Doctoral Thesis

Asymptotic behavior of random dynamical systems arising from a single neuron model

（単一の神経細胞モデルから派生する
ランダム力学系の漸近的挙動）

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Contents

1 Introduction

The non-expanding piecewise linear map on interval $[0, 1]$, known as the Nagumo-Sato (NS) model [21], is described as

$$S_{\alpha,\beta}(x) = \alpha x + \beta \pmod 1, \tag{1.1}$$

where $0 < \alpha, \beta < 1$. The NS model corresponds to a special case of Caianiello's model [7], and it describes the simplified dynamics of a single neuron. It is known that the system (1.1) shows periodic behavior of the trajectory for almost every (α, β). The transformation has one discontinuous point when $\alpha + \beta > 1$, and it leads to a complicated structure for a periodicity on the parameter space. This structure is presented graphically in Fig.1(pp.20) which shows regions in which $S_{\alpha,\beta}$ has a periodic point. An important feature of the structure is that there exists a region in which $S_{\alpha,\beta}$ has a periodic point with period $(m+n)$ between the region with period m and n. The structure is known as Arnold tongues or phase-loking regions which were originally studied in circle map models of cardiac arrhythmias [9, 12, 20], and gives a layered structure which is obtained by classifying the parameter space by the rotation number of the map. Such regions have been also observed for other models [5, 13, 25].

The first main result in this paper (Theorem 4.3) states the definition of the phase-loking region for the NS model as a Farey structure, and gives a detailed analysis of these regions. Indeed, considering properties of a rational characteristic sequence (introduced in section 3) which is one of our mathematical techniques, we calculate boundaries of each region on the parameter space in which the NS model has a periodic point for any period. Then we succeeded to find explicit parameter regions in which $S_{\alpha,\beta}$ has rational rotation number l/n for all l/n in $(0, 1)$. In 1987, Ding and Hemmer [8] studied similar parameter regions for same piecewise linear maps, and our result gives more detailed analysis of their works. These contents for the Farey structure are based on [22].

Next, we consider a perturbed dynamical system in which noise is applied to the NS model $S_{\alpha,\beta}$, that is,

$$x_{t+1} = S_{\alpha,\beta}(x_t) + \xi_t \pmod 1 \quad \text{for} \quad (\alpha, \beta) \in (0, 1)^2, \tag{1.2}$$

where $\{\xi_t\}$ are independent random variables each having the same density g satisfying $\mathrm{supp}\{g\} = [0, \theta]$ with $\theta \in [0, 1]$. We discuss two important asymptotic properties for the Markov operator [18, 19] corresponding to the model (1.2). It is well known that the Markov operator describes asymptotic behaviors of a trajectory. Especially, we focus on properties of **asymptotic**

periodicity and **asymptotic stability** which are introduced in section 2. These asymptotic behaviors for the NS model are also observed and discussed in [10, 11, 19]. Asymptotic periodicity with period 1 is equivalent to asymptotic stability, and it is important problem to classify the system as the case with period 1 or more than 1 since these two cases give different mixing properties of the system.

The second main result in this paper (Theorem 5.1) shows that a sufficient condition for which the Markov operator corresponding to the perturbed NS system has either asymptotic periodicity (period > 1) or asymptotic stability (period $= 1$). More precisely, for almost all $(\alpha, \beta) \in (0, 1)^2$, there exists a critical value $\theta_*(\alpha, \beta)$ such that the Markov operator generated by the system (1.2) displays asymptotic periodicity if θ is less than $\theta_*(\alpha, \beta)$. On the other hand, if θ is greater than $\theta_*(\alpha, \beta)$, the Markov operator shows asymptotic stability. In 1991, Provatas and Mackey [26] have already showed the same result as Theorem 5.1 in the case that rotation numbers is $1/n$. Thus, our theorem extends their result to all cases of rotation numbers l/n by using Theorem 4.3. Recently, Inoue [14] succeeded to give a sufficient condition for Markov operators associated with certain random interval maps. Also, his result can apply to random maps generalized by expanding transformations, but it is not clear for perturbed non-expanding maps. We insist that our result can apply to certain perturbed non-expanding maps.

Moreover, our third result (Theorem 5.5) leads to the asymptotic behavior of the perturbed system for the special parameters $\alpha = 1/2$, $\beta = 17/30$ and $\theta = 1/15$. Although Lasota and Mackey [19] discussed the special case numerically and they mentioned asymptotic periodicity with period 3 for the system, asymptotic stability for the system is shown by our result. Also, Kaijser [16] showed that it displays asymptotic stability in this special case. These second and third results which show asymptotic behaviors of Markov operator corresponding to the NS model are based on [23].

The organization of this paper is as follows. In §2, we prepare some mathematical tools including Markov operator and its asymptotic properties. In §3, we introduce a rational characteristic sequence and its properties which play an important role to establish our main results. In §4, a Farey structure on the parameter space for the deterministic NS model is derived. In §5, a sufficient condition for asymptotic behaviors of the Markov operator corresponding to the perturbed NS model is shown for each parameters and noise. In §6, we show the numerical illustrations of our results which describe asymptotic behaviors of the Markov operator generated by the perturbed NS model. In appendix, we prove an important inequality (Lemma

3.9) which is necessary to lead our second main result (Theorem 5.1) by using a special case of induction based on a Farey series. Finally, we attach a flowchart of this paper to help readers understand.

2 Preliminary

In this section, we prepare some mathematical tools, Markov operator, Perron-Frobenius operator and their asymptotic properties, to state our main results (Theorem 5.1). The Markov operator is an important tool which describes a density evolution generated by a system, and its convergence gives existence of an invariant measure for the system. All topics in this section are mainly based on [19].

2.1 Markov operator

Let $(X, \mathcal{A}, \mu)$ be a measure space, that is, $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a measure on $\mathcal{A}$. Moreover, assume that the measure space $(X, \mathcal{A}, \mu)$ is finite, $\mu(X) < \infty$. We first introduce the definition of Markov operator and its basic properties. Let $L^1(X) = L^1(X, \mathcal{A}, \mu)$ be the space of all integrable functions on X, i.e. $\|f\|_{L^1} := \int_X |f(x)| d\mu(x) < \infty$.

Definition 2.1. A linear operator $P : L^1(X) \to L^1(X)$ is called a **Markov operator** if P satisfies $Pf \geq 0$ and $\|Pf\|_{L^1} = \|f\|_{L^1}$ for $f \in L^1$ with $f \geq 0$.

Proposition 2.2. *If P is a Markov operator, then, for every $f \in L^1$,*

(i) $(Pf(x))^+ \leq Pf^+(x)$

(ii) $(Pf(x))^- \leq Pf^-(x)$

(iii) $|Pf(x)| \leq P|f(x)|$

(iv) $\|Pf\|_{L^1} \leq \|f\|_{L^1}$

where $f^+(x) := \max\{0, f(x)\}$ and $f^-(x) := \max\{0, -f(x)\}$.

Note that a Markov operator is a contraction operator from the property (iv) of the above proposition. Next we prepare some classical definitions in a measure theory.

Definition 2.3. We define $D(X, \mathcal{A}, \mu) := \{f \in L^1(X, \mathcal{A}, \mu) \mid f \geq 0, \|f\|_{L^1} = 1\}$. Any function $f \in D(X, \mathcal{A}, \mu)$ is called a **density**. The set $D(X, \mathcal{A}, \mu)$ is sometimes denoted by D simply.

Definition 2.4. For $f \in L^1(X, \mathcal{A}, \mu)$ with $f \geq 0$, we define a measure μ_f by

$$\mu_f(A) = \int_A f(x) d\mu(x), \quad \text{for} \quad A \in \mathcal{A}.$$

We say that μ_f is **absolutely continuous** with respect to μ, and f is called a **Radon-Nikodym derivative** of μ_f with respect to μ.

Definition 2.5. Let P be a Markov operator. Any $f \in D$ satisfying $Pf = f$ is called a **stationary density** of P.

Definition 2.6. A transformation $S : X \to X$ is **measurable** if $S^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$.

Definition 2.7. A measurable transformation $S : X \to X$ is **nonsingular** if $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ with $\mu(A) = 0$.

Under these definitions, a Perron-Frobenius operator corresponding to a nonsingular transformation can be defined as follows, which plays an important role to consider the evolution of density functions generated by nonsingular transformations.

Definition 2.8. Let $S : X \to X$ be a nonsingular transformation, the operator $P : L^1 \to L^1$ defined by

$$\int_A Pf(x) d\mu(x) = \int_{S^{-1}(A)} f(x) d\mu(x), \quad \text{for} \quad A \in \mathcal{A}, \tag{2.1}$$

is called the **Perron-Frobenius operator** corresponding to S.

Proposition 2.9. *If P is a Perron-Frobenius operator, then,*

(i) *P is a linear and a Markov operator,*

(ii) $\int_X Pf(x) d\mu(x) = \int_X f(x) d\mu(x)$,

(iii) *if P is a Perron-Frobenius operator corresponding to S, then P^n is a Perron-Frobenius operator corresponding to S^n.*

We explain the Perron-Frobenius operator from an intuitive point of view. Let S be a transformation on $[0, 1]$ and pick a large number N. We first prepare initial points on $[0, 1]$, $x_1^0, x_2^0, \cdots, x_N^0$, and then we apply the map S to each initial points, that is, one can obtain new N states denoted by

$$x_1^1 = S(x_1^0), x_2^1 = S(x_2^0), \cdots, x_N^1 = S(x_N^0).$$

Loosely speaking, we say that a function $f_0(x)$ is the density for the initial points when we have, for every interval $\Delta_0 \subset [0,1]$,

$$\int_{\Delta_0} f_0(u)du \simeq \frac{1}{N}\sum_{j=1}^{N} 1_{\Delta_0}(x_j^0),$$

where 1_Δ is a characteristic function on Δ. Similarly, the density $f_1(x)$ for the states satisfies, for $\Delta_1 \subset [0,1]$,

$$\int_{\Delta_1} f_1(u)du \simeq \frac{1}{N}\sum_{j=1}^{N} 1_{\Delta_1}(x_j^1). \quad -(*)$$

Now note that, for $\Delta_1 \subset [0,1]$, $x_j^1 \in \Delta_1$ if and only if $x_j^0 \in S^{-1}(\Delta_1)$. Then we have $1_{\Delta_1}(S(x)) = 1_{S^{-1}(\Delta)}(x)$ and may rewrite the equation $(*)$ as

$$\begin{aligned}\int_{\Delta_1} f_1(u)du &\simeq \frac{1}{N}\sum_{j=1}^{N} 1_{S^{-1}(\Delta_1)}(x_j^0). \\ &\simeq \int_{S^{-1}(\Delta_1)} f_0(u)du.\end{aligned}$$

This relationship between f_0 and f_1 implies the equation (2.1), that is, $Pf_0 = f_1$, and it tells us how the initial density f_0 is transformed into a new density f_1 by a Perron-Frobenius operator corresponding to the given map S.

In case when S is an interval map on $[a,b]$, the Perron-Frobenius operator associated with S allows us to obtain an explicit form Pf. If one takes the interval $[a,x]$ as A in the equation (2.1), then we have

$$Pf(x) = \frac{d}{dx}\int_{S^{-1}([a,x])} f(s)ds.$$

Example 2.10. (Dyadic transformation)

Consider the dyadic transformation

$$S(x) = 2x \pmod 1, \tag{2.2}$$

on a measure space $([0,1], \mathcal{A}([0,1]), \mu)$. Note that, for any interval $[0,x] \subset [0,1]$,

$$S^{-1}([0,x]) = [0, x/2] \cup [1/2, (x+1)/2].$$

Then we see that the Perron-Frobenius operator P corresponding to S is given by

$$Pf(x) = \frac{1}{2}f\left(\frac{x}{2}\right) + \frac{1}{2}f\left(\frac{x+1}{2}\right),$$

and $P1 = 1$. □

Example 2.11. (Logistic map)

Consider the logistic map defined by

$$S(x) = 4x(1-x) \tag{2.3}$$

on a measure space $([0,1], \mathcal{A}([0,1]), \mu)$. For any interval $[0,x] \subset [0,1]$, we have

$$S^{-1}([0,x]) = \left[0, \frac{1}{2} - \frac{1}{2}\sqrt{1-x}\right] \cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{1-x}, 1\right],$$

and the Perron-Frobenius operator is given by

$$Pf(x) = \frac{1}{4\sqrt{1-x}} \left\{ f\left(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}\right) + f\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right) \right\}.$$

□

Definition 2.12. A measurable transformation $S : X \to X$ is said to be **measure preserving** if $\mu(S^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$. Moreover, we say that the measure μ is **invariant** under S if S is measure preserving.

The next result gives a characterization of the invariant measure μ in terms of P.

Proposition 2.13. *Let $S : X \to X$ be a nonsingular transformation and P be the Perron-Frobenius operator corresponding to S. Then, for nonnegative $f \in L^1$, a measure μ_f defined by*

$$\mu_f(A) = \int_A Pf(x) d\mu(x), \quad \text{for} \quad A \in \mathcal{A},$$

is invariant if and only if f is a fixed point of P, that is, $Pf = f$.

In the end of Section 2.1, we recall that ergodic properties of S (e.g. ergodicity, mixing and exactness) can be represented in terms of Perron-Frobenius operator P corresponding to S.

Definition 2.14. Let $(X, \mathcal{A}, \mu)$ be a measure space and $S : X \to X$ be a nonsingular transformation. Then S is called **ergodic** if either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$ holds for every invariant set $A \in \mathcal{A}$, $S^{-1}(A) = A$.

