On Moduli Spaces of Quasi-Maps and Gromov-Witten Invariants

(擬写像のモジュライ空間とグロモフ・ウィッテン不変量に関する研究)

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1 Introduction

1.1 Overview of Mirror Symmetry.

In *Superconformal field theory* in Superstring Theory, it gives two physical theories to 3-dimensional Calabi-Yau manifold X. They are called by **A-model** and **B-model**. *Mirror Symmetry* is a existence of a pair of Calabi-Yau manifolds X, X° such that A-model of X and B-model of X° are equivalent, via **mirror map** ([4]).

$$(\mathrm{A-model\,theory\,of}\,X) \overset{\text{mirror map}}{\Longleftrightarrow} (\mathrm{B-model\,theory\,of}\,X^\circ).$$

A-model of X depends on Kähler forms of X, and B-model of X° depends on complex structure of X°. Roughly speaking, A-model has informations of symplectic geometry or algebraic geometry such as enumerative geometry. Furthermore, B-model has informations of complex geometry such as Hodge theory. At a glance, these concepts are different each other, however, Mirror Symmetry claims that they are very similar theories.

Mirror Symmetry gives a lot of surprising results or conjectures in mathematics. For example, this theory has been in the limelight since it has computed enumerative geometric numbers, **Gromov-Witten invariants** (GW invariants) in terms of Hodge theory (see [2]). Although GW invariants are rational numbers which relate to the number of rational curves, they are defined by intersection numbers on *the moduli space of stable maps.* The stable map is a map from 1-dimensional curve (it is called by *semi-stable curve*) to variety X, which some conditions hold. Hence, it may be compute the number of stable maps, but actually the GW invariant can take on rational numbers. As [2], by using a equality of the *Yukawa couplings* of A- and B-model, a generating functions of GW invariants are computed by solutions of *Picard-Fuchs equations* for family of complex structures of X°.

For instance, let us demonstrate it in the case of quintic threefold X and its mirror variety

$$X_z^\circ = \left\{ [x_1 : x_2 : x_3 : x_4 : x_5] \mid \sum_{i=1}^{5} x_i^5 + \psi \prod_{i=1}^{5} x_i = 0 \right\} / \mathbb{Z}_5.$$

($z = \psi^{-5}$ is a local coordinate of complex moduli.) In [2], we can see an equality

$$\kappa_{ttt} = \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \kappa_{\psi\psi\psi} \left(\frac{q}{z} \frac{dz}{dq} \right)^3, \tag{1.1}$$

where $q = e^{2\pi\sqrt{-1}t}$ means a local coordinate of *Kähler moduli* of X. Let $\langle \mathcal{O}_h \mathcal{O}_h \mathcal{O}_h \rangle_{0,d}$ be a 3-pointed GW invariant of degree d, which relates to the number of 3-pointed stable maps of degree d. Then, κ_{ttt} and $\kappa_{\psi\psi\psi}$ are A- and B-model Yukawa couplings respectively:

$$\kappa_{ttt} = 5 + \sum_{d=1}^{\infty} \langle \mathcal{O}_h \mathcal{O}_h \mathcal{O}_h \rangle_{0,d}\, q^d,\ \kappa_{\psi\psi\psi} = \frac{5/(2\pi i)^3}{(1+5^5 z) y_0(z)^2},$$

where $y_0(z)$ is a solution of Picard-Fuchs equation

$$\left(\left(z\frac{d}{dz} \right)^4 + 5z \left(5z\frac{d}{dz} + 4 \right) \left(5z\frac{d}{dz} + 3 \right) \left(5z\frac{d}{dz} + 2 \right) \left(5z\frac{d}{dz} + 1 \right) \right) f = 0$$

for 1-parameter family of X_z°, which is holomorphic at $z=0$. The mirror map is a map between q and z. It is also computed by Picard-Fuchs equation as follows:

$$q = \exp\left(2\pi\sqrt{-1}\frac{y_1(z)}{y_0(z)}\right),$$

where $y_1(z)$ is a solution of Picard-Fuchs equation which has a singularity of $\log(z)$ at $z=0$. The story of Mirror Symmetry is that we can compute GW invariants by solving Picard-Fuchs equation and Taylor expansion on RHS of (1.1).

This story is interesting because algebraic geometrical numbers can be computed by Picard-Fuchs equation. In [2], however, there is no strict proof since this paper based on physics. The main theorem of [2] was proven by Givental et al ([9, 14] etc) as a generalized statement.

Now, Mirror Symmetry provides stimulus to not only Hodge theory and enumerative geometry, but also representation theory, number theory, topological recursion theory, etc. Therefore, this Mirror Symmetry which computes GW invariants is also called by *classical* Mirror Symmetry.

To compute GW invariants has been a centered problem since the beginning of Mirror Symmetry. Our research is to find some fascinated formulas in moduli spaces viewpoint.

In Part I, we will consider in the case of projective hypersurfaces $\mathbb{P}^{N-1}$. In [10], Prof. Jinzenji provides moduli spaces of quasi-maps $\widetilde{Mp}_{0,2}(N,d)$. Although it is another compactification of stable map moduli space, it has a lot of informations of B-model theory. Indeed, *generating functions of intersection numbers* $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ *on* $\widetilde{Mp}_{0,2}(N,d)$ *give solutions of Picard-Fuchs equations or mirror maps.* Hence, we can obtain formulas which compute GW invariant in terms of intersection numbers on $\widetilde{Mp}_{0,2}(N,d)$. We will research this moduli space in detail.

Furthermore, we will show that there is an injective homomorphism from the Chow ring of $\widetilde{Mp}_{0,2}(N,d)$ to Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$ in $d=1$ and $d=2$. We will compute its Gromov-Witten invariants by using the data of Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$. This formula is similar to a formula of $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ in terms of Chow ring of $\widetilde{Mp}_{0,2}(N,d)$. *It seems that these results lead another formulation of mirror map.*

In Part II[*1], we will generalize the moduli spaces of quasi-maps to weighted projective space $\mathbb{P}(1,1,1,3)$. The background of this study is j-invariant of elliptic curves. In the case of $\widetilde{Mp}_{0,2}(N,d)$, its intersection numbers are coefficients of power expansion of mirror maps for projective hypersurfaces, as [10]. Moreover, it is well-known that the j-invariant is a mirror map of Calabi-Yau surface of $\mathbb{P}(1,1,1,3)$ (see [15]). Therefore, the followings is expected: the coefficients of power series expansion of inverse function of j-invariant are intersection numbers on the moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$. We will prove this interesting claim by giving a concrete toric data of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

[*1] This is a joint work with Prof. Masao Jinzenji.

1.2 Moduli Spaces of (Quasi-)Stable Maps.

For a variety X, let us consider some kinds of maps p from $\mathbb{P}^1$ to X. For example, when X is a projective space $\mathbb{P}^{N-1}$, then it forms the following expression:

$$p([s:t]) = [f_1(s,t) : f_2(s,t) : \cdots : f_N(s,t)]. \tag{1.2}$$

where $f_i(s,t)$ is a homogeneous polynomial of degree d for all $i = 1, 2, \dots, N$. Strictly speaking, p may not be map since if all $f_i(s,t)$ are divided by a polynomial $\phi(s,t)$, then p is not defined on the zero locus of $\phi(s,t)$. For this reason, the moduli space of such maps is non-compact.

In order to compactify the moduli space, we usually introduce *stable maps*. They are maps from semi-stable curves C to variety X. The moduli space of them is written by $\overline{M}_{g,n}(X,\beta)$, where g is genus of C, $n \geq 0$ is the number of marked points on C, and $\beta \in H_2(X,\mathbb{Z})$ is degree. Although the *Gromov-Witten Invariants* is defined by integrations on $\overline{M}_{g,n}(X,\beta)$, $\overline{M}_{0,n}(X,\beta)$ has extremely complicated structure. Accordingly, there hardly exist results which compute the GW invariants from the Chow ring of the moduli space of stable maps.

On the other hand, a simpler compactification $\widetilde{Mp}_{0,2}(N,d)$ was introduced in [10]. It is a space of parameters which come from coefficients of polynomials $f_i(s,t)$ of (1.2). This moduli involves some bad loci which the map (1.2) is not a map. However, it has important informations for GW invariants of projective hypersurfaces from mirror symmetry point of view. For example, when we consider quintic threefolds, then its mirror map can be written by

$$t(x) = x + \sum_{d=1}^{\infty} \frac{w(\mathcal{O}_{h^0}\mathcal{O}_{h^2})_{0,d}}{5} e^{dx},$$

where

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(5,d)} \mathrm{ev}_1^*(h^a) \wedge \mathrm{ev}_2^*(h^2) \wedge c_{top}(\mathcal{E}_d^5).$$

Here, ev_i is evaluation map along i-th marked point, $\mathcal{E}_d^5$ is a vector bundle corresponding that the quasi-maps are contained in quintic threefold. Furthermore, there exist some formulas which compute GW invariants in terms of $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$. The merit of it is that $\widetilde{Mp}_{0,2}(N,d)$ is toric variety, hence it has simpler structure than $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$. However, there is no strict proof that $\widetilde{Mp}_{0,2}(N,d)$ is a toric variety. In this paper, *we will not only give a proof of it, but also show that $\widetilde{Mp}_{0,2}(N,d)$ is a compact orbifold.*

Some generalizations of $\widetilde{Mp}_{0,2}(N,d)$ has already attempted in [10], for example, some toric varieties which have linearly independent two Kähler forms. This direction of study is still open. On the other hand, there is a overview of generalization to weighted projective space $\mathbb{P}(1,1,1,3)$ in [10] (we will denote it $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$). It is interesting since the Picard-Fuchs equation of 1-parameter family of K3 surfaces containing $\mathbb{P}(1,1,1,3)$ relates to the *j-invariant* of elliptic curves. In the last part of [10] (preprint ver. on arXiv:1006.0607), there is a conjecture that

$$w_d = \frac{1}{2} w(\mathcal{O}_1\mathcal{O}_h)_{0,d}$$

where w_d is a d-th coefficient of inverse j-invariant. In this paper, we will discuss it in Part II.

1.3 Acknowledgements.

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Part I

The Case of Projective Space $\mathbb{P}^{N-1}$

2 Introduction for Part I.

In this part, we will discuss about $\widetilde{Mp}_{0,2}(N, d)$ and computing GW invariants for projective hypersurfaces ([19]). First, we will prove that the moduli space of quasi-maps $\widetilde{Mp}_{0,2}(N, d)$ is toric compact orbifold by providing a concrete toric data.

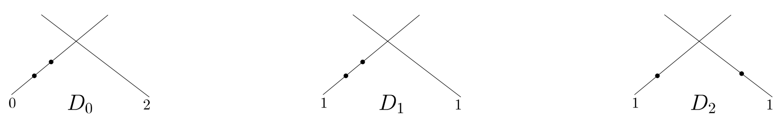
The detailed toric data gives a structure of Chow ring $A^*(\widetilde{Mp}_{0,2}(N, d))$ (See [7], etc.). It is expected that $\widetilde{Mp}_{0,2}(N, d)$ is closely related to the moduli space of stable maps $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$. We will write down this claim more explicitly by comparing Chow rings, and prove it in the case of $d = 1$ and $d = 2$.

Moreover, we will compute GW invariants of hypersurfaces by using the structures of Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ in the case of $d = 1, 2$. In order to compute Gromov-Witten invariants, we usually use *classical mirror symmetry* or *fixed point localization theorem* ([4] or [13], etc.) especially when we are dealing with basic examples such as projective hypersurfaces. Furthermore, typical proofs of classical Mirror theorem for toric complete intersections were done by using fixed point localization technique ([4], [9], [14]). Since localization technique does not need detailed structure of Chow ring of the corresponding moduli space, it is still unclear how Gromov-Witten invariants are written in terms of Chow ring of the moduli space.

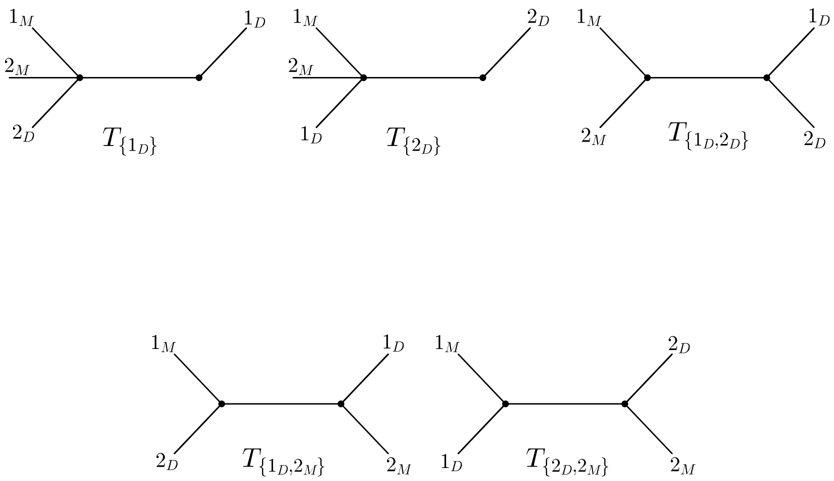
2.1 On the Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$.

In order to accomplish our program, we need to know detailed structure of the Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$. We mainly refer to Mustaţăs' results [16, 17], and Cox's results [6] to obtain information we need.

In [6], Cox computed Chow ring of $\overline{M}_{0,2}(\mathbb{P}^1, 2)$. Its structure is described by using natural basis $D_0, D_1, D_2, H_1, H_2, \psi_1, \psi_2$. H_1 and H_2 are pullbacks of hyperplane class with respect to evaluation maps $\mathrm{ev}_1,\ \mathrm{ev}_2 : \overline{M}_{0,2}(\mathbb{P}^1, 2) \to \mathbb{P}^1$. ψ_1 and ψ_2 are so called ψ-classes of universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, 2)$. D_i's are classes that correspond to loci which parametrize stable maps from nodal curves. The stable maps that belong to each D_i are represented by the following graphs.



In [16], Mustaţă and Mustaţă determined $A^*(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ in general case. They constructed intermediate moduli spaces $\overline{M}_{0,1}(\mathbb{P}^{N-1}, d, k)$ and their substrata $\overline{M}^k_I$, and computed extended Chow rings $B^*(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$. $\overline{M}^k_I$ is defined for integer k $(0 \le k \le d)$ and *nested set* $I \subset \mathcal{P}\backslash\{\emptyset, D\}$, and it parametrizes *$k$-stable maps of I-split type* (where $\mathcal{P}$ is a power set of $D = \{1, 2, \ldots, d\}$). The extended Chow ring $B^*(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ is generated by classes associated with $\overline{M}_I^k$'s, and $A^*(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ is given as a subring of $B^*(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ that are invariant under action of symmetric group S_d. In [17], they extended their strategy to compute $A^*(\overline{M}_{0,m}(\mathbb{P}^{N-1}, d))$. For example, generators of $B^*(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 2))$ are given by H, ψ, $T_{\{1_D\}}$, $T_{\{2_D\}}$, $T_{\{1_D,2_D\}}$, $T_{\{1_D,2_M\}}$ and $T_{\{2_D,2_M\}}$. H is a pullback of hyperplane class via evaluation map ev_1. ψ coincides with the ψ_1 used by Cox. The others correspond to the following graphs:



$T_{\{1_D\}}$ and $T_{\{2_D\}}$ correspond to D_1 which was used by Cox. Similarly, $T_{\{1_D,2_D\}}$ corresponds to D_0, $T_{\{1_D,2_M\}}$ and $T_{\{2_D,2_M\}}$ correspond to D_2.

Then, basis of $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 2))$ are given by,

$$H,\ \psi,\ S_0 := T_{\{1_D,2_D\}},\ S_1 := T_{\{1_D\}} + T_{\{2_D\}},\ S_2 := T_{\{1_D,2_M\}} + T_{\{2_D,2_M\}},$$
$$P_1 := T_{\{1_D\}}T_{\{2_D\}},\ P_2 := T_{\{1_D,2_M\}}T_{\{2_D,2_M\}},\ P_3 := T_{\{2_D\}}T_{\{1_D,2_M\}} + T_{\{1_D\}}T_{\{2_D,2_M\}}.$$

2.2 Main Result 1 of Part I: Comparing $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ and $\widetilde{Mp}_{0,2}(N, d)$.

Our study is influenced by concept of the moduli space $\widetilde{Mp}_{0,2}(N, d)$ of *quasi-maps* from $\mathbb{P}^1$ with two marked points to projective space $\mathbb{P}^{N-1}$, which was introduced by Jinzenji ([10]). This moduli space was also constructed rigorously by Fontanine and Kim [3]. In [10], he presented outline of construction of the moduli space $\widetilde{Mp}_{0,2}(N, d)$, implied that it is a toric variety and conjectured generators and relations of its Chow ring. Although he mentioned the fact that the moduli space $\widetilde{Mp}_{0,2}(N, d)$ is a toric variety, no explicit proof was given. So, in Section 3 of this paper, we prove the following proposition:

Proposition 2.1 *The space $\widetilde{Mp}_{0,2}(N,d)$ is a simplicial complete toric variety, and its Chow ring is isomorphic to a quotient ring of polynomial ring $\mathbb{C}[H_0,H_1,\ldots,H_d]$ modulo an ideal generated by*

$$H_0^N, H_1^N(H_0-2H_1+H_2), H_2^N(H_1-2H_2+H_3), \ldots, H_{d-1}^N(H_{d-2}-2H_{d-1}+H_d), H_d^N. \tag{2.1}$$

We prove it by giving a concrete toric data and using standard theory of toric variety.

From early stage of our study, we have been interested in *bad loci of* $\widetilde{Mp}_{0,2}(N,d)$. Although $\widetilde{Mp}_{0,2}(N,d)$ parametrizes degree d holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}^{N-1}$ with two marked points (see section 3 of this paper, or [10]), it has some loci which do *not* correspond to holomorphic maps. It is given as follows: let us consider a "rational map" $p:\mathbb{P}^1\rightarrow\mathbb{P}^{N-1}$ given by

$$p([s:t]) = [\sum_{j=0}^{d} a_j^0 s^{d-j}t^j : \sum_{j=0}^{d} a_j^1 s^{d-j}t^j : \cdots : \sum_{j=0}^{d} a_j^N s^{d-j}t^j].$$

If all the polynomials $\sum_{j=0}^{d} a_j^i s^{d-j}t^j$ are divisible by one polynomial $f(s,t)$, then image of p at zero points of $f(s,t)$ cannot be defined. However, we thought that $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$ can be constructed from $\widetilde{Mp}_{0,2}(N,d)$ by successive blow up along these bad loci. In the $d=1$ case, it is well known that $\overline{M}_{0,2}(\mathbb{P}^{N-1},1)$ is isomorphic to blow-up of $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ along its diagonal subset Δ. Moreover, it can be shown that $\widetilde{Mp}_{0,2}(N,1)$ is isomorphic to $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$, and its "bad locus" is given by the diagonal set Δ. In this case, $\overline{M}_{0,2}(\mathbb{P}^{N-1},1)$ *is blow-up of* $\widetilde{Mp}_{0,2}(N,1)$ *along its bad locus.*

Also in general degree d, similar claim may be true, but we haven't obtained rigorous proof of this kind of result. However, we show that Chow ring of $\widetilde{Mp}_{0,2}(N,2)$ is closely related to Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1},2)$. To state our result, we transform the basis of $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1},2))$ as follows:

Definition 2.1 (Key transformation)

$$h_0 := H,\ h_1 := H+\psi,\ h_2 := H+2\psi+S_2.$$

Then, we prove the following lemma:

Lemma 2.1 *In the ring $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1},2))$, the following relations hold.*

$$h_0^N=0,\quad h_1^N(h_0-2h_1+h_2)=0,\quad h_2^N=0.$$

These relations are nothing but the relations of the Chow ring of $\widetilde{Mp}_{0,2}(N,2)$!

2.3 Main Result 2 of Part I: Computing GW invariants.

For degree k projective hypersurface $M_N^k\subset\mathbb{P}^{N-1}$, the Gromov-Witten invariants $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d}$ is defined by the formula:

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d} = \int_{\overline{M}_{0,2}(\mathbb{P}^{N-1},d)} \mathrm{ev}_1^*(h^a)\wedge\mathrm{ev}_2^*(h^b)\wedge c_{\mathrm{top}}(R^0\pi_*\mathrm{ev}_3^*\mathcal{O}_{\mathbb{P}^{N-1}}(k)), \tag{2.2}$$

where a, b are nonnegative integers, ev_i is the evaluation map at the i-th marked point, $\pi : \overline{M}_{0,3}(\mathbb{P}^{N-1}, d) \to \overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ is the forgetful map, and h is a hyperplane class of $\mathbb{P}^{N-1}$.

To study GW invariants, Jinzenji introduced intersection number $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ of $\widetilde{Mp}_{0,2}(N, d)$ defined as follows.

Definition 2.2

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(N,d)} \mathrm{ev}_1^*(h^a) \wedge \mathrm{ev}_2^*(h^b) \wedge c_{\mathrm{top}}(\mathcal{E}_d^k),$$

where h is a hyperplane class of $\mathbb{P}^{N-1}$ and $\mathcal{E}_d^k$ is an orbi-bundle on $\widetilde{Mp}_{0,2}(N, d)$, which corresponds to the condition that the images of quasi maps are contained in degree k hypersurface M_N^k. This orbi-bundle is constructed in [10]. In [10], he obtained the result that mirror map of hypersurface M_N^k used in mirror computation of Gromov-Witten invariants is reconstructed as generating function of these intersection numbers, and generalized this framework to the case of toric manifolds with two Kähler forms. He also obtained a formula that represents $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ in terms of Chow ring of $\widetilde{Mp}_{0,2}(N, d)$. Let $e^k(x, y) = \prod_{j=0}^{k}(jx+(k-j)y)$. Then $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ has the following expression.

