{u_1}^{\alpha_1} \cdots \partial_{u_m}^{\alpha_m}$, $u^\alpha = u_1^{\alpha_1} \cdots u_m^{\alpha_m}$ and so on. We also define

$$G^{(k,i)}(u, v, w) = \sum_{\alpha+|\beta|+|\gamma|=k} \partial_{u_i}^\alpha \partial_{v_i}^\beta \partial_{w_i}^\gamma G(0, 0, 0) \frac{u_i^\alpha v_i^\beta w_i^\gamma}{\alpha!\beta!\gamma!},$$

where $v_i = (v_{i,a})_{a=0,\cdots,n}$, $w_i = (w_{i,ja})_{\substack{j=1,\cdots,n \\ a=0,\cdots,n}}$, and $\partial_{v_i}^\beta = \partial_{v_{i,0}}^{\beta_0} \cdots \partial_{v_{i,n}}^{\beta_n}$ for a multi-index $\beta = (\beta_0, \cdots, \beta_n)$. $\partial_{w_i}^\gamma$ is defined similarly.

For a positive constant c, we define

$$N(c) = \left\{ X = (X_0, \cdots, X_n) \in \mathbb{R}^{n+1} \,\middle|\, X_0^2 - c^2 \sum_{j=1}^n X_j^2 = 0 \right\},$$

$$L(c) = \left\{ (u, v, w) \,\middle|\, \text{there exist } \mu = (\mu_j), \nu = (\nu_j) \in \mathbb{R}^m \text{ and } X \in N(c) \text{ such that } v = (\mu_j X_a)_{\substack{j=1,\cdots,m \\ a=0,\cdots,n}},\ w = (\nu_j X_k X_a)_{\substack{j=1,\cdots,m \\ k=1,\cdots,n \\ a=0,\cdots,n}} \right\}.$$

Theorem 1.1 (Klainerman [10], the author [8]). *Let $c_1 = \cdots = c_m = c$. Suppose that $(n, p) = (3, 2)$ or $(2, 3)$. If $F^{(p)} \equiv 0$ on $L(c)$, then* (GE) *holds for* (1.1).

See also Christodoulou [2], Godin [3], Hoshiga [4] and the author [7].

The above condition, $F^{(p)} \equiv 0$ on $L(c)$, is known as the "Null Condition".

Now we consider the general case $c_1 \leq c_2 \leq \cdots \leq c_m$. For simplicity of exposition, we assume $c_1 < c_2 < \cdots < c_m$ here.

When $F = F(v, w)$ (namely F is independent of u), Agemi – Yokoyama ([1]) introduced the Null Condition for systems with different speeds of propagation.

Theorem 1.2 (Agemi –Yokoyama [1], Hoshiga – Kubo [5], Yokoyama [16]). *Let $c_1 < \cdots < c_m$. Suppose that $(n, p) = (3, 2)$ or $(2, 3)$. Moreover we assume that F is independent of u, namely $F = F(v, w)$. If*

$$F_i^{(p,i)} \equiv 0 \text{ on } L(c_i) \text{ for each } i \in \{1, \cdots, m\},$$

then (GE) *holds for* (1.1).

Now we want to consider the case where $F = F(u, Du, D_x Du)$.

In general, estimates for u are not so good as those for its derivatives Du and $D_x Du$. Especially, we note here that the energy inequality does not give a natural estimate for u itself in general. These facts make the analysis complicated.

However, for the special case where F has a divergent form, the situation is much simpler. Roughly speaking, the solution u for this case can be written in terms of solutions to some wave equations. Hence the necessary tools for the analysis are no more than those necessary for the case where F depends only on derivatives of u. From this observation, we can see that a similar proof to that of Theorem 1.2 gives us the following result.

Theorem 1.3. *Let $c_1 < \cdots < c_m$. Suppose that $(n,p) = (3,2)$ or $(2,3)$. Assume that there exist functions $G_a(u,v) = \big(G_{a,i}(u,v)\big)_{i=1,\cdots,m}$ $(a = 0, \cdots, n)$ such that*

$$F(u, Du, D_x Du) = \sum_{a=0}^{n} \partial_a \big\{ G_a(u, Du) \big\} \text{ for any } u \in C^2.$$

If $G_{a,i}^{(p,i)} \equiv 0$ on $L(c_i)$ for any $a \in \{0, \cdots, n\}$ and any $i \in \{1, \cdots, m\}$, then (GE) *holds for the Cauchy problem* (1.1).

Note that nonlinear terms of order greater than $p+1$ is also assumed to have a divergent form in Theorem 1.3.

Our main purpose is to show that we need no restriction on terms of order greater than $p+1$ in order to get (GE) when $(n,p) = (3,2)$. More precisely, our main result is the following:

Theorem 1.4. *Suppose that $c_1 < \cdots < c_m$ and $(n,p) = (3,2)$. Assume that there exist functions $G_a(u,v) = \big(G_{a,i}(u,v)\big)_{i=1,\cdots,m}$ $(a = 0,1,2,3)$ such that*

$$F^{(2)}(u, Du, D_x Du) = \sum_{a=0}^{3} \partial_a \big\{ G_a(u, Du) \big\} \text{ for any } u \in C^2.$$

If $G_{a,i}^{(2,i)} \equiv 0$ on $L(c_i)$ for any $a \in \{0,1,2,3\}$ and $i \in \{1, \cdots, m\}$, then (GE) *holds for the Cauchy problem* (1.1).

In order to prove Theorem 1.4, we need some estimates for L^∞- and L^2-norms of u in addition to estimates used in the proof of Theorem 1.2, because $F - F^{(2)}$ does not necessarily have a divergent form.

2. L^2-ESTIMATES

Let f and g be functions in $\mathscr{S}$, where $\mathscr{S}$ denotes the class of rapidly decreasing functions, and c a positive constant. We define a mapping $U^*[f, g; c]$ by

$$U^*[f, g; c](t, x) = u(t, x) \text{ for } t > 0 \text{ and } x \in \mathbb{R}^3, \tag{2.1a}$$

where u is the unique classical solution to

$$\begin{cases} (\partial_t^2 - c^2\Delta_x)u(t,x) = 0 & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^3, \\ u(0,x) = f(x),\ (\partial_t u)(0,x) = g(x) & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{2.1b}$$

For a given function $\phi = \phi(t,x)$, we define another mapping $U[\phi;c]$ by

$$U[\phi;c](t,x) = v(t,x) \text{ for } (t,x) \in (0,\infty)\times\mathbb{R}^3\ , \tag{2.2a}$$

where v is the unique classical solution to

$$\begin{cases} (\partial_t^2 - c^2\Delta_x)v(t,x) = \phi(t,x) & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^3, \\ v(0,x) = (\partial_t v)(0,x) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{2.2b}$$

To estimate L^2-norms of $U[F_i - F_i^{(2)};c_i]$, we use the following.

Lemma 2.1 (von Wahl [15]). *We have*

$$\left\|U[\phi;c](t,\cdot)\right\|_{L^2} \le C\int_0^t \|\phi(\tau,\cdot)\|_{L^{6/5}}d\tau \quad \textit{for } t>0, \tag{2.3}$$

and

$$\left\|U^*[f,g;c](t,\cdot)\right\|_{L^2} \le C(\|f\|_{L^2} + \|g\|_{L^{6/5}}) \quad \textit{for } t>0. \tag{2.4}$$

For the estimate of L^2-norms of $U[F_i^{(2)};c_i]$, we can make use of a better estimate because $F^{(2)} = \sum_a \partial_a G_a$.

Since

$$U[\partial_a\phi;c] = \partial_a U[\phi;c] - \delta_{0,a}U_0[0,\phi(0,\cdot);c], \quad a = 0,1,2,3 \tag{2.5}$$

with Kronecker's delta

$$\delta_{i,j} = \begin{cases} 1, & i=j \\ 0, & i\ne j, \end{cases}$$

the energy inequality and (2.4) imply the following:

Lemma 2.2. *Let $a \in \{0,1,2,3\}$. Then we have*

$$\|U[\partial_a\phi;c](t,\cdot)\|_{L^2} \le C\left(\int_0^t \|\phi(\tau,\cdot)\|_{L^2}d\tau + \delta_{0,a}\|\phi(0,\cdot)\|_{L^{6/5}}\right) \tag{2.6}$$

for $t>0$.

3. L^∞-DECAY ESTIMATES

We introduce some weights which are concerned with decay of solutions to wave equations. Let c be a non-negative constant. We define

$$w^{\mu,\nu}(\tau,\rho;c) = (1+\rho)^\mu(1+|c\tau-\rho|)^\nu, \tag{3.1}$$

$$w_*^{\mu,\nu}(\tau,\rho;c) = (1+\tau+\rho)^\mu(1+|c\tau-\rho|)^\nu \text{ for } \tau\geq 0 \text{ and } \rho\geq 0. \tag{3.2}$$

For given positive constants $c_1,\cdots,c_m$, we define

$$W(\tau,\rho) = W(\tau,\rho;c_1,\cdots,c_m) = \min_{j=1,\cdots,m} w^{1,1}(\tau,\rho;c_j). \tag{3.3}$$

For functions $\phi=\phi(t,x)$ and $w=w(\tau,\rho)$ which are defined on $\mathbb{R}_+\times\mathbb{R}^3$ and $\mathbb{R}_+\times\mathbb{R}_+$ respectively, and for a non-negative integer k, we define

$$|\!|\!|\phi|\!|\!|_{w;k,t} = \sum_{|\alpha|\leq k}\sup_{0\leq\tau\leq t}\sup_{y\in\mathbb{R}^3}\Big\{|y|\,w\big(\tau,|y|\big)\,|\partial^\alpha\phi(\tau,y)|\Big\}. \tag{3.4}$$

We also define

$$\|\phi\|_{w;k,t} = \sum_{|\alpha|+|\beta|\leq k}\sup_{0\leq\tau\leq t}\big\|w\left(\tau,|\cdot|\right)\partial^\alpha\Omega^\beta\phi(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)}, \tag{3.5}$$

where $\Omega^\beta = \Omega_{12}^{\beta_1}\Omega_{23}^{\beta_2}\Omega_{31}^{\beta_3}$ with $\Omega_{ij}=x_i\partial_j-x_j\partial_i$.

