博士学位論文

Flat surfaces associated with framed base curves

（枠付け可能曲線に付随する平坦な曲面）

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Flat surfaces associated with framed base curves

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Introduction

Geometric information of Frenet curves in the Euclidean 3-space are described by two invariants up to congruence. However we cannot investigate non-regular curves and straight lines by using the invariants because such cureves are not Frenet curves. Moreover we remark that the invariants are not invariants as "movement" but invariants as "shape". Notwithstanding, we have many objects which have singular points in classical differential geometry. On the other hand, there are many articles concerning flat surfaces associated with space curves (in particular, Frenet curves) in the Euclidean 3-space which are called developable surfaces. Developable surfaces are surfaces with vanishing Gaussian curvature in the Euclidean 3-space. Applications of developable surfaces cover several areas - from ship-building to manufacturing of clothing - as they are suitable to the modeling of surfaces which can be made out of leather , paper, fiber, and sheet metal (cf. [39]). We have important 3-kinds of developable surfaces associated with a space curve. These are the tangent developable surface (i.e. the envelope of osculating planes), the focal developable surface (i.e. the envelope of normal planes) and the rectifying developable surface (i.e. the envelope of rectifying planes). Their singularities and geometric properties are investigated in [5, 7, 17, 23–26, 33, 34, 36, 37, 40]. However the author could not find any articles concerning singularities of the focal developable surfaces and of the rectifying developable surfaces associated with non-regular curves.

In this thesis we investigate singularities of flat surfaces associated with non-regular curves in Euclidean 3-space and Euclidean 3-sphere as an application of the singularity theory. In order to consider non-regular curves, we first introduce the following notion of framed curves in the direct product of the Euclidean n-space $\mathbb{R}^n$ and the $(n, n-1)$-type frame field $V_{n,n-1}$:

Definition (Framed curve, [22]) We say that $(\gamma, \boldsymbol{N}) : I \to \mathbb{R}^n \times V_{n,n-1}$ is a *framed curve* if $\dot{\gamma}(t) \cdot \boldsymbol{n}_i(t) = 0$ for all $t \in I$ and $i = 1, \ldots, n-1$, where $\boldsymbol{N}(t) = (\boldsymbol{n}_1(t), \boldsymbol{n}_2(t),$ ..., $\boldsymbol{n}_{n-1}(t))$. If $(\gamma, \boldsymbol{N})$ is an immersion, we call $(\gamma, \boldsymbol{N})$ a *framed immersion*.

Definition (Framed base curve, [22]) We say that $\gamma : I \to \mathbb{R}^n$ is a *framed base curve* if there exists $\boldsymbol{N} : I \to V_{n,n-1}$ such that $(\gamma, \boldsymbol{N})$ is a framed curve.

The author and Masatomo Takahashi introduced the notion of framed curves and framed base curves in [22]. Framed curves are natural generalizations not only of Frenet curves, but also of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$. For a framed curve, we define invariants which are called the curvatures of the framed curve, similar to the curvatures of a Frenet curve and of a Legendre curve. The curvatures of the framed curve is quite useful to analyze the framed curve and its singularities. In fact, we have existence and uniqueness for the framed curve by using its curvatures (cf. Theorems 1.1.4 and 1.1.5). We must consider how to take a parameter and a moving frame for a framed base curve. Therefore we investigate properties of parameter changes and of frame changes in Section 1.3.1. As applications, we consider a contact between framed curves (cf. Section 1.3.2), and relationships between projections of framed curves and Legendre curves (cf. Section 1.3.3).

Secondly we apply the theory of framed curves to study of geometric properties of flat surfaces associated with non-regular curves in the Euclidean 3-space $\mathbb{R}^3$. In Chapter 2, we investigate the focal developable surface (i.e. the envelope of normal planes) and the rectifying developable surface (i.e. the envelope of rectifying planes) of a Frenet type framed base curve. A Frenet type framed base curve is defined to be a non-regular space curve which has the regular unit tangent vector field. It is a natural generalization of a Frenet curve. In order to define each developable surface, we apply the notion of support functions (cf. [28]). We introduce new invariants which related to characterizations as developable surfaces and singularities of each developable surface associated with a Frenet type framed base curve. The torsion of a Frenet type framed base curve is constantly equal to zero if and only if the focal surface is a cylindrical surface (cf. Theorem 2.2.2, (1)). The one of new invariants is constantly equal to zero under a certain condition if and only if the focal surface is a conical surface (cf. Theorem 2.2.2, (2)). We give characterizations of singularities of the focal developable surface of a Frenet framed base curve by using those invariants and criteria for the recognitions of wave fronts (cf. Theorem 2.2.3 and [32]). The set of singular values of the focal developable surface of a Frenet type framed base curve is called the evolute. The evolute of a Frenet framed base curve is given as the locus of the centers of osculating spheres and is also a Frenet type framed base curve (cf. Proposition 2.2.5). We give relationships between singularities of the evolute and of the focal developable surface (cf. Corollary 2.2.6). For the rectifying developable surface of a Frenet type framed base curve, we have theorems which correspond with the above (cf. Theorems 2.3.2 and 2.3.3). In Section 2.3.2, we define a framed helix and consider relationships between the rectifying developable surface.

Thirdly we consider two kinds of developable surfaces along a frontal curve on an embedded surface in the Euclidean 3-space. One is called the osculating developable surface, and the other is called the normal developable surface. The notions of these developable surfaces are generalizations the notions in [18, 28]. We discovered new invariants of the frontal curve which characterize singularities of the developable surfaces (cf. Theorems 3.2.4 and 3.3.4). In particular, a frontal curve is a contour generator with respect to an orthogonal projection or a central projection if and only if one of these invariants constantly equal to zero (cf. Theorem 3.2.2). We have some interesting examples of each developable surface in Section 3.4.

At last we apply the theory of framed curves to spherical geometry. Izumiya, Nagai and Saji introduced extrinsic flat great circular surfaces and their singularities in [27]. Extrinsic flat great circular surfaces are surfaces with vanishing extrinsic Gaussian curvature in the Euclidean 3-sphere. By using criteria in [27], we consider dual, tangent and focal extrinsic flat great circular surfaces associated to spherical framed base curves and their singularities. Moreover we investigate dual extrinsic flat great circular surfaces of evolutes.

Throughout this thesis, all mappings and manifolds are differentiable of class C^∞.

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Chapter 1

Basic concepts of framed curves

The study of curves in the Euclidean space is a classical and important subject in differential geometry. Classical approach by using the Frenet frame associated with a Frenet curve (i.e. a regular curve with a certain regularity condition) is quite useful to analyze the Frenet curve and its geometric properties. However, most of curves appearing in applications have singularities. Therefore, it is important to establish a method to handle curves with singularities. We introduce how to consider curves with singularities in this chapter. This chapter is based on [22].

A framed curve in the Euclidean space is a curve with a moving frame. It is a natural generalization not only of a Frenet curve, but also of a Legendre curve in the unit tangent bundle over a plane. We define functions for a framed curve which are called the curvature of the framed curve, similar to the curvature of a Frenet curve and of a Legendre curve. The curvature of the framed curve are quite useful to analyze the framed curve and its singularities. In fact, we have the existence theorem (cf. Theorem 1.1.4) and the uniqueness theorem (cf. Theorem 1.1.5) for the framed curve by using its curvature. We must consider how to take a parameter and a moving frame for a given framed curve. Therefore, we investigate properties of parameter changes and of frame changes in Section 1.3.1. As applications, we consider a contact between framed curves (cf. Section 1.3.2), and give a relationship between projections of framed curves and Legendre curves (cf. Section 1.3.3).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space equipped with the canonical inner product $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^n a_i b_i$ for any $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$. Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1} \in \mathbb{R}^n$ be vectors $\boldsymbol{a}_i = (a_{i,1}, \dots, a_{i,n})$ for $i = 1, \dots, n-1$. We define the vector product

$$\boldsymbol{a}_1 \times \dots \times \boldsymbol{a}_{n-1} = \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \end{vmatrix} = \sum_{i=1}^n \det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}, \boldsymbol{e}_i)\, \boldsymbol{e}_i,$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n$. Then $(\boldsymbol{a}_1 \times \dots \times \boldsymbol{a}_{n-1}) \cdot \boldsymbol{a}_i = 0$ for $i = 1, \dots, n-1$. We remark that for the case of $n = 3$,