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} = \int_{\widetilde{Mp}_{0,2}(N,d)} H_0^a H_d^b \frac{\prod_{i=0}^{d-1} e^k(H_i, H_{i+1})}{\prod_{j=1}^{d-1}(kH_j)}. \tag{2.3}$$

In [12], he proved mirror formulas that express Gromov-Witten invariant $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_{0,d}$ of hypersurface M_N^k in terms $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,f}$ $(f \le d)$ and Gromov-Witten invariants of degree lower than d in the $d = 1, 2, 3$ cases. For example, in $d = 1, 2$ cases, GW invariants $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_{0,d}$ of degree k hypersurface $M_N^k \subset \mathbb{P}^{N-1}$ have the following expression.

$$\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_{0,1} = w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,1} - w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,1}, \tag{2.4}$$
$$\text{(where } a, b \ge 0, a + b = 2N - k - 3),$$

$$\begin{aligned}\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_{0,2} &= w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,2} - w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,2} \\ &\quad - \frac{1}{k} \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\mathcal{O}_{h^{1+k-N}} \rangle_{0,1} w(\mathcal{O}_{h^{a+b-N+k}}\mathcal{O}_{h^0})_{0,1}, \end{aligned} \tag{2.5}$$
$$\text{(where } a, b \ge 0, a + b = 3N - 2k - 3).$$

We use these formulas to prove our main theorems.

We expected that the equation (2.3) *can be extended to formulas that express the Gromov-Witten invariant* $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_{0,d}$ *in terms of Chow ring of* $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$, because Chow rings of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ and $\widetilde{Mp}_{0,2}(N, d)$ are closely related to each other, as can be seen in Lemma 2.1.

In Section 5 of this paper, we first review the structure of the Chow ring $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 1))$ presented in [17], and show that three classes h_0, h_1, t are generators of $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 1))$. Then we prove the following theorem:

Theorem 2.1 *For the Gromov-Witten invariant* $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,1}$ *of a hypersurface* M_N^k*, the following formula holds:*

$$\int_{\overline{M}_{0,2}(\mathbb{P}^{N-1},1)} h_0^a h_1^b e^k(h_0, h_1+t) = \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,1}\,.$$

In Section 6, we review the structure of the Chow ring $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1},2))$ presented in [17] again, and show that it is generated by five classes h_0, h_1, h_2, S_0, S_1 of degree 2 and a class P_1 of degree 4. Furthermore, by using the ring structure in detail, we prove the following:

Theorem 2.2 *For the Gromov-Witten invariants* $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,2}$ *of hypersurface* M_N^k*, the following formula holds:*

$$\int_{\overline{M}_{0,2}(\mathbb{P}^{N-1},2)} (h_0^a - (h_1+S_0)^a)h_2^b \frac{e^k(h_0, h_1+S_0)e^k(h_1+S_0, h_2+2S_0+S_1)}{k(h_1+S_0)}$$
$$= \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,2}\,.$$

Comparing our results with the definition of GW invariants (2.2), Theorem 2.1 seems to be satisfying to us but Theorem 2.2 includes somewhat strange factor $-(h_1+S_0)^a$. In fact, $\int_{\overline{M}_{0,2}(\mathbb{P}^{N-1},2)} h_0^a h_2^b \frac{e^k(h_0,h_1+S_0)e^k(h_1+S_0,h_2+2S_0+S_1)}{k(h_1+S_0)}$ gives us the value quite close to $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,2}$, but does not coincide with it. In this case, there might be some possibilities to obtain more satisfying formulas.

3 The Moduli Spase $\widetilde{Mp}_{0,2}(N,d)$

In this section we construct toric data of $\widetilde{Mp}_{0,2}(N,d)$ and prove Proposition 2.1. First, we discuss what $\widetilde{Mp}_{0,2}(N,d)$ is (see Section 2.1.1. of [10]).

$\widetilde{Mp}_{0,2}(N,d)$ is a compactified moduli space that parametrizes degree d polynomial maps from 2-pointed $\mathbb{P}^1$ to $\mathbb{P}^{N-1}$. A degree d polynomial map is given by,

$$p([s:t]) = [\mathbf{a}_0 s^d + \mathbf{a}_1 s^{d-1}t + \cdots + \mathbf{a}_d t^d],$$

where $\mathbf{a}_i$'s are vectors in $\mathbb{C}^N$. Then we introduce uncompactified moduli space $Mp_{0,2}(N,d)$ defined by,

$$Mp_{0,2}(N,d) := \{\ (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d) \mid \mathbf{a}_0, \mathbf{a}_d \neq \mathbf{0}\ \}/(\mathbb{C}^*)^2.$$

We set the 2-marked points in $\mathbb{P}^1$ as $0 := [1:0]$ and $\infty := [0:1]$. Then the condition $\mathbf{a}_0, \mathbf{a}_d \neq \mathbf{0}$ comes from requirement that images of these two marked points are well-defined in $\mathbb{P}^{N-1}$. The $(\mathbb{C}^*)^2$ action is induced from the automorphisms of $\mathbb{P}^1$ which fixes 0 and ∞, and equivalence relation used in the definition of projective space $\mathbb{P}^{N-1}$, and it is explicitly written as follows.

$$(\mu,\nu)\cdot(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d) = (\mu\mathbf{a}_0, \mu\nu\mathbf{a}_1, \ldots, \mu\nu^{d-1}\mathbf{a}_{d-1}, \mu\nu^d\mathbf{a}_d). \tag{3.1}$$

Note that with $\mu = (\mu')^d$ and $\nu = (\mu')^{-1}(\nu')^d$, the action (3.1) is equivalent to

$$(\mu',\nu')\cdot(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d)$$
$$= ((\mu')^d\mathbf{a}_0, (\mu')^{d-1}(\nu')\mathbf{a}_1, \ldots, (\mu')(\nu')^{d-1}\mathbf{a}_{d-1}, (\nu')^d\mathbf{a}_d).. \tag{3.2}$$

In order to compactify $Mp_{0,2}(N,d)$, we should add infinite loci corresponding to $\mathbf{a}_i=\infty$, $(1\leq i\leq d-1)$. Thus, we should introduce coordinates u_i's which satisfies that the locus $\mathbf{a}_i=\infty$ is described as zero locus of u_i. Since we added the parameters $u_1,\dots,u_{d-1}$, we need to make $(\mathbb{C}^*)^{d-1}$-action. This action should reproduce the action (3.2) in the case of $u_i\neq 0$ for all $i=1,2,\dots,d-1$. In order to reproduce this action, we should define the moduli space $\widetilde{Mp}_{0,2}(N,d)$ as follows:

Definition 3.1

$$
\begin{aligned}
&\widetilde{Mp}_{0,2}(N,d)\\
&:=\{(\mathbf{a}_0,\dots,\mathbf{a}_d,u_1,\dots,u_{d-1})\in\mathbb{C}^{N(d+1)+d-1}\\
&\qquad\qquad \mid \mathbf{a}_0\neq 0,(\mathbf{a}_i,u_i)\neq 0\ (1\leq i\leq d-1),\mathbf{a}_d\neq 0\}/(\mathbb{C}^*)^{d+1},
\end{aligned}
$$

where the $(\mathbb{C}^*)^{d+1}$-action is given by

$$
\begin{aligned}
&(\lambda_0,\dots,\lambda_d)\cdot(\mathbf{a}_0,\dots,\mathbf{a}_d,u_1,\dots,u_{d-1})\\
&=(\lambda_0\mathbf{a}_0,\lambda_1\mathbf{a}_1,\dots,\lambda_d\mathbf{a}_d,\\
&\lambda_0^{-1}\lambda_1^2\lambda_2^{-1}u_1,\lambda_1^{-1}\lambda_2^2\lambda_3^{-1}u_2,\dots,\lambda_{d-2}^{-1}\lambda_{d-1}^2\lambda_d^{-1}u_{d-1}).
\end{aligned}
\tag{3.3}
$$

In fact, if $u_i\neq 0$ for all $i=1,2,\dots,d-1$, then we can set

$$
\lambda_i=\Big(\frac{\lambda_0}{u_1}\Big)^{x_{i,1}}\Big(\frac{1}{u_2}\Big)^{x_{i,2}}\Big(\frac{1}{u_3}\Big)^{x_{i,3}}\cdots\Big(\frac{1}{u_{d-2}}\Big)^{x_{i,d-2}}\Big(\frac{\lambda_d}{u_{d-1}}\Big)^{x_{i,d-1}},
$$

where the matrix $(x_{i,j})_{i,j=1}^{d-1}$ is an inverse matrix of Cartan matrix $A_{d-1}=$

$$
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0\\
-1 & 2 & -1 & \cdots & 0\\
0 & -1 & 2 & \cdots & 0\\
0 & 0 & -1 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & -1\\
0 & 0 & 0 & \cdots & 2
\end{pmatrix}.
$$

Note that $x_{i,1}=(d-i)/d$, $x_{i,d-1}=i/d$. Therefore, let $\lambda_0=(\mu')^d$ and $\lambda_d=(\nu')^d$, then

$$
\begin{aligned}
&(\lambda_0,\dots,\lambda_d)\cdot(\mathbf{a}_0,\dots,\mathbf{a}_d,u_1,\dots,u_{d-1})\\
&=((\mu')^d\mathbf{a}_0,(\mu')^{d-1}(\nu')\mathbf{a}'_1,\dots,(\mu')(\nu')^{d-1}\mathbf{a}'_{d-1},(\nu')^d\mathbf{a}_d,1,1,\dots,1),
\end{aligned}
$$

where $\mathbf{a}'_i$ is

$$
\frac{1}{u_1^{x_{i,1}}}\frac{1}{u_2^{x_{i,2}}}\cdots\frac{1}{u_{d-1}^{x_{i,d-1}}}\mathbf{a}_i.
$$

We also note that $x_{i,j}>0$ for all $i,j=1,2,\dots,d-1$.

In order to see that it is compact orbifold in explicit, constructing a fan corresponding to $\widetilde{Mp}_{0,2}(N,d)$ makes short work of its proof. In addition, we can easily compute Chow ring of $\widetilde{Mp}_{0,2}(N,d)$ and its orbifold Euler character and Chern classes from the toric data. In this manner, writing down the toric data will give some benefits.

3.1 Construction of Toric data of $\widetilde{Mp}_{0,2}(N,d)$

In this subsection, we will construct a fan $\Sigma_{N,d}$ and prove that it corresponds to $\widetilde{Mp}_{0,2}(N,d)$. Especially, we will introduce $\Sigma_{N,d}$ as it realizes the action (3.3) in homogeneous representation of toric variety.

Let

$$p_1, p_2, \ldots, p_N \in \mathbb{Z}^{N-1}$$

be column vectors which are the 1-skelton of the fan of $\mathbb{P}^{N-1}$, i.e.

$$(p_1, p_2, \ldots, p_{N-1}, p_N) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in M_{N-1,N}(\mathbb{Z}).$$

Next, we introduce $(d+1)$ column vectors

$$v'_0, v'_1, \ldots, v'_d \in \mathbb{Z}^{d-1},$$

defined by,

$$(v'_0, v'_1, \ldots, v'_{d-1}, v'_d) = \begin{pmatrix} -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \end{pmatrix} \in M_{d-1,d+1}(\mathbb{Z}).$$

The $(d-1)\times(d-1)$-submatrix in center of this matrix is Cartan matrix A_{d-1}. Finally, we define column vectors,

$$v_{i,j} \;\; (0 \le i \le d, 1 \le j \le N), \quad u_k \;\; (1 \le k \le d-1)$$

as follows:
for $j \neq N$,

$$v_{i,j} = \begin{pmatrix} \mathbf{0}_{N-1} \\ \vdots \\ p_j \\ \vdots \\ \mathbf{0}_{N-1} \\ \mathbf{0}_{d-1} \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \in \mathbb{Z}^{(d+1)(N-1)+(d-1)},$$

for $j = N$,

$$v_{i,N} = \begin{pmatrix} \mathbf{0}_{N-1} \\ \vdots \\ p_N \\ \vdots \\ \mathbf{0}_{N-1} \\ v'_i \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \in \mathbb{Z}^{(d+1)(N-1)+(d-1)}$$

and for $k = 1, \ldots, d-1$,

$$u_k = \begin{pmatrix} \mathbf{0}_{N-1} \\ \vdots \\ \mathbf{0}_{N-1} \\ -e_k \end{pmatrix} \quad \in \mathbb{Z}^{(d+1)(N-1)+(d-1)}$$

where $\mathbf{0}_{N-1}$(resp. $\mathbf{0}_{d-1}$) is the zero vector in $\mathbb{Z}^{N-1}$ (resp. $\mathbb{Z}^{d-1}$) and e_k is the k-th standard basis of $\mathbb{Z}^{d-1}$.

Example 3.1 ($N = 2$, $d = 2$.)

$$v_{0,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{0,2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, v_{1,1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_{1,2} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix},$$

$$v_{2,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_{2,2} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, u_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We want to construct a fan corresponding to $\widetilde{Mp}_{0,2}(N,d)$ with cones generated by $\{v_{i,j}\}_{i=0,\ldots,d,\ j=1,\ldots,N}$ and $\{u_k\}_{k=1,\ldots,d-1}$. Note that cones should be *strongly convex*, i.e. the cones contain no linear subspaces. Therefore, we do not consider the cone generated by vectors which are linearly dependent. Let us detect linear relations among $\{v_{i,j}\}_{i=0,\ldots,d,\ j=1,\ldots,N}$ and $\{u_k\}_{k=1,\ldots,d-1}$: Since we have the relations,

$$p_1 + p_2 + \cdots + p_N = \mathbf{0}_{N-1},$$

and

$$\begin{aligned} v'_0 &= -e_1, \\ v'_1 &= -2(-e_1) + (-e_2), \\ v'_i &= (-e_{i-1}) - 2(-e_i) + (-e_{i+1}) \quad (2 \le i \le d-2), \\ v'_{d-1} &= (-e_{d-2}) - 2(-e_{d-1}), \\ v'_d &= (-e_{d-1}), \end{aligned}$$

we obtain the following relations among the vectors in $\Sigma(1)$:

$$\begin{aligned} \Sigma_{j=1}^N v_{0,j} &= u_1 \\ \Sigma_{j=1}^N v_{1,j} &= -2u_1 + u_2, \\ \Sigma_{j=1}^N v_{i,j} &= u_{i-1} - 2u_i + u_{i+1} \quad (2 \le i \le d-2), \\ \Sigma_{j=1}^N v_{d-1,j} &= u_{d-2} - 2u_{d-1}, \\ \Sigma_{j=1}^N v_{d,j} &= u_{d-1}. \end{aligned}$$

We can unify these relations into the form:

$$\Sigma_{j=1}^{N} v_{i,j} = u_{i-1} - 2u_i + u_{i+1} \quad (0 \leq i \leq d) \tag{3.4}$$

by setting $u_{-1} = u_0 = u_d = u_{d+1} = 0$.

These relations are necessary and sufficient. In fact, it is clear that they are linearly independent. Furthermore, let us assume,

$$\sum_{i=0}^{d} \sum_{j=1}^{N} a_{i,j} v_{i,j} + \sum_{k=1}^{d-1} b_k u_k = 0.$$

Then $a_{i,1} = a_{i,2} = \cdots = a_{i,N}$ holds for all $i = 0, 1, 2, \ldots, d$ since the equations $a_{i,j} - a_{i,N} = 0$ $(1 \leq j \leq N-1)$ follow from comparing the $((N-1)i + j)$-th element of both sides. Moreover, by comparing the $((N-1)(d+1) + k)$-th element of both sides, we obtain the equation $-a_{k-1,N} + 2a_{k,N} - a_{k+1,N} - b_k = 0$ for all $k = 1, 2, \ldots, d-1$. Hence we obtain,

$$\sum_{i=0}^{d} \sum_{j=1}^{N} a_{i,j} v_{i,j} + \sum_{k=1}^{d-1} b_k u_k = \sum_{i=0}^{d} a_{i,N} (\Sigma_{j=1}^{N} v_{i,j} - u_{i-1} + 2u_i - u_{i+1}),$$

which means that any linear relation among $\{v_{i,j}\}_{i=0,\ldots,d,\ j=1,\ldots,N}$ and $\{u_k\}_{k=1,\ldots,d-1}$ is written as linear combination of the relations presented in (3.4).

Let us go back to the definition of cones.

Definition 3.2 *Let*

$$\begin{aligned}
&P_0 := \{v_{0,1}, v_{0,2}, \ldots, v_{0,N}\}, \\
&P_1 := \{v_{1,1}, v_{1,2}, \ldots, v_{1,N}, u_1\} \\
&P_2 := \{v_{2,1}, v_{2,2}, \ldots, v_{2,N}, u_2\} \\
&\vdots \\
&P_{d-1} := \{v_{d-1,1}, v_{d-1,2}, \ldots, v_{d-1,N}, u_{d-1}\} \\
&P_d := \{v_{d,1}, v_{d,2}, \ldots, v_{d,N}\}.
\end{aligned}$$

Then, we define

$$\Sigma_{N,d}$$

is a set of cones generated by the union of proper subsets (involving empty set) of $P_0, P_1, \ldots, P_d$. (A cone corresponding to empty set is $\{0\}$).

For $l = 0, 1, \ldots, N(d+1) - 2$, let

$$\Sigma_{N,d}(l) := \{\sigma \in \Sigma_{N,d} \mid \sigma \text{ is generated by } l \text{ vectors}\}.$$

In the following, we show that this set $\Sigma_{N,d}$ is a fan. For this purpose, we have to check the following two conditions.

(i) $\dim(\sigma) = N(d+1) - 2$ for all $\sigma \in \Sigma_{N,d}(N(d+1) - 2)$.

(ii) If $\sigma \neq \tau$ $(\sigma, \tau \in \Sigma_{N,d})$, then $\sigma \cap \tau$ is a face of σ and τ.

First, we show that for all $v \in \mathbb{R}^{N(d+1)-2}$, there exists $\sigma \in \Sigma_{N,d}$ such that $v \in \sigma$. Namely, we prove the following lemma:

Lemma 3.1 *For all $v \in \mathbb{R}^{N(d+1)-2}$, there uniquely exist $a_{i,j} \in \mathbb{R}$ and $b_i \in \mathbb{R}$ that satisfy the following conditions.*

$$\begin{aligned}
&\textit{(a)} \qquad v = \sum_{i=0}^{d}\sum_{j=1}^{N} a_{i,j}v_{i,j} + \sum_{i=1}^{d-1} b_i u_i,\\
&\textit{(b)} \qquad \min(\{a_{i,j} \mid j = 1,2,\dots,N\}) = 0 \ \ (i = 0, d),\\
&\textit{(c)} \qquad \min(\{a_{i,j} \mid j = 1,2,\dots,N\} \cup \{b_i\}) = 0 \ \ (i = 1,2,\cdots,d-1).
\end{aligned}$$

Proof: It is clear that $v \in \mathbb{R}^{N(d+1)-2}$ is uniquely expressed as

$$v = \sum_{i=0}^{d}\sum_{j=1}^{N-1} a_{i,j}v_{i,j} + \sum_{i=1}^{d-1} b_i u_i \tag{3.5}$$

for some $a_{i,j} \in \mathbb{R}$ and $b_i \in \mathbb{R}$. Of course, $a_{i,j}$ and b_i can be negative. Our goal is to construct an expression of v which satisfy the claim of this lemma by using relations (3.4).

For $i = 0, 1, \dots, d$, let

$$a_i := \min\{a_{i,j} \mid j = 1,2,\dots,N\},$$

where $a_{i,N} := 0$. If we consider

$$v = v - a_0\Big(\sum_{j=1}^{N} v_{0,j} - u_1\Big) - a_d\Big(\sum_{j=1}^{N} v_{d,j} - u_{d-1}\Big), \tag{3.6}$$

by using the relations (3.4) for $i = 0, d$, then coefficients of $v_{0,j}$ and $v_{d,j}$ become $a_{0,j} - a_0, a_{d,j} - a_d$, respectively. These numbers satisfy,

$$\min\{a_{0,j} - a_0 \mid j = 1,2,\dots,N\} = \min\{a_{d,j} - a_d \mid j = 1,2,\dots,N\} = 0,$$

by definitions of a_0 and a_d. Then, coefficients of u_1 and u_{d-1} turn into $b_1 + a_0$ and $b_{d-1} + a_d$, respectively. At this stage, we introduce the notation:

$$a'_{i,j} := \begin{cases} a_{i,j} & (i \neq 0, d) \\ a_{i,j} - a_i & (i = 0, d) \end{cases}$$

$$b'_i := \begin{cases} b_1 + a_0 & (i = 1) \\ b_i & (i \neq 1, d-1) \\ b_{d-1} + a_d & (i = d-1), \end{cases}$$

then

$$\begin{aligned} v = &\sum_{j=1}^{N} a'_{0,j}v_{0,j} + \sum_{j=1}^{N} a'_{d,j}v_{d,j} \\ &+ \sum_{i=1}^{d-1}\sum_{j=1}^{N-1} a'_{i,j}v_{i,j} + \sum_{i=1}^{d-1} b'_i u_i. \end{aligned}$$

For $i = 0, 1, \ldots, d$, let

$$a'_i := \min\{a'_{i,j} \mid j = 1, 2, \ldots, N\},$$

again, where $a'_{i,N} = 0$ if $i \neq 0, d$.