First we state known decay estimate for homogeneous wave equations.

Lemma 3.1. *Let c be a positive constant. Suppose that $f,\,g\in\mathscr{S}(\mathbb{R}^3)$. Then there exists some constant C, depending on c, f and g, such that*

$$w_*^{1,1}(t,|x|;c)\big|U^*[f,g;c](t,x)\big| \leq C \tag{3.6}$$

for $t>0$ and $x\in\mathbb{R}^3$.

Since $F^{(2)}=\sum_a\partial_a G_a$, we can apply the following estimates due to Yokoyama [16] to estimate L^∞ norms of $U[F_i^{(2)};c_i]$, according to (2.5).

Lemma 3.2 (Yokoyama [16]). *Let c_0 be a positive constant and c be a non-negative constant. We define*

$$\Phi_\theta(t) = \begin{cases}\log(2+t), & \textit{when } \theta=0,\\ 1, & \textit{when } \theta>0.\end{cases} \tag{3.7}$$

(i) *If $c\neq c_0$, we have*

$$|\partial_a U[\phi;c_0](t,x)| \leq Cw^{-1,-\mu}(t,|x|;c_0)\Phi_{\nu-1}(t)|\!|\!|\phi|\!|\!|_{w_*^{\mu,\nu}(\cdot,\cdot;c);1,t} \tag{3.8}$$

for $\mu>0$ and $\nu\geq 1$,

(ii) *We have*

$$\begin{aligned}|\partial_a U[\phi;c_0](t,x)| \leq & Cw^{-1,-\mu}(t,|x|;c_0)\Phi_{\nu-1}(t)|\!|\!|\phi|\!|\!|_{w_*^{\mu,\nu}(\cdot,\cdot;c_0);1,t}\\ &+Cw^{-1,-\nu}(t,|x|;c_0)\Phi_{\mu-1}(t)|\!|\!|\phi|\!|\!|_{w_*^{\mu,\nu}(\cdot,\cdot;c_0);1,t}\end{aligned} \tag{3.9}$$

for $\mu\geq 1$ and $\nu\geq 1$.

Here the constant C may depend on c_0 and c, but is independent of other quantities.

As a corollary of the above lemma we have the following.

Corollary 3.3. *Let c and c_0 be positive constants. Then for $\mu \geq 1$, we have*

$$w^{1,1}(t,|x|;c)|\partial_a U[\phi;c_0](t,x)| \leq C\Phi_{\mu-1}(t)|||\phi|||_{w^{\mu,\mu}(\cdot,\cdot;c);1,t} \tag{3.10}$$

where Φ_θ was defined in Lemma 3.2.

Corollary 3.3 and a kind of Sobolev's inequality (see [11]) imply

$$w^{1,1}(t,|x|;c_0)|\partial_a U[\phi;c_0](t,x)| \leq C\log(2+t)\|\phi\|_{W;4,t}, \tag{3.11}$$

where $c_0 > 0$ and $W(\tau,\rho) = W(\tau,\rho;c_1,\cdots,c_m)$ with $0 < c_1 < \cdots < c_m$.

To estimate L^∞-norms of $U[F_i - F_i^{(2)};c_i]$, we need a new estimate.

Theorem 3.4. *Let $c_0 > 0$ and $0 < c_1 < \cdots < c_m$ be given. Then we have*

$$w_*^{1,1}(t,|x|;c_0)|U[\phi;c_0](t,x)| \leq C\|\phi\|_{W^2;3,t}, \tag{3.12}$$

where $W(\tau,\rho) = W(\tau,\rho;c_1,\cdots,c_m)$ as before.

Outline of proof. We may assume $c_0 = 1$. After elementary but complicated calculations, we have

$$\frac{1}{r}\int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{d\rho}{(1+\rho)^2(1+|cs-\rho|)^2} \leq C(1+t+r)^{-1}(1+|t-r|)^{-1}. \tag{3.13}$$

As a consequence, by writing the solution explicitly, we obtain

$$w_*^{1,1}(t,|x|;c_0)\big|U[\phi;c_0](t,x)\big| \leq C|||\phi|||_{w^{2,2}(\cdot,\cdot;c);0,t}. \tag{3.14}$$

Now the result follows from appropriate partition of unity and a kind of Sobolev's inequality.

4. Outline of proof of Theorem 1.4

We introduce some vector fields:

$$\begin{aligned} \Gamma_0 = t\partial_t + \sum_{j=1}^3 x_j\partial_j,\ \Gamma_1 = \partial_t,\ \Gamma_2 = \partial_1,\ \Gamma_3 = \partial_2,\ \Gamma_4 = \partial_3, \\ \Gamma_5 = \Omega_{12},\ \Gamma_6 = \Omega_{13},\ \Gamma_7 = \Omega_{23}, \end{aligned} \tag{4.1}$$

where $\Omega_{ij} = x_i\partial_j - x_j\partial_i$. Γ_0 is introduced to treat the null forms, which are associated with the Null Condition.

We write Γ^α for $\Gamma_1^{\alpha_1}\cdots\Gamma_6^{\alpha_6}$ using a multi-index α.

For a non-negative integer s and a function v for which the following definitions make sense, we define

$$|v(t,x)|_s = \sum_{|\alpha|\le s} |\Gamma^\alpha v(t,x)|, \tag{4.2}$$

$$\|v(t,\cdot)\|_{s,p} = \left\| |v(t,\cdot)|_s \right\|_{L^p(\mathbb{R}^3)}, \quad (1 \le p \le \infty). \tag{4.3}$$

Because of the classical local existence theorem, it suffices to get some *a priori* estimate to prove global existence of the solution.

Let $T > 0$ and $u_\varepsilon = (u_{\varepsilon,1}, \cdots, u_{\varepsilon,m})$ be the solution to (1.1) for $0 \le t < T$, where ε is the parameter appeared in (1.1).

We define

$$E_\varepsilon(t) = \sup_{0\le\tau<t} \sum_{i=1}^m e_{\varepsilon,i}(\tau), \tag{4.4}$$

where

$$\begin{aligned} e_{\varepsilon,i}(t) = &\left\| w^{1,1}(t,|\cdot|;c_i)\, |u_{\varepsilon,i}(t,\cdot)|_{K+2} \right\|_{L^\infty(\mathbb{R}^3)} \\ &+ (1+t)^{-\lambda} \left(\|u_{\varepsilon,i}(t,\cdot)\|_{2K,2} + \|Du_{\varepsilon,i}(t,\cdot)\|_{2K,2} \right) \\ &+ \left\| w^{1-2\lambda,1-2\lambda}(t,|\cdot|;c_i)\, |u_{\varepsilon,i}(t,\cdot)|_{2K-4} \right\|_{L^\infty(\mathbb{R}^3)} \\ &+ \left\| w^{1,1-2\lambda}(t,|\cdot|;c_i)\, |u_{\varepsilon,i}(t,\cdot)|_{2K-8} \right\|_{L^\infty(\mathbb{R}^3)} + \|u_{\varepsilon,i}(t,\cdot)\|_{2K-6,2}. \end{aligned} \tag{4.5}$$

In the above, K is a sufficiently large integer and λ is a positive and sufficiently small constant.

Applying the estimates in Sections 2 and 3, we can prove the following.

Proposition 4.1. *There exist positive constants M_0 and C_0, which are independent of T and of small ε, such that $E_\varepsilon(T) < M_0$ implies*

$$E_\varepsilon(T) \le C_0 \left(\varepsilon + E_\varepsilon(T)^2 \right). \tag{4.6}$$

By Proposition 4.1, standard arguments imply Theorem 1.4.

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ON THE COUPLED SYSTEM OF NONLINEAR WAVE EQUATIONS WITH DIFFERENT PROPAGATION SPEEDS IN TWO SPACE DIMENSIONS

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1. Introduction and Main results

In the present paper, we treat the coupled system of wave equations with different propagation speeds:

$$(\partial_t^2 - \Delta)f = F(f, \partial f, g, \partial g), \qquad x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{1.1}$$

$$(\partial_t^2 - s^2\Delta)g = G(f, \partial f, g, \partial g), \qquad x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{1.2}$$

$$f(x,0) = f_0(x) \in H^a, \qquad \partial_t f(x,0) = f_1(x) \in H^{a-1}, \qquad x \in \mathbb{R}^n, \tag{1.3}$$

$$g(x,0) = g_0(x) \in H^a, \qquad \partial_t g(x,0) = g_1(x) \in H^{a-1}, \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where $\partial = \partial_{x_j} (1 \le j \le n)$ or ∂_t and s is a propagation speed of (1.2) with $s > 1$. The nonlinear terms are as follows:

$$F = \Sigma_{j=1}^3 \alpha_j F_j, \qquad \alpha_j \in \mathbb{C},$$
$$G = \Sigma_{j=1}^3 \beta_j G_j, \qquad \beta_j \in \mathbb{C},$$
$$F_1 = g\partial g, \quad F_2 = f\partial g, \quad F_3 = g\partial f,$$
$$G_1 = f\partial f, \quad G_2 = f\partial g, \quad G_3 = g\partial f.$$

Our aim is to prove the time local well-posedness with the low regularity initial datas. Physically, this system describe the Klein-Gordon-Zakharov equations (K-G-Z) and the coupled system of complex scalar field and Maxwell equations (C-M). we can derive the time local well-posedness of (K-G-Z) and (C-M) from the time local well-posednwss of this system.