$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{vmatrix} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{vmatrix}.$$

The set

$$\begin{aligned} V_{n,n-1} &= \{\boldsymbol{N} = (\boldsymbol{n}_1, \dots, \boldsymbol{n}_{n-1}) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \mid \boldsymbol{n}_i \cdot \boldsymbol{n}_j = \delta_{i,j}, i, j = 1, \dots, n-1\} \\ &= \{\boldsymbol{N} = (\boldsymbol{n}_1, \dots, \boldsymbol{n}_{n-1}) \in S^{n-1} \times \dots \times S^{n-1} \mid \boldsymbol{n}_i \cdot \boldsymbol{n}_j = 0, i \neq j, i, j = 1, \dots, n-1\} \end{aligned}$$

is called the $(n,n-1)$-*type Stiefel manifold* (or, the set of all orthonormal $(n-1)$-frames in $\mathbb{R}^n$). This is an $n(n-1)/2$-dimensional smooth manifold.

If $\boldsymbol{N} = (\boldsymbol{n}_1, \dots, \boldsymbol{n}_{n-1}) \in V_{n,n-1}$, we define a unit vector $\boldsymbol{t} = \boldsymbol{n}_1 \times \dots \boldsymbol{n}_{n-1}$ of $\mathbb{R}^n$. It follow that $(\boldsymbol{N}, \boldsymbol{t}) \in SO(n)$, where $SO(n)$ is the set of all special orthogonal matrices.

1.1 Framed curves in $\mathbb{R}^n \times V_{n,n-1}$

Definition 1.1.1 (Framed curve, [22]) We say that $(\gamma, \boldsymbol{N}) : I \to \mathbb{R}^n \times V_{n,n-1}$ is a *framed curve* if $\dot{\gamma}(t) \cdot \boldsymbol{n}_i(t) = 0$ for all $t \in I$ and $i = 1, \dots, n-1$, where $\boldsymbol{N}(t) = (\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \dots, \boldsymbol{n}_{n-1}(t))$. If $(\gamma, \boldsymbol{N})$ is an immersion, we call $(\gamma, \boldsymbol{N})$ a *framed immersion*.

Definition 1.1.2 (Framed base curve, [22]) We say that $\gamma : I \to \mathbb{R}^n$ is a *framed base curve* if there exists $\boldsymbol{N} : I \to V_{n,n-1}$ such that $(\gamma, \boldsymbol{N})$ is a framed curve.

We define smooth functions for a framed curve similar to the curvature of a Frenet curve and of a Legendre curve. Let $(\gamma, \boldsymbol{N}) : I \to \mathbb{R}^n \times V_{n,n-1}$ be a framed curve. We define $\boldsymbol{t} : I \to S^{n-1}$ by $\boldsymbol{t}(t) = \boldsymbol{n}_1(t) \times \dots \times \boldsymbol{n}_{n-1}(t)$. By definition, $(\boldsymbol{N}(t), \boldsymbol{t}(t)) \in SO(n)$ for each $t \in I$ and we call $\{\boldsymbol{N}(t), \boldsymbol{t}(t)\}$ a *moving frame* along the framed base curve $\gamma(t)$. Then we have the Frenet-Serret type formula

$$\begin{pmatrix} \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{t}}(t) \end{pmatrix} = A(t) \begin{pmatrix} \boldsymbol{N}(t) \\ \boldsymbol{t}(t) \end{pmatrix},$$

where $A(t) = (\alpha_{i,j}(t)) \in \mathfrak{o}(n)$ for $i, j = 1, \dots, n$ and $\mathfrak{o}(n)$ is the set of all skew-symmetric matrices. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = \alpha(t)\boldsymbol{t}(t).$$

We call the mapping $(A, \alpha) : I \to \mathfrak{o}(n) \times \mathbb{R}$ the *curvature* of the framed curve (with respect to the parameter t). Clearly, t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$. The curvature is quite useful to analyze the framed curve and its singularities (cf. Theorems 1.1.4 and 1.1.5).

Definition 1.1.3 ([22]) Let $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}}) : I \to \mathbb{R}^n \times V_{n,n-1}$ be framed curves. We say that $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}})$ are (positive) *congruent* as framed curves if there exists a matrix $X \in SO(n)$ and a vector $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$\widetilde{\gamma}(t) = X(\gamma(t)) + \boldsymbol{x}, \; \widetilde{\boldsymbol{N}}(t) = X(\boldsymbol{N}(t))$$

for all $t \in I$.

We have the following theorems.

Theorem 1.1.4 (The Existence Theorem, [22]) *Let $(A, \alpha) : I \to \mathfrak{o}(n) \times \mathbb{R}$ be a smooth mapping. Then there exists a framed curve $(\gamma, \boldsymbol{N}) : I \to \mathbb{R}^n \times V_{n,n-1}$ whose associated curvature is (A, α).*

Theorem 1.1.5 (The Uniqueness Theorem, [22]) *Let $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}}) : I \to \mathbb{R}^n \times V_{n,n-1}$ be framed curves whose curvatures (A, α) and $(\widetilde{A}, \widetilde{\alpha})$ coincide. Then $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}})$ are congruent as framed curves.*

We shall prove these theorems in Section 1.2.

1.2 Proofs for the theorems existence and uniqueness

First we prove the existence theorem by using the existence theorem of a solution of a system of linear ordinary differential equations.

Proof of Theorem 1.1.4. Choose any fixed value $t = t_0$ of the parameter. We consider the initial value problem

$$\frac{d}{dt}F(t) = A(t)F(t),\ F(t_0) = I_n,$$

where $F(t) \in M(n)$; here $M(n)$ is the set of all $n \times n$ matrices and I_n is the identity matrix. By the existence theorem of a solution of a system of linear ordinary differential equations, there exists a solution $F(t)$. Since $A(t) \in \mathfrak{o}(n)$,

$$\frac{d}{dt}({}^tF(t)F(t)) = \left(\frac{d}{dt}{}^tF(t)\right)F(t) + {}^tF(t)\left(\frac{d}{dt}F(t)\right) = {}^tF(t)\left({}^tA(t) + A(t)\right)F(t) = O.$$

It follows that ${}^tF(t)F(t)$ is constant. Thus ${}^tF(t)F(t) = {}^tF(t_0)F(t_0) = I_n$, and $F(t)$ is an orthogonal matrix. Set $F(t) = {}^t(\boldsymbol{n}_1(t), \ldots, \boldsymbol{n}_{n-1}(t), \boldsymbol{t}(t))$. Since $(d/dt)(\det F(t)) = 0$, we have

$$\det F(t) = \det F(t_0) = \det I_n = 1$$

and $\boldsymbol{t}(t) = \boldsymbol{n}_1(t) \times \ldots \times \boldsymbol{n}_{n-1}(t)$. Next we consider the initial value problem

$$\dot{\gamma}(t) = \alpha(t)\boldsymbol{t}(t),\ \gamma(t_0) = \boldsymbol{x},$$

where $\boldsymbol{x}$ is a point in $\mathbb{R}^n$. By the existence theorem of a solution of a system of linear ordinary differential equations, there exists a solution $\gamma(t)$. Therefore, there exists a framed curve $(\gamma, \boldsymbol{N}) : I \to \mathbb{R}^n \times V_{n,n-1}$ whose associated curvature is (A, α). □

In order to prove the uniqueness theorem, we prepare two lemmas.