To finish the proof, we have to show that there exist $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$ which satisfy

$$\min(\{a'_{i,j} + \alpha_i \mid j = 1, 2, \ldots, N\} \cup \{b'_i - \alpha_{i-1} + 2\alpha_i - \alpha_{i+1}\}) = 0 \qquad (3.7)$$

for all $i = 1, 2, \ldots, d-1$, where we set $\alpha_0 = \alpha_d = 0$. If they are found, then by using relations (3.4) of $i = 1, 2, \ldots, d-1$,

$$v = v + \sum_{i=1}^{d-1} \alpha_i \left(\sum_{j=1}^{N} v_{i,j} - u_{i-1} + 2u_i - u_{i+1}\right) \qquad (3.8)$$

gives the expression we need. In fact, the coefficient of $v_{i,j}$ of right hand side is $a'_{i,j} + \alpha_i$, and the coefficient of u_i of right hand side is $b'_i - \alpha_{i-1} + 2\alpha_i - \alpha_{i+1}$. By definitions of $a'_1, a'_2, \ldots, a'_{d-1}$, the condition (3.7) can be replaced by simpler condition:

$$\min\{a'_i + \alpha_i,\ b'_i - \alpha_{i-1} + 2\alpha_i - \alpha_{i+1}\} = 0 \qquad (3.9)$$

for all $i = 1, 2, \ldots, d-1$, where $\alpha_0 = \alpha_d = 0$. The problem at hand is equivalent to existence of unique solution of piecewise linear equations (3.9) for variables $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$. Now, we introduce the following piecewise linear functions

$$F_i(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}) := \min\{a'_i + \alpha_i,\ b'_i - \alpha_{i-1} + 2\alpha_i - \alpha_{i+1}\} \quad (i = 1, 2, \ldots, d-1)$$

over $\mathbb{R}^{d-1} = \{(\alpha_1, \alpha_2, \ldots, \alpha_{d-1})\}$.

In order to show existence and uniqueness of solution of the piecewise linear equations $F_i = 0$ $(i = 1, \cdots, d-1)$, we only have to prove that the piecewise linear map:

$$F : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1};\ \alpha \mapsto (F_1(\alpha), F_2(\alpha), \ldots, F_{d-1}(\alpha)) \qquad (3.10)$$

is bijective.

Case of $d = 1$: In this case, $\Sigma_{N,d}$ becomes a fan corresponding to $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ by definition 3.2. This fan clearly satisfies this lemma.

Case of $d = 2$: In this case, the equation we have to solve is $\min\{a'_1 + \alpha_1, b'_1 + 2\alpha_1\} = 0$. The solution is given by $\max\{-a'_1, -\frac{b'_1}{2}\}$.

Case of $d = 3$: In this case, let us transform $\alpha'_i := \alpha_i + a'_i$. Then F is bijective if and only if the equation:

$$\min\{\alpha'_1, B_1 + 2\alpha'_1 - \alpha'_2\} = C_1,$$
$$\min\{\alpha'_2, B_2 - \alpha'_1 + 2\alpha'_2\} = C_2,$$

has unique solution for arbitrary C_1 and C_2. But this equation is explicitly solved as follows.

$$\alpha'_1 = \max\{C_1, \frac{C_1 + C_2 - B_1}{2}, \frac{2C_1 + C_2 - 2B_1 - B_2}{3}\}$$
$$\alpha'_2 = \max\{C_2, \frac{C_1 + C_2 - B_2}{2}, \frac{C_1 + 2C_2 - B_1 - 2B_2}{3}\}.$$

Case of $d > 3$: In this case, it might be too difficult to solve the system of piecewise linear equations explicitly. Therefore, we refer to [1] which studies solving piecewise linear equations in abs-normal form. In [1], Griewank et al. proved the following proposition. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise linear map.

Proposition 3.1 (Proposition 4.1. of [1]) *If F is simply switched in that $L = 0$ and its kinks satisfy LIKQ then F is bijective if and only if it is coherently oriented.*

Abs-normal form means piecewise linear equations which can be represented by the absolute value function. Our system of piecewise linear equations can be rewritten in abs-normal form since we have the following identity.

$$\min\{x, y\} = \frac{1}{2}(x + y - |x - y|).$$

Let $z_i := b'_i - a'_i - \alpha_{i-1} + \alpha_i - \alpha_{i+1}$. Then $F_i = \frac{1}{2}(a'_i + b'_i - \alpha_{i-1} + 3\alpha_i - \alpha_{i+1} - |z_i|)$, and the abs-normal form representation of the equation is given by,

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{d-1} \\ F_1 \\ F_2 \\ \vdots \\ F_{d-1} \end{bmatrix} = \begin{bmatrix} b'_1 - a'_1 \\ b'_2 - a'_2 \\ \vdots \\ b'_{d-1} - a'_{d-1} \\ \frac{1}{2}(a'_1 + b'_1) \\ \frac{1}{2}(a'_2 + b'_2) \\ \vdots \\ \frac{1}{2}(a'_{d-1} + b'_{d-1}) \end{bmatrix} + \begin{bmatrix} Z & L \\ J & Y \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d-1} \\ |z_1| \\ |z_2| \\ \vdots \\ |z_{d-1}| \end{bmatrix}, \tag{3.11}$$

where $Z = A_{d-1} - E_{d-1}$, $L = 0$, $J = \frac{1}{2}(A_{d-1} + E_{d-1})$, $Y = -\frac{1}{2}E_{d-1}$. ($A_{d-1} \in M_{d-1,d-1}(\mathbb{R})$ is the A_{d-1} Cartan matrix and $E_{d-1} \in M_{d-1,d-1}(\mathbb{R})$ is the identity matrix.)

In the abs-normal form (3.11),

$$L = 0$$

means *simply switched*.

Kink is union of hyperplanes $\{\alpha \in \mathbb{R}^{d-1} \mid z_i(\alpha) = 0\}$. On simply switched abs-normal form, the kink satisfies *linear independence kink qualification (LIKQ)* if the normal vectors of the hyperplanes intersecting at some point α are always linearly independent. In our case, it holds since the matrix $Z = A_{d-1} - E_{d-1}$ is nonsingular whenever $d > 3$.

A piecewise linear map F is *coherently oriented* if for all region on which F becomes affine, its Jacobians have the same nonzero determinant sign. Note that on a region where F becomes affine, there exists $I := \{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, d-1\}$ $(0 < i_1 < i_2 < \cdots < i_r < d)$ such that F is represented as follows.

$$F(\alpha) = A_I\alpha + \mathbf{c}.$$

Here, c is some constant vector and A_I is given by,

$$\left(\begin{array}{c|c|c|c|c|c|c}
 & 0 & & & & & \\
 & \vdots & & & & & \\
A_{i_1-1} & 0 & & & & & \\
 & -1 & & & & & \\
\hline
0 \quad \cdots \quad 0 & 1 & 0 \quad \cdots \quad 0 & 0 & & & \\
\hline
 & -1 & & 0 & & & \\
 & & & \vdots & & & \\
 & 0 & & & & & \\
 & \vdots & A_{i_2-i_1-1} & 0 & & & \\
 & 0 & & -1 & & & \\
\hline
0 \quad \cdots \quad 0 & 0 & 0 \quad \cdots \quad 0 & 1 & 0 \cdots & & \\
\hline
 & & & -1 & & & \\
 & & & \vdots & \ddots & & \\
\hline
 & & & & & 1 & \\
\hline
 & & & & & -1 & \\
 & & & & & 0 & \\
 & & & & & \vdots & A_{d-i_r-1} \\
 & & & & & 0. &
\end{array}\right). \tag{3.12}$$

(blank boxes are zero matrices.)
We can easily compute determinant of its Jacobian A_I as follows.

$$\det A_I = (\det A_{i_1-1})(\det A_{i_2-i_1-1}) \cdots (\det A_{d-i_r-1}) = i_1(i_2 - i_1) \cdots (d - i_r) > 0$$

Hence $F(\alpha) = (F_1(\alpha), F_2(\alpha), \ldots, F_{d-1}(\alpha))$ is coherently oriented.

By the proposition 4.1. of [1], F is bijective, and there exists unique solution of the system of piecewise linear equations (3.9). □

Existence of $a_{i,j}$, b_i in Lemma 3.1 implies that $\dim(\sigma) = N(d+1) - 2$ whenever $\sigma \in \Sigma_{N,d}(N(d+1)-2)$. This fact can be also shown by computing determinants of matrices A_I of (3.12). Uniqueness of $a_{i,j}$, b_i means that for all $\sigma, \tau \in \Sigma_{N,d}$, $\sigma \cap \tau$ is a face of σ and τ. Hence the conditions (i) and (ii) are satisfied and we can conclude that $\Sigma_{N,d}$ is a fan.

The lemma 3.1 also means that the fan $\Sigma_{N,d}$ is simplicial and complete since all maximal cones $\Sigma_{N,d}$ is spanned by $N(d+1)-2$ vectors and $|\Sigma_{N,d}| := \bigcup_{\sigma \in \Sigma_{N,d}} \sigma = \mathbb{R}^{N(d+1)-2}$.

In this way, we have proved the following theorem:

Theorem 3.1 *The set $\Sigma_{N,d}$ in Definition 3.2 is a simplicial and complete fan. Accordingly, the toric variety $X_{\Sigma_{N,d}}$ corresponding to the fan $\Sigma_{N,d}$ is complete (i.e. compact) orbifold.*

Let us show that toric variety X_Σ corresponding to the fan $\Sigma := \Sigma_{N,d}$ realizes the variety $\widetilde{Mp}_{0,2}(N,d)$. To see it, we use a standard fact of general theory of toric variety, which describes X_Σ as a quotient space of an open subset of $\mathbb{C}^{\Sigma(1)}$. We review here some facts on general fan Σ in lattice set $N = \mathbb{Z}^n$ (see Chapter 3 of [4] or [7]). Let $\Sigma(1)$ be a set of 1-skeltons of Σ. Although $\rho \in \Sigma(1)$ is a ray on $N_{\mathbb{R}} := N \otimes \mathbb{R} = \mathbb{R}^n$, we identify this ray as generator vector v_ρ of a semi-group

$N \cap \rho$. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We have the following exact sequence for the Chow group $A_{n-1}(X_\Sigma)$:

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma) \to 0. \tag{3.13}$$

where $M \to \mathbb{Z}^{\Sigma(1)}$ is defined by $m \mapsto (\langle m \cdot v_\rho \rangle)_{\rho \in \Sigma(1)}$ ($\langle m \cdot v_\rho \rangle$ is standard inner product),
and $\mathbb{Z}^{\Sigma(1)} \to A_{\dim(X_\Sigma)-1}(X_\Sigma)$ is defined by $(m_\rho) \mapsto \sum_\rho m_\rho [D_\rho]$ ($[D_\rho]$ is a divisor class associated to $\rho \in \Sigma(1)$). Next, we define a closed subset $Z(\Sigma)$ of $\mathbb{C}^{\Sigma(1)}$. The *primitive collection* $\mathcal{S}$ is a subset of $\Sigma(1)$ which is not the set of 1-dimensional cones of some cone $\sigma \in \Sigma$ but every proper subset of $\mathcal{S}$ is contained in some cone in the fan. Then, let

$$Z(\Sigma) := \bigcup_{\mathcal{S}:\text{prim.coll.}} \mathbf{V}(\mathcal{S}),$$

where $\mathbf{V}(\mathcal{S}) = \{x \in \mathbb{C}^{\Sigma(1)} \mid x_\rho = 0, \rho \in \mathcal{S}\} \subset \mathbb{C}^{\Sigma(1)}$. Finally, let $G := \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{C}^*)$. The group G acts on $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ as $g \cdot (x_\rho)_{\rho \in \Sigma(1)} := (g([D_\rho])x_\rho)$. Then the following theorem holds:

Theorem 3.2 *If the1-dimensional cones of Σ span $N_{\mathbb{R}}$, then:*

1. *X_Σ is the categorical quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by G,*
2. *X_Σ is the geometric quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by G if and only if X_Σ is simplicial.*

See [5] for more detail.

We use this theorem in the case of $\Sigma = \Sigma_{N,d}$.

Lemma 3.2 *The primitive collections of the fan $\Sigma = \Sigma_{N,d}$ are*

$$P_0, P_1, \ldots, P_d.$$

Proof. By Definition 3.2, it is clear that they are primitive collections of $\Sigma_{N,d}$. If P is a primitive collection of $\Sigma_{N,d}$, then since P does not generate any cone of $\Sigma_{N,d}$, P has to contain P_i for some $i = 0, 1, \ldots, d$. If P_i is proper subset of P, then P_i does not generate any cone of $\Sigma_{N,d}$, thus P should not be a primitive collection. Therefore, $P = P_i$. $\Box$

We introduce the following notation,

$$\mathbb{C}^{\Sigma(1)} = \{x = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, u_1, u_2, \ldots, u_{d-1}) \mid \mathbf{a}_i \in \mathbb{C}^N,\ u_i \in \mathbb{C}\},$$

where $\mathbf{a}_i$ and u_i represent $(x_{v_{i,1}}, \cdots, x_{v_{i,N}})$ and x_{u_i} respectively. Then we can easily see that

$$Z(\Sigma) = \{x \in \mathbb{C}^{\Sigma(1)} \mid \mathbf{a}_0 = 0,\ \mathbf{a}_d = 0,\ (\mathbf{a}_i, u_i) = 0\,(1 \le i \le d-1)\}.$$

Let us check the $G = (\mathbb{C}^*)^{d+1}$-action on $\mathbb{C}^{\Sigma(1)}$ (note that $|\Sigma(1)| - \mathrm{rank}(M) = d+1$. then we can see $\mathrm{rank}(A_{\dim(X_\Sigma)-1}(X_\Sigma)) = d+1$ from the exact sequence (3.13)). Let $[D_{i,j}]$ (resp. $[U_k]$) be a divisor class that corresponds to 1-skelton

$v_{i,j}$ (resp. u_k). By the exact sequence (3.13) and definition of $v_{i,j}$ and u_k, we obtain the following relations on a Chow group $A_{\dim(X_\Sigma)-1}(X_\Sigma)$:

$$[D_{i,1}] = [D_{i,2}] = \cdots = [D_{i,N}] \, (0 \le i \le d),$$
$$[U_k] = -[D_{k-1,N}] + 2[D_{k,N}] - [D_{k+1,N}] \, (1 \le k \le d-1).$$

Let $\lambda_i := g([D_{i,1}])$ ($g \in G$). The above relation tells us that $g([U_k]) = \lambda_{k-1}^{-1}\lambda_k^2\lambda_{k+1}^{-1}$, and the $(\mathbb{C}^*)^{d+1}$-action turns out to be,

$$\begin{aligned}&(\lambda_0, \ldots, \lambda_d) \cdot (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, u_1, u_2, \ldots, u_{d-1})\\ =&(\lambda_0\mathbf{a}_0, \lambda_1\mathbf{a}_1, \ldots, \lambda_d\mathbf{a}_d, \lambda_0^{-1}\lambda_1^2\lambda_2^{-1}u_1, \lambda_1^{-1}\lambda_2^2\lambda_3^{-1}u_2, \ldots, \lambda_{d-2}^{-1}\lambda_{d-1}^2\lambda_d^{-1}u_{d-1}).\end{aligned}$$

This action is the same as the $(\mathbb{C}^*)^{d+1}$-action (3.3) in Definition 3.1.

By combining Theorem 3.1 and the fact that $\widetilde{Mp}_{0,2}(N,d)$ is the toric variety corresponding to the fan $\Sigma_{N,d}$ of Definition 3.2, we obtain a strict proof of the following theorem:

Theorem 3.3 *The toric variety $\widetilde{Mp}_{0,2}(N,d)$ is complete (i.e. compact) orbifold.*

3.2 The Chow ring of $\widetilde{Mp}_{0,2}(N,d)$.

In this subsection, we prove Proposition 2.1.

The Chow ring of $\widetilde{Mp}_{0,2}(N,d)$ is computed by its toric data. To illustrate the recipe of computation, we review general theory of toric varieties (Chap.3 of [4] or [7], etc.). Let Σ be a simplicial complete fan on a lattice set N, and

$$\Sigma(1) := \{\rho_1, \rho_2, \ldots, \rho_r\}$$

be an 1-skelton (i.e. a collection of 1-dimensional cones of Σ). We identify ρ_i with the generator of semi-group $\rho_i \cap N$ in the same way as the previous subsection, and we denote it by v_i. We define two ideals of $\mathbb{C}[x_1, x_2, \ldots, x_r]$. First one is given by,

$$I(\Sigma) := (\sum_{i=1}^{r} \langle m, v_i \rangle \cdot x_i \mid m \in M).$$

The other one is the ideal called *Stanley-Reisner ideal* of the corresponding fan Σ and is defined by,

$$SR(\Sigma) := (x_{i_1}x_{i_2}\cdots x_{i_j} \mid \{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\} \in \mathcal{S}),$$

where $\mathcal{S}$ is the primitive collection of the fan Σ (see the previous subsection). Chow ring of a toric variety X_Σ is isomorphic to the following quotient ring:

$$A^*(X_\Sigma) \cong \mathbb{C}[x_1, x_2, \ldots, x_r]/(I(\Sigma) + SR(\Sigma)).$$

This isomorphism enables us to prove Proposition 2.1:

Proposition 3.2 *$A^*(\widetilde{Mp}_{0,2}(N,d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]/\mathcal{I}$,*
where $\mathcal{I} = (H_0^N, H_d^N, H_k^N(-H_{k-1} + 2H_k - H_{k+1}))$

proof. Let $x_{i,j}$ (resp. y_k) be a variable corresponding to a cone $v_{i,j}$ (resp. u_k) of the fan $\Sigma_{N,d}$. Let $\{e_\alpha\}_{\alpha=1}^{dN+N-2}$ be standard basis of $\mathbb{Z}^{dN+N-2}$. When $1 \le \alpha \le N-1$, we have,

$$\langle e_\alpha, v_{i,j}\rangle = \begin{cases} 1 \quad (i=0, j=\alpha) \\ -1 \quad (i=0, j=N) \\ 0 \quad (\text{otherwise}) \end{cases}$$
$$\langle e_\alpha, u_k\rangle = 0 \text{ (for all } k).$$

Hence we obtain $x_{0,\alpha} - x_{0,N} \in I(\Sigma)$, and we can identify $x_{0,\alpha}$ with $x_{0,N}$ in $A^*(\widetilde{Mp}_{0,2}(N,d))$. In the same manner, when $\ell(N-1)+1 \le \alpha \le (\ell+1)(N-1)$ (where $0 \le \ell \le d$), we have,

$$\langle e_\alpha, v_{i,j}\rangle = \begin{cases} 1 \quad (i=\ell, j=\alpha-\ell(N-1)) \\ -1 \quad (i=\ell, j=N) \\ 0 \quad (\text{otherwise}) \end{cases}$$
$$\langle e_\alpha, u_k\rangle = 0 \text{ (for all } k).$$

Therefore, $x_{\ell,\alpha-\ell(N-1)} = x_{\ell,N}$ in $A^*(\widetilde{Mp}_{0,2}(N,d))$ for $0 \le \ell \le d$, $\ell(N-1)+1 \le \alpha \le (\ell+1)(N-1)$. Consequently, if we denote $x_{\ell,N}$ by H_ℓ, we obtain the relation $x_{\ell,j} = H_\ell$ for $0 \le \ell \le d$, $1 \le j \le N-1$. If $(d+1)(N-1)+1 \le \alpha \le (d+1)(N-1)+(d-1)$, we have,

$$\langle e_\alpha, v_{i,j}\rangle = \begin{cases} -1 \quad (i=\alpha-(d+1)(N-1)-1, j=N) \\ 2 \quad (i=\alpha-(d+1)(N-1), j=N) \\ -1 \quad (i=\alpha-(d+1)(N-1)+1, j=N) \\ 0 \quad (\text{otherwise}), \end{cases}$$
$$\langle e_\alpha, u_k\rangle = \begin{cases} -1 \quad (k=\alpha-(d+1)(N-1)) \\ 0 \quad (\text{otherwise}). \end{cases}$$

Hence we obtain the relation $-H_{k-1}+2H_k-H_{k+1}-y_k = 0$, which yields $y_k = -H_{k-1}+2H_k-H_{k+1}$ $(1 \le k \le d-1)$. Accordingly, we have shown that

$$\mathbb{C}[x_\rho \mid \rho \in \Sigma_{N,d}(1)]/I(\Sigma_{N,d}) = \mathbb{C}[H_0, H_1, \ldots, H_d].$$

Next, we compute the Stanley-Reisner ideal $SR(\Sigma_{N,d})$. It is clear that $SR(\Sigma_{N,d})$ coincides with $\mathcal{I} = (H_0^N, H_d^N, H_k^N(-H_{k-1}+2H_k-H_{k+1}))$ since we have obtained the primitive collection $\mathcal{S}$ in the previous subsection. $\Box$

3.3 The (Top) Chern Class of $\widetilde{Mp}_{0,2}(N,d)$.

In this subsection, we will deal with a result for Chern class and orbifold Euler characteristic of $\widetilde{Mp}_{0,2}(N,d)$. We will compute orbifold Euler characteristic of $\widetilde{Mp}_{0,2}(N,d)$ as intersection number of the top Chern class.