In the case of $n \ge 4$, we can prove the time local well-posedness with $a \ge (n-1)/2$ by the Strichartz estimate. This proof is indepent of the difference of the speeds. In the case of $n = 3$, we can prove the time local well-posedness with $a > 1$ by the Strichartz estimate. To prove the time local well-posedness with $a = 1$ in this argument, we need the limiting case of the Strichartz estimate, which fails. But, Ozawa, Tsutaya and Tsutsumi[5] proved the time local well-posedness in the case of $F = F_1, G = G_3$ with $a = 1$ by using the difference of speeds and Fourier restriction norm method. By this result and the energy conservation, they proved the time global well-posedness of (K-G-Z). By the same argument, T[7] proved the time local well-posedness in the case of $F = F_2, F = F_3, G = G_1$ with $a = 1$. By this result and the energy conservation, we had the time global well-posedness of (C-M).

Fourier restriction norm method was developed by Bourgain [1] and [2] to study the nonlinear Schrödinger equation and the KdV equation, and it was improved for the one

dimensional case by Kenig, Ponce and Vega[4]. The related method was developed by Klainerman and Machedon [3] for the nonlinear wave equations.

In the case of $n = 2$, it seem to be difficult to prove the time local well-posedness with $a < 3/4$ by the Strichartz estimate. But, in the present paper, we have the time local well-posedness with $a > 1/2$ by using the difference of speeds and Fourier restriction norm method.

Before we state the theorem, we give several notations. For a function $u(t,x)$, we denote by $\widetilde{u}(\tau,\xi)$ the Fourier transform in both x and t variables of u. For $a,b \in \mathbb{R}, s > 0$ and $l = +$ or $-$, we define the spaces $X^{a,b}_{s,l}$ as follows:

$$X^{a,b}_{s,l} = \{u \in \mathcal{S}'(\mathbb{R}^3) \big| \|u\|_{X^{a,b}_{s,l}} < \infty\}$$

$$\|u\|_{X^{a,b}_{s,l}} = \| < \xi >^a P^b_{s,l}(\tau,\xi)\widetilde{u}\|$$

where $P_{s,l}(\tau,\xi) = (1 + |\tau + sl|\xi||), < \xi >= \sqrt{1+|\xi|^2}$ and $\|\cdot\| = \|\cdot\|_{L^2_{\tau,\xi}}$. For $T > 0$, we denote the cut function $\chi(t), \chi_T(t) \in C^\infty_0$ as follows:

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \le 1/2, \\ 0 & \text{for } |t| > 1, \end{cases}$$

$$\chi_T(t) = \chi(t/T).$$

For $s > 0$, we define $W_{s,\pm}(t) = e^{\mp ist\omega}$, where $\omega = \sqrt{1+\Delta}$. We put

$$\langle f, g\rangle = \int_{\mathbb{R}^3} f(t,x)\overline{g(t,x)}dtdx.$$

Theorem 1.1. *Let $s > 1$ or $1 > s > 0, a > 1/2$ and $2a - 1/2 > b > 1/2$, then there exist $T > 0$ and problem (1.1)-(1.4) has time local unique solution satisfying*

$$f, g \in C([-T,T] : H^a(\mathbb{R}^2)) \cap C^1([-T,T] : H^{a-1}(\mathbb{R}^2)), \tag{1.5}$$

$$\chi_T(t)(f \pm i\omega^{-1}\partial_t f) \in X^{a,b}_{1,\pm},$$

$$\chi_T(t)(g \pm i(s\omega)^{-1}\partial_t g) \in X^{a,b}_{s,\pm}.$$

Furthermore, this solution depends continuously on initial datas in the topology of (1.5).

2. The proof of the Theorem

We first put

$$f_\pm = f \pm i\omega^{-1}\partial_t f,$$

$$g_\pm = g \pm i(s\omega)^{-1}\partial_t g.$$

Then, (1.1)-(1.4) are rewritten as follows:

$$(i\partial_t \mp D)f_\pm = \mp\omega^{-1}F \mp (D-\omega)f_\pm, \tag{2.1}$$

$$(i\partial_t \mp sD)g_\pm = \mp(s\omega)^{-1}G \mp s(D-\omega)g_\pm, \tag{2.2}$$

$$f_\pm(0) = f_{\pm 0}, \quad g_\pm(0) = g_{\pm 0}, \tag{2.3}$$

where

$$f_{\pm 0} = f_0 \pm i\omega^{-1}f_1 \in H^a,$$

$$g_{\pm 0} = f_0 \pm i(s\omega)^{-1}f_1 \in H^a.$$

We try to solve (2.1)-(2.3) locally in time. For that purpase, we consider the following integral equations associated with (2.1)-(2.3):

$$f_\pm(t) = \chi(t)W_{1,\pm}(t)f_{\pm 0} \mp i\chi_T(t)\int_0^t W_{1,\pm}(t-s)\{\omega^{-1}F + (D-\omega)f_\pm\}ds, \tag{2.4}$$

$$g_\pm(t) = \chi(t)W_{s,\pm}(t)g_{\pm 0} \mp i\chi_T(t)\int_0^t W_{s,\pm}(t-s)\{\omega^{-1}G + (D-\omega)g_\pm\}ds. \tag{2.5}$$

If we try to apply the Fourier restriction norm method to (2.4)-(2.5), we have only to prove the following estimates:

$$\|F_1\|_{X_{1,l}^{a-1,b-1+\epsilon}} \le C\Sigma_{j,k}\|g_j\|_{X_{s,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \tag{2.6}$$

$$\|F_2\|_{X_{1,l}^{a-1,b-1+\epsilon}} \le C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \tag{2.7}$$

$$\|F_3\|_{X_{1,l}^{a-1,b-1+\epsilon}} \le C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \tag{2.8}$$

$$\|G_1\|_{X_{s,l}^{a-1,b-1+\epsilon}} \le C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|f_k\|_{X_{1,k}^{a,b}}, \tag{2.9}$$

$$\|G_2\|_{X_{s,l}^{a-1,b-1+\epsilon}} \le C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \tag{2.10}$$

$$\|G_3\|_{X_{s,l}^{a-1,b-1+\epsilon}} \le C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \tag{2.11}$$

where $a > 1/2, 2a - 1/2 > b > 1/2$ and $\epsilon > 0$ which is sufficiently small and j, k and l denote either of $+$ or $-$ sign. Without loss of generality, we can assume $\widetilde{f_j}$ and $\widetilde{g_k} > 0$. Here,we note that

$$\begin{aligned} f &= 1/2(f_+ + f_-), \\ \partial_t f &= \frac{\omega}{2i}(f_+ - f_-), \\ g &= 1/2(g_+ + g_-), \\ \partial_t g &= \frac{s\omega}{2i}(g_+ - g_-). \end{aligned}$$

Therefore, the left hand side of (2.6) is bounded by

$$\Sigma_{j,k}\|g_j\omega g_k\|_{X_{1,l}^{a-1,b-1+\epsilon}}. \tag{2.12}$$

To prove (2.6), we have only to prove

$$\|g_j\omega g_k\|_{X_{1,l}^{a-1,b-1+\epsilon}} \le C\|g_j\|_{X_{s,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}},$$

which is equivalent to

$$\langle g_j g_k, h\rangle \le C\|g_j\|_{X_{s,j}^{a,b}}\|g_k\|_{X_{s,k}^{a-1,b}}\|h\|_{X_{1,l}^{1-a,1-b-\epsilon}}$$

by duality argument.We obtain this inequality by interpolating between (2.13) and (2.14). In the same manner, we obtain (2.7)-(2.11) from Proposition 2.1.

Proposition 2.1. *Assume that $a > 1/2$,$b > 1/4$,$4a + 2b > 3$ and $s > 1$ or $0 < s < 1$. Then the following inequalities hold.*

$$|\langle f, gh\rangle| \le C\|f\|_{X_{1,j}^{-a,b}}\|g\|_{X_{s,k}^{a,b}}\|h\|_{X_{1,l}^{a,b}} \tag{2.13}$$

$$|\langle f, gh\rangle| \le C\|f\|_{X_{s,j}^{-a,b}}\|g\|_{X_{1,k}^{a,b}}\|h\|_{X_{1,l}^{a,b}} \tag{2.14}$$

where j,k and l denote either of $+$ or $-$ sign.

Remark 2.1. This inequalities hold with $a = 1/2, b > 1/2$.But, because of $b > 1/2$, we can't apply Proposition 2.1 to (1.1)-(1.4).

Before we prove Proposition 2.1, we mention an essential lemma.

Lemma 2.1. *Assume that $a > 1/2, b > 1/4, 4a + 2b > 3$ and $s > 1$ or $0 < s < 1$. Then, there is a positive constant C and the following inequalities hold.*

$$\sup_{\tau,\xi} < \xi >^{2a} P_{1,j}^{-2b}(\tau,\xi)\big(< \xi >^{-2a} P_{s,k}^{-2b}(\tau,\xi) *_{\tau,\xi} < \xi >^{-2a} P_{1,l}^{-2b}(\tau,\xi)\big) < C$$

where j, k and l denote either of $+$ or $-$ sign.