Lemma 1.2.1 ([22]) *Let $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}})$ are congruent as framed curves. Then their curvatures coincide.*

Proof. Since $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}})$ are congruent as framed curves, there exist a matrix $X \in SO(n)$ and a vector $\boldsymbol{x} \in \mathbb{R}^n$ which satisfy

$$\widetilde{\gamma}(t) = X(\gamma(t)) + \boldsymbol{x},\ \widetilde{\boldsymbol{N}}(t) = X(\boldsymbol{N}(t)).$$

By the definition of $\boldsymbol{t}$, we have $\widetilde{\boldsymbol{t}}(t) = X(\boldsymbol{t}(t))$ for all $t \in I$. By straightforward calculations, we have

$$\widetilde{\alpha}_{i,j}(t) = \dot{\widetilde{\boldsymbol{n}}}_1(t) \cdot \widetilde{\boldsymbol{n}}_j(t) = X(\dot{\boldsymbol{n}}_i(t)) \cdot X(\boldsymbol{n}_j(t)) = \dot{\boldsymbol{n}}_i(t) \cdot \boldsymbol{n}_j(t) = \alpha_{i,j}(t),$$

$$\dot{\widetilde{\gamma}} = X(\dot{\gamma}(t)) = X(\alpha(t)\boldsymbol{t}(t)) = \alpha(t)X(\boldsymbol{t}(t)) = \alpha(t)\widetilde{\boldsymbol{t}}(t).$$

Hence we have $A(t) = \widetilde{A}(t)$ and $\alpha(t) = \widetilde{\alpha}(t)$. □

Lemma 1.2.2 ([22]) *Let $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}}) : I \to \mathbb{R}^n \times V_{n,n-1}$ be framed curves with a common curvature, that is, $(A(t), \alpha(t)) = (\widetilde{A}(t), \widetilde{\alpha}(t))$ for all $t \in I$. If there exists a parameter $t = t_0$ for which $(\gamma(t_0), \boldsymbol{N}(t_0)) = (\widetilde{\gamma}(t_0), \widetilde{\boldsymbol{N}}(t_0))$, then $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}})$ coincide.*

Proof. Here we put $\boldsymbol{n}_n(t) = \boldsymbol{t}(t)$. Define a smooth function $f : I \to \mathbb{R}$ by $f(t) = \sum_{i=1}^{n} \boldsymbol{n}_i(t) \cdot \widetilde{\boldsymbol{n}}_i(t)$. Since $\alpha_{i,j}(t) = \widetilde{\alpha}_{i,j}(t)$ and $\alpha_{i,j}(t) = -\alpha_{j,i}(t)$, we have

$$\begin{aligned}
\dot{f}(t) &= \sum_{i=1}^{n} \left(\dot{\boldsymbol{n}}_i(t) \cdot \widetilde{\boldsymbol{n}}_i(t) + \boldsymbol{n}_i(t) \cdot \dot{\widetilde{\boldsymbol{n}}}_i(t) \right) \\
&= \sum_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} \alpha_{i,j}(t) \boldsymbol{n}_j(t) \right) \cdot \widetilde{\boldsymbol{n}}_i(t) + \boldsymbol{n}_i(t) \cdot \left(\sum_{j=1}^{n} \widetilde{\alpha}_{i,j}(t) \widetilde{\boldsymbol{n}}_j(t) \right) \right\} \\
&= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \left(\alpha_{i,j}(t) + \alpha_{j,i}(t) \right) \boldsymbol{n}_i(t) \cdot \widetilde{\boldsymbol{n}}_j(t) \right\} \\
&= 0.
\end{aligned}$$

It follows that f is constant. Moreover $\boldsymbol{N}(t_0) = \widetilde{\boldsymbol{N}}(t_0)$, so that $\boldsymbol{t}(t_0) = \widetilde{\boldsymbol{t}}(t_0)$. Hence $f(t_0) = n$ and the function f is constant with value n. By the Cauchy-Schwarz inequality, we have

$$\boldsymbol{n}_i(t) \cdot \widetilde{\boldsymbol{n}}_i(t) \le \|\boldsymbol{n}_i(t)\| \|\widetilde{\boldsymbol{n}}_i(t)\| = 1$$

for each $i = 1, \dots, n$. If one of these inequalities were strict, the value of $f(t)$ would be less than n. It follows that these inequalities are equalities, and we have $\boldsymbol{n}_i(t) \cdot \widetilde{\boldsymbol{n}}_i(t) = 1$ for all $t \in I$ and $i = 1, \dots, n$. Then we have

$$\|\boldsymbol{n}_i(t) - \widetilde{\boldsymbol{n}}_i(t)\|^2 = \boldsymbol{n}_i(t) \cdot \boldsymbol{n}_i(t) - 2\boldsymbol{n}_i(t) \cdot \widetilde{\boldsymbol{n}}_i(t) + \widetilde{\boldsymbol{n}}_i(t) \cdot \widetilde{\boldsymbol{n}}_i(t) = 0.$$

Hence $\boldsymbol{n}_i(t) = \widetilde{\boldsymbol{n}}_i(t)$ for all $t \in I$ and $i = 1, \dots, n$. Since $\dot{\gamma}(t) = \alpha(t)\boldsymbol{t}(t)$, $\dot{\widetilde{\gamma}}(t) = \widetilde{\alpha}(t)\widetilde{\boldsymbol{t}}(t)$ and the assumption $\alpha(t) = \widetilde{\alpha}(t)$, we obtain $(d/dt)(\gamma(t) - \widetilde{\gamma}(t)) = 0$. It follow that $\gamma(t) - \widetilde{\gamma}(t)$ is constant. By the condition $\gamma(t_0) = \widetilde{\gamma}(t_0)$, we have $\gamma(t) = \widetilde{\gamma}(t)$ for all $t \in I$. □

Proof of Theorem 1.1.5. Choose any fixed value $t = t_0$ of the parameter. By using a matrix $X \in SO(n)$ and a vector $\boldsymbol{x} \in \mathbb{R}^n$, we can assume that $\widetilde{\gamma}(t_0) = X(\gamma(t_0)) + \boldsymbol{x}$ and $\widetilde{\boldsymbol{N}}(t_0) = X(\boldsymbol{N}(t_0))$. By Lemma 1.2.1, the curvatures of the framed curves $(\gamma(t), \boldsymbol{N}(t))$ and $(X(\gamma(t)) + \boldsymbol{x}, X(\boldsymbol{N}(t)))$ coincide. By Lemma 1.2.2, we have

$$\widetilde{\gamma}(t) = X(\gamma(t)) + \boldsymbol{x},\ \widetilde{\boldsymbol{N}}(t) = X(\boldsymbol{N}(t))$$

for all $t \in I$. It follows that $(\gamma, \boldsymbol{N})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{N}})$ are congruent as framed curves. □

The uniqueness theorem can be proved also by using the uniqueness theorem of a solution of a system of linear ordinary differential equations.

1.3 Framed curves in $\mathbb{R}^3 \times V_{3,2}$

In this section we focus on framed curves in $\mathbb{R}^3 \times V_{3,2}$. One can extend the results to curves in higher-dimensional spaces. However it is rather tedious; we concentrate on the case of $n = 3$.

We use the following notations throughout this section. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ be a framed curve and $\boldsymbol{t}(t) = \boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t)$. The Frenet-Serret type formula is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}_1(t) \\ \dot{\boldsymbol{n}}_2(t) \\ \dot{\boldsymbol{t}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \\ \boldsymbol{t}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\boldsymbol{n}}_1(t) \cdot \boldsymbol{n}_2(t)$, $m(t) = \dot{\boldsymbol{n}}_1(t) \cdot \boldsymbol{t}(t)$ and $n(t) = \dot{\boldsymbol{n}}_2(t) \cdot \boldsymbol{t}(t)$. Moreover there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = \alpha(t)\boldsymbol{t}(t).$$

Example 1.3.1 A typical example of a framed curve is a Frenet curve. Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve, that is, $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linearly independent for all $t \in I$. If we take $\boldsymbol{n}_1(t) = \boldsymbol{n}(t)$ and $\boldsymbol{n}_2(t) = \boldsymbol{b}(t)$, then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve. We remark that $\boldsymbol{t}(t) = \boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t)$ is the unit tangent vector at $t \in I$. Here,

$$\boldsymbol{t}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}, \ \boldsymbol{n}(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)}{\|(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)\|}, \ \boldsymbol{b}(t) = \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}.$$

We give relationships between invariants of a Frenet curve and of a framed curve.