In general theory of toric varieties ([7]), the total Chern class of toric variety with respected to a fan Σ is

$$\prod_{\sigma\in\Sigma(1)} (1+[D_\sigma]),$$

where $[D_\sigma] \in A^*(X_\Sigma)$ is a class corresponding to a divisor with respected to 1-dimensional cone $\sigma \in \Sigma(1)$. In the case of $\Sigma = \Sigma_{N,d}$, it is written by

$$\begin{aligned}
&c(\widetilde{Mp}_{0,2}(N,d))\\
&=\prod_{i=0}^{d}(1+H_i)^N \times \prod_{i=1}^{d-1}(1+[U_i])\\
&=\prod_{i=0}^{d}(1+NH_i+\cdots+NH_i^{N-1}+H_i^N) \times \prod_{i=1}^{d-1}(1+[U_i]),
\end{aligned}$$

where $[U_i] = -H_{i-1}+2H_i-H_{i+1}$. By proposition 2.1, we obtain

$$\begin{aligned}
&c(\widetilde{Mp}_{0,2}(N,d))\\
=&(1+NH_0+\cdots+NH_0^{N-1})(1+NH_d+\cdots+NH_d^{N-1})\\
&\times\prod_{i=1}^{d-1}(1+NH_i+\cdots+NH_i^{N-1}+H_i^N)(1+[U_i])\\
=&(1+NH_0+\cdots+NH_0^{N-1})(1+NH_d+\cdots+NH_d^{N-1})\\
&\times\prod_{i=1}^{d-1}\{(1+NH_i+\cdots+NH_i^{N-1}+H_i^N)+[U_i](1+NH_i+\cdots+NH_i^{N-1})\}.
\end{aligned}$$

Let us compute the top Chern class. Note that the dimension of $\widetilde{Mp}_{0,2}(N,d)$ equals $N(d+1)-2$. Hence, it is given by

$$\begin{aligned}
&c_{top}(\widetilde{Mp}_{0,2}(N,d))\\
=&NH_0^{N-1}\cdot NH_d^{N-1}\cdot\prod_{i=1}^{d-1}(H_i^N+NH_i^{N-1}[U_i])\\
=&N^2H_0^{N-1}H_1^NH_2^N\cdots H_{d-1}^NH_d^{N-1}\prod_{i=1}^{d-1}(1+\frac{N[U_i]}{H_i})\\
=&N^2H_0^{N-1}H_1^NH_2^N\cdots H_{d-1}^NH_d^{N-1}\\
&+N^2H_0^{N-1}H_1^NH_2^N\cdots H_{d-1}^NH_d^{N-1}\sum_{j=1}^{d-1}\sum_{0<d_1<d_2<\cdots<d_j<d}\prod_{k=1}^{j}\frac{N[U_{d_k}]}{H_{d_k}}.
\end{aligned}$$

We have to compute the following class

$$H_0^{N-1}H_1^NH_2^N\cdots H_{d-1}^NH_d^{N-1}\prod_{k=1}^{j}\frac{[U_{d_k}]}{H_{d_k}}. \tag{3.14}$$

In order to compute it, we have to decide a volume form. From the fan $\Sigma_{N,d}$, the appropriate class is

Definition 3.3

$$\begin{aligned}
\mathrm{Vol}_{N,d} :=& \prod_{j=1}^{N-1}[D_{0,j}]\prod_{j=1}^{N-1}[D_{1,j}]\cdots\prod_{j=1}^{N-1}[D_{d,j}]\cdot[U_1][U_2]\cdots[U_{d-1}]\\
=&H_0^{N-1}H_1^{N-1}\cdots H_d^{N-1}\cdot[U_1][U_2]\cdots[U_{d-1}]
\end{aligned}$$

since determinant of matrix which is obtained by $\{v_{i,j} \mid 0 \le i \le d,\ 1 \le j \le N-1\} \cup \{u_k \mid 1 \le k \le d-1\}$ equals ± 1. *2

We have to compute (3.14) in terms of $\mathrm{Vol}_{N,d}$. First, let us prove the following lemma:

Lemma 3.3

$$H_0^{N-1} H_1^N H_2^N \cdots H_{d-1}^N H_d^{N-1} = \frac{1}{d}\mathrm{Vol}_{N,d}.$$

Proof. We use mathematical induction along d. Let

$$I := H_0^{N-1} H_1^{N-1} \cdots H_d^{N-1},$$

then the LHS of this Lemma is

$$I \cdot H_1 H_2 \cdots H_{d-1}.$$

Since $[U_{d-1}] = -H_{d-2} + 2H_{d-1} - H_d$, we substitute $H_{d-1} = \frac{1}{2}([U_{d-1}] + H_{d-2} + H_d)$ to $I \cdot H_1 H_2 \cdots H_{d-1}$ then

$$\begin{aligned}
&I \cdot H_1 H_2 \cdots H_{d-1} \\
=&\frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2}([U_{d-1}] + H_{d-2} + H_d) \\
=&\frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2}[U_{d-1}] + \frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2}.
\end{aligned}$$

(Note that I is divisible by H_d^{N-1} and $H_d^N = 0$.)

The term $I \cdot H_1 H_2 \cdots H_{d-2}[U_{d-1}]$ can be applied the induction hypothesis, thanks to the divisor $H_d^{N-1}[U_{d-1}]$. In fact,

$$\begin{aligned}
&H_d^{N-1}[U_{d-1}] \cdot A^*(\widetilde{Mp}_{0,2}(N,d)) \\
=&H_d^{N-1}[U_{d-1}] \cdot \mathbb{C}[H_0, H_1, \ldots, H_d]/(H_0^N, H_1^N[U_1], H_2^N[U_2], \ldots, H_{d-1}^N[U_{d-1}], H_d^N) \\
\cong&\mathbb{C}[H_0, H_1, \ldots, H_{d-1}]/(H_0^N, H_1^N[U_1], H_2^N[U_2], \ldots, H_{d-2}^N[U_{d-2}], H_{d-1}^N).
\end{aligned}$$

Accordingly,

$$\begin{aligned}
I \cdot H_1 H_2 \cdots H_{d-2}[U_{d-1}] &= \frac{1}{d-1} I \cdot [U_1][U_2] \cdots [U_{d-1}] \\
&= \frac{1}{d-1}\mathrm{Vol}_{N,d}
\end{aligned}$$

by induction hypothesis.

We should compute the other term $I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2}$. The following holds:

$$\begin{aligned}
&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-j} \\
=&\frac{j-1}{j} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-j-1} + \frac{1}{j} I \cdot H_1 H_2 \cdots H_{d-1}.
\end{aligned} \tag{3.15}$$

for $2 \le j \le d-2$. We are going to prove it below.

*2 In general theory of toric variety, a maximal cone which generated by such vectors corresponds to a smooth point on toric variety. [7], etc.

Now,

$$
\begin{aligned}
&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_1 \\
=&\frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_2.
\end{aligned}
$$

If (3.15) is proven, then

$$
\begin{aligned}
&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_1 \\
=&\frac{d-3}{2(d-2)} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_1 + \frac{1}{2(d-2)} I \cdot H_1 H_2 \cdots H_{d-1},
\end{aligned}
$$

Therefore,

$$
I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_1 = \frac{1}{d-1} I \cdot H_1 H_2 \cdots H_{d-1}.
$$

Finally, by using (3.15) repeatedly, we obtain

$$
\begin{aligned}
&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2} \\
=&\frac{1}{2} I H_1 H_2 \cdots H_{d-2} \cdot H_{d-3} + \frac{1}{2} I H_1 H_2 \cdots H_{d-1} \\
=&\frac{1}{3} I H_1 H_2 \cdots H_{d-2} \cdot H_{d-4} + \frac{2}{3} I H_1 H_2 \cdots H_{d-1} \\
=&\cdots \\
=&\frac{1}{j} I H_1 H_2 \cdots H_{d-2} \cdot H_{d-j-1} + \frac{j-1}{j} I H_1 H_2 \cdots H_{d-1} \\
=&\cdots \\
=&\frac{1}{d-2} I H_1 H_2 \cdots H_{d-2} \cdot H_1 + \frac{d-3}{d-2} I H_1 H_2 \cdots H_{d-1} \\
=&\left(\frac{1}{(d-2)(d-1)} + \frac{d-3}{d-2} \right) I \cdot H_1 H_2 \cdots H_{d-1} \\
=&\frac{d-2}{d-1} I \cdot H_1 H_2 \cdots H_{d-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&I \cdot H_1 H_2 \cdots H_{d-1} \\
=&\frac{1}{2} \frac{1}{d-1} \mathrm{Vol}_{N,d} + \frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2} \\
=&\frac{1}{2} \frac{1}{d-1} \mathrm{Vol}_{N,d} + \frac{1}{2} \frac{d-2}{d-1} I \cdot H_1 H_2 \cdots H_{d-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I \cdot H_1 H_2 \cdots H_{d-1} =& \frac{\frac{1}{2} \frac{1}{d-1}}{1 - \frac{1}{2} \frac{d-2}{d-1}} \mathrm{Vol}_{N,d} \\
=& \frac{1}{d} \mathrm{Vol}_{N,d}. \ \square
\end{aligned}
$$

Proof of the equality (3.15) : If $j = 2$, then

$$\begin{aligned}&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2} \\ =&\frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot ([U_{d-2}] + H_{d-3} + H_{d-1}).\end{aligned}$$

Note that we can regard $[U_{d-2}]$ as zero with the divisor $I \cdot H_{d-2}$, hence

$$\begin{aligned}&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2} \\ =&\frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-3} + \frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-1}.\end{aligned} \tag{3.16}$$

If $j = 3$, then

$$\begin{aligned}&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-3} \\ =&\frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-4} + \frac{1}{2} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-2}.\end{aligned}$$

By substituting (3.16), we obtain

$$\begin{aligned}&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-3} \\ =&\frac{2}{3} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-4} + \frac{1}{3} I \cdot H_1 H_2 \cdots H_{d-1}.\end{aligned}$$

Inductively, we obtain the equality

$$\begin{aligned}&I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-j} \\ =&\frac{j-1}{j} I \cdot H_1 H_2 \cdots H_{d-2} \cdot H_{d-j-1} + \frac{1}{j} I \cdot H_1 H_2 \cdots H_{d-1}.\end{aligned}$$

for $2 \leq j \leq d-2$. $\Box$

Let us calculate the class

$$H_0^{N-1} H_1^N H_2^N \cdots H_{d-1}^N H_d^{N-1} \prod_{k=1}^{j} \frac{N[U_{d_k}]}{H_{d_k}}.$$

Let $J_{d_1, d_2, \ldots, d_j}$ be an ideal of $\mathbb{C}[H_0, H_1, \ldots, H_d]$, which is generated by

$$H_{d_k}^N \text{ (for } k = 0, 1, 2, \ldots, j+1) \text{ , } H_\alpha^N [U_\alpha] \text{ (for } \alpha \neq d_k).$$

(Here, we put $d_0 := 0$ and $d_{d+1} := d$.) Then, the following isomorphism holds:

$$\prod_{k=1}^{j} [U_{d_k}] \cdot A^*(\widetilde{Mp}_{0,2}(N, d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d] / J_{d_1, d_2, \ldots, d_j}. \tag{3.17}$$

This means that we can apply Lemma 3.3 to

$$H_{d_k}^{N-1} H_{d_k+1}^N H_{d_k+2}^N \cdots H_{d_{k+1}-1}^N H_{d_{k+1}}^{N-1}$$

for $0 \leq k \leq j$ (note that $d_0 := 0$, $d_{j+1} := d$). Therefore, we obtain

$$H_0^{N-1} H_1^N H_2^N \cdots H_{d-1}^N H_d^{N-1} \prod_{k=1}^{j} \frac{[U_{d_k}]}{H_{d_k}} = \prod_{k=0}^{j} \frac{1}{d_{k+1} - d_k} \cdot \mathrm{Vol}_{N,d}.$$

Accordingly,

$$\begin{aligned}
&c_{top}(\widetilde{Mp}_{0,2}(N,d))\\
=&\frac{N^2}{d}\mathrm{Vol}_{N,d}+\sum_{j=1}^{d-1}N^2\sum_{0<d_1<d_2<\cdots<d_j<d}\prod_{k=0}^{j}\frac{N}{d_{k+1}-d_k}\cdot\mathrm{Vol}_{N,d}\\
=&\sum_{j=0}^{d-1}N^2\sum_{0<d_1<d_2<\cdots<d_j<d}\prod_{k=0}^{j}\frac{N}{d_{k+1}-d_k}\cdot\mathrm{Vol}_{N,d}.
\end{aligned}$$

This provides orbifold Euler characteristic. Since $\mathrm{Vol}_{N,d}$ corresponds to smooth point,

$$\int_{\widetilde{Mp}_{0,2}(N,d)}\mathrm{Vol}_{N,d}=1.$$

Therefore,

Theorem 3.4

$$\chi_{orb}(\widetilde{Mp}_{0,2}(N,d))=\sum_{j=0}^{d-1}N^2\sum_{0<d_1<d_2<\cdots<d_j<d}\prod_{k=0}^{j}\frac{N}{d_{k+1}-d_k}$$

where $d_0=0,\ d_{j+1}=d$.

This is not integer in many case. We should receive it as an effect of orbifold singularity.

4 The Gromov-Witten Invariants of M_N^k.

In this section, following [10, 12], we derive numerically explicit formulas of the two pointed Gromov-Witten invariants of degree 1 and 2 of M_N^k. In the beginning, we define constants which play important roles in the remaining part of this paper.

Definition 4.1 *Let ℓ_i^k be the coefficient of $x^{k-i}y^{i+1}$ in $e^k(x,y):=\prod_{j=0}^{k}(jx+(k-j)y)$.*

Then, $e^k(x,y)=\sum_{i=0}^{k-1}\ell_i^k x^{k-i}y^{i+1}$. Note that $\ell_i^k=\ell_{k-1-i}^k$ and $\ell_i^k=0$ for $i<0$, $k\le i$. We describe $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d}$ as a polynomial of ℓ_i^k in order to prove our main theorems with the aid of the formula (2.3) and Proposition 2.1.

4.1 The Case of $d=1$.

Note that $\widetilde{Mp}_{0,2}(N,1)\cong\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$. We can compute degree 1 GW invariants of M_N^k with the equation (2.4) in Section 1. We can compute $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,1}$ by using (2.3).

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,1}=\int H_0^aH_1^be^k(H_0,H_1)=\int H_0^aH_1^b\sum_{i=0}^{k-1}\ell_i^kH_0^{i+1}H_1^{k-i}$$

$$= \int \sum_{i=0}^{k-1} \ell_i^k H_0^{a+i+1} H_1^{b+k-i} = \int \ell_{N-a-2}^k H_0^{N-1} H_1^{N-1}$$
$$= \ell_{N-a-2}^k.$$

Then we obtain

$$w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,1} = \ell_{N-a-b-2}^k = \ell_{k-N+1}^k,$$

(where we used $a+b = 2N-k-3$, and shortened $\int_{\widetilde{Mp}_{0,2}(N,1)}$ to $\int$). Therefore, we have

$$\begin{aligned}\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,1} &= w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,1} - w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,1} \\ &= \ell_{N-a-2}^k - \ell_{k-N+1}^k. \end{aligned} \tag{4.1}$$

4.2 The Case of $d = 2$.

We first present the following formula:

Lemma 4.1

$$\int_{\widetilde{Mp}_{0,2}(N,2)} H_0^{\alpha} H_1^{\beta} H_2^{\gamma} = \begin{cases} \frac{1}{2^{\beta-N+1}}\binom{\beta-N}{N-\alpha-1} & (N \leq \beta \leq 3N-2,\ 0 \leq \alpha, \gamma \leq N-1) \\ 0 \quad (\text{otherwise}). \end{cases}$$

We omit the proof of it because it is easily done by using the relation $H_1^N(H_0 - 2H_1 + H_2) = 0 \Leftrightarrow H_1^{N+1} = \frac{1}{2}H_1^N(H_0 + H_2)$ and $H_0^N = H_2^N = 0$.

$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,2}$ and $w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,2}$ are computed as follows:

$$\begin{aligned} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,2} &= \int H_0^a H_2^b \frac{e^k(H_0,H_1)e^k(H_1,H_2)}{kH_1} \\ &= \int H_0^a H_2^b \frac{1}{k} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \ell_i^k \ell_j^k H_0^{k-i} H_1^{i+j+1} H_2^{k-j} \\ &= \int \frac{1}{k} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \ell_i^k \ell_j^k H_0^{a+k-i} H_1^{i+j+1} H_2^{b+k-j} \\ &= \frac{1}{k} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \binom{i+j-N+1}{N-a-k+i-1}, \\ w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,2} &= \frac{1}{k} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \binom{i+j-N+1}{N-(a+b)-k+i-1} \\ &= \frac{1}{k} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \binom{i+j-N+1}{-2N+k+i+2}. \end{aligned}$$

To compute $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,2}$ from the equation (2.5) in Section 1, we should evaluate
$w(\mathcal{O}_{h^{a+b-N+k}}\mathcal{O}_{h^0})_{0,1}$ and $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\mathcal{O}_{h^{1+k-N}}\rangle_{0,1}$. The former is already computed in the previous subsection. Since $a+b = 3N-2k-3$ in this case, we obtain,

$$w(\mathcal{O}_{h^{a+b-N+k}}\mathcal{O}_{h^0})_{0,1} = \ell_{k-N+1}^k.$$

The latter can be computed by using Theorem 1 in [11] which computes n-*pointed* degree 1 GW invariants of M_N^k. If we apply this formula to the case of $n=3$, then

$$\begin{aligned}
&\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\mathcal{O}_{h^{1+k-N}}\rangle_{0,1} \\
&= \int_{\widetilde{Mp}_{0,2}(N,1)} (H_1-H_0)\cdot e^k(H_0,H_1)\cdot H_0^a\frac{H_0^b-H_1^b}{H_0-H_1}\cdot\frac{H_0^{1+k-N}-H_1^{1+k-N}}{H_0-H_1} \\
&= \int (H_1^{1+k-N}-H_0^{1+k-N})H_0^a\sum_{j=0}^{k-1}\ell_j^k H_0^{k-j}H_1^{j+1}\cdot\sum_{i=0}^{b-1}H_0^iH_1^{b-1-i} \\
&= \int (H_1^{1+k-N}-H_0^{1+k-N})\sum_{j=0}^{k-1}\sum_{i=0}^{b-1}\ell_j^k H_0^{a+k+i-j}H_1^{b-i+j} \\
&= \int \sum_{j=0}^{k-1}\sum_{i=0}^{b-1}\ell_j^k H_0^{a+k+i-j}H_1^{k-N+b-i+j+1} - \int \sum_{j=0}^{k-1}\sum_{i=0}^{b-1}\ell_j^k H_0^{a+2k-N+i-j+1}H_1^{b-i+j} \\
&= \sum_{i=0}^{b-1}(\ell_{i+a+k-N+1}^k-\ell_{i-b+N-1}^k) \\
&= \sum_{i=0}^{b-1}(\ell_{i+a+k-N+1}^k-\ell_{i+k-N+1}^k).
\end{aligned}$$

In the last line, we used the symmetry relation $\ell_i=\ell_{k-1-i}$ and inverted the order of terms. Combining these results, we obtain the following formula.

$$\begin{aligned}
&\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,2} \\
&= w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,2} - w(\mathcal{O}_{h^{a+b}}\mathcal{O}_{h^0})_{0,2} - \frac{1}{k}\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\mathcal{O}_{h^{1+k-N}}\rangle_{0,1}\, w(\mathcal{O}_{h^{a+b-N+k}}\mathcal{O}_{h^0})_{0,1} \\
&= \frac{1}{k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\ell_i^k\ell_j^k\frac{1}{2^{i+j-N+2}}\left(\binom{i+j-N+1}{N-a-k+i-1}-\binom{i+j-N+1}{-2N+k+i+2}\right) \\
&\qquad -\frac{1}{k}\ell_{k-N+1}^k\sum_{i=0}^{b-1}(\ell_{i+a+k-N+1}^k-\ell_{i+k-N+1}^k)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\ell_i^k\ell_j^k\frac{1}{2^{i+j-N+2}}\left(\binom{i+j-N+1}{N-a-k+i-1}-\binom{i+j-N+1}{N-k+j-1}\right) \\
&\qquad -\frac{1}{k}\ell_{k-N+1}^k\sum_{i=0}^{b-1}(\ell_{i+a+k-N+1}^k-\ell_{i+k-N+1}^k). \qquad (4.2)
\end{aligned}$$

5 Proof of Theorem 1.1.

In this section, we will prove Theorem 2.1 of this paper. From now on, an integer n is sometimes used as $N-1$:

$$n := N-1.$$

In order to prove the theorem, we use the results on Chow ring of moduli space $\overline{M}_{0,m}(\mathbb{P}^n, d)$ of stable maps of degree d from genus 0 stable curve to projective space $\mathbb{P}^n$.

5.1 Review of the structure of $A^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$.

Theorem 5.1 (Theorem 5.1. of [17]) *Let $M = \{1_M, 2_M, \ldots, m_M\}$, $D = \{1_D, \ldots, d_D\}$, $M' = M \backslash \{1_M\}$, $D' = M' \sqcup D$ and $d' = |D'| = d + m - 1$.*

$B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ is a $\mathbb{Q}-$algebra generated by divisors

$$H \ , \ \psi \ , \ T_h$$

for all $h \subset D'$ such that $h \neq \emptyset$ or $\{i_M\}$ for $i_M \in M'$. Let $T_\emptyset = 1$.

The ideal of relations is generated by:

(1) H^{n+1};

(2) $T_h T_{h'}$ *unless* $h \cap h' = \emptyset$, *or* $\emptyset \neq h \subseteq h'$ *or* $\emptyset \neq h' \subseteq h$;

(3)
- $(m \geq 1)$ $T_h T_{h'}(\psi + \sum_{h \cup h' \subseteq h''} T_{h''})$ *for all* $h \neq h'$ *nonempty;*
- $(m \geq 2)$ $T_h(\psi + \sum_{h \cup \{i_M\} \subseteq h'} T_{h'})$ *for all* $h \neq \emptyset$ *and* $i_M \in M' \backslash h$;
- $(m \geq 3)$ $\psi + \sum_{\{i_M, j_M\} \subseteq h} T_h$ *for all* $i_M, j_M \in M'$;

(4) $(m > 1)$ $(H + d\psi + \sum_{i_M \in h} |h \cap M'| T_h)^{n+1}$;

(5) $T_h(\sum_{h' \neq h} P(t_{h'})|_{t_{h'}=0}^{t_{h'}=T_{h'}} + \psi^{-1}(H + |^c h_D|\psi)^{n+1})$ *for all* h,

where

$$\begin{aligned} P(t_{h'}) &= (\psi + \sum_{h'' \supset h'} T_{h''} + t_{h''})^{-1} \\ &[(H + |^c h_D|\psi + \sum_{h'' \supset h'} |h''_D \backslash h_D| T_{h''} + |h'_D \backslash h_D| t_{h'})^{n+1} - \\ &(H + |^c h_D \cap^c h'_D|\psi + \sum_{h'' \supset h'} |h''_D \backslash (h_D \cup h'_D)| T_{h''})^{n+1}]. \end{aligned}$$

Here for any $h \subset D'$, $h_D := h \cap D$ and $^c h_D := D \backslash h$.