Definition 2.15. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ be a measure preserving. Then S is called **mixing** if

$$\lim_{n\to\infty} \mu(A \cap S^{-n}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{A}.$$

Definition 2.16. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ be a measure preserving transformation such that $S(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. Then S is called **exact** if

$$\lim_{n\to\infty} \mu(S^n(A)) = 1 \quad \text{for every } A \in \mathcal{A},\ \mu(A) > 0.$$

Proposition 2.17. *([19], Theorem 4.4.1) Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ a measure preserving transformation, and P the Perron-Frobenius operator corresponding to S. Then*

(a) *S is ergodic if and only if the sequence $\{P^t f\}$ is Cesàro convergent to 1 for all $f \in D$.*

(b) *S is mixing if and only if the sequence $\{P^t f\}$ is weakly convergent to 1 for all $f \in D$.*

(c) *S is exact if and only if the sequence $\{P^t f\}$ is strongly convergent to 1 for all $f \in D$.*

Therefore, one can classify ergodicity, mixing and exactness by using the concepts of a convergence for Perron-Frobenius operator corresponding to a system.

2.2 Asymptotic behavior of Markov operator

In this subsection, we define two important properties of Markov operator P, asymptotic periodicity and asymptotic stability, and collect known sufficient conditions for P satisfying these two properties. Let $(X, \mathcal{A}, \mu)$ be a finite measure space.

Definition 2.18. $\{P^t\}$ is said to be **asymptotically periodic** if there exists an integer r, two sequences of nonnegative functions $g_i \in D$ and $h_i \in L^\infty(X)$, $i = 1, \cdots, r$, and an operator $Q : L^1(X) \to L^1(X)$ such that for every $f \in L^1(X)$, Pf can be written in the form

$$Pf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf(x),$$

where

$$\lambda_i(f) = \int_X f(x) h_i(x) \mu(dx).$$

Moreover functions g_i and operator Q satisfy the following properties;

(i) $g_i(x) g_j(x) = 0$ for all $i \neq j$;

(ii) There exists a permutation ρ of $\{1, \cdots, r\}$ such that $Pg_i = g_{\rho(i)}$.

(iii) $||P^t Qf||_{L^1} \to 0$ as $t \to \infty$ for every $f \in L^1(X)$.

Remark 2.19. When $\{P^t\}$ is an asymptotically periodic Markov operator, then P has a stationary density f_*

$$f_*(x) = \frac{1}{r}\sum_{i=1}^{r} g_i(x).$$

Definition 2.20. $\{P^t\}$ is said to be **asymptotically stable** if there exists a unique $f_* \in D$ such that $Pf_* = f_*$ and $\lim_{t\to\infty} ||P^t f - f_*||_{L^1} = 0$ for every $f \in D$.

Proposition 2.21. *$\{P^t\}$ is asymptotically stable if and only if $\{P^t\}$ is asymptotically periodic with $r = 1$.*

Next examples satisfy either asymptotic stability or periodicity for Perron-Frobenius operator corresponding to the system.

Example 2.22. (Dyadic transformation)

Let P be a Perron-Frobenius operator corresponding to the dyadic transformation (2.2), then $\{P^t\}$ is asymptotically stable. Indeed, P has stationary density $f_\star = 1_{[0,1]}$ and one can show that $P^t f$ converges to $f_\star$ for any $f \in D$. □

Example 2.23. (Logistic map)

Let P be a Perron-Frobenius operator corresponding to the logistic map (2.3), then $\{P^t\}$ is asymptotically stable. Indeed, one can show that P has stationary density $f_\star(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ and $P^t f$ converges to $f_\star$ for any $f \in D$. □

Example 2.24. (Generalized tent map)

Consider the generalized tent map defined by, for a parameter $a \in (1, 2]$,

$$S(x) = \begin{cases} ax & \text{for} \quad x \in [0, 1/2] \\ a(1-x) & \text{for} \quad x \in [1/2, 1]. \end{cases}$$

This map was considered by Provatas and Mackey [27] and they showed that $\{P^t\}$ is asymptotically periodic for the Perron-Frobenius operator P corresponding to this map S. More precisely, when the parameter a satisfies

$$2^{1/2^{n+1}} < a \le 2^{1/2^n} \quad \text{for} \quad n = 0, 1, 2, \cdots,$$

then $\{P^t\}$ is asymptotically periodic with the period 2^n. □

In order to establish our main Theorem 5.1, we use the next two results which gives us sufficient conditions for asymptotic periodicity and stability.

Definition 2.25. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\mu(X) < \infty$. A Markov operator P is called **constrictive** if there exists a $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $t_0(f)$ for which

$$\int_E P^t f(x)\mu(dx) \leq \kappa \quad \text{for all } t \geq t_0(f) \quad \text{and } E \text{ with } \mu(E) \leq \delta.$$

Proposition 2.26. *([19], Theorem 5.3.1) If P is a constrictive Markov operator, then $\{P^t\}$ is asymptotically periodic.*

Proposition 2.27. *([19], Theorem 5.6.1) Let P be a constrictive Markov operator. Assume there is a set $A \subset X$ of nonzero measure, $\mu(A) > 0$, with the property that for every $f \in D$ there is an integer $t_0(f)$ such that $P^t f(x) > 0$ for almost all $x \in A$ and all $t > t_0(f)$. Then $\{P^t\}$ is asymptotically stable.*

In the end of Section 2.2, we mention relations between asymptotically periodic and some ergodic properties as follows.

Proposition 2.28. *([19], Theorem 5.5.1) Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $P : L^1 \to L^1$ a constrictive Markov operator. Then P is ergodic if and only if the permutation ρ of $\{1, \cdots, r\}$ (see Definition 2.18) is cyclical, that is, ρ has no invariant subset.*

Proposition 2.29. *([19], Theorem 5.5.2 and Theorem 5.5.3) Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $P : L^1 \to L^1$ a constrictive Markov operator. Then following are equivalent.*

$$r = 1 \Longleftrightarrow P \text{ is mixing} \Longleftrightarrow P \text{ is exact},$$

where r is defined in Definition 2.18.

Surprisingly, the mixing and exactness are equivalent in the class of a constrictive Markov operator from the above proposition. These propositions suggest that it is important to classify the case $r > 1$ or $r = 1$ of asymptotic periodicity.

2.3 Markov operator for perturbed system

A final topic of section 2 is the result by Iwata and Ogihara[15] which shows asymptotic periodicity of a Markov operator for a perturbed system on interval $[0, 1]$. Consider a finite measure space $([0, 1], \mathcal{A}([0, 1]), \mu)$ where μ is the Lebesgue measure on $[0, 1]$ and a process on $[0, 1]$ defined by

$$x_{t+1} = T(x_t) + \xi_t \pmod 1, \tag{2.4}$$

where $T : [0,1] \to [0,1]$ is measurable and $\xi_0, \xi_1, \cdots$ are independent random variables each having the same density g. The Markov operator $\overline{P} : L^1([0,1]) \to L^1([0,1])$ of this system is defined by

$$\overline{P}f(x) = \int_{[0,1]} f(y) \sum_{i=0}^{1} g(x - T(y) + i)dy \quad \text{for} \quad f \in L^1. \tag{2.5}$$

Proposition 2.30. *([15], Theorem 2.8) The Markov operator $\overline{P} : L^1([0,1]) \to L^1([0,1])$ defined by (2.5) is constrictive, and this means that, the sequence $\{\overline{P}^t\}$ is asymptotically periodic.*

From this proposition, a Markov operator for such perturbed system always has asymptotic periodicity for any measurable transformation T, which means that an additive noise induces asymptotic periodicity for the system. The fact plays a key role to prove Theorem 5.1.

3 Rational characteristic sequence

A rational characteristic sequence is known as mechanical words, rotation words or Christoffel words [4] and good sequence in [22], and if l/n is replaced by an irrational number, then it is known as Sturmian words or characteristic sequence [2, 3, 6]. Two series of functions $A_i(\alpha)$ and $F_{n,l}(i)$ and numbers $P(A_i)$ (see Eq.(3.5), (3.18) and (3.6)) generated by the sequence are important tools to prove Farey structure on a parameter space of NS model (Theorem 4.3). Preparing some properties of the sequence, we will show the most important inequality (Lemma 3.9) for the function $F_{n,l}(i)$, which is used to prove Theorem 4.7 and consequently leads our main result (Theorem 5.1). We first introduce the definition and some useful properties of the rational characteristic sequence. Let $Pr(n)$ be a set of numbers $l < n$ satisfying l and n are mutually prime, $Pr(n) := \{l < n \mid GCD(n, l) = 1\}$ for each $n \in \mathbb{N}$.

3.1 Basic properties of rational characteristic sequence

Definition 3.1. For $n \in \mathbb{N}$ and $l \in Pr(n)$, we define a sequence $\{k_i\}_{i\in\mathbb{Z}}$ with $k_i \in \{0, 1\}$ for all $i \in \mathbb{Z}$ by

$$k_i := \left[\frac{(i+1)l}{n}\right] - \left[\frac{il}{n}\right] \quad \text{for} \quad i \in \mathbb{Z}, \tag{3.1}$$

where $[x]$ is the integer part of x. The sequence $\{k_i\}_{i\in\mathbb{Z}}$ is called a **Rational characteristic sequence** with respect to (n, l). Obviously, $k_0 = 0$ and $k_{n-1} = 1$ always hold.

In this paper, if we write (n, l), then n and l always satisfy $n \in \mathbb{N}_{\geq 2}$ and $l \in Pr(n)$.

Proposition 3.2. *([22], Proposition 2.2) Let $\{k_i\}_{i\in\mathbb{Z}}$ be a rational characteristic sequence with respect to (n, l). We then have the following properties.*

(i) $k_{i\pm n} = k_i \quad \text{for} \quad i \in \mathbb{Z},$

(ii) $k_{n-1-i} = k_i \quad \text{for} \quad i \in \mathbb{Z},\ i \notin n\mathbb{Z}, n\mathbb{Z} - 1,$

(iii) $k_{i-\hat{l}} = k_i \quad \text{for} \quad i \in \mathbb{Z},\ i \notin n\mathbb{Z}, n\mathbb{Z} - 1,$

where $\hat{l} = \min\{t \in \mathbb{N} \mid tl = 1 \pmod{n}\}$.

Proof. The property (i) follows from next calculations;

$$k_{i\pm n} = \left[\frac{(i\pm n+1)l}{n}\right] - \left[\frac{(i\pm n)l}{n}\right] = \left[\pm l + \frac{(i+1)l}{n}\right] - \left[\pm l + \frac{il}{n}\right] = k_i.$$

The property (ii) follows from the well-known property of Gaussian part for real numbers, $[a]+[-a]=-1$ if $a\notin\mathbb{Z}$, and the following calculations;

$$\begin{aligned} k_i - k_{n-1-i} &= \left(\left[\frac{(i+1)l}{n}\right] - \left[\frac{il}{n}\right]\right) - \left(\left[\frac{(n-i)l}{n}\right] - \left[\frac{(n-i-1)l}{n}\right]\right) \\ &= \left(\left[\frac{(i+1)l}{n}\right] + \left[-\frac{(i+1)l}{n}\right]\right) - \left(\left[\frac{il}{n}\right] + \left[-\frac{il}{n}\right]\right). \end{aligned}$$

To prove the property (iii), use an equality $\hat{l}l - [\frac{\hat{l}l}{n}]n = 1$. Put $N = [\frac{il}{n}]$. From the next calculation, we see that if $k_{i-\hat{l}} = 1$ and $k_i = 0$, $[\frac{(i+1)l}{n}] = [\frac{il}{n}] = [\frac{(i+1)l-1}{n}] = N$ and $[\frac{il-1}{n}] = N-1$, then an equality $il - nN = 0$ holds, and if $k_{i-\hat{l}} = 0$ and $k_i = 1$, $[\frac{il}{n}] = [\frac{(i+1)l-1}{n}] = [\frac{(il-1}{n}] = N$ and $[\frac{(i+1)l}{n}] = N+1$, then an equality $il - nN = n-l$ holds.

$$\begin{aligned} k_{i-\hat{l}} &= \left[\frac{(i-\hat{l}+1)l}{n}\right] - \left[\frac{(i-\hat{l})l}{n}\right] \\ &= \left[\frac{(i+1)l - [\frac{\hat{l}l}{n}]n - 1}{n}\right] - \left[\frac{il - [\frac{\hat{l}l}{n}]n - 1}{n}\right] \\ &= \left[\frac{(i+1)l-1}{n}\right] - \left[\frac{il-1}{n}\right]. \end{aligned}$$

However, we can show that $il - nN = 0$ if and only if $i = 0 \pmod{n}$, and $il - nN = n-l$ if and only if $i = n-1 \pmod{n}$. These imply the property (iii). □

Proposition 3.3. *([22], Proposition 2.3) Let $\{k_i\}_{i\in\mathbb{Z}}$ be a rational characteristic sequence with respect to (n,l) and $\{k'_i\}_{i\in\mathbb{Z}}$ be another rational characteristic sequence with respect to (n',l'). If $\frac{l}{n} < \frac{l'}{n'}$ and $nl' - n'l = 1$, then the sequence $\{\hat{k}_i\}_{i\in\mathbb{Z}}$ defined by*

$$\hat{k}_i := \begin{cases} k_i & \text{for} \quad i = 0,\cdots,n-1 \\ k'_{i-n} & \text{for} \quad i = n,\cdots,n+n'-1 \end{cases} \tag{3.2}$$

and $\hat{k}_m := \hat{k}_i$ if $m = i + t(n+n')$ with $i = 0,\cdots,n+n'-1$ and $t\in\mathbb{Z}\backslash\{0\}$, is the rational characteristic sequence with respect to $(n+n', l+l')$.