Proposition 1.3.2 ([22]) *With the same notations as in Example 1.3.1, the relationships between invariants of a Frenet curve $(\kappa(t), \tau(t))$ and of a framed curve $(\ell(t), m(t), n(t), \alpha(t))$ are given by*

$$\kappa(t) = \frac{\sqrt{m^2(t) + n^2(t)}}{|\alpha(t)|}, \ \tau(t) = \frac{(m^2(t) + n^2(t))\ell(t) + m(t)\dot{n}(t) - \dot{m}(t)n(t)}{(m^2(t) + n^2(t))\alpha(t)}.$$

Proof. By straightforward calculations, we have

$$\begin{aligned} \dot{\gamma}(t) &= \alpha(t)\boldsymbol{t}(t), \\ \ddot{\gamma}(t) &= -\alpha(t)m(t)\boldsymbol{n}_1(t) - \alpha(t)n(t)\boldsymbol{n}_2(t) + \dot{\alpha}(t)\boldsymbol{t}(t), \\ \dddot{\gamma}(t) &= -\left(2\dot{\alpha}(t)m(t) + \alpha(t)\dot{m}(t) - \alpha(t)\ell(t)n(t)\right)\boldsymbol{n}_1(t) \\ &\quad - \left(2\dot{\alpha}(t)n(t) + \alpha(t)\dot{n}(t) + \alpha(t)\ell(t)m(t)\right)\boldsymbol{n}_2(t) \\ &\quad + \left(\ddot{\alpha}(t) - \alpha(t)m^2(t) - \alpha(t)n^2(t)\right)\boldsymbol{t}(t). \end{aligned}$$

It follows that

$$\begin{aligned} \|\dot{\gamma}(t)\| &= |\alpha(t)|, \\ \|\dot{\gamma}(t) \times \ddot{\gamma}(t)\| &= \alpha^2(t)\sqrt{m^2(t) + n^2(t)}, \\ \det\left(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)\right) &= \alpha^3(t)\left(m(t)\dot{n}(t) - \dot{m}(t)n(t) + \left(m^2(t) + n^2(t)\right)\ell(t)\right). \end{aligned}$$

Therefore the curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3} = \frac{\sqrt{m^2(t) + n^2(t)}}{|\alpha(t)|}$$

and the torsion $\tau(t)$ is given by

$$\tau(t)=\frac{\det(\dot{\gamma}(t),\ddot{\gamma}(t),\dddot{\gamma}(t))}{\|\dot{\gamma}(t)\times\ddot{\gamma}(t)\|^2}=\frac{\left(m^2(t)+n^2(t)\right)\ell(t)+m(t)\dot{n}(t)-\dot{m}(t)n(t)}{\alpha(t)\left(m^2(t)+n^2(t)\right)}.$$

□

Let $(\gamma,\boldsymbol{n}_1,\boldsymbol{n}_2):I\to\mathbb{R}^3\times V_{3,2}$ be a framed curve with the curvature (ℓ,m,n,α). By the proof of Proposition 1.3.2, we have the following Taylor expansion of γ:

$$\begin{aligned}\gamma(t) &= \gamma(t_0)+(t-t_0)\alpha(t_0)\boldsymbol{t}(t_0)\\ &+\frac{(t-t_0)^2}{2}\left(-\alpha(t_0)m(t_0)\boldsymbol{n}_1(t_0)-\alpha(t_0)n(t_0)\boldsymbol{n}_2(t_0)+\dot{\alpha}(t_0)\boldsymbol{t}(t_0)\right)\\ &+\frac{(t-t_0)^3}{3!}(-(2\dot{\alpha}(t_0)m(t_0)+\alpha(t_0)\dot{m}(t_0)-\alpha(t_0)\ell(t_0)n(t_0))\boldsymbol{n}_1(t_0)\\ &\qquad -(2\dot{\alpha}(t_0)n(t_0)+\alpha(t_0)\dot{n}(t_0)+\alpha(t_0)\ell(t_0)m(t_0))\boldsymbol{n}_2(t_0)\\ &\qquad +(\ddot{\alpha}(t_0)-\alpha(t_0)m^2(t_0)-\alpha(t_0)n^2(t_0))\boldsymbol{t}(t_0))+o(4).\end{aligned}$$

If t_0 is a singular point of γ, then we have

$$\begin{aligned}\gamma(t) &= \gamma(t_0)+\frac{(t-t_0)^2}{2}\dot{\alpha}(t_0)\boldsymbol{t}(t_0)\\ &+\frac{(t-t_0)^3}{3!}(-2\dot{\alpha}(t_0)m(t_0)\boldsymbol{n}_1(t_0)-2\dot{\alpha}(t_0)n(t_0)\boldsymbol{n}_2(t_0)+\ddot{\alpha}(t_0)\boldsymbol{t}(t_0))+o(4).\end{aligned}$$

Let $\gamma:(\mathbb{R},0)\to(\mathbb{R}^3,\boldsymbol{0})$ be a space curve germ and write $\gamma(t)=(x(t),y(t),z(t))$. We can show that if γ is not infinitely flat (i.e. either $x(t)$, $y(t)$ or $z(t)$ does not belong to $\mathfrak{m}_1^\infty$), then γ is a framed base curve. Here $\mathfrak{m}_1^\infty$ is the ideal of infinitely flat function germs. Without loss of the generality we assume that $x(t)$ does not belong to $\mathfrak{m}_1^\infty$ and that order $x(t)\le\min\{\text{order } y(t),\text{order } z(t)\}$. Then there exist smooth function germs $a(t)$ and $b(t)$ such that $\dot{y}(t)=a(t)\dot{x}(t)$ and $\dot{z}(t)=b(t)\dot{x}(t)$. Thus if we take

$$\begin{aligned}\boldsymbol{n}_1(t) &= \frac{1}{\sqrt{1+a^2(t)}}\left(-a(t),1,0\right),\\ \boldsymbol{n}_2(t) &= \frac{1}{\sqrt{\left(1+a^2(t)\right)\left(1+a^2(t)+b^2(t)\right)}}\left(-b(t),-a(t)b(t),1+a^2(t)\right),\end{aligned}$$

then $(\gamma,\boldsymbol{n}_1,\boldsymbol{n}_2)$ is a framed curve. We remark that

$$\boldsymbol{t}(t)=\boldsymbol{n}_1(t)\times\boldsymbol{n}_2(t)=\frac{1}{\sqrt{1+a^2(t)+b^2(t)}}\left(1,a(t),b(t)\right).$$

On the other hand, constant maps are also framed base curves which do not satisfy the above sufficient condition. In particular an analytic curve germ is always a framed base curve because if it is infinitely flat, then it is constant.

We summarize the above arguments as the following proposition.

Proposition 1.3.3 *Let $\gamma:(I,t_0)\to\mathbb{R}^3$ be a real analytic curve germ. Then γ is a framed base curve, that is, there exists a mapping germ $(\boldsymbol{n}_1,\boldsymbol{n}_2):(I,t_0)\to V_{3,2}$ such that $(\gamma,\boldsymbol{n}_1,\boldsymbol{n}_2)$ is a framed curve.*

Example 1.3.4 Let n_1, n_2, n_3, k_1 and k_2 be natural numbers with $n_2 = n_1 + k_1$ and $n_3 = n_2 + k_2$, and let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : \mathbb{R} \to \mathbb{R}^3 \times V_{3,2}$ be

$$\begin{aligned}
\gamma(t) &= \left(\frac{1}{n_1}t^{n_1}, \frac{1}{n_2}t^{n_2}, \frac{1}{n_3}t^{n_3}\right),\\
\boldsymbol{n}_1(t) &= \frac{1}{\sqrt{1+t^{2k_1}}}(-t^{k_1}, 1, 0),\\
\boldsymbol{n}_2(t) &= \frac{1}{\sqrt{(1+t^{2k_1})(1+t^{2k_1}+t^{2k_1+2k_2})}}(-t^{k_1+k_2}, -t^{2k_1+k_2}, 1+t^{2k_1}).
\end{aligned}$$