Proof of Proposition 2.1. Without loss of generality, we can assume $\widetilde{f}, \widetilde{g}$ and $\widetilde{h} > 0$. We first prove (2.13). By the duality argument, (2.13) is equivalent to

$$\| < \xi >^{a} P_{1,j}^{-b}(\tau,\xi)\widetilde{gh}\|_{L_{\tau,\xi}} \leq C\| < \xi >^{a} P_{s,k}^{b}(\tau,\xi)\widetilde{g}\|\| < \xi >^{a} P_{s,l}^{b}(\tau,\xi)\widetilde{h}\|,$$

which is equivalent to

$$\| < \xi >^{a} P_{1,j}^{b}(\tau,\xi)\big(< \xi >^{-a} P_{s,k}^{-b}(\tau,\xi)\widetilde{g} *_{\tau,\xi} < \xi >^{-a} P_{s,l}^{-b}(\tau,\xi)\widetilde{h}\big)\| \leq C\|\widetilde{g}\|^2\|\widetilde{h}\|^2. \tag{2.15}$$

By the Schwartz's inequality and Lemma 2.1, the left hand side of (2.15) is bounded by

$$\begin{aligned}\int_{\mathbb{R}^3} < \xi >^{2a} P_{1,j}^{2b}(\tau,\xi)\big(< \xi >^{-2a} P_{s,k}^{-2b}(\tau,\xi) *_{\tau,\xi} < \xi >^{-2a} P_{s,l}^{-2b}(\tau,\xi)\big)\big(\widetilde{g}^2 *_{\tau,\xi} \widetilde{h}^2\big) d\tau d\xi \\ \leq C\int_{\mathbb{R}^3} \widetilde{g}^2 *_{\tau,\xi} \widetilde{h}^2 d\tau d\xi \\ \leq C\|\widetilde{g}^2\|_{L^1_{\tau,\xi}}\|\widetilde{h}^2\|_{L^1_{\tau,\xi}} \\ \leq C\|\widetilde{g}\|\|\widetilde{h}\|.\end{aligned}$$

We next prove (2.14).From (2.13), we have

$$\begin{aligned}|\langle f, gh\rangle| =& |\langle \omega^{-2a} f, \omega^{2a}(gh)\rangle| \\ \leq& |\langle \omega^{2a} g, (\omega^{-2a} f)\overline{h}\rangle| + |\langle \omega^{2a} h, (\omega^{-2a} f)\overline{g}\rangle| \\ \leq& C\|g\|_{X_{1,j}^{a,b}}\|f\|_{X_{s,k}^{-a,b}}\|h\|_{X_{1,l}^{a,b}}.\end{aligned}$$

□

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Generalized Broadwell models for the discrete Boltzmann equation with linear and quadratic terms

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Generalized Broadwell models for the discrete Boltzmann equation with linear and quadratic terms

MOS classification numbers : 35Q35, 82C40, 76P05

Summary. A generalization of the Broadwell models for the discrete Boltzmann equation with linear and quadratic terms is investigated. We prove that there exists a time-global solution to this model in one space-dimension for locally bounded initial data, using a maximum principle of solutions. The boundedness of solutions is established by analyzing the system of ordinary equations related to the linear term.

1. Introduction

In this paper, we are concerned with the global existence and sharp estimate of solutions $u = (u_i)_{i=1}^N$ to the following equations:

$$\frac{\partial u_i}{\partial t} + c_i \frac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \text{ in } \mathbf{R} \times (0, \infty) \tag{1.1}$$

$$u_i(x, 0) = u_i^0(x) \text{ in } \mathbf{R} \tag{1.2}$$

Here c_i is a constant, $Q_i(u)$ and $L_i(u)$ are following:

$$Q_i(u) = \sum_{j,k,\ell} (A_{ij}^{k\ell} u_k u_\ell - A_{k\ell}^{ij} u_i u_j); \ A_{k\ell}^{ij} \text{ are non-negative constant,}$$

$$L_i(u) = \sum_k (\alpha_i^k u_k - \alpha_k^i u_i); \ \alpha_i^k \text{ are non-negative constant.}$$

This system of equations describes the motion of particles of a rarefied gas in a thin infinite tube. The discrete models of the Boltzmann equation consists in discretization of the velocity of molecules, that is, the molecules take only a finite number of velocities $C_i \in \mathbf{R}^3$. The solution $u_i(X, t)$ represents the distribution function of the molecules animated with the velocity C_i, that is, $u_i(X, t)$ is the density of molecules with velocity C_i at time t and at point $X = (x, y, z)$. We consider only binary collisions between molecules, which are represented by the nonlinear collision term $Q_i(u)$. The coefficients A_{kl}^{ij} of $Q_i(u)$ are the proportion of collisions which transform the molecules with velocity C_k and the molecules with velocity C_ℓ into the molecules with velocity C_i and the molecules with velocity C_j. We take also into account of the reflection over the wall of tube, which can be represented by the linear term $L_i(u)$. Like coefficients A_{kl}^{ij}, the coefficients α_k^i are the proportion of reflection which transforms the molecules with velocity C_k into the molecules with velocity C_i. By the virtue of the thinness of tube, we can assume that the function $u_i(X, t)$ is homogeneous with respect to the variables y and z where we take $x-$axis as a variable along the axis of tube. Let c_i be $x-$component of the velocity C_i. For the above-mentioned considerations, the solutions $u = (u_i(x, t))_{i=1}^N$ satisfy the system of equations (1.1). Here we study the initial value problem (1.1)-(1.2) with the initial data $u_i^0(x) \geq 0$.

Taking into account of the physical theory, we impose on the coefficients $A_{k\ell}^{ij}$ and α_k^i the following assumptions: for any (i, j, k, ℓ), we have

$$A_{k\ell}^{ij} = A_{\ell k}^{ij} = A_{k\ell}^{ji} \geq 0, \ A_{k\ell}^{ij}(c_i + c_j - c_k - c_\ell) = 0 \text{ and } \alpha_k^i \geq 0. \tag{1.3}$$

The Broadwell model [3,4] of the discrete Boltzmann equation is a special case of the discrete models of the Boltzmann equation and its coefficients $A_{k\ell}^{ij}$ are given in explicit form (See [6,8]). In [11], it was shown that there exists a time-global solution to the Broadwell model in one space-dimension. In [13], the global existence of solutions to the Broadwell model in two space-dimension was proved

with the intervention of the H-theorem. In [12], under the assumption of the microreversibility of the collisions: $A^{ij}_{k\ell} = A^{k\ell}_{ij}$ for any (i,j,k,ℓ), it was shown that there exists a time-global solution for initial data sufficiently close to the stationary solution. When the initial data are arbitrary in $L^1 \cap L^\infty(\mathbf{R})$, in [1], the global existence of solutions to the Broadwell models in one space-dimension was proved under the restriction that the c_i are mutually different. Moreover, in [2], the above restrictions were eliminated in order to show a global existence of solution to a generalization of the Broadwell model, in one space-dimension. In [6] and [8], the system including only the quadratic terms were systematically studied. The system having both the quadratic term and the linear term is a classical object in physics but it has less studied in mathematics.

Our purpose of this paper is to study the generalized Broadwell models for the system having both the linear term and the quadratic term. The generalized Broadwell models were introduced as the right-left models in [2] for the system including only the quadratic terms. Both the Broadwell models [3,4] and the Cabannes' 14 velocities model [5] can be considered as a special case of the right-left models. According to [2], we shall formulate the right-left models for our system. Let $E = \{1, \cdots, N\}$ be a disjoint union of two subsets R and L such that $i \in R$ [resp. L] $\Rightarrow c_i \geq 0$ [resp. ≤ 0]. We suppose the following assumptions:

$$A^{ij}_{k\ell} \neq 0 \Rightarrow (i,j) \text{ and } (k,\ell) \in (R \times L) \cup (L \times R), \tag{1.4}$$

$$\alpha^i_k \neq 0 \Rightarrow (R \times R) \cup (L \times L). \tag{1.5}$$

The condition (1.4) means that the collisions occur between molecules which are animated in different directions along the $x-$axis. The condition (1.5) means that the sign of $x-$component of the velocity does not change by the reflection. We prove the time-global existence of solutions for locally bounded initial data to the generalized Broadwell models and we show the explicit estimate and the boundedness of solutions under the assumptions which are weaker than the version of microreversibility of the reflection and which will be explained later. It is remarkable that we do not assume the microreversibility condition: $A^{ij}_{k\ell} = A^{k\ell}_{ij}$ and that we do not use the H-theorem. In fact, in the mesonic process, the break effect does not imply the microreversibility of the collisions (See [9]), and hence we do not have any necessary reason to assume this condition.

For the later use, we define a terminology for the coefficients of the linear term. The subset $J \subset E$ is called strictly diffusive if $j \in J$ and $\alpha^k_j > 0$ imply $k \in J$ and if there exists $j \in J$ and $k \notin J$ such that $\alpha^j_k > 0$. The condition $\alpha^k_j > 0 \Rightarrow \alpha^j_k > 0$, weak version of microreversibility of the reflection, implies that there exists no strictly diffusive subset.

2. Statement of Main Results

The followiing theorem is our main result.

Theorem 2.1. For non-negative initial data $u^0 = (u_i^0) \in L^1 \cap L^\infty(\mathbf{R})$, there exists a unique time-global solution to the generalized Broadwell models of the discrete Boltzmann equation satisfying the following:

$$\|u(\cdot,t)\|_{L^\infty} \leq \|u^0\|_{L^\infty} e^{C_1(\mu+t+1)} \tag{2.1}$$

where C_1 depends only on the system and μ represents the total mass: $\mu = \sum_i \int_{\mathbf{R}} u_i^0(x)dx$. Moreover, if we suppose that there exists no strictly diffusive subset, then we have the boundedness of solutions and its estimate:

$$\sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{L^\infty} \leq \|u^0\|_{L^\infty} e^{C_2(\mu+1)} \tag{2.2}$$

where C_2 depends only on the system.
Taking account of the finite-speed propagation, we have the following:

Corollary 2.2. For non-negative initial data $u^0 = (u_i^0) \in L^\infty_{loc}(\mathbf{R})[resp. L^\infty(\mathbf{R})]$, there exists a unique time-global solution $u \in L^\infty(\mathbf{R} \times [0,\infty))$ to the generalized Broadwell models of the discrete Boltzmann equation [resp. satisfying the following:

$$u(x,t) \leq e^{C_3(1+t)}$$

where C_3 depends only on the system and on the $\|u^0\|_{L^\infty}$].