Proof. First we show the equality $\hat{k}_i = k_i$ for $i = 0,\cdots,n-1$. By our assumption $nl' - n'l = 1$, we have that

$$\begin{aligned} \hat{k}_i - k_i &= \left(\left[\frac{(i+1)(l+l')}{n+n'}\right] - \left[\frac{i(l+l')}{n+n'}\right]\right) - \left(\left[\frac{(i+1)l}{n}\right] - \left[\frac{il}{n}\right]\right) \\ &= \left[\frac{(i+1)l}{n} + \frac{i+1}{n(n+n')}\right] - \left[\frac{il}{n} + \frac{i}{n(n+n')}\right] - \left[\frac{(i+1)l}{n}\right] + \left[\frac{il}{n}\right]. \end{aligned}$$

Put $N = [\frac{il}{n}]$, then we can see that $N < l$ and have either $[\frac{(i+1)l}{n}] = N$ or $[\frac{(i+1)l}{n}] = N + 1$.
(I) In case of $[\frac{(i+1)l}{n}] = N$, one has either $[\frac{(i+1)(l+l')}{n+n'}] = N$ or $[\frac{(i+1)(l+l')}{n+n'}] = N+1$. If we assume $[\frac{(i+1)(l+l')}{n+n'}] = N+1$, then one can obtain that $[\frac{i(l+l')}{n+n'}] = N$ and the following inequality;

$$\frac{l}{n} < \frac{N+1}{i+1} \leq \frac{l+l'}{n+n'}. \tag{3.3}$$

Thus, because of $N < l$ and $0 \leq i \leq n-1$, i which satisfies the inequality (3.3) does not exist by the property of Farey series (See Theorem5.3 of [1]). Then we can see that $[\frac{(i+1)(l+l')}{n+n'}] = N$ and $[\frac{i(l+l')}{n+n'}] = N$. This implies that the equality $\hat{k}_i = k_i$ holds for $i = 0, \cdots, n-1$.
(II) In case of $[\frac{(i+1)l}{n}] = N+1$, one has either $[\frac{i(l+l')}{n+n'}] = N$ or $[\frac{i(l+l')}{n+n'}] = N+1$. If we assume $[\frac{i(l+l')}{n+n'}] = N+1$, then one can obtain that $[\frac{(i+1)(l+l')}{n+n'}] = N+1$ and the following inequality;

$$\frac{l}{n} < \frac{N+1}{i} \leq \frac{l+l'}{n+n'}. \tag{3.4}$$

From the same reasons with (I), this is contradiction. Then we can see that $[\frac{i(l+l')}{n+n'}] = N$ and $[\frac{(i+1)(l+l')}{n+n'}] = N+1$. This implies that the equality $\hat{k}_i = k_i$ holds for $i = 0, \cdots, n-1$.

Next, we show the equality $\hat{k}_i = k'_i$ for $i = 0, \cdots, n'-1$ can be shown by the same way. First one has

$$\begin{aligned}
\hat{k}_{i+n} &- k'_i \\
= &\left(\left[\frac{(i+n+1)(l+l')}{n+n'}\right] - \left[\frac{(i+n)(l+l')}{n+n'}\right]\right) - \left(\left[\frac{(i+1)l'}{n'}\right] - \left[\frac{il'}{n'}\right]\right) \\
= &\left[\frac{(i+1)l'}{n'} + \frac{n'-(i+1)}{n'(n+n')}\right] - \left[\frac{il'}{n'} + \frac{n'-i}{n'(n+n')}\right] - \left[\frac{(i+1)l'}{n'}\right] + \left[\frac{il'}{n'}\right].
\end{aligned}$$

Put $N = [\frac{il'}{n'}]$, then we can see that $N < l'$ and have either $[\frac{(i+1)l'}{n'}] = N$ or $[\frac{(i+1)l'}{n'}] = N + 1$. In case of $[\frac{(i+1)l'}{n'}] = N$, one can see that $[\frac{(i+n+1)(l+l')}{n+n'}] = N$ and $[\frac{(i+n)(l+l')}{n+n'}] = N$. In case of $[\frac{(i+1)l'}{n'}] = N + 1$, one can show that $[\frac{(i+n+1)(l+l')}{n+n'}] = N + 1$ and $[\frac{(i+n)(l+l')}{n+n'}] = N$. These imply that the equality $\hat{k}_{i+n} = k'_i$ holds for $i = 0, \cdots, n'-1$.

The inequality $\hat{k}_m := \hat{k}_i$ if $m = i + t(n+n')$ with $i = 0, \cdots, n+n'-1$ and $t \in \mathbb{Z}\backslash\{0\}$ is obvious because of the property of Gaussian. From these observations, We complete the proof. □

Example 3.4. The rational characteristic sequence for (n, l) with $n = 2, 3, 4, 5$ are followings. Here we write only $k_0, \cdots, k_{n-1}$ with the bracket $\langle\cdot\rangle$ since their 01 words are repeated.

$(2,1)$: $\langle 01\rangle$, $(3,1)$: $\langle 001\rangle$, $(3,2)$: $\langle 011\rangle$, $(4,1)$: $\langle 0001\rangle$, $(4,3)$: $\langle 0111\rangle$,
$(5,1)$: $\langle 00001\rangle$, $(5,2)$: $\langle 00101\rangle$, $(5,3)$: $\langle 01011\rangle$, $(5,4)$: $\langle 01111\rangle$,

One can immediately see that the sequence for $(5,2)$, $\langle 00101\rangle$, can be made the sequence of $(3,1)$ and $(2,1)$, that is, $\langle 00101\rangle = \langle 001\rangle + \langle 01\rangle$. Similarly, we have following examples.

$(7,3) = (5,2) + (2,1)$: $\langle 00101\rangle + \langle 01\rangle = \langle 0010101\rangle$
$(12,5) = (5,2) + (3,7)$: $\langle 00101\rangle + \langle 0010101\rangle = \langle 001010010101\rangle$ □

3.2 Properties of functions $\{A_i(\alpha)\}_{i=0}^{n-1}$

Define two sequences $\{A_i(\alpha)\}_{i=0}^{n-1}$ and $\{P(A_i)\}_{i=0}^{n-1}$ of length n as follows.

$$\begin{pmatrix} A_0(\alpha) \\ A_1(\alpha) \\ A_2(\alpha) \\ \vdots \\ A_{n-1}(\alpha) \end{pmatrix} := \frac{1}{1-\alpha^n} \begin{pmatrix} k_0 & k_1 & k_2 & \cdots & k_{n-2} & k_{n-1} \\ k_1 & k_2 & k_3 & \cdots & k_{n-1} & k_0 \\ k_2 & k_3 & & \iddots & k_0 & k_1 \\ \vdots & \iddots & & \iddots & \iddots & \vdots \\ k_{n-1} & k_0 & k_1 & \cdots & & k_{n-2} \end{pmatrix} \begin{pmatrix} \alpha^{n-1} \\ \alpha^{n-2} \\ \alpha^{n-3} \\ \vdots \\ 1 \end{pmatrix} \tag{3.5}$$

$$\begin{pmatrix} P(A_0) \\ P(A_1) \\ P(A_2) \\ \vdots \\ P(A_{n-1}) \end{pmatrix} := \begin{pmatrix} k_0 & k_1 & k_2 & \cdots & k_{n-2} & k_{n-1} \\ k_1 & k_2 & k_3 & \cdots & k_{n-1} & k_0 \\ k_2 & k_3 & & \iddots & k_0 & k_1 \\ \vdots & \iddots & & \iddots & \iddots & \vdots \\ k_{n-1} & k_0 & k_1 & \cdots & & k_{n-2} \end{pmatrix} \begin{pmatrix} n-1 \\ n-2 \\ n-3 \\ \vdots \\ 0 \end{pmatrix} \tag{3.6}$$

where $\{k_m\}_{m=0}^{n-1}$ is a rational characteristic sequence with respect to (n,l). We can rewrite these sequences for each $i = 0, \cdots, n-1$ as following forms.

$$A_i(\alpha) = \frac{1}{1-\alpha^n}\left(\sum_{m=0}^{i-1} k_m \alpha^{i-m-1} + \sum_{m=i}^{n-1} k_m \alpha^{n+i-1-m}\right) \tag{3.7}$$

$$P(A_i) = \sum_{m=0}^{i-1} k_m (i-m-1) + \sum_{m=i}^{n-1} k_m (n+i-1-m) \tag{3.8}$$

$$= n\sum_{m=i}^{n-1} k_m - \sum_{m=0}^{n-1} k_m m + (i-1)l \tag{3.9}$$

Here we used the relation $\sum_{m=0}^{n-1} k_m = l$ which is immediately derived by definition of the sequence. We next prepare a few lemmas which give some properties of sequences $\{A_i(\alpha)\}_{i=0}^{n-1}$ and $\{P(A_i)\}_{i=0}^{n-1}$. These arguments are useful in §4 for the proof of existence of Farey sturucture for our NS model (Theorem 4.3).

Lemma 3.5.

(i) $P(A_i) \neq P(A_j) \quad \text{for} \quad i \neq j.$

(ii) $0 \leq P(A_i) - P(A_0) < n \quad \text{for} \quad i = 0, \cdots, n-1.$

(iii) $P(A_0) = \frac{1}{2}(n-1)(l-1)$ and $\sum_{m=0}^{n-1} k_m m = \frac{1}{2}(n-1)(l+1)$ holds.

(iv) If $k_i = 0$, then $P(A_i) - P(A_0) < n - l$ for $i = 0, \cdots, n-1$.
If $k_i = 1$, then $P(A_i) - P(A_0) \geq n - l$ for $i = 0, \cdots, n-1$.

Proof. (i) Assume $P(A_i) = P(A_j)$ and $i < j$. We then have

$$\frac{l}{n} = \frac{\sum_{m=i}^{j-1} k_m}{j-i}.$$

This leads to a contradiction to $(n, l) = 1$ and $j - i < n$.

(ii) By the definition of $P(A_i)$, we can rewrite as

$$P(A_i) = P(A_0) + il - n \sum_{m=0}^{i-1} k_m.$$

Thus, we should prove $0 \leq il - n \sum_{m=0}^{i-1} k_m < n$. This can be changed to the next inequality.

$$\sum_{m=0}^{i-1} k_m \leq \frac{il}{n} < 1 + \sum_{m=0}^{i-1} k_m \tag{3.10}$$

We can prove this inequality (3.10) from the relation $\sum_{m=0}^{i-1} k_m = \sum_{m=0}^{i-1}([\frac{(m+1)l}{n}] - [\frac{ml}{n}]) = [\frac{il}{n}]$.

(iii) From the equation (3.9), we have

$$\begin{aligned} \sum_{i=0}^{n-1} P(A_i) &= n \sum_{i=0}^{n-1} \sum_{m=i}^{n-1} k_m - n \sum_{m=0}^{n-1} k_m m + \sum_{i=0}^{n-1} (i-1)l \\ &= n \sum_{m=0}^{n-1} k_m + \frac{l}{2} n(n-3) = \frac{l}{2} n(n-1). \end{aligned} \tag{3.11}$$

On the other hand, from the properties (i) and (ii) of this Lemma 3.5, we see that $P(A_i) - P(A_0)$ are distinct integers from 0 to $n-1$, thus we have

$$\sum_{i=0}^{n-1} P(A_i) = nP(A_0) + \frac{1}{2} n(n-1). \tag{3.12}$$

By the equations (3.11) and (3.12), we can see that $P(A_0) = \frac{1}{2}(n-1)(l-1)$ and $\sum_{m=0}^{n-1} k_m m = \frac{1}{2}(n-1)(l+1)$.

(iv) From the property (ii) of this Lemma 3.5, we should only show that $P(A_i) - P(A_0) < n - l$ if $k_i = 0$ for $i = 0, \cdots, n-1$. By the equation (3.10),

$$\left[\frac{il}{n}\right] < 1 + \sum_{m=0}^{i-1} k_m.$$

Assume $k_i = 0$, since $[\frac{il}{n}] = [\frac{(i+1)l}{n}]$ then we have $[\frac{(i+1)l}{n}] < 1 + \sum_{m=0}^{i-1} k_m$. Since $[\frac{(i+1)l}{n}]$ and $1 + \sum_{m=0}^{i-1} k_m$ are different integers, we obtain

$$\frac{(i+1)l}{n} \quad < \quad 1 + \sum_{m=0}^{i-1} k_m.$$

Thus,

$$il - n\sum_{m=0}^{i-1} k_m \quad < \quad n - l.$$

Since a left-hand side is equal to $P(A_i) - P(A_0)$, then $P(A_i) - P(A_0) < n - l$ holds.

□

Lemma 3.6. *For all $\alpha \in (0,1)$ and $i \neq j$*

$$A_i(\alpha) < A_j(\alpha) \quad \text{if and only if} \quad P(A_i) > P(A_j). \tag{3.13}$$

Proof. We can write

$$P(A_i) = P(A_0) + il - \left[\frac{il}{n}\right] n \quad \text{for} \quad i = 0, \cdots, n-1.$$

Define a map $\sigma : \{0, 1, \cdots, n-1\} \to \{0, 1, \cdots, n-1\}$ by $\sigma(i) := il - [\frac{il}{n}]n$. Note that σ is a permutation. Since σ is one to one, there exists an inverse map $\phi(i) := \sigma^{-1}(i)$. Now we have that for $i = 0, \cdots, n-1$,

$$P(A_{\phi(i+1)}) - P(A_{\phi(i)}) = \sigma(\phi(i+1)) - \sigma(\phi(i)) = 1 > 0$$

This implies that the inequality $P(A_{\phi(i+1)}) > P(A_{\phi(i)})$ holds for $i = 0, \cdots, n-2$. Next we should show that the inequality $A_{\phi(i+1)}(\alpha) > A_{\phi(i)}(\alpha)$ holds for all $\alpha \in (0,1)$ and $i = 0, \cdots, n-2$, which implies $A_i(\alpha) < A_j(\alpha)$ if and only if $P(A_i) > P(A_j)$ for all $\alpha \in (0,1)$ and $i \neq j$ since ϕ is one to one.