We can easily check that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is a framed curve. We say that γ is (n_1, n_2, n_3)-type. Since $\boldsymbol{t} : \mathbb{R} \to S^2$;

$$\boldsymbol{t}(t) = \frac{1}{\sqrt{1+t^{2k_1}+t^{2k_1+2k_2}}}(1, t^{k_1}, t^{k_1+k_2}),$$

the curvature of the framed curve $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is given by

$$\begin{aligned}
\ell(t) &= \frac{k_1 t^{2k_1+k_2-1}}{(1+t^{2k_1})\sqrt{1+t^{2k_1}+t^{2k_1+2k_2}}},\\
m(t) &= \frac{-k_1 t^{k_1-1}}{\sqrt{(1+t^{2k_1})(1+t^{2k_1}+t^{2k_1+2k_2})}},\\
n(t) &= \frac{-t^{k_1+k_2-1}(k_1+k_2+k_2t^{2k_1})}{(1+t^{2k_1}+t^{2k_1+2k_2})\sqrt{1+t^{2k_1}}},\\
\alpha(t) &= t^{n_1-1}\sqrt{1+t^{2k_1}+t^{2k_1+2k_2}}.
\end{aligned}$$

We now consider a framed curve in a plane. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ be a framed curve with the curvature (ℓ, m, n, α). We denote a plane by

$$P(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} \cdot \boldsymbol{v} = c\},$$

where $\boldsymbol{v} \in S^2$ and $c \in \mathbb{R}$. If $\gamma(t) \in P(\boldsymbol{v}, c)$, then we have $\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t)) = 0$. It follows that

$$\alpha(t)\left(\left(m^2(t)+n^2(t)\right)\ell(t) + m(t)\dot n(t) - \dot m(t)n(t)\right) = 0$$

for all $t \in I$. Conversely, we have the following result.

Proposition 1.3.5 ([22]) *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ be a framed curve with the curvature (ℓ, m, n, α).*

(1) *If $\alpha(t) = 0$ for all $t \in I$, then $\gamma(t)$ is a point.*

(2) *If $m^2(t) + n^2(t) = 0$ for all $t \in I$, then $\gamma(t)$ is a part of a straight line.*

(3) *If $(m^2(t)+n^2(t))\ell(t) + m(t)\dot n(t) - \dot m(t)n(t) = 0$ and $m^2(t)+n^2(t) \neq 0$ for all $t \in I$, then there exist a constant vector $\boldsymbol{v} \in S^2$ and a constant number $c \in \mathbb{R}$ such that $\gamma(t) \in P(\boldsymbol{v}, c)$.*

Proof. (1) Since $\dot\gamma(t) = \alpha(t)\boldsymbol{t}(t) = 0$ for all $t \in I$, $\gamma(t)$ is a point.

(2) By the Frenet-Serret type formula, $\dot{\boldsymbol{t}}(t) = 0$ for all $t \in I$ and hence $\dot\gamma(t) = \alpha(t)\boldsymbol{t}(t) = \alpha(t)\boldsymbol{v}$, where $\boldsymbol{v} \in S^2$ is a constant vector. Then there exists a constant

vector $\boldsymbol{x}$ such that $\gamma(t) = (\int \alpha(t)dt)\boldsymbol{v} + \boldsymbol{x}$. It follows that $\gamma(t)$ is a part of a straight line.

(3) We take a Bishop frame $\{\overline{\boldsymbol{n}}_1(t), \overline{\boldsymbol{n}}_2(t), \boldsymbol{t}(t)\}$ along the framed base curve $\gamma(t)$ (cf. Section 1.3.1). By straightforward calculations, we have

$$\overline{m}(t)\dot{\overline{n}}(t) - \dot{\overline{m}}(t)\overline{n}(t) = (m^2(t) + n^2(t))\ell(t) + m(t)\dot{n}(t) - \dot{m}(t)n(t) = 0$$

and

$$\overline{m}^2(t) + \overline{n}^2(t) = m^2(t) + n^2(t) = 0$$

for all $t \in I$. It follows that $\overline{m}(t)$ and $\overline{n}(t)$ are linearly dependent on I (cf. [4, 35, 44]). Thus there exists a non-zero constant vector (c_1, c_2) such that $c_1\overline{m}(t) + c_2\overline{n}(t) = 0$ for all $t \in I$. Then $\widetilde{\boldsymbol{v}} = c_1\overline{\boldsymbol{n}}_1(t) + c_2\overline{\boldsymbol{n}}_2(t)$ is a non-zero constant vector. Let $\boldsymbol{v} = \widetilde{\boldsymbol{v}}/\sqrt{c_1^2 + c_2^2}$. Since $\dot{\gamma}(t) \cdot \boldsymbol{v} = \alpha(t)\boldsymbol{t}(t) \cdot \boldsymbol{v} = 0$ for all $t \in I$, there exists a constant number $c \in \mathbb{R}$ such that $\gamma(t) \in P(\boldsymbol{v}, c)$. □

Remark 1.3.6 If $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ is an analytic mapping, then $\overline{m}(t)$ and $\overline{n}(t)$ are also analytic functions. Hence if $m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0$ for all $t \in I$, then $\overline{m}(t)$ and $\overline{n}(t)$ are linearly dependent on I (cf. [4, 44]). It follows that there exist a constant vector $\boldsymbol{v} \in S^3$ and a constant number $c \in \mathbb{R}$ such that $\gamma(t) \in P(\boldsymbol{v}, c)$.

We also define a Legendre curve on a Plane.

Definition 1.3.7 ([22]) We say that $(\gamma, \boldsymbol{n}) : I \to \mathbb{R}^3 \times S^2$ is a *Legendre curve* on the plane $P(\boldsymbol{v}, c)$ if $\gamma(t) \cdot \boldsymbol{v} = c$, $\boldsymbol{n}(t) \cdot \boldsymbol{v} = 0$ and $\dot{\gamma}(t) \cdot \boldsymbol{n}(t) = 0$ for all $t \in I$.

Proposition 1.3.8 ([22]) (1) *If $(\gamma, \boldsymbol{n}) : I \to \mathbb{R}^3 \times S^2$ is a Legendre curve on the palne $P(\boldsymbol{v}, c)$, then $(\gamma, \boldsymbol{n}, \boldsymbol{v}) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve with $\ell(t) = m(t) = 0$ for all $t \in I$. Conversely, if $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve with $\ell(t) = m(t) = 0$ for all $t \in I$, then there exist a constant vector $\boldsymbol{v} \in S^2$ and a constant number $c \in \mathbb{R}$ such that $(\gamma, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times S^2$ is a Legendre curve on the plane $P(\boldsymbol{v}, c)$.*

(2) *If $(\gamma, \boldsymbol{n}) : I \to \mathbb{R}^3 \times S^2$ is a Legendre curve on the palne $P(\boldsymbol{v}, c)$, then $(\gamma, \boldsymbol{v}, \boldsymbol{n}) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve with $\ell(t) = n(t) = 0$ for all $t \in I$. Conversely, if $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve with $\ell(t) = n(t) = 0$ for all $t \in I$, then there exist a constant vector $\boldsymbol{v} \in S^2$ and a constant number $c \in \mathbb{R}$ such that $(\gamma, \boldsymbol{n}_1) : I \to \mathbb{R}^3 \times S^2$ is a Legendre curve on the plane $P(\boldsymbol{v}, c)$.*

Proof. (1) By definition, we have $\dot{\gamma}(t) \cdot \boldsymbol{v} = 0$ and $(\boldsymbol{v}, \boldsymbol{n}(t)) \in V_{3,2}$. Since $\boldsymbol{v}$ is constant, we have $\ell(t) = m(t) = 0$ for all $t \in I$. Conversely, by the Frenet-Serret type formula, $\boldsymbol{v} = \boldsymbol{n}_1(t) \in S^2$ is a constant vector. Moreover, since $\dot{\gamma}(t) \cdot \boldsymbol{v} = \alpha(t)\boldsymbol{t}(t) \cdot \boldsymbol{n}_1(t) = 0$ for all $t \in I$, there exists a constant number $c \in \mathbb{R}$ such taht $\gamma(t) \cdot \boldsymbol{v} = c$. It follows that $(\gamma, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times S^2$ is a Legendre curve on the plane $P(\boldsymbol{v}, c)$.