(We will also use this theorem in the following section, i.e. in the case of degree 2.) To obtain the Chow ring $A^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ from $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$, we have to consider a $S_d \times S_m-$action on $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$. This action is realized as permutation of $D \cup M$. It is visualized by using the graph that corresponds to T_h in Section 1. The Chow ring $A^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ is an invariant ring of $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ under this action (see [17]). In the case of $B^*(\overline{M}_{0,2}(\mathbb{P}^n, 1))$, $A^*(\overline{M}_{0,2}(\mathbb{P}^n, 1))$ coincides with $B^*(\overline{M}_{0,2}(\mathbb{P}^n, 1))$ since this action is trivial. Its generators are

$$H, \psi, T_{\{1_D\}}.$$

Let $T := T_{\{1_D\}}$. Then its relations are the following:

$$H^{n+1}, \ T\psi, \ (H + \psi)^{n+1}, \ \frac{(H + \psi + T)^{n+1} - H^{n+1}}{\psi + T}.$$

Of course, these relations do not contradict the fact that $\overline{M}_{0,2}(\mathbb{P}^n, 1)$ is isomorphic to blow-up of $\mathbb{P}^n \times \mathbb{P}^n$ along its diagonal subset Δ. To see it, we define the following transformation:

Definition 5.1 (Key transformation)

$$h_0 := H,\ h_1 := H + \psi.$$

Then, the above relations change into the following:

$$h_0^{n+1}, T(h_1 - h_0), h_1^{n+1},$$

and

$$\frac{(h_1 + T)^{n+1} - h_0^{n+1}}{h_1 - h_0 + T}. \tag{5.1}$$

At this stage, we introduce a symbol h which satisfies $hT = h_0T = h_1T$ (we have the relation $T(h_1 - h_0)$). Let us expand the last relation:

$$\begin{aligned}
&\frac{(h_1 + T)^{n+1} - h_0^{n+1}}{h_1 - h_0 + T} \\
&= \sum_{i=0}^{n} h_0^{n-i}(h_1 + T)^i = \sum_{i=0}^{n} h_0^{n-i} \sum_{j=0}^{i} \binom{i}{j} h_1^{i-j} T^j \\
&= \sum_{i=0}^{n} h_0^{n-i} h_1^i + \sum_{i=0}^{n}\sum_{j=1}^{i} \binom{i}{j} h^{n-j} T^j = \sum_{i=0}^{n} h_0^{n-i} h_1^i + \sum_{j=1}^{n}\sum_{i=j}^{n} \binom{i}{j} h^{n-j} T^j \\
&= \sum_{i=0}^{n} h_0^{n-i} h_1^i + \sum_{j=1}^{n} \binom{n+1}{j+1} h^{n-j} T^j.
\end{aligned} \tag{5.2}$$

In the last line, we used an identity $\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}$. It can be shown by using $\binom{j}{j} + \binom{j+1}{j} = \binom{j+2}{j+1}$ and Pascal's triangle. If we regard T as exceptional divisor of blow-up of $\mathbb{P}^n \times \mathbb{P}^n$ along Δ, we can see that these relations coincides with those of the blow-up (see [8]). Hence we identify T with the exceptional divisor of the blow-up.

5.2 Proof of Theorem 2.1.

In this subsection, we prove the following theorem on degree 1 Gromov-Witten invariants of a hypersurface M_{n+1}^k:

Theorem 5.2

$$\langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0,2} = \int_{\overline{M}_{0,2}(\mathbb{P}^n,1)} h_0^a h_1^b e^k(h_0, h_1 + T),$$

where $e^k(x, y) := \prod_{j=0}^{k}(jx + (k-j)y) := \sum_{i=0}^{k-1} \ell_i^k x^{k-i} y^{i+1}$.

First, we prove the following lemma.

Lemma 5.1

$$\int_{\overline{M}_{0,2}(\mathbb{P}^n,1)} h_0^\alpha h_1^\beta (h_1+T)^\gamma = \begin{cases} 1 & (\alpha = n,\ \gamma \neq n) \\ -1 & (\alpha \neq n,\ \gamma = n) \\ 0 & (\text{otherwise}) \end{cases},$$

where $\alpha, \beta, \gamma \geq 0$*,* $\alpha + \beta + \gamma = 2n$*.*

Proof of lemma 5.1.
Note that $\int h_0^n h_1^n = 1$. If $\alpha = n$ and $\gamma \neq n$, then $\gamma = n - \beta < n$ and $\beta = n - \gamma > 0$. Hence $\alpha + \beta > n$. By the relation $T(h_1 - h_0) = 0$, the terms that contain T in $h_0^\alpha h_1^\beta (h_1+T)^\gamma$ vanish. Therefore, $h_0^\alpha h_1^\beta (h_1+T)^\gamma = h_0^n h_1^{\beta+\gamma} = h_0^n h_1^n$. If $\alpha \neq n$ and $\gamma = n$, then $\beta + \gamma > n$ as above. Therefore, $h_0^\alpha h_1^\beta (h_1+T)^\gamma = h_0^\alpha h_1^\beta T^n = h^n T^n$. We obtain the relation $h_0^n h_1^n + h^n T^n = 0$ by multiplying the relation (5.2) by h_0^n, and $h_0^\alpha h_1^\beta (h_1+T)^\gamma = -h_0^n h_1^n$.

Hereinafter, we consider cases when $h_0^\alpha h_1^\beta (h_1+T)^\gamma$ vanishes. If $\alpha = \gamma = n$ and $\beta = 0$, then $h_0^\alpha h_1^\beta (h_1+T)^\gamma = h_0^n (h_1+T)^n = h_0^n h_1^n + h^n T^n = 0$. If $\gamma < n$ and $n - \beta + 1 \leq \alpha \leq n-1$, then $\alpha + \beta \geq n+1$. Hence $h_0^\alpha h_1^\beta (h_1+T)^\gamma = h_0^\alpha h_1^{\beta+\gamma}$. However, $\beta + \gamma = 2n - \alpha \geq n+1$, and it vanishes. Finally, we have to consider a case of $\gamma > n$. We multiply the relation (5.1) by $h_1 - h_0 + T$, and obtain the relation $(h_1+T)^{n+1} = 0$. Therefore $h_0^\alpha h_1^\beta (h_1+T)^\gamma = 0$. $\square$

Proof of Theorem 2.1.

$$\begin{aligned} h_0^a h_1^b e^k(h_0, h_1+T) &= \sum_{i=0}^{k-1} \ell_i^k h_0^{i+1} (h_1+T)^{k-i} = \sum_{i=0}^{k-1} \ell_i^k h_0^{a+i+1} h_1^b (h_1+T)^{k-i} \\ &= (\ell_{n-a-1}^k - \ell_{k-n}^k) h_0^n h_1^n. \end{aligned}$$

Therefore, $\int_{\overline{M}_{0,2}(\mathbb{P}^n,1)} h_0^a h_1^b e^k(h_0, h_1+T) = \ell_{n-a-1}^k - \ell_{k-n}^k = \langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0,1}$. $\square$

6 Proof of Theorem 1.2.

In this section, we prove Theorem 1.2 of this paper. First of all, we have to apply Theorem 5.1 to determine the structure of $A^*(\overline{M}_{0,2}(\mathbb{P}^n, 2))$. In this case,

$$M = \{1_M, 2_M\},\ \ D = \{1_D, 2_D\},\ \ M' = \{2_M\},\ \ D' = \{1_D, 2_D, 2_M\}.$$

Therefore, the generators of $B^*(\overline{M}_{0,2}(\mathbb{P}^n, 2))$ are

$$H,\ \ \psi,\ \ T_1 := T_{1_D},\ \ T_2 := T_{2_D},\ \ U_1 := T_{1_D, 2_M},\ \ U_2 := T_{2_D, 2_M},\ \ S_0 := T_{1_D, 2_D}.$$

Then, the relations are given as follows:

$$H^{n+1}, \tag{6.1}$$
$$S_0 U_1,\ \ S_0 U_2,\ \ U_1 U_2, \tag{6.2}$$
$$T_1 T_2 (\psi + S_0)\ \ T_1 U_2 \psi,\ \ T_2 U_1 \psi, \tag{6.3}$$
$$T_1(\psi + U_1),\ \ T_2(\psi + U_2),\ \ S_0 \psi, \tag{6.4}$$
$$(H + 2\psi + U_1 + U_2)^{n+1}, \tag{6.5}$$

and the relations (5) in Theorem 5.1. We take a close look at the relation (5) individually since they have quite long expression.

The case of $h=\emptyset$:

$$
\begin{aligned}
&\frac{(H+2\psi+2S_0+U_1+T_1)^{n+1}-(H+\psi+S_0)^{n+1}}{\psi+S_0+U_1+T_1}\\
-&\frac{(H+2\psi+2S_0+U_1)^{n+1}-(H+\psi+S_0)^{n+1}}{\psi+S_0+U_1}\\
+&\frac{(H+2\psi+2S_0+U_2+T_2)^{n+1}-(H+\psi+S_0)^{n+1}}{\psi+S_0+U_2+T_2}\\
-&\frac{(H+2\psi+2S_0+U_2)^{n+1}-(H+\psi+S_0)^{n+1}}{\psi+S_0+U_2}\\
+&\frac{(H+2\psi+U_1)^{n+1}-(H+\psi)^{n+1}}{\psi+U_1}-\frac{(H+2\psi)^{n+1}-(H+\psi)^{n+1}}{\psi}\\
+&\frac{(H+2\psi+U_2)^{n+1}-(H+\psi)^{n+1}}{\psi+U_2}\\
-&\frac{(H+2\psi)^{n+1}-(H+\psi)^{n+1}}{\psi}\\
+&\frac{(H+2\psi+2S_0)^{n+1}-H^{n+1}}{\psi+S_0}\\
-&\frac{(H+2\psi)^{n+1}-H^{n+1}}{\psi}\\
+&\frac{(H+2\psi)^{n+1}}{\psi}.
\end{aligned}
$$

The cases of $h=\{1_D\}$ and $h=\{2_D\}$:

$$
\begin{aligned}
T_1\Big(&\frac{(H+\psi+S_0+U_2+T_2)^{n+1}-H^{n+1}}{\psi+S_0+U_2+T_2}\\
-&\frac{(H+\psi+S_0+U_2)^{n+1}-H^{n+1}}{\psi+S_0+U_2}\\
+&\frac{(H+\psi+U_2)^{n+1}-H^{n+1}}{\psi+U_2}\\
-&\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi}\\
+&\frac{(H+\psi+S_0)^{n+1}-H^{n+1}}{\psi+S_0}\\
-&\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi}\\
+&\frac{(H+\psi)^{n+1}}{\psi}\Big)
\end{aligned}
\qquad
\begin{aligned}
T_2\Big(&\frac{(H+\psi+S_0+U_1+T_1)^{n+1}-H^{n+1}}{\psi+S_0+U_1+T_1}\\
-&\frac{(H+\psi+S_0+U_1)^{n+1}-H^{n+1}}{\psi+S_0+U_1}\\
+&\frac{(H+\psi+U_1)^{n+1}-H^{n+1}}{\psi+U_1}\\
-&\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi}\\
+&\frac{(H+\psi+S_0)^{n+1}-H^{n+1}}{\psi+S_0}\\
-&\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi}\\
.+&\frac{(H+\psi)^{n+1}}{\psi}\Big).
\end{aligned}
$$

The cases of $h = \{1_D, 2_M\}$ and $h = \{2_D, 2_M\}$:

$$U_1\Big(\frac{(H+\psi+S_0+U_2+T_2)^{n+1}-H^{n+1}}{\psi+S_0+U_2+T_2} -\frac{(H+\psi+S_0+U_2)^{n+1}-H^{n+1}}{\psi+S_0+U_2} +\frac{(H+\psi+U_2)^{n+1}-H^{n+1}}{\psi+U_2} -\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi} +\frac{(H+\psi+S_0)^{n+1}-H^{n+1}}{\psi+S_0} -\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi} +\frac{(H+\psi)^{n+1}}{\psi}\Big).$$

$$U_2\Big(\frac{(H+\psi+S_0+U_1+T_1)^{n+1}-H^{n+1}}{\psi+S_0+U_1+T_1} -\frac{(H+\psi+S_0+U_1)^{n+1}-H^{n+1}}{\psi+S_0+U_1} +\frac{(H+\psi+U_1)^{n+1}-H^{n+1}}{\psi+U_1} -\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi} +\frac{(H+\psi+S_0)^{n+1}-H^{n+1}}{\psi+S_0} -\frac{(H+\psi)^{n+1}-H^{n+1}}{\psi} +\frac{(H+\psi)^{n+1}}{\psi}\Big).$$

The relation of $h = \{1_D, 2_D\}$ is trivial.

We can simplify these relations by using the relations from (1) to (4). We write down the simplest form:

$$\begin{aligned}&\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}((H+2\psi+2S_0+U_1+T_1)^i+(H+2\psi+2S_0+U_2+T_2)^i\\ &\qquad -(H+2\psi+2S_0+U_1)^i-(H+2\psi+2S_0+U_2)^i)\\ &+\sum_{i=0}^{n}(H+\psi)^{n-i}((H+2\psi+S_2)^i-(H+2\psi)^i)\\ &+2\sum_{i=0}^{n}H^{n-i}(H+2\psi+2S_0)^i,\end{aligned} \tag{6.6}$$

$$T_1\sum_{i=0}^{n}H^{n-i}((H+U_2+T_2)^i+(H+\psi+S_0)^i-H^i), \tag{6.7}$$

$$T_2\sum_{i=0}^{n}H^{n-i}((H+U_1+T_1)^i+(H+\psi+S_0)^i-H^i), \tag{6.8}$$

$$U_1\sum_{i=0}^{n}H^{n-i}((H+T_2)^i-H^i+(H+\psi)^i), \tag{6.9}$$

$$U_2\sum_{i=0}^{n}H^{n-i}((H+T_1)^i-H^i+(H+\psi)^i). \tag{6.10}$$

Generators of $A^*(\overline{M}_{0,2}(\mathbb{P}^n, 2))$ are given by,

$$H,\ \ \psi,\ \ S_0,\ \ S_1 := T_1+T_2,\ \ S_2 := U_1+U_2,$$
$$P_1 := T_1T_2,\ \ P_2 := U_1U_2,\ \ P_3 := T_1U_2+T_2U_1.$$

But we don't have to use P_2 and P_3. $P_2 = U_1U_2$ vanishes by the relation (6.2). On the other hand, by using the relation (6.4), we obtain,

$$0 = T_1(\psi + U_1) + T_1(\psi + U_1) = S_1\psi + S_1S_2 - P_3,$$

Hence P_3 is represented in terms of ψ, S_1 and S_2.

We can derive some useful relations from them by applying the following transformation:

Definition 6.1 (Key transformation)

$$h_0 := H, \;\; h_1 := H + \psi, \;\; h_2 := H + 2\psi + S_2.$$

Then the generators of $A^*(\overline{M}_{0,2}(\mathbb{P}^n, 2))$ turn into h_0, h_1, h_2, S_0, S_1 and P_1.

Lemma 6.1

$$h_0^{n+1} = 0, \;\; h_2^{n+1} = 0, \;\; h_1^{n+1}(h_0 - 2h_1 + h_2) = 0.$$

Proof. The first two are clear from (6.1) and (6.5). If we multiply by ψ the relation (6.9) and (6.10), and add the resulting relations, then we obtain the third relation as follows:

$$\begin{aligned}
&\psi \cdot (6.9) + \psi \cdot (6.10) \\
=&\psi U_1 \sum_{i=0}^{n} H^{n-i}(H+\psi)^i + \psi U_2 \sum_{i=0}^{n} H^{n-i}(H+\psi)^i \qquad \text{(by (6.3))} \\
=&S_2((H+\psi)^{n+1} - H^{n+1}) \\
=&(h_0 - 2h_1 + h_2)h_1^{n+1}. \qquad \square
\end{aligned}$$

These relations are interesting since they are the same as the ones of $A^*(\widetilde{Mp}_{0,2}(N,2))$. Hence we use the same volume form as the one of $\widetilde{Mp}_{0,2}(N,2)$.

$$\int_{\overline{M}_{0,2}(\mathbb{P}^n,2)} 2h_0^n h_1^{n+1} h_2^n = 1.$$

If there is no room for misunderstanding, we abbreviate $\int_{\overline{M}_{0,2}(\mathbb{P}^n,2)}$ as $\int$. We write down the relations from (6.1) to (6.5) in terms of new generators.

Lemma 6.2

$$\begin{aligned}
P_3 = S_1(h_2 - h_1), \;\; P_1(h_2 - h_0) = 0, \;\; P_1(h_1 - h_0 + S_0) = 0, \\
S_0(h_1 - h_0) = 0, \;\; S_0(h_1 - h_2) = 0, \;\; S_1(h_1 - h_0)(h_1 - h_2) = 0.
\end{aligned}$$

We have to compute the following intersection number in order to prove of Theorem 2.2:

$$\int h_0^\alpha (h_1 + S_0)^\beta (h_2 + 2S_0 + S_1)^\gamma h_2^\delta. \tag{6.11}$$

For this purpose, we set

$$g_0 := h_0, \;\; g_1 := h_1 + S_0, \;\; g_2 := h_2 + 2S_0 + S_1,$$

and derive relations among g_0, g_1, and g_2.

Lemma 6.3

$$g_0^{n+1}=0,\ \ g_1^{n+1}(g_0-2g_1+g_2)=0,\ \ (g_1-g_0)\sum_{i=0}^{n}(g_0^i+g_2^i)g_1^{n-i}=0,\ \ (g_1-g_0)g_2^{n+1}=0.$$

Proof. The first relation is trivial. The second relation is equivalent to

$$S_1(h_1+S_0)^{n+1}=0$$

by lemma 6.1. It is shown by adding (6.7) and (6.8), and by multiplying the resulting expression by $(\psi+S_0)$.

$$\begin{aligned}
0=&(\psi+S_0)((6.7)+(6.8))\\
=&(\psi+S_0)(T_1\sum_{i=0}^{n}H^{n-i}(H+\psi+S_0)^i+T_2\sum_{i=0}^{n}H^{n-i}(H+\psi+S_0)^i)\quad \text{(by (6.3))}\\
=&S_1((H+\psi+S_0)^{n+1}-H^{n+1})\\
=&S_1(h_1+S_0)^{n+1},
\end{aligned}$$

where we use the relations, $T_1(\psi+S_0)T_2=0$, $T_1(\psi+S_0)U_2=0$ and $T_2(\psi+S_0)U_1=0$. The third relation is obtained from multiplying the relation (6.6) by $(\psi+S_0)$.

$$\begin{aligned}
0=&(\psi+S_0)\cdot(6.6)\\
=&(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}((H+2\psi+2S_0+S_1+S_2)^i-(H+2\psi+2S_0+S_2)^i)\\
&+(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi)^{n-i}((H+2\psi+S_2)^i-(H+2\psi)^i)\\
&+2(\psi+S_0)\cdot\sum_{i=0}^{n}H^{n-i}(H+2\psi+2S_0)^i\\
=&(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}((H+2\psi+2S_0+S_1+S_2)^i-(H+2\psi+2S_0+S_2)^i)\\
&+(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}((H+2\psi+2S_0+S_2)^i-(H+2\psi+2S_0)^i)\\
&+2(\psi+S_0)\cdot\sum_{i=0}^{n}H^{n-i}(H+2\psi+2S_0)^i\\
=&(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}(H+2\psi+2S_0+S_1+S_2)^i\\
&-(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}(H+2\psi+2S_0)^i\\
&+2(\psi+S_0)\cdot\sum_{i=0}^{n}H^{n-i}(H+2\psi+2S_0)^i
\end{aligned}$$

$$
\begin{aligned}
=&(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}(H+2\psi+2S_0+S_1+S_2)^i\\
&+(H+\psi+S_0)^{n+1}-H^{n+1}\\
=&(\psi+S_0)\cdot\sum_{i=0}^{n}(H+\psi+S_0)^{n-i}((H+2\psi+2S_0+S_1+S_2)^i+H^i)\\
=&(g_1-g_0)\sum_{i=0}^{n}g_1^{n-i}(g_2^i+g_0^i).
\end{aligned}
$$

On the second equality, we used $(\psi+S_0)(U_1+T_1)(U_2+T_2)=0$ which comes from (6.3), and $U_1U_2=0$. On the third equality, we inserted S_0 and $2S_0$ to $(H+\psi)^{n-i}$, $(H+2\psi+S_2)^i$, and $(H+2\psi)^i$ since $(H+2\psi+S_2)^i-(H+2\psi)^i$ is divisible by S_2, and $S_0S_2=0$.

The last relation is shown by multiplying the third relation by (g_1-g_2) and by applying the second relation to the resulting expression.

$$
\begin{aligned}
(g_1-g_0)(g_1-g_2)\sum_{i=0}^{n}(g_0^i+g_2^i)g_1^{n-i}=&(g_1-g_0)(g_1^{n+1}-g_2^{n+1}+(g_1-g_2)\sum_{i=0}^{n}g_0^ig_1^{n-i})\\
=&(g_1-g_0)g_1^{n+1}-(g_1-g_0)g_2^{n+1}+(g_1-g_2)g_1^{n+1}\\
=&-(g_1-g_0)g_2^{n+1}.\quad\Box
\end{aligned}
$$

Next, we compute some intersection numbers on $\overline{M}_{0,2}(\mathbb{P}^n,2)$.