Now we can see $\sigma(\hat{l}) = 1$ and $\phi(i) = i\hat{l} \pmod{n}$. From the equation (3.7),

$$\begin{aligned}
&(1-\alpha^n)(A_{\phi(i)}(\alpha) - A_{\phi(i+1)}(\alpha)) \\
= &\sum_{m=0}^{\phi(i)-1} k_m \alpha^{\phi(i)-m-1} + \sum_{m=\phi(i)}^{n-1} k_m \alpha^{n+\phi(i)-m-1} \\
&- \sum_{m=0}^{\phi(i+1)-1} k_m \alpha^{\phi(i+1)-m-1} - \sum_{m=\phi(i+1)}^{n-1} k_m \alpha^{n+\phi(i+1)-m-1}.
\end{aligned}$$

By the property (i) of Proposition 3.2,

$$\begin{aligned}
= &\sum_{m=0}^{\phi(i)-1} k_{\phi(i)-m-1}\alpha^m + \sum_{m=\phi(i)}^{n-1} k_{\phi(i)-m-1}\alpha^m \\
&- \sum_{m=0}^{\phi(i+1)-1} k_{\phi(i+1)-m-1}\alpha^m - \sum_{m=\phi(i+1)}^{n-1} k_{\phi(i+1)-m-1}\alpha^m \\
= &\sum_{m=0}^{n-1} k_{\phi(i)-m-1}\alpha^m - \sum_{m=0}^{n-1} k_{\phi(i+1)-m-1}\alpha^m.
\end{aligned}$$

Since $\phi(i+1) = \phi(i) + \hat{l}$ or $\phi(i) + \hat{l} - n$, then

$$\begin{aligned}
= &\sum_{m=0}^{n-1} \left(k_{\phi(i)-m-1} - k_{\phi(i)+\hat{l}-m-1}\right)\alpha^m \\
= &\sum_{m'=\phi(i)+\hat{l}-n}^{\phi(i)+\hat{l}-1} \left(k_{m'-\hat{l}} - k_{m'}\right)\alpha^{\phi(i)+\hat{l}-m'-1} \quad - (*).
\end{aligned}$$

If $0 \in \{\phi(i) + \hat{l} - n, \cdots, \phi(i) + \hat{l} - 1\}$, from the property (iii) of Proposition 3.2,

$$\begin{aligned}
(*) &= (k_{-\hat{l}} - k_0)\alpha^{\phi(i)+\hat{l}-1} + (k_{-\hat{l}-1} - k_{-1})\alpha^{\phi(i)+\hat{l}} \\
&= \alpha^{\phi(i)+\hat{l}-1}(1-\alpha) > 0.
\end{aligned}$$

If $0 \notin \{\phi(i) + \hat{l} - n, \cdots, \phi(i) + \hat{l} - 1\}$,

$$\begin{aligned}
(*) &= (k_{-\hat{l}-n} - k_{-n})\alpha^{\phi(i)+\hat{l}+n-1} + (k_{-\hat{l}-1-n} - k_{-1-n})\alpha^{\phi(i)+\hat{l}+n} \\
&= \alpha^{\phi(i)+\hat{l}-1+n}(1-\alpha) > 0.
\end{aligned}$$

Then we complete the proof. □

Lemma 3.7. *We can write*

$$\min_{\{i|k_i=0\}} A_i(\alpha) = \frac{1}{1-\alpha^n}\sum_{i=1}^{n-1} k_i\alpha^{i-1} \quad \text{and} \tag{3.14}$$

$$\max_{\{i|k_i=1\}} A_i(\alpha) = \frac{1}{1-\alpha^n}\left(\sum_{i=1}^{n-1} k_i\alpha^{i-1} - \alpha^{n-2} + \alpha^{n-1}\right). \tag{3.15}$$

Proof. First, we show that

$$A_{\hat{l}}(\alpha) = \frac{1}{1-\alpha^n}\sum_{i=1}^{n-1} k_i\alpha^{i-1} \quad \text{and} \tag{3.16}$$

$$A_{n-1}(\alpha) = \frac{1}{1-\alpha^n}\left(\sum_{i=1}^{n-1} k_i\alpha^{i-1} - \alpha^{n-2} + \alpha^{n-1}\right). \tag{3.17}$$

We denote $(a_1, \cdots, a_n)\cdot(b_1, \cdots, b_n)$ for $\sum_{i=1}^{n} a_i b_i$. To calculate them, we use the properties of Proposition 3.2 and $k_{n-\hat{l}-1} = 0$ and $k_{n-\hat{l}} = 1$ as follows;

$$\begin{aligned}
& (1-\alpha^n)A_{\hat{l}}(\alpha) \\
= \; & (k_{n-\hat{l}-1}, k_{n-\hat{l}}, \cdots, k_{n-1}, k_0, \cdots, k_{n-2-\hat{l}})\cdot(\alpha^{n-1}, \alpha^{n-2}, \cdots, \alpha, 1) \\
= \; & (k_{n-\hat{l}-1}, k_{n-\hat{l}}, k_{1-\hat{l}}, k_{2-\hat{l}}, \cdots, k_{n-2-\hat{l}})\cdot(\alpha^{n-1}, \alpha^{n-2}, \cdots, \alpha, 1) \\
= \; & (k_0, k_{n-1}, k_1, k_2, \cdots, k_{n-2})\cdot(\alpha^{n-1}, \alpha^{n-2}, \cdots, \alpha, 1) \\
= \; & (k_0, k_{n-1}, k_{n-2}, \cdots, k_2, k_1)\cdot(\alpha^{n-1}, \alpha^{n-2}, \alpha^{n-3}, \cdots, \alpha, 1) \\
= \; & (k_{n-1}, k_{n-2}, \cdots, k_2, k_1)\cdot(\alpha^{n-2}, \alpha^{n-3}, \cdots, \alpha, 1) \\
= \; & (k_1, k_2, \cdots, k_{n-1})\cdot(1, \alpha, \cdots, \alpha^{n-2}) \\
= \; & \sum_{i=1}^{n-1} k_i\alpha^{i-1}
\end{aligned}$$

$$\begin{aligned}
& (1-\alpha^n)A_{n-1}(\alpha) \\
= \; & (k_{n-1}, k_0, k_1, k_2, \cdots, k_{n-2})\cdot(\alpha^{n-1}, \alpha^{n-2}, \alpha^{n-3}, \cdots, \alpha, 1). \\
= \; & (k_{n-1}, k_0, k_{n-2}, \cdots, k_2, k_1)\cdot(\alpha^{n-1}, \alpha^{n-2}, \alpha^{n-3}, \cdots, \alpha, 1) \\
= \; & (k_1, k_2, \cdots, k_{n-2}, k_0, k_{n-1})\cdot(1, \alpha, \cdots, \alpha^{n-3}, \alpha^{n-2}, \alpha^{n-1}) \\
= \; & (k_1, k_2, \cdots, k_{n-2}, k_{n-1})\cdot(1, \alpha, \cdots, \alpha^{n-3}, \alpha^{n-1}) \\
= \; & \sum_{i=1}^{n-1} k_i\alpha^{i-1} - \alpha^{n-2} + \alpha^{n-1}
\end{aligned}$$

Next we should compute a sum of each degree of $A_{\hat{l}}(\alpha)$ and $A_{n-1}(\alpha)$. For the equation $A_{\hat{l}}(\alpha)$, we have that

$$\sum_{i=1}^{n-1} k_i(i-1) - \frac{1}{2}(n-1)(l-1) = n-l-1.$$

For the equation $A_{n-1}(\alpha)$, we can show that a sum of the degree is $n-l$. From Lemma 3.6 and Lemma 3.5(iv), we complete the proof. □

3.3 Properties of function $F_{n,l}(i)$

To avoid messy calculations, for $i = 2, \cdots, n$ we define

$$F_{n,l}(i) = \frac{1}{1-\alpha^i} \sum_{m=1}^{i-1} k_m \alpha^m, \quad \alpha \in (0,1). \tag{3.18}$$

Lemma 3.8.

$$F_{n+n',l+l'}(i) = \begin{cases} F_{n,l}(i) \qquad (i = 1, \cdots, n) \\ \frac{1-\alpha^n}{1-\alpha^i} F_{n,l}(n) + \frac{\alpha^n(1-\alpha^{i-n})}{1-\alpha^i} F_{n',l'}(i-n) \\ \qquad (i = n+1, \cdots, n+n') \end{cases} \tag{3.19}$$

$$F_{n+n',l+l'}(i) = \begin{cases} F_{n',l'}(i) \qquad (i = 1, \cdots, n'-1) \\ F_{n',l'}(n') - \frac{\alpha^{n'-1}}{1-\alpha^{n'}} \quad (i = n') \\ \frac{1-\alpha^{n'}}{1-\alpha^i} F_{n',l'}(n') + \frac{\alpha^{n'}(1-\alpha^{i-n'})}{1-\alpha^i} F_{n,l}(i-n') - \frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^i} \\ \qquad (i = n'+1, \cdots, n'+n-1) \\ \frac{1-\alpha^{n'}}{1-\alpha^{n+n'}} F_{n',l'}(n') + \frac{\alpha^{n'}(1-\alpha^n)}{1-\alpha^{n+n'}} F_{n,l}(n) \\ \qquad -\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n+n'}} - \frac{\alpha^{n+n'-1}}{1-\alpha^{n+n'}} \quad (i = n'+n) \end{cases} \tag{3.20}$$

Proof. Eq.(3.19) comes clearly from Prop.3.3. Eq.(3.20) is also obtained immediately by applying the property (ii) in Prop.3.2 to $\{\hat{k}_i\}$.

□

The next lemma is a property of $F_{n,l}(n)$ which is the most important for the proof of Proposition 4.7, and this consequently concludes our main result (Theorem 5.1).

Lemma 3.9. *For any (n,l), the inequality*

$$\frac{\alpha^{n-1}-\alpha^n}{1-\alpha^n} < F_{n,l}(n) - F_{n,l}(i) < \frac{\alpha^i}{1-\alpha^i} \tag{3.21}$$

holds for $\alpha \in (0,1)$ and $i = 2, \cdots, n-1$.

Proof. Our main idea for the proof is the induction for (n,l) based on Farey series. See appendix. □

4 Deterministic Nagumo-Sato Model

In this section, we show that the system (1.1) possesses the Farey structure in the parameter space which is a layered structure (see Fig.1) and gives the regions of parameter space in which $S_{\alpha,\beta}$ has a periodic point. After that, we show a property for a preimage of zero for deterministic NS model (Proposition 4.7).

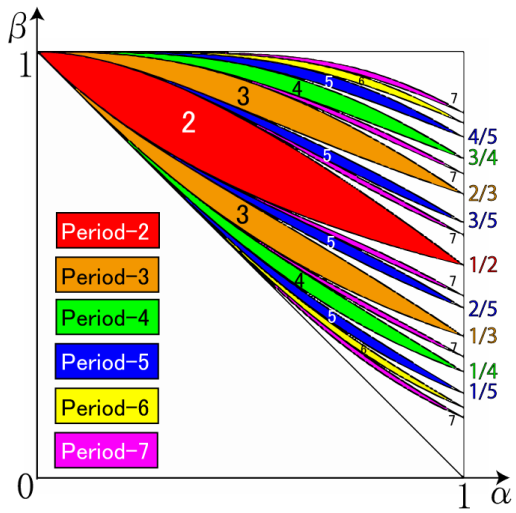

Figure 1: The region of the parameter space (α, β) in which $S_{\alpha,\beta}$ has a periodic point with period $n = 2, \cdots, 7$.

4.1 Farey structure of the NS model

Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be two projections with $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$, and define a Farey structure as follows. Let E be a bounded subset of $\mathbb{R}^2$ and let $\{D_{n,l}\}_{n\in\mathbb{N}, l\in Pr(n)}$ be a family of subsets of E satisfying the following properties. For each $\alpha \in \pi_1(E)$, there exist real numbers $B^U_{n,l}(\alpha)$ and $B^L_{n,l}(\alpha)$ such that

$$\pi_2\left(D_{n,l} \cap \pi_1^{-1}\{\alpha\}\right) = \left[B^L_{n,l}(\alpha), B^U_{n,l}(\alpha)\right). \tag{4.1}$$

We denote $D_{n,l} \prec D_{n',l'}$ if $B^U_{n,l}(\alpha) < B^L_{n',l'}(\alpha)$ holds for any $\alpha \in \pi_1(E)$. We then consider a two parameter family of transformations of $[0, 1)$, $\{T_{\alpha,\beta} : [0, 1) \to [0, 1)\}_{(\alpha,\beta)\in E}$.

Definition 4.1. $\{T_{\alpha,\beta}\}_{(\alpha,\beta)\in E}$ possesses a **Farey structure** in a parameter subspace $E \subset \mathbb{R}^2$ if there exists $\{D_{n,l}\}_{(n,l)} \subset E$ satisfying the property (4.1) such that

(i) $Leb(D_{n,l}) > 0$ for all (n,l),

(ii) $T_{\alpha,\beta}$ with $(\alpha,\beta) \in D_{n,l}$ has a periodic point with period n for each (n,l),

(iii) $D_{n+1,1} \prec D_{n,1}$ and $D_{n,n-1} \prec D_{n+1,n}$ hold for every $n \in \mathbb{N}$. If (n,l) and (n',l') satisfying $nl' - n'l = 1$ and $D_{n,l} \prec D_{n',l'}$, then $D_{n,l} \prec D_{n+n',l+l'} \prec D_{n',l'}$.

To state the next Theorem 4.3, which shows that $\{S_{\alpha,\beta}\}$ has this Farey structure, we define two functions $B^U_{n,l}(\alpha)$ and $B^L_{n,l}(\alpha)$ and sets $\{D_{n,l}\}_{(n,l)}$ as follows;

$$B^U_{n,l}(\alpha) = (1-\alpha)\left(\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1} k_m\alpha^m + 1\right), \tag{4.2}$$

$$B^L_{n,l}(\alpha) = (1-\alpha)\left(\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1} k_m\alpha^m + 1 - \frac{\alpha^{n-1}-\alpha^n}{1-\alpha^n}\right), \tag{4.3}$$

and

$$D_{n,l} = \{(\alpha,\beta) \in (0,1)^2 \mid B^L_{n,l}(\alpha) \le \beta < B^U_{n,l}(\alpha)\}, \tag{4.4}$$

where the sequence $\{k_i\}_{i\in\mathbb{Z}}$ is a rational characteristic sequence (3.1) with respect to (n,l). Note that these functions can be rewritten as follows by using the equations (3.14) and (3.15);

$$B^U_{n,l}(\alpha) = (1-\alpha)\left(\alpha \min_{\{i|k_i=0\}} A_i(\alpha) + 1\right), \tag{4.5}$$

$$B^L_{n,l}(\alpha) = (1-\alpha)\left(\alpha \max_{\{i|k_i=1\}} A_i(\alpha) + 1\right). \tag{4.6}$$

The parameter regions in Fig.1 correspond to above sets $\{D_{n,l}\}$ for $n = 2,3,\cdots,7$. The next proposition derives the property of $\{D_{n,l}\}$ defined above which implies (iii) of Farey structure.