3. A Proof of Theorem 2.1

It is easy to see that it is sufficient to prove Theorem 2.1 for the initial data $u^0 \in C_0^\infty$. According to the method in [2], we define the quantity $r(x,t) = \int_{-\infty}^{x} \sum_{i\in R} u_i(y,t)dy$. Then we have $0 \leq r(x,t) \leq \mu$ and $\dfrac{\partial r}{\partial x} = \sum_{i\in R} u_i$. By virtue of the assumption of the generalized Broadwell models, we have

$$\begin{aligned}
\frac{\partial r}{\partial t} &= \int_{-\infty}^{x} dy \sum_{i\in R}\left(-c_i \frac{\partial u_i}{\partial x}\right) + \left(\sum_{R\times L\times R\times L} + \sum_{R\times L\times L\times R}\right)(A_{ij}^{k\ell} u_k u_\ell - A_{k\ell}^{ij} u_i u_j) \\
&\quad + \sum_{R\times R}(\alpha_i^k u_k - \alpha_k^i u_i) \\
&= -\sum_{i\in R} c_i u_i.
\end{aligned}$$

For $K > 0$ which will be determined later, we put

$$S_1(t) = \sup_{x\in\mathbf{R}} \max_{p\in L} \sum_{k:c_k=c_p} u_k(x,t)\exp\{K(r(x,t)-t)\}. \tag{3.1}$$

We will show that $S_1(t)$ is a decreasing function. It will follows thereby that

$$e^{-Kt} \sup_{x\in\mathbf{R}} \max_{p\in L} \sum_{k:c_k=c_p} u_k(x,t) \leq S_1(t) \leq S_1(0) \leq e^{K\mu} \sup_{x\in\mathbf{R}} \max_{p\in L} \sum_{k:c_k=c_p} u_k^0(x).$$

By the same argument for $p \in R$, we will have

$$\max_i \|u_i(\cdot,t)\|_{L^\infty} \le N e^{K(\mu+t)} \max_i \|u_i^0\|_{L^\infty},$$

which implies the estimate (2.1).

To prove that $S_1(t)$ is a decreasing function, it is sufficient to show that $\frac{\partial}{\partial t}\sum_{k:c_k=c_p} u_k(x,t)\exp\{K(r(x,t)-t)\} \le 0$ for any (x,p) where the value $S_1(t)$ is attained. Let (x,p) be a such point and we put $P = \{p' | c_{p'} = c_p\}$. We have then

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{p'\in P} u_{p'}(x,t)\exp\{K(r(x,t)-t)\} \\
= &\left(\frac{\partial}{\partial t} + c_p\frac{\partial}{\partial x}\right)\sum_{p'\in P} u_{p'}(x,t)\exp\{K(r(x,t)-t)\} \\
= &\exp\{K(r(x,t)-t)\}\cdot\{K\sum_{i\in R, p'\in P}(c_{p'}-c_i)u_{p'}u_i + \sum_{E\times R\times L\times R}(A^{rs}_{p'q}u_r u_s - A^{p'q}_{rs}u_{p'}u_q) \\
&+ \sum_{E\times L}(\alpha^r_{p'}u_r - \alpha^{p'}_r u_{p'}) - K\sum_{p'\in P} u_{p'}\} =: \exp\{K(r(x,t)-t)\}\cdot\varphi_1(x,t)
\end{aligned}$$

If $c_p < 0$, we have $c_p - c_i < 0$ for $i \in R$, $\sum_{E\times R\times L\times R} A^{rs}_{p'q}u_r u_s \le C^{st}\sum_{i\in R, p'\in P} u_{p'}u_i$ and $\sum_{E\times L}(\alpha^r_{p'}u_r - \alpha^{p'}_r u_{p'}) - K\sum_{p'\in P} u_{p'} \le (C^{st}-K)\sum_{p'\in P} u_{p'} \le 0$ for a large K, because we have, for $r \in L$, $\sum_{r':c_{r'}=c_r} u_{r'} \le \sum_{p'\in P} u_{p'}$. Hence the function $\varphi_1(x,t)$ is non-positive for a large K.

If $c_p = 0$, putting R_0 [resp. L_0] $= \{i \in R$ [resp. L] $|c_i = 0\}$, we have $P = L_0$ and, by the same argument,

$$\begin{aligned}
\varphi_1(x,t) &= -K\sum_{i\in R, p'\in L_0} c_i u_{p'}u_i + \sum_{L_0\times R\times L\times R}(A^{rs}_{p'q}u_r u_s - A^{p'q}_{rs}u_{p'}u_q) \\
&\quad + \sum_{L_0\times L}(\alpha^r_{p'}u_r - \alpha^{p'}_r u_{p'}) - K\sum_{p'\in P} u_{p'} \\
&\le -K\sum_{i\in R, p'\in L_0} c_i u_{p'}u_i + \sum_{L_0\times R\times L\times (R\setminus R_0)}(A^{rs}_{p'q}u_r u_s - A^{p'q}_{rs}u_{p'}u_q) \le 0
\end{aligned}$$

for a large K, where we used $A^{ij}_{k\ell}(c_i + c_j - c_k - c_\ell) = 0$.

Therefore we showed $\frac{\partial}{\partial t}\sum_{p'\in P} u_{p'}(x,t)\exp\{K(r(x,t)-t)\} \le 0$ and then the estimate (2.1).

For the rest of proof, we use the following:

Lemma 3.1. Suppose that there exists no strictly diffusive subset of E. Then

there exists a vector (λ_i) in the kernel of the matrix $\mathcal{L}$ whose all components are positive where the matrix $\mathcal{L} = (\alpha_i^j - \delta_{ij}\sum_k \alpha_k^i)_{i,j}$.

By the virtue of the classical theory (for example, see [7]), the real part of any eigen-value ζ of $\mathcal{L}$ is negative unless ζ itself is equal to 0.
Let us define $f = (f_i(t))_{i\in E}$ as a solution to

$$\frac{df}{dt} = \mathcal{L}f,\ f_i(0) = f_i^0 > 0. \tag{3.2}$$

Then, by the form $\frac{df_i}{dt} + (\sum_k \alpha_k^i) f_i = \sum_k \alpha_i^k f_k$, the $f_i(t)$ is positive for any $t \geq 0$. It is clear that

$$f_i(t) = \sum_{n=0}^{n_1} a_n^{(i)} t^n + \sum_{\zeta:\Re e\zeta<0} \sum_{n=0}^{n_\zeta} b_{\zeta,n}^{(i)} t^n e^{\zeta t} \tag{3.2}$$

where $a_n^{(i)}$ and $b_{\zeta,n}^{(i)}$ are some constant. The boundedness of any $f_i(t)$ follows from the fact that $\frac{d}{dt}\sum_i f_i(t) = 0$ and that $f_i(t) > 0$. Hence $a_n^{(i)} = 0$ for $n > 0$ and each function $f_i(t)$ converge exponentially to a constant $\lambda_i \geq 0$. Cleary the vector $(\lambda_i)_{i\in E}$ is in the kernel of $\mathcal{L}$. Let J be a set of $j \in E$ such that $\lambda_j = 0$. It is easy to see that $\sum_{j\in J}\sum_{k\in E} \alpha_j^k \lambda_k = 0$ and that $\alpha_j^k = 0$ for $j \in J$ and $k \notin J$. This means that $j \in J$ and $\alpha_j^k > 0$ imply $k \in J$. By the assumption, we have then $\alpha_k^j = 0$ for $j \in J$ and $k \notin J$. Hence we have

$$\frac{d}{dt}\sum_{j\in J} f_j(t) = \sum_{j,k\in J} \alpha_j^k f_k(t) - \sum_{j,k\in J} \alpha_k^j f_j(t) = 0$$

i.e. $\sum_{j\in J} f_j(t)$ is a constant and it cannot converge to 0. This is a contradiction.
Therefore J is a empty set and any λ_i is positive. It completes the proof of Lemma 3.1.□

Now we give a proof for the rest of Theorem 2.1. We put, for a suitable constant $K' > 0$,

$$S_2(t) = \sup_{x\in\mathbf{R}} \max_{p\in L} \frac{u_p(x,t)}{\lambda_p} \exp\{K' r(x,t)\}. \tag{3.3}$$

If we show that $S_2(t)$ is a decreasing function, by the similar argument, we will have the estimate (2.2). We want to show, for any (x,p) such that the value $S_2(t)$ is attained, $\frac{\partial}{\partial t}\frac{u_p(x,t)}{\lambda_p}\exp\{K' r(x,t)\} \leq 0$. We have

$$\frac{\partial}{\partial t}\frac{u_p(x,t)}{\lambda_p}\exp\{K' r(x,t)\}$$

$$= \exp\{K'r(x,t)\} \cdot \left\{ K' \sum_{i \in R} (c_p - c_i) \frac{u_p u_i}{\lambda_p} + \sum_{R \times L \times R} \frac{A^{rs}_{pq} u_r u_s - A^{pq}_{rs} u_p u_q}{\lambda_p} \right.$$

$$\left. + \sum_{L} \frac{\alpha^r_p u_r - \alpha^p_r u_p}{\lambda_p} \right\} =: \exp\{K'r(x,t)\} \cdot \varphi_2(x,t)$$

If $c_p < 0$, we have $c_p - c_i < 0$ for $i \in R$, $\sum_{R \times L \times R} A^{rs}_{pq} u_r u_s \leq C^{st} \sum_{i \in R} u_p u_i$ and $\sum_{L} \frac{\alpha^r_p u_r - \alpha^p_r u_p}{\lambda_p} \leq \sum_{E} \frac{u_p}{\lambda^2_p} (\alpha^r_p \lambda_r - \alpha^p_r \lambda_p) = 0$, because we have, for $r \in L$, $\frac{u_r}{\lambda_r} \leq \frac{u_p}{\lambda_p}$ and the condition (1.5). Then the function $\varphi_2(x,t)$ is non-positive for a large K'.