Lemma 6.4

$$
\begin{aligned}
&\int S_1^ah_0^bh_1^ch_2^d=0\ (\text{for}\ \ a+b+c+d=3n+1,\ \ 0<a<n).,\\
&\int S_1^nh_1h_0^nh_2^n=-1.
\end{aligned}
$$

Proof. Let us discuss the first equation. If $c=0$, then $b+d=3n+1-a>2n+1$. Hence $S_1^ah_0^bh_2^d=0$ by the relation $h_0^{n+1}=h_2^{n+1}=0$. If $c\neq 0$, then thanks to the relation $S_1(h_1-h_0)(h_1-h_2)=0$ in Lemma 6.2, we have only to consider the case of $c=1$. Then $b+d=3n-a>2n$, and $S_1^ah_0^bh_1h_2^d=0$ as above. As for the second equation, we compute $H^n(H+2\psi+S_2)^n\cdot((6.9)+(6.10))$:

$$
\begin{aligned}
0=&H^n(H+2\psi+S_2)^n\cdot((6.9)+(6.10))\\
=&H^n(H+2\psi+S_2)^n(U_1T_2^n+U_2T_1^n+(U_1+U_2)(H+\psi)^n)\quad(\text{by }H^{n+1}=0)\\
=&h_0^nh_2^n(U_1T_2^n+U_2T_1^n+(h_0-2h_1+h_2)h_1^n)\\
=&h_0^nh_2^n(U_1T_2^n+U_2T_1^n)-2h_0^nh_1^{n+1}h_2^n.
\end{aligned}
$$

It is clear that $U_1T_2^n+U_2T_1^n-S_1^{n-1}P_3=U_1T_2^n+U_2T_1^n-(T_1+T_2)^{n-1}(T_1U_2+T_2U_1)$ is divisible by $P_1=T_1T_2$. Then by using the relation $P_1(h_2-h_0)=0$ of Lemma 6.2 and $h_0^{n+1}=h_2^{n+1}=0$, we obtain $h_0^nh_2^n(U_1T_2^n+U_2T_1^n)=h_0^nh_2^nS_1^{n-1}P_3$. Since $P_3=S_1(h_2-h_1)$, we obtain the following equation:

$$-S_1^nh_0^nh_1h_2^n-2h_0^nh_1^{n+1}h_2^n=0.\Box$$

Now, we compute intersection numbers of the type given in (6.11).

Lemma 6.5 *Let α, β, γ, δ be nonnegative integers which satisfies $\alpha+\beta+\gamma+\delta = 3n+1$. If $1 \le \delta \le n$, then*

$$\int g_0^\alpha g_1^\beta g_2^\gamma h_2^\delta = \begin{cases} \frac{1}{2^{\beta-n}}\left\{\binom{\beta-n-1}{n-\alpha} - \binom{\beta-n-1}{n-\gamma}\right\}, (\gamma < n \text{ or } \gamma = n,\ n-\delta+1 > \alpha) \\ -1, \qquad (\gamma = n,\ n-\delta+1 \le \alpha \le n) \\ 0, \qquad (0 \le \alpha \le n,\ n+1 \le \gamma \le 2n-\delta). \end{cases}$$

If $\delta = 0$, $0 \le \alpha \le n$, $0 \le \gamma \le 2n$, then it is zero.

Proof. First, we consider the case of $\delta = 0$, $0 \le \alpha \le n$ and $0 \le \gamma \le 2n$. If $n+1 \le \gamma \le 2n$, then $\alpha + \beta \ge n+1$. Hence it is zero since there is the relations $(g_1 - g_0)g_2^{n+1} = 0$ and $g_0^{n+1} = 0$ of Lemma 6.3. If $0 \le \gamma \le n+1$, then we have only to consider the case of $g_0^n g_1^{n+1} g_2^n$ since there is the relation $g_1^{n+1}(g_0 - 2g_1 + g_2) = 0$ and $g_0^{n+1} = 0$. To prove $g_0^n g_1^{n+1} g_2^n = 0$, we should multiply $g_0^n g_2^n$ to the relation $(g_1 - g_0)\sum_{i=0}^n (g_0^i + g_2^i) g_1^{n-i} = 0$.

Next, we consider the case of $1 \le \delta \le n$, $0 \le \alpha \le n$, $n+1 \le \gamma \le 2n-\delta$. In this case, the left hand side is zero since $\alpha + \beta \ge n+1$, and there are relations $(g_1 - g_0)g_2^{n+1} = 0$ and $g_0^{n+1} = 0$.

If $1 \le \delta \le n$, $\gamma = n$ and $n - \delta + 1 \le \alpha \le n$, then

$$\begin{aligned} & h_0^\alpha (h_1 + S_0)^\beta (h_2 + 2S_0 + S_1)^n h_2^\delta \\ =& h_0^\alpha h_1^\beta (h_2 + S_1)^n h_2^\delta \\ =& h_0^\alpha h_1^\beta S_1^n h_2^\delta && \text{(by Lemma 6.4, } n+\delta > n) \\ =& S_1^n h_0^n h_1 h_2^n && \text{(by } S_1(h_1 - h_0)(h_1 - h_2) = 0). \end{aligned}$$

Therefore, $\int g_0^\alpha g_1^\beta g_2^n h_2^\delta = -1$ by Lemma 6.4.

Now, we prove the top case. If $2 \le \delta \le n$, $n+1-\delta \le \alpha \le n$ and $n+1-\delta \le \gamma \le n-1$, then the right hand side is zero since $\beta - n - 1 = 2n - \alpha - \gamma - \delta < n - \alpha, n - \gamma$. On the left hand side, it is

$$\begin{aligned} & h_0^\alpha (h_1 + S_0)^\beta (h_2 + 2S_0 + S_1)^\gamma h_2^\delta \\ =& h_0^\alpha h_1^\beta (h_2 + S_1)^\gamma h_2^\delta && \text{(by } \alpha + \delta > n, \text{ Lemma 6.1, 6.2)} \\ =& h_0^\alpha h_1^\beta h_2^{\gamma+\delta} && \text{(by Lemma 6.4)} \\ =& 0 && \text{(by } \gamma + \delta > n). \end{aligned}$$

If $\alpha = n$, $\beta = n+1$, $\gamma = n - \delta$ and $1 \le \delta \le n$, then the right hand side is $\frac{1}{2}$. On the left hand side, it is

$$\begin{aligned} & \int h_0^n (h_1 + S_0)^{n+1} (h_2 + 2S_0 + S_1)^{n-\delta} h_2^\delta \\ =& \int h_0^n h_1^{n+1} h_2^n \end{aligned}$$

Therefore, $\int g_0^n g_1^{n+1} g_2^{n-\delta} h_2^\delta = \frac{1}{2}$.

If $\alpha = n - \delta$, $\beta = n+1$, $\gamma = n$ and $1 \le \delta \le n$, then the right hand side is $-\frac{1}{2}$. On the left hand side, we consider $0 = g_0^{n-\delta} g_2^n h_2^\delta \cdot (g_1 - g_0) \sum_{i=0} (g_0^i + g_2^i) g_1^{n-i}$

from Lemma 6.3.

$$0 = g_0^{n-\delta} g_2^n h_2^\delta \cdot (g_1 - g_0) \sum_{i=0}^{n} (g_0^i + g_2^i) g_1^{n-i}$$
$$= g_0^{n-\delta} g_2^n h_2^\delta \cdot (g_1 - g_0)(2g_1^n + \sum_{i=1}^{\delta} g_0^i g_1^{n-i}) \qquad \text{(by Lemma 6.3).}$$

Here, if $0 < i < \delta$, then

$$\begin{aligned}
&\int g_0^{n-\delta} g_2^n h_2^\delta (g_1 - g_0) g_0^i g_1^{n-i} \\
=&\int g_0^{n+i-\delta} g_1^{n-i+1} g_2^n h_2^\delta - g_0^{n+i-\delta+1} g_1^{n-i} g_2^n h_2^\delta \\
=&(-1) - (-1) && \text{(by the above case)} \\
=&0 && .
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= \int g_0^{n-\delta} g_2^n h_2^\delta \cdot (g_1 - g_0)(2g_1^n + g_0^\delta g_1^{n-\delta}) \\
&= \int (2g_0^{n-\delta} g_1^{n+1} g_2^n h_2^\delta - 2g_0^{n-\delta} g_1^n g_2^n h_2^\delta + g_0^n g_1^{n-\delta+1} g_2^n h_2^\delta) \\
&= \int 2g_0^{n-\delta} g_1^{n+1} g_2^n h_2^\delta - 2 \cdot (-1) + (-1) \\
&= \int 2g_0^{n-\delta} g_1^{n+1} g_2^n h_2^\delta + 1.
\end{aligned}$$

Accordingly, $\int g_0^{n-\delta} g_1^{n+1} g_2^n h_2^\delta = -\frac{1}{2}$.

Finally, we use induction along $\beta \geq n+1$ to prove the remaining part of this lemma. To execute induction, the following equation is needed: if $1 \leq \delta \leq n-1$, $0 \leq \alpha \leq n$ and $n+1 \leq \gamma \leq 2n+1-\delta$, then

$$g_0^\alpha g_1^\beta g_2^\gamma h_2^\delta = 0. \tag{6.12}$$

It is already proved in the above.

If $\beta = n+1$, then Lemma 6.5 is true as above. If $\beta > n+1$, by using the relation $g_1^{n+1}(g_0 - 2g_1 + g_2) = 0$, we obtain,

$$\begin{aligned}
\int g_0^\alpha g_1^\beta g_2^\gamma h_2^\delta =& \frac{1}{2} \int g_0^{\alpha+1} g_1^{\beta-1} g_2^\gamma h_2^\delta + \frac{1}{2} \int g_0^\alpha g_1^{\beta-1} g_2^{\gamma+1} h_2^\delta \\
=& \frac{1}{2} \frac{1}{2^{\beta-n-1}} \left\{ \binom{\beta-n-2}{n-\alpha-1} - \binom{\beta-n-2}{n-\gamma} \right\} \\
&+ \frac{1}{2} \frac{1}{2^{\beta-n-1}} \left\{ \binom{\beta-n-2}{n-\alpha} - \binom{\beta-n-2}{n-\gamma-1} \right\} \\
=& \frac{1}{2^{\beta-n}} \left\{ \binom{\beta-n-1}{n-\alpha} - \binom{\beta-n-1}{n-\gamma} \right\},
\end{aligned}$$

where we use the law of Pascal's triangle in the last line. The equation 6.12 is used in the case of $\gamma = n$ in the above induction. □

In this Lemma, we do not compute the intersection numbers in the cases of $0 \leq \delta \leq n$ and $\gamma > 2n - \delta$. Although the above result is not complete in this sense, it is sufficient to prove Theorem 2.2.

Proof of Theorem 2.2.

Let a, b be nonnegative integers satisfying $a+b = 3n-2k$, and k be a positive integer. First, we compute intersection number $\int h_0^a h_2^b e^k(g_0, g_1) e^k(g_1, g_2)/(kg_1)$:

$$\int h_0^a h_2^b \frac{e^k(g_0,g_1)e^k(g_1,g_2)}{kg_1} = \int h_0^a h_2^b \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \ell_i^k \ell_j^k g_0^{k-i} g_1^{i+j+1} g_2^{k-j}$$
$$= \int \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \ell_i^k \ell_j^k g_0^{a+k-i} g_1^{i+j+1} g_2^{k-j} h_2^b.$$

Here we note that it takes a form to which we can apply Lemma 6.5. Indeed, if k, which is the maximum degree of g_2 in this summation, is greater than $2n - b$, then the total degree $a + b + 2k + 1$ becomes greater than $3n + 1$, and the intersection number vanishes. We also note that the three cases in Lemma 6.5 are disjoint with each other. Therefore, we obtain,

$$\begin{aligned}
&\int h_0^a h_2^b \frac{e^k(g_0,g_1)e^k(g_1,g_2)}{kg_1} \\
=&\frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=k-n+1}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+j-n+1}} \left(\binom{i+j-n}{n-a-k+i} - \binom{i+j-n}{n-k+j} \right) \\
&+ \frac{1}{k} \sum_{i=k-n+a+b}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+k-2n+1}} \left(\binom{i+k-2n}{n-a-k+i} - \binom{i+k-2n}{0} \right) \\
&- \frac{1}{k} \sum_{i=k-n+a}^{k-n+a+b-1} \ell_i^k \ell_{k-n}^k \\
=&\frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=k-n+1}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+j-n+1}} \left(\binom{i+j-n}{n-a-k+i} - \binom{i+j-n}{n-k+j} \right) \\
&+ \frac{1}{k} \sum_{i=0}^{k-1} \ell_i^k \ell_{k-n}^k \frac{1}{2^{i+k-2n+1}} \left(\binom{i+k-2n}{n-a-k+i} - \binom{i+k-2n}{0} \right) \\
&- \frac{1}{k} \sum_{i=k-n+a}^{k-n+a+b-1} \ell_i^k \ell_{k-n}^k \\
=&\frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=k-n}^{k-1} \ell_i^k \ell_j^k \frac{1}{2^{i+j-n+1}} \left(\binom{i+j-n}{n-a-k+i} - \binom{i+j-n}{n-k+j} \right) \\
&- \frac{1}{k} \sum_{i=k-n+a}^{k-n+a+b-1} \ell_i^k \ell_{k-n}^k
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\ell_i^k\ell_j^k\frac{1}{2^{i+j-n+1}}\left(\binom{i+j-n}{n-a-k+i}-\binom{i+j-n}{n-k+j}\right)\\
&-\frac{1}{k}\ell_{k-n}^k\sum_{i=0}^{b-1}\ell_{a+k-n+i}^k.
\end{aligned}
$$

On the second equality, we used $i+k-2n<0$ when $i<k-n+a+b$, and on the fourth equality, we used $n-k+j<0$ and $(i+j-n)-(n-a-k+i)=j-2n+a+k<-3n+2k+a=-b\leq 0$ when $j<k-n$.

Finally, we compute $\int g_1^a h_2^b e^k(g_0,g_1)e^k(g_1,g_2)/(kg_1)$:

$$
\begin{aligned}
&\int g_1^a h_2^b\frac{e^k(g_0,g_1)e^k(g_1,g_2)}{kg_1}\\
=&\int\frac{1}{k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\ell_i^k\ell_j^k g_0^{k-i}g_1^{a+i+j+1}g_2^{k-j}h_2^b.
\end{aligned}
$$

At this stage, we use the fact that the expression in Lemma 6.5, which contains binomial coefficients, is anti-symmetric under interchange of α and γ. Then we obtain,

$$
\int g_1^a h_2^b\frac{e^k(g_0,g_1)e^k(g_1,g_2)}{kg_1}=\frac{1}{k}\ell_{k-n}^k\sum_{i=k-n}^{k-n+b-1}\ell_i^k=\frac{1}{k}\ell_{k-n}^k\sum_{i=0}^{b-1}\ell_{i+k-n}^k.
$$

Combining these results, we reach the final expression:

$$
\begin{aligned}
&\int(h_0^a-g_1^a)h_2^b\frac{e^k(g_0,g_1)e^k(g_1,g_2)}{kg_1}\\
=&\frac{1}{k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\ell_i^k\ell_j^k\frac{1}{2^{i+j-n+1}}\left(\binom{i+j-n}{n-a-k+i}-\binom{i+j-n}{n-k+j}\right)\\
&-\frac{1}{k}\ell_{k-n}^k\sum_{i=0}^{b-1}(\ell_{a+k-n+i}^k-\ell_{i+k-n}^k),
\end{aligned}
$$

which coincides with the r.h.s. of (4.2). $\square$

Part II

The Case of Weighted Projective Space $\mathbb{P}(1,1,1,3)$

In this part, we will construct the moduli space of quasi-maps from $\mathbb{P}^1$ to weighted projective space $\mathbb{P}(1,1,1,3)$ by using the analogy for previous technique. In addition, we compute intersection numbers and show that they appear in the coefficients of power expansion of inverse function of j-invariant. The results of this part is a joint work with Prof. Masao Jinzenji.

7 Introduction for Part II.

In the previous part, we constructed a concrete toric data of $\widetilde{Mp}_{0,2}(N,d)$. By the result of ([10]), coefficients of mirror maps of 1-parameter families of projective hypersurfaces is intersection numbers over the moduli space of quasi-maps. In the same way, we will write coefficients of inverse function of j-invariant of elliptic curve as an intersection numbers of moduli space of quasi-maps on a weighted projective space $\mathbb{P}(1,1,1,3)$.

The j-invariant is a weight zero modular function of τ which is a coordinate of complex structures of elliptic curves over $\mathbb{C}$:

$$\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau),$$

where τ is in the upper half plane. j-invariant is one of the most fundamental tools for studying elliptic curves. For example, it determines a group structure of elliptic curves.

When $q=e^{2\pi\sqrt{-1}\tau}$, Fourier expansion of j-invariant is given by

$$\begin{aligned} j(q) &= q^{-1}+744+196884q+21493760q^2+864299970q^3+20245856256q^4+\cdots \\ &=: q^{-1}+\sum_{d=1}^{\infty} j_d q^{d-1}. \end{aligned} \tag{7.1}$$

These coefficients were found to be related to ranks of irreducible representations of the Monster group (the largest sporadic simple group), which known as monstrous moonshine.

In this part, we deal with the expansion coefficients of *inverse* function of $-\log(j(\tau))$,

$$\begin{aligned} 2\pi\sqrt{-1}\tau &= -\log(j)+744j^{-1}+473652j^{-2}+451734080j^{-3}+510531007770j^{-4}+\cdots \\ &=: -\log(j)+\sum_{d=1}^{\infty} w_d j^{-d}. \end{aligned} \tag{7.2}$$

This function appeared as the mirror map of the K3 surface in $\mathbb{P}(1,1,1,3)$. The expansion coefficient j_d is reconstructed by expansion coefficient w_d via the following relation:

$$j_d=\sum_{\sigma_d\in OP_d}(-(d-1))^{l(\sigma_d)-1}\frac{1}{(l(\sigma_d))!}\prod_{j=1}^{l(\sigma_d)} w_{d_j},$$

where OP_d is set of partitions of positive integer d,

$$P_d := \{\sigma_d = (d_1, d_2, \dots, d_{l(\sigma_d)}) | \sum_{j=1}^{l(\sigma_d)} = d,\ d_1 \geq d_2 \geq \cdots \geq d_{l(\sigma_d)} \geq 1\}$$

and $l(\sigma_d)$ is length of a partition σ_d. In [10], Prof. Jinzenji conjectured that **the coefficient w_d is written as an intersection number of moduli space of quasi-maps from $\mathbb{P}^1$ to $\mathbb{P}(1,1,1,3)$ of degree d with two marked points.** The aim of this part is to prove this conjecture.

7.1 Picard-Fuchs equation for j-invariant.

It is known that the Picard-Fuchs equation for 1-parameter deformation of algebraic K3 surfaces is solved in terms of the j-invariant ([15]). Let us demonstrate it in the case of 1-parameter deformation of algebraic K3 containing $\mathbb{P}(1,1,1,3)$. We mainly refer section 5.4. of [15].

The 1-parameter deformation of K3 which we need is

$$x_1^6 + x_2^6 + x_3^6 + x_4^2 + z^{-1/6} x_1 x_2 x_3 x_4 = 0$$

embedded in $\mathbb{P}(1,1,1,3) = \{(x1, x2, x3, x4)\}$. Its Picard-Fuchs equation is given by

$$(\Theta^3 - 8z(6\Theta+1)(6\Theta+3)(6\Theta+5))f(z) = 0, \tag{7.3}$$

where Θ is a differential operator $\Theta = z\frac{d}{dz}$. It is obtained from standard techniques, for example, Griffiths-Dwork method, or A-Hypergeometric equations (see [4], etc.).

In the following, we briefly review the process of solving the above equation. Let us assume that $f(z)$ is expanded as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n(\epsilon) z^{n+\epsilon},$$

where ϵ is a parameter and $a_0(\epsilon) = 1.$. Furthermore, we assume that the coefficients $a_n(\epsilon)$ are functions along ϵ. By applying it to (7.3), we obtain

$$\sum_{n=0}^{\infty} (a_n(\epsilon)(n+\epsilon)^3 - 8(6n-5+6\epsilon)(6n-3+6\epsilon)(6n-1+6\epsilon)a_{n-1}(\epsilon))z^{n+\epsilon} = 0,$$

where $a_{-1}(\epsilon) = 0$. Since it holds for any z, we obtain

$$\begin{aligned} a_n(\epsilon) &= \frac{8(6n-5+6\epsilon)(6n-3+6\epsilon)(6n-1+6\epsilon)}{(n+\epsilon)^3} a_{n-1}(\epsilon) \\ &= \frac{\Gamma(6n+6\epsilon+1)}{\Gamma(n+\epsilon+1)^3 \Gamma(3n+3\epsilon+1)}, \end{aligned}$$

where $\Gamma(x)$ is the Gamma function. Hence, let

$$f(z,\epsilon) := \sum_{n=0}^{\infty} \frac{\Gamma(6n+6\epsilon+1)}{\Gamma(n+\epsilon+1)^3 \Gamma(3n+3\epsilon+1)} z^{n+\epsilon}.$$

We can obtain the solutions of Picard-Fuchs equation (7.3) from this expression. By setting $\epsilon = 0$, it gives a solution which is holomorphic at $z = 0$:

$$f_0(z) := \sum_{d=0}^{\infty} \frac{2^{3d}(6d-1)!!}{(d!)^3} z^d. \tag{7.4}$$

If we differentiate $f(z, \epsilon)$ by ϵ and set $\epsilon = 0$, we obtain the solution which has a log-singularity at $z = 0$:

$$f_1(z) := f_0(z)(\log(z)) + \sum_{d=0}^{\infty} (\sum_{j=1}^{3d} \frac{6}{2j-1} - \sum_{j=1}^{d} \frac{3}{j}) \frac{2^{3d}(6d-1)!!}{(d!)^3} z^d. \tag{7.5}$$

The mirror map for K3 surface embedded in $\mathbb{P}(1,1,1,3)$ is given by

$$f_1(z)/f_0(z).$$

It gives the inverse function of $-\log(j(\tau))$ ([15]).