Proposition 4.2. *Let $D_{n,l}$ be defined by (4.2), (4.3) and (4.4). Then, for every $n \in \mathbb{N}_{\ge 2}$, relations $D_{n+1,1} \prec D_{n,1}$ and $D_{n,n-1} \prec D_{n+1,n}$ hold. Moreover, if $D_{n,l} \prec D_{n',l'}$ and satisfy $nl' - n'l = 1$, then there exists a region $D_{n+n',l+l'}$ such that $D_{n,l} \prec D_{n+n',l+l'} \prec D_{n',l'}$.*

Proof. If we choose $l = 1$, a rational characteristic sequence with respect to $(n, 1)$ is satisfied $k_i = 0$ for $i = 0, \cdots, n-2$ and $k_{n-1} = 1$ for an arbitrary $n \in \mathbb{N}_{\geq 2}$. By using this property, for $\alpha \in (0, 1)$,

$$\begin{aligned} & B^L_{n,1}(\alpha) - B^U_{n+1,1}(\alpha) \\ = \quad & (1-\alpha)\left(\frac{\alpha^{n-1}}{1-\alpha^n} + 1 - \frac{\alpha^{n-1} - \alpha^n}{1-\alpha^n}\right) - (1-\alpha)\left(\frac{\alpha^n}{1-\alpha^{n+1}} + 1\right) \\ = \quad & \frac{\alpha^{2n}(1-\alpha)^2}{(1-\alpha^n)(1-\alpha^{n+1})} > 0. \end{aligned}$$

By the same way, if we choose $l = n-1$, then a rational characteristic sequence with respect to $(n, n-1)$ is satisfied $k_0 = 0$ and $k_i = 1$ for $i = 1, \cdots, n-1$ for an arbitrary $n \in \mathbb{N}_{\geq 2}$. Hence, for $\alpha \in (0, 1)$,

$$\begin{aligned} & B^L_{n+1,n}(\alpha) - B^U_{n,n-1}(\alpha) \\ = \quad & (1-\alpha)\left(\frac{1}{1-\alpha^{n+1}}\sum_{i=1}^{n}\alpha^i + 1 - \frac{\alpha^n - \alpha^{n+1}}{1-\alpha^{n+1}}\right) \\ & -(1-\alpha)\left(\frac{1}{1-\alpha^n}\sum_{i=1}^{n-1}\alpha^i + 1\right) = \frac{\alpha^{2n}(1-\alpha)^2}{(1-\alpha^n)(1-\alpha^{n+1})} > 0. \end{aligned}$$

Next, let $\{k_i\}_{i\in\mathbb{Z}}$ and $\{k'_i\}_{i\in\mathbb{Z}}$ be the rational characteristic sequences with respect to (n, l) and (n', l') respectively. From Proposition 3.3, if (n, l) and (n', l') satisfy $nl' - n'l = 1$, we can write a rational characteristic sequence $\{\hat{k}_i\}_{i\in\mathbb{Z}}$ with respect to $(n+n', l+l')$ as

$$\hat{k}_i := \begin{cases} k_i & \text{for} \quad i = 0, \cdots, n-1 \\ k'_{i-n} & \text{for} \quad i = n, \cdots, n+n'-1 \end{cases} \tag{4.7}$$

and $\hat{k}_m := \hat{k}_i$ if $m = i + t(n+n')$ with $i = 0, \cdots, n+n'-1$ and $t \in \mathbb{Z}\backslash\{0\}$.

Thus, for $\alpha \in (0, 1)$,

$$\begin{aligned} & \frac{B^L_{n+n',l+l'}(\alpha) - B^U_{n,l}(\alpha)}{1-\alpha} \\ = \quad & \frac{1}{1-\alpha^{n+n'}}\sum_{i=1}^{n+n'-1}\hat{k}_i\alpha^i - \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1-\alpha^{n+n'}} - \frac{1}{1-\alpha^n}\sum_{i=1}^{n-1}k_i\alpha^i \\ = \quad & \frac{\alpha^n(1-\alpha^{n'})}{1-\alpha^{n+n'}}\left(\frac{1}{1-\alpha^{n'}}\sum_{i=1}^{n'-1}k'_i\alpha^i - \frac{1}{1-\alpha^n}\sum_{i=1}^{n-1}k_i\alpha^i\right) \\ & -\frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1-\alpha^{n+n'}}. \end{aligned}$$

Here, from the assumption $D_{n,l} \prec D_{n',l'}$, that is, $B^U_{n,l}(\alpha) < B^L_{n',l'}(\alpha)$ for $\alpha \in (0, 1)$, we can see a following inequality;

$$\frac{1}{1-\alpha^{n'}}\sum_{i=1}^{n'-1}k'_i\alpha^i - \frac{1}{1-\alpha^n}\sum_{i=1}^{n-1}k_i\alpha^i > \frac{\alpha^{n'-1} - \alpha^{n'}}{1-\alpha^{n'}}. \tag{4.8}$$

Thus, we have that

$$\begin{aligned}\frac{B^L_{n+n',l+l'}(\alpha)-B^U_{n,l}(\alpha)}{1-\alpha} &> \frac{\alpha^n(1-\alpha^{n'})}{1-\alpha^{n+n'}}\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}-\frac{\alpha^{n+n'-1}-\alpha^{n+n'}}{1-\alpha^{n+n'}}\\ &= 0.\end{aligned}$$

At last, to clarify an inequality $B^U_{n+n',l+l'}(\alpha) < B^L_{n',l'}(\alpha)$ for $\alpha \in (0,1)$, we use the property of a rational characteristic sequence, (ii) of Proposition 3.2. Then, we can obtain a following equation;

$$\sum_{i=1}^{n+n'-1}\hat{k}_i\alpha^i = \sum_{i=1}^{n'-1}k'_i\alpha^i + \alpha^{n'}\sum_{i=1}^{n-1}k_i\alpha^i - \alpha^{n'-1}+\alpha^{n'}. \tag{4.9}$$

By using this equation, evaluate the last inequality as follows;

$$\begin{aligned}&\frac{B^L_{n',l'}(\alpha)-B^U_{n+n',l+l'}(\alpha)}{1-\alpha}\\ =\ &\frac{1}{1-\alpha^{n'}}\sum_{i=1}^{n'-1}k'_i\alpha^i-\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}-\frac{1}{1-\alpha^{n+n'}}\sum_{i=1}^{n+n'-1}\hat{k}_i\alpha^i\\ =\ &\frac{\alpha^{n'}(1-\alpha^n)}{1-\alpha^{n+n'}}\left(\frac{1}{1-\alpha^{n'}}\sum_{i=1}^{n'-1}k'_i\alpha^i-\frac{1}{1-\alpha^n}\sum_{i=1}^{n-1}k_i\alpha^i\right)\\ &-\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}+\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n+n'}}\\ >\ &\frac{\alpha^{n'}(1-\alpha^n)}{1-\alpha^{n+n'}}\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}-\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}+\frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n+n'}}\\ =\ &0.\end{aligned}$$

□

Theorem 4.3. *$\{S_{\alpha,\beta}\}_{(\alpha,\beta)\in E}$ possesses the Farey structure in $E = \{(\alpha,\beta) \in (0,1)^2 \mid \alpha+\beta > 1\}$ with $\{D_{n,l}\}_{n\in\mathbb{N},l\in Pr(n)}$ defined by (4.4).*

Proof. The property (i) in Definition 4.1 is obvious and (iii) has already proved in Proposition 4.2. Hence, we shall clarify (ii). $S_{\alpha,\beta}$ can be written as follows;

$$S_{\alpha,\beta}(x) = \begin{cases}\alpha x+\beta & \text{for} \quad x\in[0,\frac{1-\beta}{\alpha})\\ \alpha x+\beta-1 & \text{for} \quad x\in[\frac{1-\beta}{\alpha},1).\end{cases} \tag{4.10}$$

Now, to compute a solution of an equation $S^n_{\alpha,\beta}(x) = x$, rewrite the equation (4.10) as follows;

$$S_{\alpha,\beta}(x) = \alpha x+\beta-k \quad (k\in\{0,1\}),$$

where if $x \in [0, \frac{1-\beta}{\alpha})$, then $k = 0$, and if $x \in [\frac{1-\beta}{\alpha}, 1)$, then $k = 1$. And let $x_{n+1} = S(x_n) = \alpha x_n + \beta - k_n$ for $n \geq 0$. We then have that

$$\begin{aligned} S^n_{\alpha,\beta}(x) &= \alpha(\cdots(\alpha(\alpha x + \beta - k_0) + \beta - k_1) + \cdots)\beta - k_{n-1} \\ &= \alpha^n x + \frac{1-\alpha^n}{1-\alpha}\beta - \sum_{i=0}^{n-1} k_i \alpha^{n-1-i}. \end{aligned} \tag{4.11}$$

Thus, the solution of the equation $S^n_{\alpha,\beta}(x_0) = x_0$ is

$$x_0 = \frac{\beta}{1-\alpha} - \frac{1}{1-\alpha^n}\sum_{i=0}^{n-1} k_i \alpha^{n-1-i} \tag{4.12}$$

In order for x_0 to be periodic point with period n, $\{k_i\}_{i=0}^{n-1}$ should be satisfied that if $k_i = 0$, then $S^i_{\alpha,\beta}(x_0) \in [0, \frac{1-\beta}{\alpha})$, and if $k_i = 1$, then $S^i_{\alpha,\beta}(x_0) \in [\frac{1-\beta}{\alpha}, 1)$ for each $i \in \{0, \ldots, n-1\}$. Rewrite these conditions as follows;

$$S^i_{\alpha,\beta}(x_0) \in \left[k_i \frac{1-\beta}{\alpha}, (1-k_i)\frac{1-\beta}{\alpha} + k_i\right) \quad \text{for} \quad i = 0, \cdots, n-1. \tag{4.13}$$

From the formulas (4.11) and (4.12), the inequality (4.13) can be written as

$$\frac{(1-\alpha)(\alpha A_i(\alpha) + k_i)}{\alpha + k_i(1-\alpha)} \leq \beta < \frac{(1-\alpha)(\alpha(A_i(\alpha) + k_i) + 1 - k_i)}{\alpha + (1-k_i)(1-\alpha)}, \tag{4.14}$$

for $i = 0, \cdots, n-1$, where

$$A_i(\alpha) := \frac{1}{1-\alpha^n}\left(\sum_{m=0}^{i-1} k_m \alpha^{i-1-m} + \sum_{m=i}^{n-1} k_m \alpha^{n+i-1-m}\right).$$

Thus, if there exist an intersection of above equations (4.14) for $i = 0, \cdots, n-1$, then x_0 can be periodic point with period n. Now let functions $B_i(\alpha)$ and $C_i(\alpha)$ be

$$\begin{aligned} B_i(\alpha) &= \frac{(1-\alpha)(\alpha A_i(\alpha) + k_i)}{\alpha + k_i(1-\alpha)}, \\ C_i(\alpha) &= \frac{(1-\alpha)(\alpha(A_i(\alpha) + k_i) + 1 - k_i)}{\alpha + (1-k_i)(1-\alpha)}. \end{aligned}$$

A condition for an existence of the intersection is

$$\min_i C_i(\alpha) - \max_i B_i(\alpha) > 0.$$

Next, $B_i(\alpha)$ and $C_i(\alpha)$ can be rewritten as follows;

$$B_i(\alpha) = \begin{cases} (1-\alpha)A_i(\alpha) & \text{if} \quad k_i = 0 \\ (1-\alpha)(\alpha A_i(\alpha) + 1) & \text{if} \quad k_i = 1, \end{cases} \tag{4.15}$$

$$C_i(\alpha) = \begin{cases} (1-\alpha)(\alpha A_i(\alpha) + 1) & \text{if} \quad k_i = 0 \\ (1-\alpha)(A_i(\alpha) + 1) & \text{if} \quad k_i = 1, \end{cases} \tag{4.16}$$

and we see that

$$\max_{\{i|k_i=0\}} B_i(\alpha) < \max_{\{i|k_i=1\}} B_i(\alpha), \quad \min_{\{i|k_i=0\}} C_i(\alpha) < \min_{\{i|k_i=1\}} C_i(\alpha).$$

Thus,

$$\max_i B_i(\alpha) = \max_{\{i|k_i=1\}} B_i(\alpha), \quad \min_i C_i(\alpha) = \min_{\{i|k_i=0\}} C_i(\alpha).$$

From (4.15) and (4.16), if $k_i = 1$, then $B_i(\alpha) = (1-\alpha)(\alpha A_i(\alpha)+1)$, and if $k_i = 0$, then $C_i(\alpha) = (1-\alpha)(\alpha A_i(\alpha)+1)$. Since these are coincided, we should compare just $A_i(\alpha)$ for the existence of the intersection. In other words, we should clarify that there exist $\{k_i\}_{i=0}^{n-1}$ such that $\{A_i(\alpha)\}_{i=0}^{n-1}$ is satisfied

$$\min_{\{i|k_i=0\}} A_i(\alpha) - \max_{\{i|k_i=1\}} A_i(\alpha) > 0. \tag{4.17}$$

Now, from Lemma 3.7, if we choose a rational characteristic sequence as $\{k_i\}_{i=0}^{n-1}$, then this inequality (4.17) holds. Furthermore we can see that

$$\begin{aligned} D_{n,l} = \{(\alpha,\beta) \in\ & (0,1)^2 \mid \\ & (1-\alpha)(\alpha \max_{\{i|k_i=1\}} A_i(\alpha)+1) \le \beta < (1-\alpha)(\alpha \min_{\{i|k_i=0\}} A_i(\alpha)+1)\}. \end{aligned}$$

Therefore, if take $(\alpha,\beta) \in D_{n,l}$, then $S_{\alpha,\beta}$ has a periodic point with period n. □

The next corollary leads a zero Lebesgue measure of parameter sets for which $S_{\alpha,\beta}$ has no periodic point.