If $c_p = 0$, we have, by the same argument,

$$\begin{aligned} \varphi_2(x,t) &= -K' \sum_{i \in R} c_i \frac{u_p u_i}{\lambda_p} + \sum_{R \times L \times R} \frac{A^{rs}_{pq} u_r u_s - A^{pq}_{rs} u_p u_q}{\lambda_p} \\ &\quad + \sum_{L} \frac{\alpha^r_p u_r - \alpha^p_r u_p}{\lambda_p} \\ &\leq -K' \sum_{i \in R} c_i \frac{u_p u_i}{\lambda_p} + \sum_{R \times L \times (R \setminus R_0)} \frac{A^{rs}_{pq} u_r u_s - A^{pq}_{rs} u_p u_q}{\lambda_p} \leq 0 \end{aligned}$$

for a large K', where we used $A^{ij}_{k\ell}(c_i + c_j - c_k - c_\ell) = 0$.

Therefore we showed $\frac{\partial}{\partial t} \frac{u_p(x,t)}{\lambda_p} \exp\{K'(r(x,t))\} \leq 0$ and then the estimate (2.2). It completes a proof of Theorem 2.1.

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THE NONRELATIVISTIC LIMIT OF THE NONLINEAR KLEIN–GORDON EQUATION

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October 23, 1999

ABSTRACT. In this paper we consider the nonrelativistic limit of the nonlinear Klein–Gordon equation. We study how the solutions of the nonlinear Klein–Gordon equation converge toward the corresponding solutions of the nonlinear Schrödinger equation when the speed of light tends to infinity. Especially we consider the rate of convergence. We use Strichartz's estimate for the Klein–Gordon equation.

1. INTRODUCTION

We consider the nonlinear (and linear) Klein–Gordon equation in space-time $\mathbb{R}^{n+1}$

$$\frac{\hbar^2}{2mc^2}u'' - \frac{\hbar^2}{2m}\Delta u + \frac{mc^2}{2}u + \lambda|u|^{\gamma-1}u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.1}$$

where $\hbar$ is the Planck constant, m is the mass of particle, c is the speed of light, and u'' is the second time derivative, and $\lambda > 0$. When $n = 3$ and $\gamma = 3$, the equation (1.1) was introduced by Schiff [1] as the equation of classical neutral scalar mesons. If $\lambda = 0$, the equation (1.1) is the linear Klein–Gordon equation.

Substituting

$$u = ve^{-imc^2t/\hbar},$$

we obtain from (1.1) the following nonlinear Klein–Gordon equation for v:

$$\frac{\hbar}{2mc^2}v'' - i\hbar v' - \frac{\hbar^2}{2m}\Delta v + \lambda|v|^{\gamma-1}v = 0.$$

The aim of this paper is to study this equation, particularly in the limit $c \to \infty$. We regard the procedure of taking limit $c \to \infty$ as "nonrelativistic limit." Formally, the limit equation is

$$-i\hbar v' - \frac{\hbar^2}{2m}\Delta v + \lambda|v|^{\gamma-1}v = 0.$$

This is the nonlinear Schrödinger equation. So we expect that solutions of the nonlinear Klein–Gordon equation converge as $c \to \infty$ toward the corresponding solutions of the nonlinear Schrödinger equation. We may think of the Klein–Gordon equation as a relativistic generalization for the Schrödinger equation. From this relation, we have a particular interest in the convergence of solutions of two equations. In this paper we study

this problem in detail. For simplicity, we set $A = -\Delta$, $\varepsilon = 1/c^2$, $f(v) = \lambda|v|^{\gamma-1}v$, and $\hbar = 2m = 1$. Given initial data, we rewrite the equations in question as

$$\varepsilon v'' - iv' + Av + f(v) = 0,\ v(0) = v_{0\varepsilon},\ v'(0) = v_{1\varepsilon}, \tag{1.2}$$

$$-iv' + Av + f(v) = 0,\ v(0) = v_{00}. \tag{1.3}$$

We denote by v_ε and v_0 the solution of (1.2) and (1.3), respectively.

We investigate how v_ε converges to v_0 as $\varepsilon \to 0$. There are a few results on the problem. The convergence in several modes has been proved, see [2] [3]. In [15], we have proved the convergence in $L^\infty(0,T;L^2)$. In this paper, we consider the rate of this convergence. When ε tends to 0, how rapidly does v_ε converge toward v_0 ? We show in Theorem 1 the upper bound of the order for nonlinear case. For linear case, we give the upper bound as well as the lower bound in Theorem 2.

This paper is constructed as follows. In Section 2, we state the main theorem. In Section 3, we give Strichartz's estimate for the Klein-Gordon equation. Using this estimate, we prove the main theorem in Section 4.

We close this section by giving several notation. We abbrevitate $L^q(\mathbb{R}^n)$ to L^q and $L^r(I;L^q(\mathbb{R}^n))$ to L^rL^q, where I is a time interval. We denote by $H^{s,p}$ and $B^s_{p,l}$ the Sobolev space and Besov space of order s, respectively. For any p with $1 < p < \infty$, p' stands for its Hölder conjugate, i.e. $p' = p/(p-1)$.

2. Main Theorem

We state our main theorem.

Theorem 1. (Nonlinear Case)
Let $n = 3$, $\lambda > 0$ and $1 < \gamma < 21/5$. We assume that

$$v_{0\varepsilon} \in H^1,\ v_{1\varepsilon} \in L^2, \tag{2.1}$$

$$v_{00} \in H^1, \tag{2.2}$$

$$\sup_{\varepsilon>0}(\|v_{0\varepsilon}\|_{H^1} + \varepsilon^{1/2}\|v_{1\varepsilon}\|_{L^2}) < \infty, \tag{2.3}$$

$$\|v_{0\varepsilon} - v_{00}\|_{L^2} \le c\varepsilon^{1/4}. \tag{2.4}$$

Then for every $T > 0$, there exists c such that

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \le c\varepsilon^{1/4}. \tag{2.5}$$

Remark 1.
In [15], we have shown only convergence of the LHS of (2.5) without specific rate. Theorem 1 gives an upper bound of the rate of this convergence.

Theorem 2. (Linear Case)

Let $\lambda = 0$. We assume (2.1),(2.2),(2.3), and (2.4).

Then for every $T > 0$, there exists c such that

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \le c\varepsilon^{1/4}. \tag{2.6}$$

Moreover, for any $\alpha \ge 1/4$, $\delta > 0$, there exist $v_{0\varepsilon}$ and v_{00} such that

$$\|v_{0\varepsilon} - v_{00}\|_{L^2} \le c\varepsilon^{\alpha}, \tag{2.7}$$

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \ge c\varepsilon^{1/4+\delta}. \tag{2.8}$$

3. Strichartz's type estimate for the Klein–Gordon equation

In this section we study the space–time integrability properties of solutions of the free Klein–Gordon equation for the proof of Theorem 1. To this end we construct Strichartz's estimate involving the parameter ε for equation (1.2). From Duhamel principle, the solution v_ε of (1.2) satisfies the integral equation.

$$v_\varepsilon(t) = I_\varepsilon(t)v_{0\varepsilon} + J_\varepsilon(t)v_{1\varepsilon} - \frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)f(v_\varepsilon(s))ds, \tag{3.1}$$

where

$$I_\varepsilon(t) = e^{\frac{it}{2\varepsilon}}(\cos tA_\varepsilon - \frac{i}{2\varepsilon}A_\varepsilon^{-1}\sin tA_\varepsilon),$$

$$J_\varepsilon(t) = e^{\frac{it}{2\varepsilon}}A_\varepsilon^{-1}\sin tA_\varepsilon,$$

$$A_\varepsilon = \frac{1}{\varepsilon}\Big(\varepsilon A + \frac{1}{4}\Big)^{1/2}.$$

We investigate the operator $J_\varepsilon(t)$.

Proposition 3. For any interval $I \subset \mathbb{R}$ with $0 \in \bar{I}$, $u \in C_0(I \times \mathbb{R}^n)$ and pair (q', r') such that

$$1 - \frac{1}{r'} = \frac{n}{2}\Big(\frac{1}{q'} - \frac{1}{2}\Big) \quad , \quad \frac{1}{2} \le \frac{1}{q'} \le \frac{1}{2} + \frac{2}{n+2}. \tag{3.2}$$

the following estimate holds :

$$\Big\|\int_0^t \frac{1}{\varepsilon}J_\varepsilon(t-s)u(s)ds\Big\|_{L^\infty(I;L^2)} \le c\|u\|_{L^{r'}(I;L^{q'})}, \tag{3.3}$$

where c is independent of u, I, and ε.

Key estimate for the proof of Proposition 3.