Assertion (2) can be proved by similar way. □

1.3.1 Parameter changes and frame changes

Let I and $\overline{I}$ be intervals. A smooth function $s : \overline{I} \to I$ is said to be a *change of parameter* if s is surjective and has a positive derivative at every point. It follows that s is a diffeomorphism. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ and $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2) : \overline{I} \to \mathbb{R}^3 \times V_{3,2}$ be framed curves with the curvatures (ℓ, m, n, α) and $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$, respectively.

Suppose that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)$ are parametrically equivalent via the change of parameter $s : \overline{I} \to I$, that is, $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)(t) = (\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(s(t))$ for all $t \in \overline{I}$. Then we have

$$\overline{\ell}(t) = \dot{s}(t)\ell(s(t)),\ \overline{m}(t) = \dot{s}(t)m(s(t)),\ \overline{n}(t) = \dot{s}(t)n(s(t)),\ \overline{\alpha}(t) = \dot{s}(t)\alpha(s(t)).$$

Therefore, the curvature depends on a parametrization.

Generally, we cannot consider the arc-length parameter of the framed base curve γ. However, if $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is an immersion, we can introduce the arc-length parameter for the framed immersion $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$. The *speed* of the framed immersion at the parameter t is defined to be the length of the tangent vector at t, namely,

$$s(t) = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\boldsymbol{n}}_1(t) \cdot \dot{\boldsymbol{n}}_1(t) + \dot{\boldsymbol{n}}_2(t) \cdot \dot{\boldsymbol{n}}_2(t)}.$$

Given scalars $a, b \in I$, we define the arc-length from $t = a$ to $t = b$ to be the integral of the speed,

$$L_{(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)}(t) = \int_a^b s(t)dt.$$

By the same method for the arc-length parameter of a regular curve, one can prove the following (cf. [5, 16, 17]).

Proposition 1.3.9 ([22]) *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ be a framed immersion and $t_0 \in I$. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is parametrically equivalent to a unit speed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2) : \widetilde{I} \to \mathbb{R}^3 \times V_{3,2}$ under a change of parameter $t : \widetilde{I} \to I$ with $t(0) = t_0$ and $t'(s) > 0$.*

We call the parameter s in Proposition 1.3.9 the *arc-length parameter* for the framed immersion. Let s be the arc-length parameter for $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$. By definition, we have $\gamma'(s) \cdot \gamma'(s) + \boldsymbol{n}_1'(s) \cdot \boldsymbol{n}_1'(s) + \boldsymbol{n}_2'(s) \cdot \boldsymbol{n}_2'(s) = 1$, where $'$ is the derivation with respect to s. It follows that $2\ell^2(s) + m^2(s) + n^2(s) + \alpha^2(t) = 1$.

For the normal planes of $\gamma(t)$, which are spanned by $\boldsymbol{n}_1(t)$ and $\boldsymbol{n}_2(t)$, there are other frames by rotations and reflections. We define $(\overline{\boldsymbol{n}}_1(t), \overline{\boldsymbol{n}}_2(t)) \in V_{3,2}$ by

$$\begin{pmatrix} \overline{\boldsymbol{n}}_1(t) \\ \overline{\boldsymbol{n}}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve and $\overline{\boldsymbol{t}}(t) = \overline{\boldsymbol{n}}_1(t) \times \overline{\boldsymbol{n}}_2(t) = \boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t) = \boldsymbol{t}(t)$. The curvature of $(\gamma, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)$ is given by

$$\begin{aligned} \overline{\ell}(t) &= \ell(t) - \dot{\theta}(t), \\ \overline{m}(t) &= m(t)\cos\theta(t) - n(t)\sin\theta(t), \\ \overline{n}(t) &= m(t)\sin\theta(t) + n(t)\cos\theta(t), \\ \overline{\alpha}(t) &= \alpha(t). \end{aligned}$$

We call $\{\overline{\boldsymbol{n}}_1(t), \overline{\boldsymbol{n}}_2(t), \boldsymbol{t}(t)\}$ a *rotated frame* along $\gamma(t)$ by $\theta(t)$. If we take a smooth function $\theta : I \to \mathbb{R}$ which satisfies $\dot{\theta}(t) = \ell(t)$, then we call $\{\overline{\boldsymbol{n}}_1(t), \overline{\boldsymbol{n}}_2(t), \boldsymbol{t}(t)\}$ a *Bishop frame* along $\gamma(t)$ by $\theta(t)$ (cf. [3, 21, 22]). For a Bishop frame, we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \dot{\overline{\boldsymbol{n}}}_1(t) \\ \dot{\overline{\boldsymbol{n}}}_2(t) \\ \dot{\boldsymbol{t}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \overline{m}(t) \\ 0 & 0 & \overline{n}(t) \\ -\overline{m}(t) & -\overline{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{\boldsymbol{n}}_1(t) \\ \overline{\boldsymbol{n}}_2(t) \\ \boldsymbol{t}(t) \end{pmatrix}.$$

On the other hand, we define $(\widetilde{\boldsymbol{n}}_1(t), \widetilde{\boldsymbol{n}}_2(t)) \in V_{3,2}$ by

$$\begin{pmatrix} \widetilde{\boldsymbol{n}}_1(t) \\ \widetilde{\boldsymbol{n}}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ is a framed curve and $\widetilde{\boldsymbol{t}}(t) = \widetilde{\boldsymbol{n}}_1(t) \times \widetilde{\boldsymbol{n}}_2(t) = -\boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t) = -\boldsymbol{t}(t)$. The curvature of $(\gamma, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ is given by

$$\begin{aligned} \widetilde{\ell}(t) &= -\ell(t) + \dot{\theta}(t), \\ \widetilde{m}(t) &= -m(t)\cos\theta(t) + n(t)\sin\theta(t), \\ \widetilde{n}(t) &= m(t)\sin\theta(t) + n(t)\cos\theta(t), \\ \widetilde{\alpha}(t) &= -\alpha(t). \end{aligned}$$

We call $\{\widetilde{\boldsymbol{n}}_1(t), \widetilde{\boldsymbol{n}}_2(t), -\boldsymbol{t}(t)\}$ a *reflected frame* along $\gamma(t)$ by $\theta(t)$.

By straightforward calculations, we have the following equations:

$$\begin{aligned} m^2(t) + n^2(t) &= \overline{m}^2(t) + \overline{n}^2(t) \\ &= \widetilde{m}^2(t) + \widetilde{n}^2(t) \end{aligned}$$

and

$$\begin{aligned} & \ell(t)(m^2(t) + n^2(t)) + m(t)\dot{n}(t) - \dot{m}(t)n(t) \\ = \ & \overline{\ell}(t)(\overline{m}^2(t) + \overline{n}^2(t)) + \overline{m}(t)\dot{\overline{n}}(t) - \dot{\overline{m}}(t)\overline{n}(t) \\ = \ & \widetilde{\ell}(t)(\widetilde{m}^2(t) + \widetilde{n}^2(t)) + \widetilde{m}(t)\dot{\widetilde{n}}(t) - \dot{\widetilde{m}}(t)\widetilde{n}(t). \end{aligned}$$

1.3.2 Contact between framed curves

In this section, we discuss contact between framed curves. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2};\ t \mapsto (\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2) : \widetilde{I} \to \mathbb{R}^3 \times V_{3,2};\ u \mapsto (\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(u)$ be framed curves, and let k be a natural number. We say that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ *and* $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ *have k-th order contact* at $t = t_0, u = u_0$ if

$$(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = (\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0),\ \frac{d}{dt}(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0),\ \dots,$$

$$\frac{d^{k-1}}{dt^{k-1}}(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0),\ \frac{d^k}{dt^k}(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) \neq \frac{d^k}{du^k}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0)$$

(cf. [10, 17]). Moreover, we say that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ *and* $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ *have at least k-th order contact* at $t = t_0, u = u_0$ if

$$(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = (\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0),\ \frac{d}{dt}(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0),\ \dots,$$

$$\frac{d^{k-1}}{dt^{k-1}}(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0).$$

Generally, we may assume that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ have at least first order contact at any point $t = t_0$, $u = u_0$ up to congruence as framed curves. We denote $\mathcal{F}(t) = (\ell(t), m(t), n(t), \alpha(t))$ and $\widetilde{\mathcal{F}}(t) = (\widetilde{\ell}(t), \widetilde{m}(t), \widetilde{n}(t), \widetilde{\alpha}(t))$ for convenience.