Corollary 4.4. *For Leb − a.e. $(\alpha,\beta) \in E$, $S_{\alpha,\beta}$ has a periodic point.*

Proof. Since $S_{\alpha,\beta}$ has a Farey structure in the parameter subspace $E = \{(\alpha,\beta) \in (0,1)^2 \mid \alpha+\beta > 1\}$, then $\{D_{n,l}\}_{n\in\mathbb{N}_{\ge 2}, l\in Pr(n)}$ are disjoint, and we can compute a following measure;

$$Leb\left(\bigcup_{n\in\mathbb{N}_{\ge 2}} \bigcup_{l\in Pr(n)} D_{n,l}\right) = \int_0^1 \sum_{n=2}^{\infty} \varphi(n) \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n} d\alpha \tag{4.18}$$

Here $\varphi(n)$ is a number of $Pr(n)$ for $n \in \mathbb{N}$ called Euler's totient function. Now we use a property of the Lambert series,

$$\sum_{n=2}^{\infty} \varphi(n) \frac{\alpha^n}{1-\alpha^n} = \left(\frac{\alpha}{1-\alpha}\right)^2. \tag{4.19}$$

By using this, we can calculate

$$Leb\left(\bigcup_{n\in\mathbb{N}_{\ge 2}} \bigcup_{l\in Pr(n)} D_{n,l}\right) = \frac{1}{2}.$$

Then, since this measure is corresponded to the measure $Leb(E)$ with $E = \{(\alpha, \beta) \in (0,1)^2 \mid \alpha + \beta > 1\}$, a proof of this corollary is completed. $\square$

Remark 4.5. The Proposition 4.2 show that the following relations hold for rational characteristic sequences $\{k_m\}$, $\{k'_m\}$ and $\{\hat{k}_m\}$ with respect to (n, l), (n', l') and $(n + n', l + l')$, respectively, with $nl' - n'l = 1$;

$$\begin{aligned} B^L_{n',l'}(\alpha) &> B^U_{n,l}(\alpha), \\ B^L_{n',l'}(\alpha) &> B^U_{n+n',l+l'}(\alpha), \quad \text{for } \alpha \in (0,1). \\ B^L_{n+n',l+l'}(\alpha) &> B^U_{n,l}(\alpha), \end{aligned}$$

By using explicit formulas (4.2), (4.3) and (3.18), these can be rewritten as follows respectively; for $\alpha \in (0, 1)$,

$$F_{n',l'}(n') - F_{n,l}(n) > \frac{\alpha^{n'-1} - \alpha^{n'}}{1 - \alpha^{n'}}, \tag{4.20}$$

$$F_{n',l'}(n') - F_{n+n',l+l'}(n + n') > \frac{\alpha^{n'-1} - \alpha^{n'}}{1 - \alpha^{n'}}, \tag{4.21}$$

$$F_{n+n',l+l'}(n + n') - F_{n,l}(n) > \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}}. \tag{4.22}$$

These inequalities will be useful for proving the Lemma 3.9 whose proof are written in appendix. Note that these inequalities (4.20),(4.21) and (4.22) are the properties of a rational characteristic sequence, which is not necessary to the NS model.

Remark 4.6. The system $S_{\alpha,\beta}$ has a periodic point with period n when $(\alpha, \beta) \in D_{n,l}$. The set of these periodic points with period n is given by

$$\mathrm{Per}_n(S_{\alpha,\beta}) = \left\{ \frac{\beta}{1 - \alpha} - A_i(\alpha) \;\middle|\; i = 0, \cdots, n - 1 \right\}. \tag{4.23}$$

4.2 Preimage of zero for NS model

In [17], Keener showed that if the set of preimages of a discontinuous point is finite, then the map has a periodic solution. The next proposition gives a new property of the preimage of zero for the NS model, which concludes that the set of preimages is finite for any parameter $(\alpha, \beta) \in D_{n,l}$. Remark that zero is a preimage of the discontinuity point of NS model. This result is used for the proof of Lemma 5.3.

Proposition 4.7. *Assume that $(\alpha, \beta) \in D_{n,l}$, then*

$$S^{-i}_{\alpha,\beta}(0) = \sum_{m=1}^{i} \frac{k_{n-i+m-1} - \beta}{\alpha^m} \in [0, 1], \qquad (i = 1, \cdots, n - 1), \tag{4.24}$$

where $\{k_m\}$ is a rational characteristic sequence with respect to (n,l). Moreover, for $i=n$, $S_{\alpha,\beta}^{-n}(0)$ is not in $[0,1]$.

Proof. Since the range of the map $S_{\alpha,\beta}$ is $[0,\alpha+\beta-1]\cup[\beta,1]$, it is obvious that, for any $x\in[0,1]$, there exists $S_{\alpha,\beta}^{-(i+1)}(x)$ in [0,1] when $S_{\alpha,\beta}^{-i}(x)\in[0,\alpha+\beta-1]\cup[\beta,1]$. Moreover, if there exists $S_{\alpha,\beta}^{-i}(0)$ in $[0,1]$, this is unique. Next, for $t=0,1,2,\cdots$, we write the map $S_{\alpha,\beta}$ as $x_{t+1}=S_{\alpha,\beta}(x_t)=\alpha x_t+\beta-\bar{k}_t$ for an initial point x_0, where $\bar{k}_t=0$ if $x_t\in[0,\frac{1-\beta}{\alpha})$, and $\bar{k}_t=1$ if $x_t\in[\frac{1-\beta}{\alpha},1)$. Then, $S_{\alpha,\beta}^{-1}(0)=\frac{\bar{k}_0-\beta}{\alpha}$ and the following equation follows inductively,

$$S_{\alpha,\beta}^{-i}(0)=\sum_{m=1}^{i}\frac{\bar{k}_{i-m+1}-\beta}{\alpha^m}\quad\text{for}\quad i=1,\cdots,n-1,$$

where $\{\bar{k}_m\}_{m=0}^{i-1}\in\{0,1\}^i$. Now we will show that there exists a sequence $\{\bar{k}_m\}_{m=0}^{n-1}\in\{0,1\}^n$ such that above $S_{\alpha,\beta}^{-i}(0)$ is in $[0,1]$ for $i=1,\cdots,n-1$. To show this, we choose a rational characteristic sequence $\{k_m\}_{m=0}^{n-1}$ with respect to (n,l) as $\{\bar{k}_m\}_{m=0}^{n-1}$. More precisely, it is enough to put $\bar{k}_{i-m+1}=k_{n-i+m-1}$ for $i=1,\cdots,n-1$ and $m=1,\cdots,i$. Applying this, we will show

$$S_{\alpha,\beta}^{-i}(0)\in[\beta,1)\ \text{if}\ \ k_{n-i-1}=0, \tag{4.25}$$

$$S_{\alpha,\beta}^{-i}(0)\in[0,\alpha+\beta-1)\ \text{if}\ \ k_{n-i-1}=1. \tag{4.26}$$

for $i=2,\cdots,n-1$.

In the case that $k_{n-i-1}=0$, we can rewrite (4.25) as follows by using the properties of rational characteristic sequences (Proposition 3.2);

$$\frac{1-\alpha}{1-\alpha^i}\left(\sum_{m=1}^{i-1}k_m\alpha^m+1-\alpha^i\right)<\beta\le\frac{1-\alpha}{1-\alpha^{i+1}}\left(\sum_{m=1}^{i-1}k_m\alpha^m+1\right).$$

Since β satisfies $B_{n,l}^L(\alpha)\le\beta<B_{n,l}^U(\alpha)$, it suffices to show that

$$\frac{1-\alpha}{1-\alpha^{i+1}}\left(\sum_{m=1}^{i-1}k_m\alpha^m+1\right)-B_{n,l}^U(\alpha)>0,$$

$$B_{n,l}^L(\alpha)-\frac{1-\alpha}{1-\alpha^i}\left(\sum_{m=1}^{i-1}k_m\alpha^m+1-\alpha^i\right)>0.$$

By using the explicit formulas (4.2) and (4.3), these inequalities can be rewritten as

$$\frac{1}{1-\alpha^{i+1}}\sum_{m=1}^{i}k_m\alpha^m-\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1}k_m\alpha^m+\frac{\alpha^{i+1}}{1-\alpha^{i+1}}>0, \tag{4.27}$$

$$\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1}k_m\alpha^m-\frac{1}{1-\alpha^i}\sum_{m=1}^{i-1}k_m\alpha^m-\frac{\alpha^{n-1}-\alpha^n}{1-\alpha^n}>0, \tag{4.28}$$

for any $i = 2, \cdots, n-1$. These inequalities (4.27) and (4.28) are allowed by Lemma 3.9.

On the other hand, in the case that $k_{n-i-1} = 1$, by similar arguments one can find that we should show next inequalities;

$$\frac{1}{1-\alpha^i}\sum_{m=1}^{i-1} k_m\alpha^m - \frac{1}{1-\alpha^n}\sum_{m=1}^{n-1} k_m\alpha^m + \frac{\alpha^i}{1-\alpha^i} > 0, \tag{4.29}$$

$$\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1} k_m\alpha^m - \frac{1}{1-\alpha^{i+1}}\sum_{m=1}^{i} k_m\alpha^m - \frac{\alpha^{n-1}-\alpha^n}{1-\alpha^n} > 0, \tag{4.30}$$

for any $i = 2, \cdots, n-2$. These inequalities (4.29) and (4.30) are allowed by Lemma 3.9.

Finally, we can show that $S^{-n}_{\alpha,\beta}(0) < 0$ by using the inequality $B^L_{n,l}(\alpha) < \beta$. Thus, $S^{-i}_{\alpha,\beta}(0)$ is in $[0, 1]$ uniquely for $i = 1, \cdots, n-1$. These complete the proof. $\square$

5 Perturbed Nagumo-Sato Model

In this section, we prove our second main theorem which states that a Markov operator generated by the system (1.2) has one of two different asymptotic properties depending on the maximum value θ of the noise.

We come to consider our random dynamical system (1.2). From Proposition 2.30, we know that the Markov operator $\overline{P} : L^1([0,1]) \to L^1([0,1])$ defined by (2.5) generated by (1.2) is asymptotically periodic. Therefore, the next main theorem gives a sufficient condition for $r > 1$ (asymptotic periodicity) and for $r = 1$ (asymptotic stability).

Theorem 5.1. *Let $\overline{P}$ be the Markov operator corresponding to system (1.2). Fix $n \in \mathbb{N}$ and $l \in Pr(n)$. Assume that $(\alpha, \beta) \in D_{n,l}$. Then there exists $\theta_*(\alpha, \beta) = \theta_*(\alpha, \beta, n, l) \in [0, 1]$ such that*

(i) *if $\theta \leq \theta_*(\alpha, \beta)$, then $r = n > 1$, and $\{\overline{P}^t\}$ is asymptotically periodic with period n,*

(ii) *if $\theta > \theta_*(\alpha, \beta)$, then $r = 1$, and $\{\overline{P}^t\}$ is asymptotically stable,*

where r is the number defined in Definition 2.18, and

$$\theta_*(\alpha, \beta) = \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n} - \beta + B^L_{n,l}(\alpha). \tag{5.1}$$

Remark 5.2. Note that the inequality $\theta \leq \theta_*(\alpha, \beta)$ means $(\alpha, \beta+\xi) \in D_{n,l}$ with arbitrary $\xi \in [0, \theta]$.

We first show the next result which is a key lemma for the proof of Theorem 5.1(i).

Lemma 5.3. *Assume that $(\alpha, \beta), (\alpha, \beta+\theta) \in D_{n,l}$ for $n \in \mathbb{N}$ and $l \in Pr(n)$. For $i = 0, \cdots, n-1$, let G_i be an interval defined by*

$$G_i = \left[\frac{\beta}{1-\alpha} - A_i(\alpha),\ \frac{\beta+\theta}{1-\alpha} - A_i(\alpha)\right], \tag{5.2}$$

where $A_i(\alpha)$ is defined by (3.7). Then, for $i = 0, 1, \cdots, n-2$,

$$x_{t+1} \in G_{i+1} \ \text{ if } \ x_t \in G_i \quad \text{and} \quad x_{t+1} \in G_0 \ \text{ if } \ x_t \in G_{n-1}, \tag{5.3}$$

where x_{t+1} is determined by the system (1.2). Moreover, there exists a number $N \in \mathbb{N}$ such that $x_{t+1} \in \bigcup_{i=0}^{n-1} G_i$ for $t > N$ and a.e.$x_0 \in [0,1] \setminus \bigcup_{i=0}^{n-1} G_i$.

Proof. First we show the statement (5.3). Let $\{k_i\}_{i\in\mathbb{Z}}$ be a rational characteristic sequence with respect to (n, l). From the definition of $A_i(\alpha)$, the following relations hold;

$$A_{i+1} = \alpha A_i(\alpha) + k_i \quad \text{for} \quad i = 0, \cdots, n-2 \quad \text{and} \tag{5.4}$$

$$A_0 = \alpha A_{n-1}(\alpha) + k_{n-1}. \tag{5.5}$$

Since $x_{t+1} = S_{\alpha,\beta}(x_t) + \xi_t \pmod 1 = \alpha x_t + \beta - k_i + \xi_t$ for all $x_t \in G_i$ and any noise $\xi_t \in [0,\theta]$, by using equation (5.4), we have

$$\begin{aligned} x_{t+1} &\in \left[\alpha\left(\frac{\beta}{1-\alpha} - A_i(\alpha)\right) + \beta - k_i + \xi_t \right. \\ &\qquad\qquad \left. , \alpha\left(\frac{\beta+\theta}{1-\alpha} - A_i(\alpha)\right) + \beta - k_i + \xi_t\right] \\ &= \left[\frac{\beta}{1-\alpha} - A_{i+1}(\alpha) + \xi_t, \frac{\beta+\theta}{1-\alpha} - A_{i+1}(\alpha) - (\theta - \xi_t)\right] \\ &\subset \left[\frac{\beta}{1-\alpha} - A_{i+1}(\alpha), \frac{\beta+\theta}{1-\alpha} - A_{i+1}(\alpha)\right] = G_{i+1} \end{aligned}$$

Therefore, $x_{t+1} \in G_{i+1}$ holds for $x_t \in G_i$. In the same fashion, we can show that $x_{t+1} \in G_0$ holds for all $x_t \in G_{n-1}$ and any noise $\xi_t \in [0,\theta]$.