We introduce the results on decay of solution of Klein–Gordon equation. (see [13]). For any $1 < q' \le 2 \le q < \infty$, the following inequality holds :

$$\|\sin((I-\Delta)^{1/2}t)u\|_{L^q(\mathbb{R}^n)} \le ct^{-n(1/2-1/q)}\|u\|_{H^{(n+2)(1/2-1/q),q'}(\mathbb{R}^n)}. \tag{3.4}$$

4. Proof of The Main Theorem

At first, we recall some properties of the solutions of nonlinear Klein–Gordon equation and nonlinear Schrödinger equation. From the assumption (2.1), there exists a unique solution v_ε of (1.2) such that (see [14])

$$v_\varepsilon \in L^\infty(0,T;H^1) \cap L^{r(q)}(0,T;B^{1-\sigma(q)}_{q,2}),$$

with

$$\frac{2\sigma(q)}{n+1} = \frac{2}{r(q)(n-1)} = \frac{1}{2} - \frac{1}{q}, \qquad 2 \le q < \infty, \quad n \le 3.$$

Moreover by the assumption (2.3) and the energy conservation for (1.2), we obtain

$$\sup_{\varepsilon>0}(\|v_\varepsilon\|_{L^\infty(0,T;H^1)} + \|v_\varepsilon\|_{L^{r(q)}(0,T;B^{1-\sigma(q)}_{q,2})}) < \infty. \tag{4.1}$$

For the case of equation (1.3), there exists a unique solution (see [8])

$$v_0 \in L^\infty(0,T;H^1) \cap L^{s(p)}(0,T;W^{1,p}),$$

with

$$\frac{2}{s(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right), \qquad 2 \le p < \frac{2n}{n-2}.$$

From the conservation laws of energy and charge for (1.3), we obtain

$$\|v_0\|_{L^\infty(0,T;H^1)} + \|v_0\|_{L^{s(p)}(0,T;W^{1,p})} < \infty. \tag{4.2}$$

Proof of Theorem 1.

We consider the case of space dimension 3. The solution v_ε of (1.2) satisfies (3.1). The solution v_0 of (1.3) satisfies

$$v_0(t) = I_0(t)v_{00} - i\int_0^t I_0(t-s)f(v_0(s))ds, \tag{4.3}$$

with

$$I_0(t) = e^{-iAt}.$$

We study $v_\varepsilon - v_0$. Subtracting (4.3) from (3.1) yields

$$v_\varepsilon(t) - v_0(t) = \sum_{i=1}^{5} P_\varepsilon^{(i)}(t), \tag{4.4}$$

with

$$P_\varepsilon^{(1)}(t) = (I_\varepsilon(t) - I_0(t))v_{00}, \tag{4.5}$$

$$P_\varepsilon^{(2)}(t) = I_\varepsilon(t)(v_{0\varepsilon} - v_{00}), \tag{4.6}$$

$$P_\varepsilon^{(3)}(t) = J_\varepsilon(t)v_{1\varepsilon}, \tag{4.7}$$

$$P_\varepsilon^{(4)}(t) = \int_0^t (iI_0(t-s) - \frac{1}{\varepsilon}J_\varepsilon(t-s))f(v_0(s))ds, \tag{4.8}$$

$$P_\varepsilon^{(5)}(t) = \frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)(f(v_0(s)) - f(v_\varepsilon(s)))ds. \tag{4.9}$$

We investigate $\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)}$.

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \le \sum_{i=1}^{5} \|P_\varepsilon^{(i)}\|_{L^\infty(0,T;L^2)}. \tag{4.10}$$

With respect to $P_\varepsilon^{(5)}$, we use Proposition 3 to have

$$\|P_\varepsilon^{(5)}\|_{L^\infty(0,T;L^2)} \le c\|f(v_\varepsilon) - f(v_0)\|_{L^{r'}(0,T;L^{q'})}, \tag{4.11}$$

where

$$1 - \frac{1}{r'} = \frac{3}{2}\left(\frac{1}{q'} - \frac{1}{2}\right), \qquad \frac{1}{2} \le \frac{1}{q'} \le \frac{9}{10}.$$

The Hölder inequality implies

$$\|f(v_\varepsilon) - f(v_0)\|_{L^{r'}L^{q'}} \le c(\|v_\varepsilon\|_{L^aL^b}^{\gamma-1} + \|v_0\|_{L^aL^b}^{\gamma-1})\|v_\varepsilon - v_0\|_{L^\infty L^2}, \tag{4.12}$$

where

$$\frac{1}{q'} = \frac{\gamma-1}{b} + \frac{1}{2} \qquad , \qquad \frac{1}{r'} = \frac{\gamma-1}{a}. \tag{4.13}$$

We use the following embedding results,

$$B_{q,2}^{1-\sigma} \subset L^b, \qquad \frac{1}{b} = \frac{1}{q} - \frac{1-\sigma}{n},$$

$$W^{1,q} \subset L^b, \qquad \frac{1}{b} = \frac{1}{q} - \frac{1}{n} \quad .$$

From this results and (4.1),(4.2), we estimate

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^8L^8} + \|v_0\|_{L^8L^8} < \infty. \tag{4.14}$$

Considering (4.13), if $\gamma < 21/5$, we can take $a < 8$, and

$$\|v_\varepsilon\|_{L^aL^8} + \|v_0\|_{L^aL^8} \le T^{1/a-1/8}(\|v_\varepsilon\|_{L^8L^8} + \|v_0\|_{L^8L^8}). \tag{4.15}$$

Thus we obtain

$$\|P_\varepsilon^{(5)}\|_{L^\infty(0,T;L^2)} \le cT^{(1/a-1/8)(\gamma-1)}\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)}.$$

We have from (4.10),

$$(1 - cT^{(1/a-1/8)(\gamma-1)})\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \le \sum_{i=1}^{4} \|P_\varepsilon^{(i)}\|_{L^\infty(0,T;L^2)} \tag{4.16}$$

For sufficiently small T, we have

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \le c\sum_{i=1}^{4} \|P_\varepsilon^{(i)}\|_{L^\infty(0,T;L^2)}. \tag{4.17}$$

So we have to study the rate of convergence for $P_\varepsilon^{(i)}, i = 1, 2, 3, 4$.

For $P_\varepsilon^{(1)}$, we rewrite $\cos tA_\varepsilon, \sin tA_\varepsilon$ with $e^{itA_\varepsilon}, e^{-itA_\varepsilon}$ and rearrange,

$$
\begin{aligned}
\|(I_\varepsilon(t) - I_0(t))v_{00}\|_{L^\infty L^2} &\le \|\{(1/2)(1+(4\varepsilon A+1)^{-1/2})e^{\frac{it}{2\varepsilon}-itA_\varepsilon} - e^{-itA}\}v_{00}\|_{L^\infty L^2} \\
&\quad + \|(1/2)(1-(4\varepsilon A+1)^{-1/2})e^{\frac{it}{2\varepsilon}+itA_\varepsilon}v_{00}\|_{L^\infty L^2} \\
&\le \|(e^{\frac{it}{2\varepsilon}-itA_\varepsilon+itA}-1)v_{00}\|_{L^\infty L^2} \\
&\quad + \|(1-(4\varepsilon A+1)^{-1/2})v_{00}\|_{L^2} \\
&= \|(e^{ita_\varepsilon}-1)v_{00}\|_{L^\infty L^2} + \|b_\varepsilon v_{00}\|_{L^2},
\end{aligned}
\tag{4.18}
$$

here we have set

$$
\begin{aligned}
a_\varepsilon &= 1/(2\varepsilon) - A_\varepsilon + A, \\
b_\varepsilon &= 1-(4\varepsilon A+1)^{-1/2}.
\end{aligned}
$$

We study the operator $a_\varepsilon, b_\varepsilon$. From the Parseval relation, we have

$$\|Au\|_{L^2} = \||\xi|^2\tilde{u}\|_{L^2}.$$

We set $\tilde{a}_\varepsilon = 1/(2\varepsilon) - 1/(2\varepsilon)(4\varepsilon|\xi|^2+1)^{1/2} + |\xi|^2$ and estimate

$$
\begin{aligned}
|e^{it\tilde{a}_\varepsilon}-1| &\le 2, \\
|e^{it\tilde{a}_\varepsilon}-1| &= |i\tilde{a}_\varepsilon \int_0^t e^{is\tilde{a}_\varepsilon}ds| \\
&\le |\tilde{a}_\varepsilon t| \\
&= t\left|4\varepsilon|\xi|^4/((4\varepsilon|\xi|^2+1)^{1/2}+1)^2\right| \\
&\le 4t\varepsilon|\xi|^4.
\end{aligned}
$$

Thus

$$|e^{it\tilde{a}_\varepsilon}-1| \le 2^{1-\theta}(4t\varepsilon|\xi|^4)^\theta, \qquad 0 \le \theta \le 1.$$

Considering assumption (2.2) , we set $\theta = 1/4$,

$$
\begin{aligned}
\|(e^{it\tilde{a}_\varepsilon}-1)v_{00}\|_{L^\infty L^2} &\le c\|t^{1/4}\varepsilon^{1/4}|\xi|v_{00}\|_{L^\infty L^2} \\
&\le cT^{1/4}\varepsilon^{1/4}\||\xi|v_{00}\|_{L^2} \\
&\le c\varepsilon^{1/4}.
\end{aligned}
\tag{4.19}
$$

Similarly, we have

$$
\begin{aligned}
|1-(4\varepsilon|\xi|^2+)^{-1/2}| &\le 2, \\
|1-(4\varepsilon|\xi|^2+1)^{-1/2}| &= \left|4\varepsilon|\xi|^2/((4\varepsilon|\xi|^2+1)^{-1/2}+1)(4\varepsilon|\xi|^2+1)\right| \\
&\le 4\varepsilon|\xi|^2,
\end{aligned}
$$

then

$$|\tilde{b}_\varepsilon| = |1-(4\varepsilon|\xi|^2+1)^{-1/2}| \le (4\varepsilon|\xi|^2)^{1/2}2^{1/2}.$$