Theorem 1.3.10 ([22]) *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ as above. If $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$, then*

$$\mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0),\ \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\widetilde{\mathcal{F}}(u_0),\ \dots,\ \frac{d^{k-1}}{dt^{k-1}}\mathcal{F}(t_0) = \frac{d^{k-1}}{du^{k-1}}\widetilde{\mathcal{F}}(u_0). \tag{1.1}$$

Conversely, if condition (1.1) holds, then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$ up to congruence as framed curves.

Proof. Suppose that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ have at least second order contact at $t = t_0$, $u = u_0$. Since $\boldsymbol{n}_1(t_0) = \widetilde{\boldsymbol{n}}_1(u_0)$ and $\boldsymbol{n}_2(t_0) = \widetilde{\boldsymbol{n}}_2(u_0)$, we have $\boldsymbol{t}(t_0) = \widetilde{\boldsymbol{t}}(u_0)$. By the Frenet-Serret type formula, we have

$$\begin{aligned}
\frac{d}{dt}(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t) &= (\alpha(t)\boldsymbol{t}(t), \ell(t)\boldsymbol{n}_2(t) + m(t)\boldsymbol{t}(t), -\ell(t)\boldsymbol{n}_1(t) + n(t)\boldsymbol{t}(t)),\\
\frac{d}{du}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u) &= (\widetilde{\alpha}(u)\widetilde{\boldsymbol{t}}(u), \widetilde{\ell}(u)\widetilde{\boldsymbol{n}}_2(u) + \widetilde{m}(u)\widetilde{\boldsymbol{t}}(u), -\widetilde{\ell}(u)\widetilde{\boldsymbol{n}}_1(u) + \widetilde{n}(u)\widetilde{\boldsymbol{t}}(u)).
\end{aligned}$$

It follows that $\mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0)$. Therefore, the first assertion of Theorem 1.3.10 holds in the case of $k = 1$.

Suppose that the assumption is true up to the k-th order contact. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$. Then these have at least k-th order contact, so that

$$\mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0),\ \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\widetilde{\mathcal{F}}(u_0),\ \dots,\ \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t_0) = \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u_0).$$

By the Frenet-Serret type formula, we have

$$\begin{aligned}
\frac{d^k}{dt^k}\gamma(t) =& \left(\frac{d^{k-1}}{dt^{k-1}}\alpha(t)\right)\boldsymbol{t}(t) + f_1\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{n}_1(t)\\
&+ f_2\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{n}_2(t) + f_3\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{t}(t),\\
\frac{d^k}{dt^k}\boldsymbol{n}_1(t) =& \left(\frac{d^{k-1}}{dt^{k-1}}\ell(t)\right)\boldsymbol{n}_2(t) + \left(\frac{d^{k-1}}{dt^{k-1}}m(t)\right)\boldsymbol{t}(t) + g_1\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{n}_1(t)\\
&+ g_2\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{n}_2(t) + g_3\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{t}(t),\\
\frac{d^k}{dt^k}\boldsymbol{n}_2(t) =& -\left(\frac{d^{k-1}}{dt^{k-1}}\ell(t)\right)\boldsymbol{n}_1(t) + \left(\frac{d^{k-1}}{dt^{k-1}}n(t)\right)\boldsymbol{t}(t) + h_1\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{n}_1(t)\\
&+ h_2\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{n}_2(t) + h_3\left(\mathcal{F}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\right)\boldsymbol{t}(t)
\end{aligned}$$

for some smooth functions f_i, g_i, h_i $(i = 1, 2, 3)$. By the same calculations,

$$\begin{aligned}
\frac{d^k}{du^k}\widetilde{\gamma}(u) =& \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\alpha}(u)\right)\widetilde{\boldsymbol{t}}(u) + f_1\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{n}}_1(u) \\
&+ f_2\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{n}}_2(u) + f_3\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{t}}(u), \\
\frac{d^k}{du^k}\widetilde{\boldsymbol{n}}_1(u) =& \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\ell}(u)\right)\widetilde{\boldsymbol{n}}_2(u) + \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{m}(u)\right)\widetilde{\boldsymbol{t}}(u) + g_1\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{n}}_1(u) \\
&+ g_2\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{n}}_2(u) + g_3\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{t}}(u), \\
\frac{d^k}{du^k}\widetilde{\boldsymbol{n}}_2(u) =& -\left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\ell}(u)\right)\widetilde{\boldsymbol{n}}_1(u) + \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{n}(u)\right)\widetilde{\boldsymbol{t}}(u) + h_1\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{n}}_1(u) \\
&+ h_2\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{n}}_2(u) + h_3\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\right)\widetilde{\boldsymbol{t}}(u).
\end{aligned}$$

It follows that $(d^{k-1}/dt^{k-1})\mathcal{F}(t_0) = (d^{k-1}/du^{k-1})\widetilde{\mathcal{F}}(u_0)$. By the induction, we have the first assertion.

Conversely, suppose that condition (1.1) holds. By the above calculations, we have $(d^i/dt^i)(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)(t_0) = (d^i/du^i)(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)(u_0)$ for $i = 1, \dots, k$. Therefore, $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$ up to congruence as framed curves. □

1.3.3 Projections to planes and Legendre curves

We quickly review Legendre curves in the unit tangent bundle over a plane; for more detail, see [10]. Legendre curves correspond with framed curves when $n = 2$. We say that $(\gamma, \boldsymbol{n}) : I \to \mathbb{R}^2 \times S^1$ is a *Legendre curve* if $(\gamma, \boldsymbol{n})^*\theta = 0$ for all $t \in I$, where θ is the canonical contact 1-form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [1,2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \boldsymbol{n}(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a *frontal* if there exists a smooth mapping $\boldsymbol{n} : I \to S^1$ such that $(\gamma, \boldsymbol{n})$ is a Legendre curve.

Let $(\gamma, \boldsymbol{n}) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet type formula of the frontal γ as follows. We put $\boldsymbol{t}(t) = J(\boldsymbol{n}(t))$, where J is the anti-clockwise rotation by $\pi/2$ on $\mathbb{R}^2$. We call the pair $\{\boldsymbol{n}(t), \boldsymbol{t}(t)\}$ a *moving frame* along the frontal $\gamma(t)$ in $\mathbb{R}^2$. The Frenet type formula of the frontal (or, the Legendre curve) is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{t}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}(t) \\ \boldsymbol{t}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\boldsymbol{n}}(t) \cdot \boldsymbol{t}(t)$. Moreover, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{t}(t).$$

We call the pair $(\ell(t), \beta(t))$ the *curvature* of the Legendre curve (with respect to the parameter t).

Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ be a framed curve with the curvature (ℓ, m, n, α). For a fixed point $t_0 \in I$, we consider three orthogonal projections from $\mathbb{R}^3$ along the directions $\boldsymbol{n}_1(t_0)$, $\boldsymbol{n}_2(t_0)$ and $\boldsymbol{t}(t_0)$.