Next we will show that, for $a.e. x_0 \in [0,1] \setminus \bigcup_{i=0}^{n-1} G_i$, there exists a number $N \in \mathbb{N}$ such that $x_{t+1} \in \bigcup_{i=0}^{n-1} G_i$ for $t > N$. Let the set M be the interval $[S^{-1}_{\alpha,\beta+\theta}(0), S^{-1}_{\alpha,\beta}(0)]$. Since the ξ_t have density g with $\mathrm{supp}\{g\} = [0,\theta]$, by using Proposition 4.7, we have

$$\bigcup_{(\xi_0\cdots\xi_{i-1})\in[0,\theta]^i} S^{-1}_{\alpha,\beta+\xi_{i-1}} \circ \cdots \circ S^{-1}_{\alpha,\beta+\xi_0}(M) \subset [S^{-i}_{\alpha,\beta+\theta}(0), S^{-i}_{\alpha,\beta}(0)] \subset [0,1], \tag{5.6}$$

for $i = 2, \cdots, n-1$. Moreover, we have the following relation from Proposition 4.7,

$$\bigcup_{(\xi_0\cdots\xi_{n-1})\in[0,\theta]^{n-1}} S^{-1}_{\alpha,\beta+\xi_{n-1}} \circ \cdots \circ S^{-1}_{\alpha,\beta+\xi_0}(M) \not\subset [0,1]. \tag{5.7}$$

This means that there do not exist any initial points which belong to M after the n-th iteration by the system (1.2). For any interval I, if $I \cap M \neq \emptyset$, then $\mathrm{supp}\{\overline{P}1_I\}$ may be divided into two intervals since $S_{\alpha,\beta}$ has a discontinuity point in M. However, (5.6) and (5.7) shows that such divisions occur less than $(n-1)$ times. If some interval I develops by this system without divisions, the Lebesgue measure of iterated sets goes to $\frac{\theta}{1-\alpha}$ after many iterations. From these facts, consider the iteration of an entire space $[0,1]$, one have

$$\lim_{t\to\infty} \left|\mathrm{supp}\{\overline{P}^t 1_{[0,1]}\}\right| \leq \frac{n\theta}{1-\alpha}. \tag{5.8}$$

On the other hand, since the set $\cup G_i$ is invariant for the iteration, we have that

$$\left|\mathrm{supp}\{\overline{P}^t 1_{[0,1]}\}\right| \geq \left|\bigcup_{i=0}^{n-1} G_i\right| = \sum_{i=0}^{n-1} |G_i| = \frac{n\theta}{1-\alpha} \quad \text{for} \quad t \geq 1. \tag{5.9}$$

Therefore, we have

$$\lim_{t\to\infty} \left|\mathrm{supp}\{\overline{P}^t 1_{[0,1]}\}\right| = \frac{n\theta}{1-\alpha}. \tag{5.10}$$

Finally, since $|\cup_i G_i| = \sum_i |G_i| = \frac{n\theta}{1-\alpha}$, we can find that there exists a number $N \in \mathbb{N}$ such that $x_{t+1} \in \bigcup_{i=0}^{n-1} G_i$ for $t > N$ and *a.e.* $x_0 \in [0,1] \setminus \bigcup_{i=0}^{n-1} G_i$. This completes the proof. □

Let c be the discontinuity point of NS model, i.e. $c = \frac{1-\beta}{\alpha}$. Then we have the following corollary of Lemma 5.3.

Corollary 5.4. *Assume that $(\alpha, \beta) \in D_{n,l}$ and $\theta \leq \theta_*(\alpha, \beta)$. Then the rotation number of the perturbed NS model (1.2) is given by*

$$\rho = \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} 1_{[c,1)}(x_i) = \frac{l}{n}, \tag{5.11}$$

for μ-a.e.x_0 and almost every realization of the system.

Proof. Let $\{k_i\}_{i\in\mathbb{Z}}$ be a rational characteristic sequence with respect to (n, l). When $\theta \leq \theta_*(\alpha, \beta)$, there exists $\tilde{t}$ such that $x_{\tilde{t}} \in G_0$ and $x_{\tilde{t}+i}$ moves in $\{G_i\}_{i=0}^{n-1}$ periodically. Since $k_i = 1$ if and only if $x_i \in [c, 1)$, and $\sum_{i=0}^{n-1} k_i = l$, one can immediately calculate $\rho(x_0) = \frac{l}{n}$ for any initial point x_0. □

Proof of Theorem 5.1(i)

Proof. From Proposition 2.30, we have shown that $\{\overline{P}^t\}$ is asymptotically periodic that $r = n$ and that the period equals n. This follows by using Lemma 5.3, since the permutation in the definition of asymptotic periodicity for $\{\overline{P}^t\}$, (ii) in Definition 2.18, becomes

$$\rho = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & 0 \end{pmatrix}. \tag{5.12}$$

The proof of part (i) of the theorem is completed. □

Proof of Theorem 5.1(ii)

Proof. We use the idea based on the method in [26] which is for the case (α, β) is chosen from $D_{n,1}$. First consider the case $\beta = B_{n,l}^L(\alpha)$ and $\theta > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n}$ for every $\alpha \in (0, 1)$. In this case, the attracting region of phase space is given by

$$G := \bigcup_{i=0}^{n-1} G_i \pmod 1, \tag{5.13}$$

where G_i is defined by Eq.(5.2). Note that $\{G_i\}$ are not always disjoint for $\theta > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n}$. Suppose f_0 is supported on G. If not, then $P^t f_0$ will be supported on G for sufficient large t. Next take an arbitrary interval $(a_0, b_0) \in G$ and show that the restriction of f_0 to (a_0, b_0), written as $f_0' = f_0 1_{(a_0,b_0)}$, will eventually spread out to fill the entire attracting region G. Now assume $(a_0, b_0) \in G_i(\text{mod } 1)$. Then there is the part of $P^n f_0'$ which returns to the set $G_i(\text{mod } 1)$ after the n-th iteration of f_0'. Then iterate this remainder n times more, again retaining only that part in the set $G_i(\text{mod } 1)$. By continuing this procedure and considering the times $t = nk$, $k = 1, 2, 3, \cdots$, some algebra yields

$$\text{supp}\{\overline{P}^{nk} f_0'\} = [\max\{\min G_i, q_1\}, \min\{\max G_i, q_2\}] \tag{5.14}$$

where

$$\begin{aligned}
\min G_i &= \frac{\beta}{1-\alpha} - A_i(\alpha), \quad \max G_i = \frac{\beta+\theta}{1-\alpha} - A_i(\alpha), \\
q_1 &= \alpha^{nk} a_0 + \beta \sum_{j=0}^{nk-1} \alpha^j - \sum_{j=0}^{nk-1-j} k_{i+j}, \\
q_2 &= \alpha^{nk} b_0 + (\beta+\theta) \sum_{j=0}^{nk-1} \alpha^j - \sum_{j=0}^{nk-1-j} k_{i+j}.
\end{aligned}$$

Taking the limit $k \to \infty$, we have

$$\begin{aligned}
\lim_{k\to\infty} q_1 &= \frac{\beta}{1-\alpha} - \left(\sum_{j=0}^{n-1} \alpha^{n-1-j} k_{i+j}\right) \sum_{k'=0}^{\infty} \alpha^{nk'} \\
&= \frac{\beta}{1-\alpha} - \frac{1}{1-\alpha} \sum_{j=0}^{n-1} \alpha^{n-1-j} k_{i+j} \\
&= \frac{\beta}{1-\alpha} - A_i(\alpha),
\end{aligned}$$

and

$$\lim_{k\to\infty} q_2 = \frac{\beta+\theta}{1-\alpha} - A_i(\alpha).$$

Therefore (5.14) becomes

$$\lim_{k\to\infty} \text{supp}\{\overline{P}^{nk} f_0'\} = G_i. \tag{5.15}$$

Next we will show that there exists $\epsilon > 0$ such that

$$\text{supp}\{\overline{P} 1_{G_{\phi(n-1)-1}}\} = G_{\phi(n-1)} \cup [0, \epsilon], \tag{5.16}$$

where $\phi(i) = i\hat{l}(\text{mod } n)$. The integer $\hat{l}$ is defined in Proposition 3.2. Note that $G_{\phi(n-1)}$ is the closest interval to 1 in all $\{G_i\}$. To obtain equation

(5.16), we need to demonstrate that

$$\frac{\beta+\theta}{1-\alpha} - A_{\phi(n-1)} > 1. \tag{5.17}$$

Use the assumptions $\beta = B^L_{n,l}(\alpha)$ and $\theta > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n}$, we have

$$\begin{aligned} &\frac{\beta+\theta}{1-\alpha} - A_{\phi(n-1)} \\ &\quad > \quad 1 + \frac{1}{1-\alpha^n}\left(\sum_{m=0}^{n-1} k_m\alpha^m - \sum_{m=0}^{\phi(n-1)-1} k_m\alpha^{\phi(n-1)-m-1}\right. \\ &\qquad\qquad \left. - \sum_{m=\phi(n-1)}^{n-1} k_m\alpha^{n+\phi(n-1)-m-1}\right). \end{aligned} \tag{5.18}$$

By using the properties of a rational characteristic sequence (Proposition3.2), we can see

$$\sum_{m=0}^{n-1} k_m\alpha^m - \sum_{m=0}^{\phi(n-1)-1} k_m\alpha^{\phi(n-1)-m-1} - \sum_{m=\phi(n-1)}^{n-1} k_m\alpha^{n+\phi(n-1)-m-1} = 0.$$

Thus equation (5.17) holds. From equation (5.15), there exists a large number N_0 such that $\mathrm{supp}\{\overline{P}^{N_0} f_0'\} = G_i$. In particular, we can choose the number $N_1 \geq N_0$ such that $\mathrm{supp}\{\overline{P}^{N_1} f_0'\} = G_{\phi(n-2)}$. From (5.16), $\mathrm{supp}\{\overline{P}^{N_1+1} f_0'\} = G_{\phi(n-1)} \cup [0,\epsilon]$. Allowing $[0,\epsilon] \in G_0$ to play the role of $[a_0, b_0]$ above, eventually $[0,\epsilon]$ will spread over the entire interval G_0. Thus, there exists a large number N_2 such that

$$\mathrm{supp}\{\overline{P}^{nN_2}\overline{P}^{N_1+1} f_0'\} = G_{\phi(n-1)} \cup G_0. \tag{5.19}$$

Continuing this argument, (a_0, b_0) will spread over the entire G, i.e.,

$$\lim_{t\to\infty} \mathrm{supp}\{\overline{P}^t f_0'\} = G. \tag{5.20}$$

Since (a_0, b_0) was an arbitrary component of a density on $[0,1)$, any component of an initial density will spread to cover the whole attracting part G, i.e.,

$$\lim_{t\to\infty} \mathrm{supp}\{\overline{P}^t f_0\} = G \quad \text{for} \quad f_0 \in D. \tag{5.21}$$

Lemma 2.27 shows that if the condition (5.21) holds for the operator P, then the iterates $\{P^t f_0\}$ will be asymptotically stable for all $f_0 \in D$.

Next, we consider the case $\beta > B^L_{n,l}(\alpha)$ and $\theta + \beta - B^L_{n,l}(\alpha) > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n}$. In this case, since there exists (n', l') such that $(\alpha, \beta+\theta) \in D_{n',l'}$, replacing n and l by $n+n'$ and $l+l'$ respectively in the argument of the previous case $\beta = B^L_{n,l}(\alpha)$ is enough to prove this case $\beta > B^L_{n,l}(\alpha)$. And we can show that any component of an initial density spreads to cover the whole attracting part of the phase space. This completes the proof. □

In addition to Theorem 5.1, the argument used in the proof of Theorem 5.1 (ii) plays a role to obtain the following result which shows an asymptotic behavior for the parameter satisfying $\beta = B^U_{n,l}(\alpha)$.

Theorem 5.5. *Let $\overline{P}$ be the Markov operator corresponding to system (1.2) and give a parameter $(\alpha, \beta) \in [0,1]^2$ satisfying $\beta = B^U_{n,l}(\alpha)$. Then, $\{\overline{P}^t\}$ is asymptotically stable for any $\theta > 0$.*

Proof. If we take $\beta = B^U_{n,l}(\alpha)$, then, for any $\theta > 0$, there exists (n', l') such that $(\alpha, \beta + \theta) \in D_{n',l'}$. Replacing n and l by $n + n'$ and $l + l'$ respectively in the argument of the previous case $\beta = B^L_{n,l}(\alpha)$ in Theorem 5.1 (ii) is enough to prove this case $\beta = B^U_{n,l}(\alpha)$ and $\theta > 0$. Then we can derive the equation (5.21) for any initial density $f_0 \in D$, and show that any component of an initial density spreads to cover the whole attracting part of the phase space. □

Remark 5.6. The condition $\beta = B^U_{n,l}(\alpha)$ implies that $S_{\alpha,\beta}$ does not have periodic point. The case $\alpha = 1/2$, $\beta = B^U_{4,1}(1/2) = 17/30$, $\theta = 1/15$ of behavior was observed numerically in [19, 22]. Although these observations showed us a periodic behavior with period 3, Theorem 5.5 indicates asymptotic stability for the case. And recently, Kaijser [16] showed that it displays asymptotic stability in this special case $\alpha = 1/2$, $\beta = 17/30$ and $\theta = 1/15$.

6 Numerical results

In this section, we show the numerical illustrations of our results (Theorem 5.1 and 5.5), which describe asymptotic behaviors of the Markov operator corresponding to system (1.2). In Fig.2, illustrated histograms approximately describe evolutions of densities by Markov operator $\overline{P}$ of system (1.2), $\{\overline{P}^t f_0\}$, with $\alpha = 1/2$, $\beta = B^L_{3,1}(1/2) = 4/7$ and $\theta = \theta_*(1/2, 4/7) = 1/14$ for an initial density $f_0 = 1_{[0,1]}$. In this case, $\overline{P}$ generated by this system has asymptotic periodicity with period 3 from Theorem 5.1(i). Indeed, Fig.2 indicates that the sequence $\{\overline{P}^t f_0\}$ has period 3, and these densities are repeated even after $t = 100,000$.