Theorem 7.1 (B. Lian, S. T. Yau, 1996)

$$\begin{aligned} \frac{f_1(j^{-1})}{f_0(j^{-1})} &= 2\pi\sqrt{-1}\tau \\ &= -\log(j) + \sum_{d=1}^{\infty} w_d j^{-d}. \end{aligned}$$

7.2 The Goal of This Part.

In order to prove the conjecture, we explicitly construct the conpactified moduli space of degree d quasi-maps from $\mathbb{P}^1$ to $\mathbb{P}(1,1,1,3)$ with two marked points, which we denote by $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$. In section 8, we provide a toric data of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ and we prove that

Theorem 7.2 *$\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ is compact toric orbifold.*

Furthermore, we show that the Chow ring of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ is given by

Proposition 7.1 *$A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]/\mathcal{I}$, where $\mathcal{I} = (H_0^4(2H_0+H_1), H_1^4(H_0+2H_1)(2H_1+H_2)(-H_0+2H_1-H_2), \ldots, H_j^4(H_{j-1}+2H_j)(2H_j+H_{j+1})(-H_{j-1}+2H_j-H_{j+1}), \ldots, H_{d-1}^4(H_{d-2}+2H_{d-1})(2H_{d-1}+H_d)(-H_{d-2}+2H_{d-1}-H_d), H_d^4(H_{d-1}+2H_d))$.*

With these preparations, we define the intersection number that corresponds to w_d by

Definition 7.1

$$w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)} H_0^a H_1^b \cdot \frac{\prod_{i=1}^{d} e^6(H_{i-1}, H_i)}{\prod_{i=1}^{d-1} 6H_i},$$

where

$$e^6(x, y) := \prod_{j=0}^{6} ((6-j)x + y).$$

In this definition, $H_0, H_1, \ldots, H_d$ are generators of Chow rings of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.
In section 9, we prove the main result of part II.

Theorem 7.3 (Main Theorem of Part II)

$$w_d = \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d}. \tag{7.6}$$

8 The moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

In this section, we provide definition of quasi-map moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ and prove that

Theorem 8.1 *$\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ is compact toric orbifold.*

Furthermore, we compute its Chow ring:

$$A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]/\mathcal{I},$$

where $\mathcal{I} = (H_0^4(2H_0+H_1), H_1^4(H_0+2H_1)(2H_1+H_2)(-H_0+2H_1-H_2), \ldots, H_j^4(H_{j-1}+2H_j)(2H_j+H_{j+1})(-H_{j-1}+2H_j-H_{j+1}), \ldots, H_{d-1}^4(H_{d-2}+2H_{d-1})(2H_{d-1}+H_d)(-H_{d-2}+2H_{d-1}-H_d), H_d^4(H_{d-1}+2H_d))$.

In addition, we define an intersection numbers $w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d}$ on the quasi-map moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

8.1 The fan of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

The generic quasi-map from $\mathbb{P}^1$ to $\mathbb{P}(1,1,1,3)$ is given by

$$\begin{aligned}
&\mathbb{P}^1 \to \mathbb{P}(1,1,1,3)\\
&[s:t] \mapsto [f_0(s.t) : f_1(s,t) : f_2(s,t) : f_3(s,t)],
\end{aligned}$$

where

$$\begin{aligned}
f_i(s,t) &:= \sum_{j=0}^{d} a_{i,j} s^{d-j} t^j, \qquad (0 \le j \le 2),\\
f_3(s,t) &:= \sum_{j=0}^{3d} a_{3,j} s^{3d-j} t^j.
\end{aligned}$$

The following $(\mathbb{C}^*)^2$-action on $(a_{i,j})$ is induced from projective equivalence of $\mathbb{P}(1,1,1,3)$ and automorphism group of $\mathbb{P}^1$ which keeps $(1:0), (0:1) \in \mathbb{P}^1$ fixed.

$$\begin{aligned}
&(\mu,\nu)\cdot(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \ldots, a_{3,3d})\\
=&(\mu\mathbf{a}_0, \mu\nu\mathbf{a}_1, \mu\nu^2\mathbf{a}_2, \ldots, \mu\nu^d\mathbf{a}_d, \mu^3 a_{3,0}, \mu^3\nu a_{3,1}, \mu^3\nu^2 a_{3,2}, \ldots, \mu^3\nu^{3d} a_{3,3d}).
\end{aligned}$$

Here $\mathbf{a}_j \in \mathbb{C}^3$ represents a vector $(a_{0,j}, a_{1,j}, a_{2,j})$. This is equivalent to the following action:

$$\begin{aligned}
&(\mu',\nu')\cdot(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \ldots, a_{3,3d})\\
=&((\mu')^d\mathbf{a}_0, (\mu')^{d-1}(\nu')\mathbf{a}_1, (\mu')^{d-2}(\nu')^2\mathbf{a}_2, \ldots, (\nu')^d\mathbf{a}_d,\\
&(\mu')^{3d}a_{3,0}, (\mu')^{3d-1}(\nu')a_{3,1}, (\mu')^{3d-2}(\nu')^2 a_{3,2}, \ldots, (\nu')^{3d}a_{3,3d}). \qquad (8.1)
\end{aligned}$$

In the following, we construct a fan which is complete and simplicial, and realizes this $(\mathbb{C}^*)^{d+1}$-action.

Let $p_0, p_1, p_2 \in \mathbb{Z}^2$ be integer vectors given by

$$(p_0, p_1, p_2) = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Next, we introduce $(d+1)$ column vectors

$$v'_0, v'_1, \ldots, v'_d \in \mathbb{Z}^{d-1},$$

defined by,

$$(v'_0, v'_1, \ldots, v'_{d-1}, v'_d) = \begin{pmatrix} -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \end{pmatrix} \in M_{d-1,d+1}(\mathbb{Z}).$$

in the same way as the case of $\widetilde{Mp}_{0,2}(N, d)$ in Section 3.

In addition, we have to introduce the following vectors:

$$(w_0, w_1, w_2, \ldots, w_d) := \begin{pmatrix} 3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 3 \end{pmatrix} \in M_{3d+1,d+1}(\mathbb{Z})$$

Finally, we define column vectors,

$$\begin{aligned} &v_{i,j} \;\; (0 \le i \le 2, 0 \le j \le d), \\ &v_{3,j} \;\; (0 \le j \le 3d), \\ &u_k \;\; (1 \le k \le d-1) \end{aligned}$$

as follows:

for $i \neq 0$,

$$v_{i,j} = \begin{pmatrix} \mathbf{0}_2 \\ \vdots \\ p_i \\ \vdots \\ \mathbf{0}_2 \\ \mathbf{0}_{3d+1} \\ \mathbf{0}_{d-1} \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \\ \\ \end{array} \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

for $i = 0$,

$$v_{0,j} = \begin{pmatrix} \mathbf{0}_2 \\ \vdots \\ p_0 \\ \vdots \\ \mathbf{0}_2 \\ -w_j \\ v'_j \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \\ \\ \end{array} \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

for $0 \leq j \leq 3d$,

$$v_{3,j} = \begin{pmatrix} \mathbf{0}_{2(d+1)} \\ e_j^{3d+1} \\ \mathbf{0}_{d-1} \end{pmatrix} \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

and for $k = 1, \ldots, d-1$,

$$u_k = \begin{pmatrix} \mathbf{0}_{2(d+1)} \\ \mathbf{0}_{3d+1} \\ -e_k^{d-1} \end{pmatrix} \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)}$$

where $\mathbf{0}_\alpha$ is the zero vector in $\mathbb{Z}^\alpha$ and e_k^β is the k-th standard basis of $\mathbb{Z}^\beta$.

Definition 8.1 *Let*

$$\begin{aligned} P_0 &:= \{v_{0,0}, v_{1,0}, v_{2,0}, v_{3,0}, v_{3,1}\}, \\ P_d &:= \{v_{0,d}, v_{1,d}, v_{2,d}, v_{3,3d-1}, v_{3,3d}\}, \\ P_i &:= \{v_{0,i}, v_{1,i}, v_{2,i}, v_{3,3i-1}, v_{3,3i}, v_{3,3i+1}, u_i\} \ (1 \leq i \leq d-1). \end{aligned}$$

Then, we define

$$\Sigma_d$$

as a set of cones generated by the union of proper subsets (involving empty set) of $P_0, P_1, \ldots, P_d$. (A cone corresponding to empty set is $\{0\}$).

We have to show that Σ_d is a fan.

Theorem 8.2 *For all d, Σ_d is simplicial complete fan.*

In order to prove that Σ_d is a simplicial complete fan, we should check the following claim:

Lemma 8.1 *for all $v \in \mathbb{R}^{6d+2}$, there uniquely exist $a_{i,j} \in \mathbb{R}$ and $b_k \in \mathbb{R}$ that satisfy the following conditions.*

$$
\begin{aligned}
&\text{(a)} \qquad v = \sum_{i=0}^{2}\sum_{j=0}^{d} a_{i,j}v_{i,j} + \sum_{j=0}^{3d} a_{3,j}v_{3,j} + \sum_{k=1}^{d-1} b_k u_k,\\
&\text{(b)} \qquad \min(\{a_{0,0}, a_{1,0}, a_{2,0}, a_{3,0}, a_{3,1}\}) = 0,\\
&\qquad\qquad \min(\{a_{0,d}, a_{1,d}, a_{2,d}, a_{3,3d-1}, a_{3,3d}\}) = 0,\\
&\text{(c)} \qquad \min(\{a_{0,i}, a_{1,i}, a_{2,i}, a_{3,3i-1}, a_{3,3i}, a_{3,3i+1}, b_i\}) = 0 \ \ (i = 1, 2, \cdots, d-1).
\end{aligned}
$$

Proof. The following relations for $\{v_{i,j}\}, \{u_k\}$ hold:

$$
v_{0,0} + v_{1,0} + v_{2,0} + 3v_{3,0} + 2v_{3,1} + v_{3,2} - u_1 = 0, \tag{8.2}
$$

$$
\begin{aligned}
&v_{0,i} + v_{1,i} + v_{2,i} + v_{3,3i-2} + 2v_{3,3i-1} + 3v_{3,3i} + 2v_{3,3i+1} + v_{3,3i+2}\\
&\qquad - u_{i-1} + 2u_i - u_{i+1} = 0, \quad (1 \le i \le d-1)
\end{aligned} \tag{8.3}
$$

$$
v_{0,d} + v_{1,d} + v_{2,d} + v_{3,3d-2} + 2v_{3,3d-1} + 3v_{3,3d} - u_d = 0. \tag{8.4}
$$

We can easily show them by definition of Σ_d.

For all $v \in \mathbb{R}^{6d+2}$, it is clear that there uniquely exist real numbers $x_{i,j}$, $(i = 1, 2,\ 0 \le j \le d)$, $x_{3,j}$, $(0 \le j \le 3d)$, y_k, $(1 \le k \le d-1)$ such that

$$
v = \sum_{i=0}^{d}(x_{1,i}v_{1,i} + x_{2,i}v_{2,i}) + \sum_{j=0}^{3d} x_{3,j}v_{3,j} + \sum_{k=1}^{d-1} y_k u_k.
$$

Then, we obtain:

$$
\begin{aligned}
&v\\
=&v + \alpha_0(v_{0,0} + v_{1,0} + v_{2,0} + 3v_{3,0} + 2v_{3,1} + v_{3,2} - u_1)\\
&+ \alpha_d(v_{0,d} + v_{1,d} + v_{2,d} + v_{3,3d-2} + 2v_{3,3d-1} + 3v_{3,3d} - u_d)\\
&+ \sum_{i=1}^{d-1} \alpha_i(v_{0,i} + v_{1,i} + v_{2,i} + v_{3,3i-2} + 2v_{3,3i-1} + 3v_{3,3i} + 2v_{3,3i+1} + v_{3,3i+2})\\
=&(\alpha_0 + x_{0,0})v_{0,0} + (\alpha_0 + x_{1,0})v_{1,0} + (\alpha_0 + x_{2,0})v_{2,0} + (3\alpha_0 + x_{3,0})v_{3,0}\\
&\qquad + (2\alpha_0 + \alpha_1 + x_{3,1})v_{3,1}\\
&+ (\alpha_d + x_{0,d})v_{0,d} + (\alpha_d + x_{1,d})v_{1,d} + (\alpha_d + x_{2,d})v_{2,d} + (3\alpha_d + x_{3,3d})v_{3,3d}\\
&\qquad + (2\alpha_d + \alpha_{d-1} + x_{3,3d-1})v_{3,3d-1}\\
&+ \sum_{i=1}^{d-1}(\alpha_i v_{0,i} + (\alpha_i + x_{1,i})v_{1,i} + (\alpha_i + x_{2,i})v_{2,i}\\
&\qquad + (\alpha_{i-1} + 2\alpha_i + x_{3,3i-1})v_{3,3i-1} + (3\alpha_i + x_{3,3i})v_{3,3i}\\
&\qquad + (2\alpha_i + \alpha_{i+1} + x_{3,3i+1})v_{3,3i+1} + (-\alpha_{i-1} + 2\alpha_i - \alpha_{i+1} + y_i)u_i).
\end{aligned}
$$

Hence, we should show the following claim:

Claim 8.1 *A map F_d from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{d+1}$:*

$$\begin{pmatrix} \min\{\alpha_0 + z_0, 2\alpha_0 + \alpha_1 + x_{3,1}\} \\ \min\{\alpha_1 + z_1, \alpha_0 + 2\alpha_1 + x_{3,2}, 2\alpha_1 + \alpha_2 + x_{3,4}, -\alpha_0 + 2\alpha_1 - \alpha_2 + y_1\} \\ \min\{\alpha_2 + z_2, \alpha_1 + 2\alpha_2 + x_{3,5}, 2\alpha_2 + \alpha_3 + x_{3,7}, -\alpha_1 + 2\alpha_2 - \alpha_3 + y_2\} \\ \vdots \\ \min\{\alpha_{d-1} + z_{d-1}, \alpha_{d-2} + 2\alpha_{d-1} + x_{3,3d-4}, 2\alpha_{d-1} + \alpha_d + x_{3,3d-2}, \\ -\alpha_{d-2} + 2\alpha_{d-1} - \alpha_d + y_{d-1}\} \\ \min\{\alpha_d + z_d, \alpha_{d-1} + 2\alpha_d + x_{3,3d-1}\} \end{pmatrix}$$

is bijective, where $z_i := \max\{0, x_{0,i}, x_{1,i}, x_{2,i}, x_{3,3i}/3\}$.

The following two lemmas lead us to proof of the above claim:

Lemma 8.2 *$F_d(\alpha)$ is coherently oriented piecewise affine map (i.e. for any component $P \subset \mathbb{R}^{d+1}$ which $F(\alpha)$ is linear,* $\det(F|_P)$ *is positive).*

Lemma 8.3 *The* recession function *of F_d*

$$F_d^\infty(\alpha) := \begin{pmatrix} \min\{\alpha_0, 2\alpha_0 + \alpha_1\} \\ \min\{\alpha_1, \alpha_0 + 2\alpha_1, 2\alpha_1 + \alpha_2, -\alpha_0 + 2\alpha_1 - \alpha_2\} \\ \min\{\alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_2 + \alpha_3, -\alpha_1 + 2\alpha_2 - \alpha_3\} \\ \vdots \\ \min\{\alpha_{d-1}, \alpha_{d-2} + 2\alpha_{d-1}, 2\alpha_{d-1} + \alpha_d, -\alpha_{d-2} + 2\alpha_{d-1} - \alpha_d\} \\ \min\{\alpha_d, \alpha_{d-1} + 2\alpha_d\} \end{pmatrix}$$

is bijective.

When these lemmas are proven, we can use the following theorem:

Theorem 8.3 (Theorem 2.5.1. of [18]) *A coherently oriented piecewise affine function is a homeomorphism if and only if its recession function is a homeomorphism.*

Hence, $F_d(\alpha)$ also homeomorphism (i.e. bijective).

8.2 Proof of Lemma 8.2.

The coefficient matrix (8.5) of F_d is given by Table 1 in the next page.

In the Table 1, we can assume that for $1 \le i \le d-1$,

$$(\epsilon_i, \epsilon_i') = \begin{cases} (1,0),\ (0,1),\ \text{or}\ (-1,-1), & (i \notin K) \\ (0,0), & (i \in K) \end{cases}$$

and

$$(\epsilon_0, \epsilon_d) \in \{0,1\}^2,$$

where $K := \{k_1, k_2, \ldots, k_r\}$.

Table 1: **Coefficient Matrix of F_d**

$$
\left(\begin{array}{cccccc|c|ccccc|c|c|ccccc}
1+\epsilon_0 & \epsilon_0 & 0 & \cdots & 0 & 0 & 0 & & & & & & & & & & & & \\
\epsilon_1 & 2 & \epsilon_1' & \cdots & 0 & 0 & 0 & & & & & & & & & & & & \\
0 & \epsilon_2 & 2 & \cdots & 0 & 0 & 0 & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & & & & & & & & & & & \\
0 & 0 & 0 & \cdots & 2 & \epsilon_{k_1-2}' & 0 & & & & & & & & & & & & \\
0 & 0 & 0 & \cdots & \epsilon_{k_1-1} & 2 & \epsilon_{k_1-1}' & & & & & & & & & & & & \\
\hline
 & & & & & & 1 & & & & & & & & & & & & \\
\hline
 & & & & & & \epsilon_{k_1+1} & 2 & \epsilon_{k_1+1}' & \cdots & 0 & 0 & & & & & & & \\
 & & & & & & 0 & \epsilon_{k_1+2} & 2 & \cdots & 0 & 0 & & & & & & & \\
 & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & & & & & \\
 & & & & & & 0 & 0 & 0 & \cdots & 2 & \epsilon_{k_2-2}' & & & & & & & \\
 & & & & & & 0 & 0 & 0 & \cdots & \epsilon_{k_2-1} & 2 & \epsilon_{k_2-1}' & & & & & & \\
\hline
 & & & & & & & & & & & & 1 & & & & & & \\
\hline
 & & & & & & & & & & & & & \ddots & & & & & \\
\hline
 & & & & & & & & & & & & & & 2 & \epsilon_{k_r+1}' & \cdots & 0 & 0 \\
 & & & & & & & & & & & & & & \epsilon_{k_r+2} & 2 & \cdots & 0 & 0 \\
 & & & & & & & & & & & & & & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & & & & & & & & & 0 & 0 & \cdots & 2 & \epsilon_{d-1}' \\
 & & & & & & & & & & & & & & 0 & 0 & \cdots & \epsilon_d & 1+\epsilon_d
\end{array}\right) \tag{8.5}
$$

where $0 < k_1 < k_2 < \cdots < k_r = d$,
$(\epsilon_0, \epsilon_1, \epsilon_1', \ldots, \epsilon_{d-1}, \epsilon_{d-1}', \epsilon_d) \in \{0, \pm 1\}^{2d}$.

Then what we have to show is that determinant of the matrix is positive. From a formula

$$\det \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix} = \det(X)\det(Y),$$

and elementary operations of matrix, we can reduce the problem to check positivity of determinant of the following matrix:

$$B_k := \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix} \in M_{k+1,k+1},$$

where $k > 0$. We can easily compute its determinant:

$$\det B_k = 9k - 6 > 0.$$

□

8.3 Proof of Lemma 8.3.

It is clear that

$$F_d^{\infty}(t\alpha) = tF_d^{\infty}(\alpha)$$

for all $t \in \mathbb{R}_{\geq 0}$. Hence, in order to prove that F_d^{∞} is injective, we only have to show that

$$G := \pi \circ F_d^{\infty}|_{S^d} : S^d \to S^d$$

is injective. where $\pi : \mathbb{R}^{d+1} - \{0\} \to S^d$ is a projection.

Obviously G is continuous. Let $\tilde{G}$ be a smoothing of G. Note that we can take volume of smoothing locus of G as small as we like because non-smooth locus of G is measure 0. Then, $G(\alpha) \neq -\alpha$ holds for all $\alpha \in S^d$ since the diagonal elements of F_d^{∞} are all positive. Therefore,

$$H(t, \alpha) := \pi(t\alpha + (1-t)\tilde{G}(\alpha))$$

gives us a homotopy from $\tilde{G}$ to the identity mapping id_{S^d}. Thus, the mapping degree of $\tilde{G}$ is 1. Hence, $\tilde{G}$ is injective since Jacobian of $\tilde{G}$ is always positive. Accordingly, G is also injective. □

8.4 Some Properties of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$.

Since the fan Σ_d which we constructed in previous section is complete and simplicial, the corresponding toric variety X_{Σ_d} is a compact orbifold.

In this subsection, we will show that the toric variety X_{Σ_d} realizes the action (8.1), and prove that $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ is a compact toric orbifold.

First, we determine the primitive collections of the fan Σ_d.

Lemma 8.4 *The primitive collections of the fan Σ_d are*

$$P_0, P_1, \ldots, P_d.$$

It can be proven in the same way of section 3.