From this, we have

$$\|b_\varepsilon v_{00}\|_{L^2} \le c\varepsilon^{1/2}. \tag{4.20}$$

Thus we have from (4.18),(4.19) and (4.20),

$$\|P_\varepsilon^{(1)}\|_{L^\infty L^2} \le c\varepsilon^{1/4}. \tag{4.21}$$

From (2.4) and $\sup_{t\in[0,T],\varepsilon>0} \|I_\varepsilon(t)\|_{\mathfrak{L}(L^2)} < \infty$, we have

$$\begin{aligned} \|P_\varepsilon^{(2)}\|_{L^\infty L^2} &\le C\|v_{0\varepsilon} - v_{00}\|_{L^2} \\ &\le c\varepsilon^{1/4}. \end{aligned} \tag{4.22}$$

The assumption (2.3) especially for $v_{1\varepsilon}$ implies

$$\begin{aligned} \|P_\varepsilon^{(3)}\|_{L^\infty L^2} &= \|2e^{\frac{it}{2\varepsilon}} \sin tA_\varepsilon\ \varepsilon(4\varepsilon A+1)^{-1/2} v_{1\varepsilon}\|_{L^\infty L^2} \\ &= \|2\varepsilon(4\varepsilon A+1)^{-1/2} v_{1\varepsilon}\|_{L^2} \\ &\le c\varepsilon\|v_{1\varepsilon}\|_{L^2} \\ &\le c\varepsilon^{1/2}. \end{aligned} \tag{4.23}$$

In order to estimate $P_\varepsilon^{(4)}$, we show that $f(v_0) = \lambda|v_0|^{\gamma-1}v_0 \in L^1H^1$.

From $|\nabla f(v_0)| \le c|v_0|^{\gamma-1}|\nabla v_0|$, we have

$$\begin{aligned} \|f(v_0)\|_{H^1} &\le c(\|f(v_0)\|_{L^2} + \|\nabla f(v_0)\|_{L^2}) \\ &\le c(\|v_0\|_{L^2} + \|\nabla v_0\|_{L^2})\ \|v_0\|_{L^\infty}^{\gamma-1}. \end{aligned} \tag{4.24}$$

From (4.2), v_0 satifies

$$v_0 \in L^{r(q)}W^{1,q} \subset L^{r(q)}L^\infty, \qquad q > 3. \tag{4.25}$$

We continue the estimate as

$$\begin{aligned} \|f(v_0)\|_{L^1H^1} &\le c\int_0^t (\|v_0\|_{L^2} + \|\nabla v_0\|_{L^2})\ \|v_0\|_{L^\infty}^{\gamma-1} ds \\ &\le c(\|v_0\|_{L^\infty L^2} + \|\nabla v_0\|_{L^\infty L^2}) \int_0^t \|v_0\|_{L^\infty}^{\gamma-1} ds \\ &\le c(\|v_0\|_{L^\infty L^2} + \|\nabla v_0\|_{L^\infty L^2})\ \|v_0\|_{L^r L^\infty}, \end{aligned} \tag{4.26}$$

provided

$$\gamma - 1 \le r = \frac{4q}{3(q-2)}.$$

Considering $q > 3$, we have for $1 < \gamma < 5$,

$$\|f(v_0)\|_{L^1H^1} < \infty. \tag{4.27}$$

We rewrite $P_\varepsilon^{(4)}$ as

$$\begin{aligned} P_\varepsilon^{(4)} = &\int_0^t (ie^{-iA(t-s)} - i(4\varepsilon A+1)^{-1/2} e^{i(\frac{1}{2\varepsilon} - A_\varepsilon)(t-s)}) f(v_0(s)) ds \\ &+ i\int_0^t (4\varepsilon A+1)^{-1/2} e^{i(\frac{1}{2\varepsilon} + A_\varepsilon)(t-s)} f(v_0(s)) ds \end{aligned} \tag{4.28}$$

$$= I_1 + iI_2. \tag{4.29}$$

Regarding I_1 , the same arguement with $P_\varepsilon^{(1)}$ and (4.27) proves

$$\|I_1\|_{L^\infty L^2} \le c\varepsilon^{1/4}. \tag{4.30}$$

The convergence of $\|I_1\|_{L^\infty L^2}$ is obtained by a technique from the *Riemann–Lebesgue* Theorem. We define, with the characteristic function $X_{[0,t]}(s)$,

$$g(s) = X_{[0,t]}(s)(4\varepsilon A+1)^{-1/2}e^{i(\frac{1}{2\varepsilon}+A_\varepsilon)t}f(v_0(s)).$$

We have

$$\begin{aligned}
(4.31)\qquad I_2 &= \int_{-\infty}^{\infty} e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)s}g(s)ds \\
&= \int_{-\infty}^{\infty} e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)(s+\pi\varepsilon)}g(s+\pi\varepsilon)ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty}(g(s)+g(s+\pi\varepsilon)e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)\pi\varepsilon})e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)s}ds \\
&= \frac{1}{2}(\int_{-\infty}^{\infty}(g(s)-g(s+\pi\varepsilon))e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)s}ds \\
&\quad + \int_{-\infty}^{\infty} g(s+\pi\varepsilon)(1+e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)\pi\varepsilon})e^{-i(\frac{1}{2\varepsilon}+A_\varepsilon)s}ds) \\
&= \frac{1}{2}(I_{2,1}+I_{2,2}).
\end{aligned}$$

For $I_{2,2}$, we have

$$\left|1+e^{-i(\frac{1}{2\varepsilon}+\frac{1}{2\varepsilon}(4\varepsilon|\xi|^2+1)^{1/2})\pi\varepsilon}\right| \le c\varepsilon|\xi|^2,$$

then

$$(4.32)\qquad \|I_{2,2}\|_{L^\infty L^2} \le c\varepsilon^{1/2}.$$

We utilize Proposition 3 for $I_{2,1}$,

(4.33)

$$\begin{aligned}
&\|I_{2,1}\|_{L^\infty L^2} \\
&= \Big\|\int_{\infty}^{\infty}(4\varepsilon A+1)^{-1/2}e^{i(\frac{1}{2\varepsilon}+A_\varepsilon)(t-s)}\Big(X_{[0,t]}(s)f(v_0(s))-X_{[0,t]}(s+\pi\varepsilon)f(v_0(s+\pi\varepsilon))\Big)ds\Big\|_{L^\infty L^2} \\
&\le c\Big\|X_{[0,t]}(s)f(v_0(s))-X_{[0,t]}(s+\pi\varepsilon)f(v_0(s+\pi\varepsilon))\Big\|_{L^{r'}(0,T;L^{q'})} \\
&= c\|f(v_0(s))-f(v_0(s+\pi\varepsilon))\|_{L^{r'}(0,t-\pi\varepsilon;L^{q'})}+c\|f(v_0(s))\|_{L^{r'}(t-\pi\varepsilon,t;L^{q'})} \\
&= I_{2,1,1}+I_{2,1,2}.
\end{aligned}$$

Concerning $I_{2,1,2}$, we estimate

$$\begin{aligned}
I_{2,1,2} &= \Big(\int_{t-\pi\varepsilon}^{t}\|v_0(s)\|_{\gamma q'}^{\gamma r'}\Big)^{1/r'} \\
&\le c\|v_0\|_{L^\infty L^{\gamma q'}}^{\gamma}\varepsilon^{1/r'}.
\end{aligned}$$

Considering (3.2), we have $1/r' > 1/4$. For arbitrary $1<\gamma<5$, there exists q' such that $2\le\gamma q'\le 6$. Therefore we obtain

$$(4.34)\qquad I_{2,1,2} \le c\varepsilon^{1/4}.$$

By the Hölder inequality, we have

$$(4.35)\qquad I_{2,1,1} \le c(\|v_0(s)\|_{L^{10}L^{10}}^{\gamma-1}+\|v_0(s+\pi\varepsilon)\|_{L^{10}L^{10}}^{\gamma-1})\|v_0(s)-v_0(s+\pi\varepsilon)\|_{L^aL^b},$$

where

$$\frac{1}{q'} = \frac{\gamma - 1}{10} + \frac{1}{b}, \quad \frac{1}{r'} = \frac{\gamma - 1}{10} + \frac{1}{a}, \tag{4.36}$$

with $a = 4(\gamma + 1)/(\gamma - 1)$, $b = 3(\gamma + 1)/(\gamma + 2)$. Investigating under (3.2), there exist (q', r'), for $1 < \gamma < 5$, satisfying (4.36). From (4.2) and embedding results, we have

$$I_{2,1,1} \leq c\|v_0(s) - v_0(s + \pi\varepsilon)\|_{L^a L^b}. \tag{4.37}$$

We now introduce another property of the solution of the nonlinear Schröndinger equation (see [12])

$$v_0 \in B^{1/2,a}_{\infty}(I; L^b). \tag{4.38}$$

An equivalent norm of the space is

$$\|u\|_{B^{1/2,a}_{\infty}(I;L^b)} = \sup_{0<\tau<\delta} \tau^{-1/2}\|u(s) - u(s + \tau)\|_{L^r(I_{\delta'};L^b)}, \tag{4.39}$$

where δ and δ' are sufficiently small and

$$I_{\delta'} = \{s|\ s, s + \tau \in I\}.$$

Therefore we have obtained

$$I_{2,1,1} \leq c\varepsilon^{1/2}, \tag{4.40}$$

and therefore

$$\|P^{(4)}_{\varepsilon}\|_{L^\infty L^2} \leq c\varepsilon^{1/4} \tag{4.41}$$

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