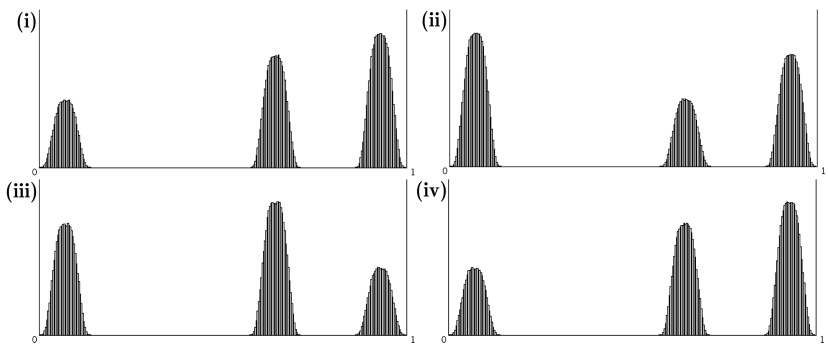

Figure 2: Asymptotic periodicity illustrated. Here we show histograms obtain after iterating 1,000,000 initial points uniformly distributed on $[0, 1]$ with $\alpha = 1/2$, $\beta = 4/7$, and $\theta = 1/14$ in Equation (1.2) for (i) $t = 1,000$; (ii) $t = 1001$; (iii) $t = 1002$; and (iv) $t = 1003$. A correspondence of the histograms for $t = 1000$ and $t = 1003$ indicates that the sequence of densities has period 3.

On the other hand, in Fig.3, we pick $\alpha = 1/2$, $\beta = B^L_{3,1}(1/2) = 4/7$ and $\theta = \theta_*(1/2, 4/7) + 0.02 = 1/14 + 0.02$ for system (1.2). In this case, $\overline{P}$ has asymptotic stability from Theorem 5.1(ii). Indeed, after t-th iterations ($t \geq 1000$), the sequence $\{\overline{P}^t f_0\}$ goes to the density of Fig.3 for the initial density $f_0 = 1_{[0,1]}$.

Finally, in Fig.4, we show the case $\alpha = 1/2$, $\beta = B^U_{4,1}(1/2) = 17/30$ and $\theta = 1/15$ for system (1.2). In this case, $\overline{P}$ has asymptotic stability from Theorem 5.5. Indeed, after 50,000th iterations, the sequence $\{\overline{P}^t f_0\}$ goes to the density of Fig.4(iv) for the initial density $f_0 = 1_{[0,1]}$

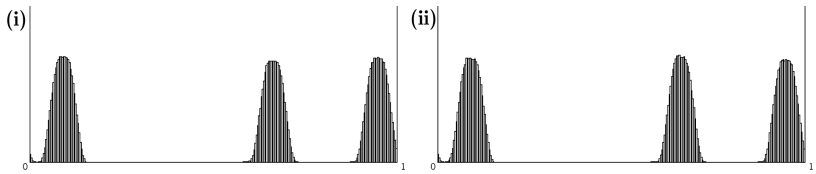

Figure 3: Asymptotic stability illustrated. Here we show histograms obtained after (i)1000th; (ii)1001th iterating 1,000,000 initial points uniformly distributed on $[0, 1]$ with $\alpha = 1/2$, $\beta = 4/7$, and $\theta = 1/14 + 0.02$ in Equation (1.2).

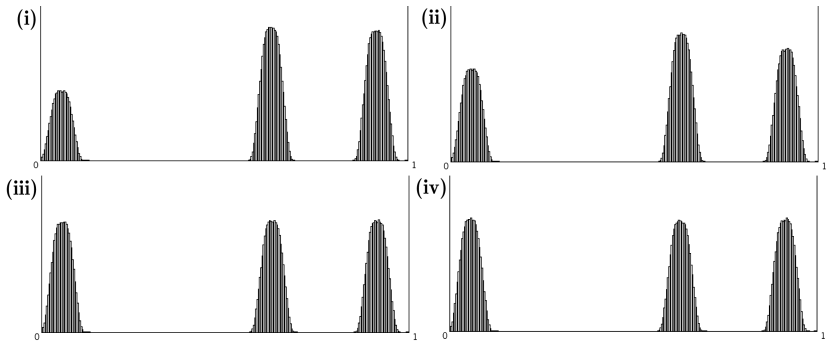

Figure 4: Asymptotic stability illustrated. Here we show histograms obtained after (i)1000th; (ii)10000th; (iii)50000th and (iv)50001th iterating 1,000,000 initial points uniformly distributed on $[0, 1]$ with $\alpha = 1/2$, $\beta = 17/30$, and $\theta = 1/15$ in Equation (1.2).

Appendix Special type of induction based on Farey series

We give the proof of Lemma 3.9 by using a special type of induction as follows;

Step(1) The inequality holds for $(n, 1)$ and $(n, n-1)$ for $n \in \mathbb{N}_{\geq 2}$.

Step(2) Assume that the inequality holds for (n, l) and (n', l') with $nl' - n'l = 1$. Then, the inequality holds for $(n + n', l + l')$.

By the definition of the Farey series, it is obvious that the above induction shows the inequality holds for all (n, l). The details of the Farey series is to be found in [1].

Definition A.1. The set of **Farey series of order** n, denoted by F_n, is the set of reduced fractions in the closed interval $[0, 1]$ with denominators $\leq n$, listed in increasing order of magnitude.

The following theorem is well known.

Theorem A.2. *([1], Theorem 5.5) The set F_{n+1} includes F_n. Each fraction in F_{n+1} which is not in F_n is the mediant of a pair of consecutive fractions in F_n. Moreover, if $\frac{a}{b} < \frac{c}{d}$ are consecutive in any F_n, then they satisfy the unimodular relation $bc - ad = 1$.*

Proof of Lemma 3.9

Proof. **(I)**First we consider the following inequality;

$$F_{n,l}(n) - F_{n,l}(i) > \frac{\alpha^{n-1} - \alpha^n}{1 - \alpha^n}. \tag{A.1}$$

We used a special type of induction for (n, l), which is associated to the Farey series.

(I-1) We first show the inequality (A.1) in the case $(n, 1)$ and $(n, n-1)$ for any $n \geq 2$.

(I-1-1) When $\{k_m\}_{m \in \mathbb{Z}}$ is a rational characteristic sequence with respect to $(n, 1)$, one can see that $k_m = 0$ for $m = 0, \cdots, n-2$ and $k_{n-1} = 1$, and obtain inequality (A.1) immediately for any $i = 2, \cdots, n-1$.

(I-1-2) When $\{k_m\}_{m \in \mathbb{Z}}$ is a rational characteristic sequence with respect to $(n, n-1)$, one can see that $k_m = 1$ for $m = 1, \cdots, n-1$. Thus the inequality (A.1) can be also shown by elementary calculations.

(I-2) Next, assume that the inequality (A.1) holds for (n, l) and (n', l') with $nl' - n'l = 1$. That is, next two inequalities hold;

$$F_{n,l}(n) - F_{n,l}(i) > \frac{\alpha^{n-1} - \alpha^n}{1 - \alpha^n} \quad \text{for} \quad i = 2, \cdots, n-1, \tag{A.2}$$

$$F_{n',l'}(n') - F_{n',l'}(i) > \frac{\alpha^{n'-1} - \alpha^{n'}}{1 - \alpha^{n'}} \quad \text{for} \quad i = 2, \cdots, n'-1, \tag{A.3}$$

where $\{k_m\}_{m\in\mathbb{Z}}$ and $\{k'_m\}_{m\in\mathbb{Z}}$ are rational characteristic sequences with respect to (n, l) and (n', l') respectively. Then, we will show that the inequality (A.1) holds for $(n + n', l + l')$. That is, we will show that the next value is positive for any $i = 2, \cdots, n + n' - 1$ and $\alpha \in (0, 1)$;

$$F_{n+n',l+l'}(n + n') - F_{n+n',l+l'}(i) - \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}}, \tag{A.4}$$

where the sequence $\{\hat{k}_m\}_{m\in\mathbb{Z}}$ is the rational characteristic sequence with respect to $(n + n', l + l')$. From the Proposition 3.3, $\{\hat{k}_m\}_{m\in\mathbb{Z}}$ can be made from $\{k_m\}_{m\in\mathbb{Z}}$ and $\{k'_m\}_{m\in\mathbb{Z}}$.

(I-2-1) For $i = 2, \cdots, n - 1$, by using (4.22) and (A.2),

$$\begin{aligned} \text{Eq.(A.4)} &= F_{n+n',l+l'}(n + n') - F_{n,l}(i) - \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}} \\ &> F_{n,l}(n) - F_{n,l}(i) > 0. \end{aligned}$$

(I-2-2) For $i = n, n + 1$, it can be obtained from (4.22) and (3.19).

(I-2-3) For $i = n + 2, \cdots, n + n' - 1$, by using (3.19),

$$\begin{aligned} \text{Eq.(A.4)} = \; & \frac{1 - \alpha^n}{1 - \alpha^{n+n'}} F_{n,l}(n) + \frac{\alpha^n(1 - \alpha^{n'})}{1 - \alpha^{n+n'}} F_{n',l'}(n') \\ & - \frac{1 - \alpha^n}{1 - \alpha^i} F_{n,l}(n) - \frac{\alpha^n(1 - \alpha^{i-n})}{1 - \alpha^i} F_{n',l'}(i - n) \\ & - \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}} \end{aligned} \tag{A.5}$$

Applying the assumption (A.3) to the fourth term of (A.5), we can calculate as follows;

$$\text{Eq.(A.4)} > \frac{\alpha^i(1 - \alpha^n)(1 - \alpha^{n+n'-1})}{(1 - \alpha^i)(1 - \alpha^{n+n'})} \left\{ F_{n',l'}(n') - F_{n,l}(n) - \frac{\alpha^{n'-1} - \alpha^{n'}}{1 - \alpha^{n'}} \right\}.$$

Then it was proven that Eq.(A.4) is positive for $i = n + 2, \cdots, n + n' - 1$ from the inequality (4.20). From these arguments (I-1) and (I-2), for any (n, l), the inequality (A.1) is proved.

(II) Next we consider another inequality;

$$F_{n,l}(n) - F_{n,l}(i) < \frac{\alpha^i}{1-\alpha^i}. \tag{A.6}$$

This proof is similar to the previous inequality (A.1). The key idea is to switch the roles of n and n'.

(II-1) In the case $(n,1)$ and $(n,n-1)$ for any integer $n \geq 2$, the inequality (A.6) can be clarified by elementary calculations because of same reasons with (I-1).

(II-2) Next, assume that the inequality (A.6) holds for (n,l) and (n',l') with $nl' - n'l = 1$. That is, next two inequalities hold;

$$F_{n,l}(n) - F_{n,l}(i) < \frac{\alpha^i}{1-\alpha^i} \quad \text{for} \quad i = 2, \cdots, n-1, \tag{A.7}$$

$$F_{n',l'}(n') - F_{n',l'}(i) < \frac{\alpha^i}{1-\alpha^i} \quad \text{for} \quad i = 2, \cdots, n'-1, \tag{A.8}$$

where $\{k_m\}_{m\in\mathbb{Z}}$ and $\{k'_m\}_{m\in\mathbb{Z}}$ are rational characteristic sequences with respect to (n,l) and (n',l') respectively. Then, we will show that the inequality (A.6) holds for $(n+n', l+l')$. That is, we will show that the next value is positive for any $i = 2, \cdots, n+n'-1$ and $\alpha \in (0,1)$;

$$\frac{\alpha^i}{1-\alpha^i} - F_{n+n',l+l'}(n+n') + F_{n+n',l+l'}(i), \tag{A.9}$$

where the sequence $\{\hat{k}_m\}_{m\in\mathbb{Z}}$ is the rational characteristic sequence with respect to $(n+n', l+l')$.

(II-2-1) For $i = 2, \cdots, n'-1$, by using (3.20), (A.8) and (4.21),

$$\begin{aligned} \text{Eq.(A.9)} &= \frac{\alpha^i}{1-\alpha^i} - F_{n+n',l+l'}(n+n') + F_{n',l'}(i) \\ &> F_{n',l'}(n') - F_{n+n',l+l'}(n+n') > 0. \end{aligned}$$

(II-2-2) For $i = n', n'+1$, it can be obtained from (4.21) and (3.20).

(II-2-3) For $i = n'+2, \cdots, n'+n-1$, by using (3.20),

$$\begin{aligned} \text{Eq.(A.9)} &= \frac{\alpha^i}{1-\alpha^i} - \frac{1-\alpha^{n'}}{1-\alpha^{n+n'}} F_{n',l'}(n') - \frac{\alpha^{n'}(1-\alpha^n)}{1-\alpha^{n+n'}} F_{n,l}(n) \\ &\quad + \frac{1-\alpha^{n'}}{1-\alpha^i} F_{n',l'}(n') + \frac{\alpha^{n'}(1-\alpha^{i-n'})}{1-\alpha^i} F_{n,l}(i-n'). \end{aligned} \tag{A.10}$$

Applying the assumption (A.7) to the fifth term of (A.10), we can calculate as follows;

$$\text{Eq.(A.9)} > \frac{\alpha^i(1-\alpha^{n'})(1-\alpha^{n+n'-1})}{(1-\alpha^i)(1-\alpha^{n+n'})} \left\{ F_{n',l'}(n') - F_{n,l}(n) \right\}.$$

Then it is proven that Eq.(A.9) is positive for $i = n' + 2, \cdots, n' + n - 1$ from the inequality (4.20). From these arguments (II-1) and (II-2), for any rational characteristic sequence with respect to (n, l), the inequality (A.6) is proved.

□

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Flowchart

We show the main flow to lead our main results as follows.