We introduce the following notation,

$$\mathbb{C}^{\Sigma_d(1)} = \{(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \ldots, a_{3,3d}, u_1, u_2, \ldots, u_{d-1}) \mid \mathbf{a}_i \in \mathbb{C}^3,\ a_{3,j}, u_i \in \mathbb{C}\}.$$

where $\Sigma_d(1)$ is a collection of 1-dimensional cones of Σ_d. Then, we define a subset $Z(\Sigma_d)$ of $\mathbb{C}^{\Sigma_d(1)}$ as follows.

$$Z(\Sigma_d) := \left\{ x \in \mathbb{C}^{\Sigma_d(1)} \mid \begin{array}{l} (\mathbf{a}_0, a_{3,0}, a_{3,1}) = 0, \\ (\mathbf{a}_i, a_{3,3i-1}, a_{3,3i}, a_{3,3i+1}, u_i) = 0, \quad (1 \le i \le d-1) \\ (\mathbf{a}_d, a_{3,3d-1}, a_{3,3d}) = 0, \end{array} \right\}.$$

Let $[D_{i,j}]$ (resp. $[U_k]$) be a divisor class that corresponds to 1-dimensional cone $v_{i,j}$ (resp. u_k). By the exact sequence (3.13) and definition of $v_{i,j}$ and u_k, we obtain the following relations on a Chow group $A_{\dim(X_\Sigma)-1}(X_\Sigma)$:

$$\begin{cases} [D_{0,j}] = [D_{1,j}] = [D_{2,j}]\,(0 \le j \le d), \\ [D_{3,3j}] = 3[D_{0,j}],\ (0 \le j \le d) \\ [D_{3,3j+1}] = 2[D_{0,j}] + [D_{0,j+1}],\ (0 \le j \le d-1) \\ [D_{3,3j+2}] = [D_{0,j}] + 2[D_{0,j+1}],\ (0 \le j \le d-1) \\ [U_k] = -[D_{0,k-1}] + 2[D_{0,k}] - [D_{0,k+1}]\,(1 \le k \le d-1). \end{cases} \tag{8.6}$$

Let $\lambda_i := g([D_{i,1}])$ $(g \in G := \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_{\Sigma_d}), (\mathbb{C}^*)))$. The above relation tells us that $g([U_k]) = \lambda_{k-1}^{-1}\lambda_k^2\lambda_{k+1}^{-1}$, and the $(\mathbb{C}^*)^{d+1}$-action turns out to be,

$$\begin{aligned} &(\lambda_0, \ldots, \lambda_d) \cdot (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \ldots, a_{3,3d}, u_1, u_2, \ldots, u_{d-1}) \\ =&(\lambda_0\mathbf{a}_0, \lambda_1\mathbf{a}_1, \ldots, \lambda_d\mathbf{a}_d, \lambda_0^3 a_{3,0}, \lambda_0^2\lambda_1 a_{3,1}, \lambda_0\lambda_1^2 a_{3,2}, \lambda_1^3 a_{3,3}, \lambda_1^2\lambda_2 a_{3,4}, \ldots, \lambda_d^3 a_{3,3d}, \\ &\lambda_0^{-1}\lambda_1^2\lambda_2^{-1}u_1, \lambda_1^{-1}\lambda_2^2\lambda_3^{-1}u_2, \ldots, \lambda_{d-2}^{-1}\lambda_{d-1}^2\lambda_d^{-1}u_{d-1}). \end{aligned}$$

When $u_k = 1$ for all $k = 1, 2, \ldots, d-1$, by setting $\lambda_i = (\mu')^{d-i}(\nu')^i$, we obtain the action which is similar to (8.1).

Accordingly, we can identify $X_{\Sigma_d} = \widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$. Hence, we obtain the theorem 7.2.

8.5 The Chow Ring of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ (proof of proposition 7.1).

In this subsection, we will compute the Chow ring of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$. The recipe of computation is same to section 3.

Let $H_i := [D_{0,i}]$ for $i = 0, 1, \ldots, d$, then we can prove that

$$A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]/\mathcal{I}$$

where $\mathcal{I} = (H_0^4(2H_0+H_1), H_1^4(H_0+2H_1)(2H_1+H_2)(-H_0+2H_1-H_2), \ldots, H_{d-1}^4(H_{d-2}+2H_{d-1})(2H_{d-1}+H_d)(-H_{d-2}+2H_{d-1}-H_d), H_d^4(H_{d-1}+2H_d))$.

Proof. It is easily see that

$$\mathbb{C}[x_{i,j}]/I(\Sigma_d) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]$$

by the relations (8.6).

The Stanley-Reisner ideal is

$$\begin{aligned} SR(\Sigma_d) =&(H_0^4(2H_0+H_1), H_1^4(H_0+2H_1)(2H_1+H_2)(-H_0+2H_1-H_2),\\ &\ldots, H_{d-1}^4(H_{d-2}+2H_{d-1})(2H_{d-1}+H_d)(-H_{d-2}+2H_{d-1}-H_d), H_d^4(H_{d-1}+2H_d)) \end{aligned}$$

since primitive collections are $P_0, P_1, \ldots, P_d$ as definition 8.1. □

8.6 The Intersection Numbers $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$.

We use

$$\begin{aligned} \mathrm{Vol}_d := &\left(\prod_{i=0}^{d}[D_{1,i}][D_{2,i}]\right)\cdot\left(\prod_{j=0}^{3d}[D_{3,j}]\right)\cdot\left(\prod_{k=1}^{d-1}[U_k]\right)\\ =& 3^{d+1}\left(\prod_{i=0}^{d}H_i^3\right)\cdot\left(\prod_{i=0}^{d-1}(2H_i+H_{i+1})(H_i+2H_{i+1})\right)\\ &\times\left(\prod_{k=1}^{d-1}(-H_{k-1}+2H_k-H_{k+1})\right). \end{aligned}$$

as a volume form of $A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d))$ since it corresponds to smooth point on toric variety.

Let us explain the definition 7.1 of intersection numbers of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ again:

Definition 8.2 *Let*

$$e^6(x,y) := \prod_{j=0}^{6}((6-j)x+y).$$

Then, we define the intersection number $w(\mathcal{O}_a\mathcal{O}_b)_{0,d}$ over $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ as

$$w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)} H_0^a H_d^b \cdot \frac{\prod_{i=1}^{d} e^6(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}.$$

This intersection number is a quasi-map analogue of genus 0 degree d two pointed GW invariants of K3 surface in $\mathbb{P}(1,1,1,3)$.

When we compute them, we will use the following fact: an integration of $\Omega \in A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d))$ can be computed by the following residue integration:

$$\prod_{i=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j} dz_j\right)\frac{\tilde{\Omega}}{I}$$

where $I = 3^{d+1}(\prod_{i=0}^{d}(z_i)^4)(2z_0+z_1)(\prod_{i=1}^{d-1}(z_{i-1}+2z_i)(2z_i+z_{i+1}))(z_{d-1}+2z_d)(\prod_{i=1}^{d-1}6z_j(2z_j-z_{j-1}-z_{j+1}))$, and $\oint_{C_j}$ means residue at $z_j=0$, $z_j=-\frac{z_{j-1}}{2}$, $z_j=-\frac{z_{j+1}}{2}$ and $z_j=\frac{z_{j-1}+z_{j+1}}{2}$. $\tilde{\Omega}$ is a polynomial which is obtained by turning H_i into z_i on Ω. We can easily translate this residue arguments in terms of Chow ring arguments.

9 The Mirror Map for K3 surface in $\mathbb{P}(1,1,1,3)$ and j-invariant.

In this section, we will show the following theorem which is main theorem of Part II:

Theorem 9.1 *Let w_d be the d-th coefficient of inverse function of j-invariant. Then*

$$w_d = \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d}.$$

In order to prove it, we should show that

$$\frac{f_1(e^x)}{f_0(e^x)} = x + \sum_{d=1}^{\infty} w_d e^{dx} = x + \sum_{d=1}^{\infty} \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d} e^{dx}, \tag{9.1}$$

where

$$f_0(z) := \sum_{d=0}^{\infty} \frac{2^{3d}\cdot(6d-1)!!}{(d!)^3} z^d,$$

$$f_1(z) := f_0(z)(\log(z)) + \sum_{d=0}^{\infty}\left(\sum_{j=1}^{3d} \frac{6}{2j-1} - \sum_{k=1}^{d}\frac{3}{j}\right)\frac{(6d-1)!!}{(d!)^3} z^d$$

are two solutions of Picard-Fuchs equation (7.3) as subsection 7.1.

Let us introduce the following generating functions.

$$L_0(e^x) := 1 + \sum_{d=1}^{\infty} \frac{d}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} e^{dx},$$

$$L_1(e^x) := 1 + \sum_{d=1}^{\infty} \frac{d}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d} e^{dx}.$$

The first lemma claims that $L_0(e^x)$ gives us the solution $f_0(e^x)$ of Picard-Fuchs equation (7.3).

Lemma 9.1

$$f_0(e^x) = L_0(e^x).$$

Proof. We have to prove that

$$\frac{2^{3d}(6d-1)!!}{(d!)^3} = \frac{d}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} \Leftrightarrow \frac{1}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} = \frac{1}{d}\cdot\frac{2^{3d}(6d-1)!!}{(d!)^3}$$

for all d. It follows from

$$\begin{aligned}
&\frac{1}{2\cdot 3^{d+1}} \prod_{j=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right) z_0^2 \left(\prod_{j=1}^{d}\tilde{e}(z_{j-1},z_j)\right) \\
&\times \left(\prod_{j=1}^{d-1}\frac{1}{6z_j(2z_j - z_{j-1} - z_{j+1})}\right)\frac{1}{z_d} \\
&. = \frac{1}{d}\cdot\frac{2^{3d}(6d-1)!!}{(d!)^3},
\end{aligned}$$

where

$$\tilde{e}(x,y) := 2^4 \cdot 3^2 xy \prod_{i=0}^{2}((2i+1)x + (5-2i)y)$$

Note that the integral in the LHS does not depend on order of integration if we take at $z_j = 0, \frac{z_{j-1}+z_{j+1}}{2}$ for $j = 1, 2, \ldots, d-1$ and at $z_j = 0$ for $j = 0, d$. Therefore, we integrate the LHS in ascending order of subscript j.

First, we integrate out z_0 variable. By picking up the factors containing z_0, integration is done as follows:

$$\begin{aligned}&\frac{1}{2\cdot 3^{d+1}}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(0)}}\frac{dz_0}{z_0}2^4\cdot 3^2 z_1\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\frac{1}{2z_1-z_0-z_2}\\ =&\frac{2^3\cdot 5!!}{3^{d-1}}\frac{z_1^4}{2z_1-z_2}.\end{aligned}$$

Then we integrate z_1 variable. Since the integrand is holomorphic at $z_1 = 0$, we only have to take residue at $z_1 = z_2/2$.

$$\begin{aligned}&\frac{1}{2\cdot 3^{d+1}}\prod_{j=0}^{d}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{z_2/2}}dz_1\cdot 2^3\cdot 3z_2\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2)\frac{1}{2z_2-z_1-z_3}\\ =&\frac{1}{2}\cdot\frac{2^6\cdot 11!!}{3^{d-2}\cdot(2!)^3}\cdot\frac{z_2^4}{\frac{3}{2}z_2-z_2}.\end{aligned}$$

Here, we use the identity:

$$\prod_{i=0}^{2}((2i+1)\frac{z_2}{2}+(5-2i)z_2) = (z_2)^3\prod_{i=0}^{2}\frac{11-2i}{2} = z_2^3\frac{11!!}{5!!\cdot 2^3}.$$

Integration of z_i $(i = 1, 2, \ldots, d-1)$ goes in the same way. We only have to take residue at $z_j = \frac{j}{j+1}z_{j+1}$. After finishing integration of z_{d-1}, what remains to do is the following integration.

$$\frac{1}{d}\cdot\frac{2^{3d}\cdot(6d-1)!!}{(d!)^3}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(0)}}\frac{dz_d}{z_d}.$$

Hence we obtain the assertion of the lemma. □

The following is the second lemma:

Lemma 9.2

$$\begin{aligned}&\frac{1}{2\cdot 3^{d+1}}\prod_{j=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right)z_0 z_1\left(\prod_{j=1}^{d}\tilde{e}(z_{j-1},z_j)\right)\\ &\times\left(\prod_{j=1}^{d-1}\frac{1}{6z_j(2z_j-z_{j-1}-z_{j+1})}\right)\frac{1}{z_d}\\ .=&\frac{1}{d}\cdot\frac{2^{3d}(6d-1)!!}{(d!)^3}\left(1-\frac{1}{d}+\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j}\right).\end{aligned}$$

Proof. In the same way as the proof previous lemma, we begin by integrating out z_0 variable.

$$
\begin{aligned}
&\frac{1}{2\cdot 3^{d+1}}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(0)}}\frac{dz_0}{z_0^2}2^4\cdot 3^2 z_1\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\frac{1}{2z_1-z_0-z_2}\\
=&\frac{2^3\cdot 5!!}{3^{d-1}}\left((a_1)\frac{z_1^3}{2z_1-z_2}+\frac{z_1^4}{(2z_1-z_2)^2}\right),
\end{aligned}\tag{9.2}
$$

where

$$
a_1 := \sum_{i=0}^{2}\frac{2i+1}{5-2i}.
$$

In deriving (9.2), we used the following equality:

$$
\begin{aligned}
&\frac{\partial}{\partial z_0}\left(\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\frac{1}{2z_1-z_0-z_2}\right)\\
=&\left(\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\right)\\
&\times\frac{1}{2z_1-z_0-z_2}\left(\sum_{i=0}^{2}\frac{2i+1}{(2i+1)z_0+(5-2i)z_1}+\frac{1}{2z_1-z_0-z_2}\right).
\end{aligned}
$$

Since we have another z_1 factor in the integrand, it becomes holomorphic at $z_1=0$ after integration of z_0. Hence integration of z_1 variable is done by taking residue at $z_1=z_2/2$.

$$
\begin{aligned}
&\frac{3^3\cdot 5!!}{3^{d-1}}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(z_2/2)}}dz_1(a_1\frac{1}{2z_1-z_2}+\frac{z_1}{(2z_1-z_2)^2})\\
&\times 2^4\cdot 3^2 z_2\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2)\frac{1}{2z_1-z_0-z_2}\\
=&\frac{1}{2}\cdot\frac{2^6\cdot 11!!}{3^{d-2}(2!)^3}\left(a_2\frac{z_2^4}{\frac{3}{2}z_2-z_3}+\frac{1}{4}\frac{z_2^5}{(\frac{3}{2}z_2-z_3)^2}\right).
\end{aligned}\tag{9.3}
$$

Here, a_2 is given by

$$
a_2 := a_1+\frac{1}{2\cdot 1}+\frac{1}{4}\sum_{i=0}^{2}\frac{4i+2}{11-2i}.
$$

In deriving (9.3), we used the following equality:

$$
\begin{aligned}
&\frac{\partial}{\partial z_1}\left(z_1(\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2))\frac{1}{2z_2-z_1-z_3}\right)\\
=&z_1\left(\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2)\right)\frac{1}{2z_2-z_1-z_3}\\
&\times\left(\frac{1}{z_1}+\sum_{i=0}^{2}\frac{2i+1}{(2i+1)z_1+(5-2i)z_2}+\frac{1}{2z_2-z_1-z_3}\right).
\end{aligned}
$$

Integration of z_j ($j = 2, 3, \ldots, d-2$) goes in the same way. After finishing integration of z_{d-1}, the LHS of this lemma becomes

$$\frac{1}{d} \cdot \frac{2^3 d \cdot (6d-1)!!}{(d!)^3} a_d \frac{1}{2\pi\sqrt{-1}} \oint_{C_{(0)}} \frac{dz_d}{z_d},$$

where

$$\begin{aligned} a_d &= \sum_{j=2}^{d} \frac{1}{j(j-1)} + \sum_{j=1}^{d} \frac{1}{j^2} \sum_{i=0}^{2} \frac{j(2i+1)}{6j-1-2i} \\ &= \sum_{j=2}^{d} \left(\frac{1}{j-1} - \frac{1}{j} \right) + \sum_{j=1}^{d} \sum_{i=0}^{2} \frac{(2i+1)}{(6j-1-2i)j} \\ &= 1 - \frac{1}{d} + \sum_{j=1}^{d} \sum_{i=0}^{2} \left(\frac{6}{6j-1-2i} - \frac{6}{6j} \right) \\ &= 1 - \frac{1}{d} + \sum_{j=1}^{3d} \frac{6}{2j-1} - \sum_{j=1}^{d} \frac{3}{j}. \end{aligned}$$

Integration of z_d immediately leads us to the assertion of the lemma. $\square$

The following is last lemma:

Lemma 9.3

$$\begin{aligned} &\frac{1}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,f} \cdot w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d-f} \\ =& \frac{1}{3^{d+1}} \prod_{j=0}^{d} \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_j} \frac{dz_j}{z_j^4} \right) z_0 (2z_{d-f} - z_{d-f-1} - z_{d-f+1}) \\ &\times \left(\prod_{j=1}^{d} \tilde{e}(z_{j-1}, z_j) \right) \left(\prod_{j=1}^{d-1} \frac{1}{6z_j(2z_j - z_{j-1} - z_{j+1})} \right) \frac{1}{z_d} \end{aligned}$$

for all $1 \le f \le d-1$.

Proof. Since the integrand is holomorphic at $z_{d-f} = (z_{d-f-1} + z_{d-f+1})/2$, it follows from integration of z_j's in ascending order of the subscript i. $\square$

Proof of the Main Theorem 7.3.

Let

$$\int L_1(e^x)dx = x + \sum_{i=1}^{\infty} \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d} e^{dx}$$

be primitive function of $L_1(e^x)$. Assertion of the theorem 7.3 is equivalent to the following equality:

$$f_1(e^x) = f_0(e^x) \int L_1(e^x)dx = L_0(e^x) \int L_1(e^x)dx, \tag{9.4}$$

where we used Lemma 9.1. Expanding RHS of (9.4), we obtain,

$$x \cdot L_0(e^x) + \sum_{d=1}^{\infty} \left(\frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d} + \sum_{f=1}^{d-1} \left(\frac{f}{4} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,f} \cdot w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d-f} \right) \right) e^{dx}.$$

We can easily derive the equality:$\sum_{f=1}^{d-1} f(2z_{d-f}-z_{d-f-1}-z_{d-f+1}) = d(z_1-z_0)+z_0-z_d$. Hence application of Lemma 9.3 and Lemma 9.1 to the RHS of (9.4) results in,

$$x\cdot f_0(e^x)+\sum_{d=1}^{\infty}R_d e^{dx}. \tag{9.5}$$

where

$$\begin{aligned}R_d :=& \frac{1}{2\cdot 3^{d+1}}\prod_{j=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right) z_0(d(z_1-z_0)+z_0)\\ &\times\left(\prod_{j=1}^{d}\tilde{e}(z_{j-1},z_j)\right)\left(\prod_{j=1}^{d-1}\frac{1}{6z_j(2z_j-z_{j-1}-z_{j+1})}\right)\frac{1}{z_d}.\end{aligned}$$

By combining Lemma 9.1 and 9.2, we can derive

$$R_d = \frac{2^{3d}(6d-1)!!}{(d!)^3}\left(\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j}\right).$$

Therefore, the RHS of (9.4) becomes

$$x\cdot f_0(e^x)+\sum_{d=1}^{\infty}(\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j})\frac{2^{3d}(6d-1)!!}{(d!)^3}e^{dx}. \tag{9.6}$$

The formula (7.5) tells us that it is nothing but $f_1(e^x)$. $\Box$

Remark 9.1

In the proof of main theorem of Part II, we used the residue argument. In this remark, we demonstrate how to translate it to Chow ring argument by quoting Lemma 9.1.

We have to compute

$$\int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)}\frac{H_0^2}{H_d}E_d, \tag{9.7}$$

where.

$$\begin{aligned}E_d :=& \frac{\prod_{i=1}^{d}e^6(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}\\ =& \frac{1}{3^{d+1}}\cdot\frac{\prod_{i=1}^{d}\tilde{e}(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}\cdot\left(3^{d+1}\prod_{i=0}^{d-1}(2H_i+H_{i+1})(H_i+2H_{i+1})\right).\end{aligned}$$

Since E_d has a factor $3^{d+1}\prod_{i=0}^{d-1}(2H_i+H_{i-1})(H_i+2H_{i+1})$, we can compute (9.7) as

$$\frac{H_0^2}{H_d}\frac{1}{3^{d+1}}\cdot\frac{\prod_{i=1}^{d}\tilde{e}(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}$$

in $\mathbb{C}[H_0,\ldots,H_d]/(H_0^4, H_1^4(-H_0+2H_1-H_2),\ldots,H_{d-1}^4(-H_{d-2}+2H_{d-1}-H_d),H_d^4)$.

We can prove that

$$H_0^3H_1^4H_2^4\cdots H_{j-1}^4H_j^5=\frac{j}{j+1}H_0^3H_1^4H_2^4\cdots H_j^4H_{j+1}$$

for all $j=0,1,2,\ldots,d-1$ by mathematical induction. Then, we obtain

$$\begin{aligned}&\frac{H_0^3H_1^4H_2^4\cdots H_{j-2}^4H_{j-1}^3\tilde{e}(H_{j-1},H_j)}{H_j}\\ =&2^4\cdot 3^2H_0^3H_1^4H_2^4\cdots H_{j-2}^4H_{j-1}^4(5H_{j-1}+H_j)(3H_{j-1}+3H_j)(H_{j-1}+H_j)\\ =&2^4\cdot 3^2H_0^3H_1^4H_2^4\cdots H_{j-2}^4H_{j-1}^4(\frac{5(j-1)}{j}+1)(\frac{3(j-1)}{j}+3)(\frac{j-1}{j}+5)H_j^3\\ =&2^4\cdot 3^2\frac{(6j-1)!!}{j^3\cdot(6j-7)!!}H_0^3H_1^4H_2^4\cdots H_{j-2}^4H_{j-1}^4H_j^3.\end{aligned}$$

This gives us the above successive argument.

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