First, we consider the projection of γ along the direction $\boldsymbol{n}_1(t_0)$ given by $\gamma_{n_1} : I \to \mathbb{R}^2$; $\gamma_{n_1}(t) = (\gamma(t) \cdot \boldsymbol{n}_2(t_0), \gamma(t) \cdot \boldsymbol{t}(t_0))$. Then $\dot{\gamma}_{n_1}(t) = \alpha(t)(\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0), \boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0))$. There is a subinterval I_1 of I around t_0 such that $(\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0))^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0))^2 \neq 0$ for all $t \in I_1$. We define a smooth mapping $\boldsymbol{n}_{n_1} : I_1 \to S^1$ by

$$\boldsymbol{n}_{n_1}(t) = \frac{1}{\sqrt{(\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0))^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0))^2}} \left(\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0), -\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0) \right).$$

Then $(\gamma_{n_1}, \boldsymbol{n}_{n_1}) : I_1 \to \mathbb{R}^2 \times S^1$ is a Legendre curve. Since $\boldsymbol{t}_{n_1} : I_1 \to S^1$ is given by

$$\boldsymbol{t}_{n_1}(t) = J(\boldsymbol{n}_{n_1}(t)) = \frac{1}{\sqrt{(\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0))^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0))^2}} \left(\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0), \boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0) \right),$$

the curvatures of $(\gamma_{n_1}, \boldsymbol{n}_{n_1})$ are

$$\begin{aligned} \ell_{n_1}(t) \;=\; & (m(t) \left((\boldsymbol{n}_1(t) \cdot \boldsymbol{n}_2(t_0)) (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0)) - (\boldsymbol{n}_1(t) \cdot \boldsymbol{t}(t_0)) (\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0)) \right) \\ & + n(t) \left((\boldsymbol{n}_2(t) \cdot \boldsymbol{t}(t_0)) (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0)) - (\boldsymbol{n}_2(t) \cdot \boldsymbol{t}(t_0)) (\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0)) \right))/ \\ & \left((\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0))^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0))^2 \right) \end{aligned}$$

and

$$\beta_{n_1}(t) = \alpha(t) \sqrt{(\boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0))^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{t}(t_0))^2}.$$

We remark that $\ell_{n_1}(t_0) = n(t_0)$ and $\beta_{n_1}(t_0) = \alpha(t_0)$. The projection of γ along the direction $\boldsymbol{n}_2(t_0)$ is similar to the case of the direction $\boldsymbol{n}_1(t_0)$.

Next, we consider the projection of γ along the direction $\boldsymbol{t}(t_0)$ given by $\boldsymbol{t}_{\boldsymbol{t}} : I \to \mathbb{R}^2$; $\boldsymbol{t}_{\boldsymbol{t}}(t) = (\gamma(t) \cdot \boldsymbol{n}_1(t_0), \gamma(t) \cdot \boldsymbol{n}_2(t_0))$. Then $\dot{\gamma}_{\boldsymbol{t}}(t) = \alpha(t)(\boldsymbol{t}(t) \cdot \boldsymbol{n}_1(t_0), \boldsymbol{t}(t) \cdot \boldsymbol{n}_2(t_0))$. In this case, $\gamma_{\boldsymbol{t}}$ is not always a frontal, that is, there does not exist a smooth mapping $\boldsymbol{n}_{\boldsymbol{t}} : I \to S^1$ such that $(\gamma_{\boldsymbol{t}}, \boldsymbol{n}_{\boldsymbol{t}}) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve, see Example 1.3.12. However, if $\gamma_{\boldsymbol{t}}$ is not infinitely flat around t_0 (i.e. either $\gamma(t) \cdot \boldsymbol{n}_1(t_0)$ or $\gamma(t) \cdot \boldsymbol{n}_2(t_0)$ does not belong to $\mathfrak{m}_1^\infty$), then $\gamma_{\boldsymbol{t}}$ is a frontal (cf. [10]).

Generally, we fix a positive orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ on $\mathbb{R}^3$ with $\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2$ and $\boldsymbol{v}_3 \neq \pm \boldsymbol{t}(t_0)$. We consider the orthogonal projection along the direction $\boldsymbol{v}_3$ given by $\gamma_{v_3} : I \to \mathbb{R}^2$; $\gamma_{v_3}(t) = (\gamma(t) \cdot \boldsymbol{v}_1, \gamma(t) \cdot \boldsymbol{v}_2)$. Then $\dot{\gamma}_{v_3}(t) = \alpha(t)(\boldsymbol{t}(t) \cdot \boldsymbol{v}_1, \boldsymbol{t}(t) \cdot \boldsymbol{v}_2)$. By the assumption, there is a subinterval $\widetilde{I}$ of I around t_0 such that $(\boldsymbol{t}(t) \cdot \boldsymbol{v}_1)^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2)^2 \neq 0$ for all $t \in \widetilde{I}$. We define a smooth mapping $\boldsymbol{n}_{v_3} : \widetilde{I} \to S^1$ by

$$\boldsymbol{n}_{v_3}(t) = \frac{1}{\sqrt{(\boldsymbol{t}(t) \cdot \boldsymbol{v}_1)^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2)^2}} \left(\boldsymbol{t}(t) \cdot \boldsymbol{v}_2, -\boldsymbol{t}(t) \cdot \boldsymbol{v}_1 \right).$$

Then $(\gamma_{v_3}, \boldsymbol{n}_{v_3}) : \widetilde{I} \to \mathbb{R}^2 \times S^1$ is a Legendre curve. Since $\boldsymbol{t}_{v_3} : I \to S^1$ is given by

$$\boldsymbol{t}_{v_3}(t) = J(\boldsymbol{n}_{v_3}(t)) = \frac{1}{\sqrt{(\boldsymbol{t}(t) \cdot \boldsymbol{v}_1)^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2)^2}} \left(\boldsymbol{t}(t) \cdot \boldsymbol{v}_1, \boldsymbol{t}(t) \cdot \boldsymbol{v}_2 \right),$$

the curvatures of the Legendre curve $(\gamma_{v_3}, \boldsymbol{n}_{v_3})$ are

$$\begin{aligned} \ell_{v_3}(t) \;=\; & (m(t) \left((\boldsymbol{n}_1(t) \cdot \boldsymbol{v}_1) (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2) - (\boldsymbol{n}_1(t) \cdot \boldsymbol{v}_2) (\boldsymbol{t}(t) \cdot \boldsymbol{v}_1) \right) \\ & + n(t) \left((\boldsymbol{n}_2(t) \cdot \boldsymbol{v}_1) (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2) - (\boldsymbol{n}_2(t) \cdot \boldsymbol{v}_2) (\boldsymbol{t}(t) \cdot \boldsymbol{v}_1) \right))/ \\ & \left((\boldsymbol{t}(t) \cdot \boldsymbol{v}_1)^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2)^2 \right) \end{aligned}$$

and

$$\beta_{v_3}(t) = \alpha(t) \sqrt{(\boldsymbol{t}(t) \cdot \boldsymbol{v}_1)^2 + (\boldsymbol{t}(t) \cdot \boldsymbol{v}_2)^2}.$$

Remark 1.3.11 If we take a positive orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ on $\mathbb{R}^3$ with $\boldsymbol{v}_3 \in S^2 \setminus \{\pm \boldsymbol{t}(I)\}$, then we may consider $\widetilde{I} = I$. In this case, the Legendre curve $(\gamma_{\boldsymbol{v}_3}, \boldsymbol{n}_{\boldsymbol{v}_3})$ can be defined globally.

Example 1.3.12 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be

$$\gamma(t) = \begin{cases} (t, 0, e^{-1/t^2}) & \text{if} \quad t > 0, \\ (0,0,0) & \text{if} \quad t = 0, \\ (t, e^{-1/t^2}, 0) & \text{if} \quad t < 0, \end{cases}$$

(cf. Figure 1.1). The curve γ is regular but does not satisfy the linearly independent condition at $t = 0$. However, γ is a framed base curve. We have the smooth mapping $(\boldsymbol{n}_1, \boldsymbol{n}_2) : \mathbb{R} \to V_{3,2}$;

$$\begin{aligned}
\boldsymbol{n}_1(t) &= \begin{cases} \left(1/\sqrt{2+\dot{f}(t)^2}\right)(\dot{f}(t), -1, -1) & \text{if} \quad t \neq 0, \\ \left(1/\sqrt{2}\right)(0, -1, -1) & \text{if} \quad t = 0, \end{cases} \\
\boldsymbol{n}_2(t) &= \begin{cases} \left(1/\sqrt{(1+\dot{f}(t)^2)(2+\dot{f}(t)^2)}\right)(\dot{f}(t), 1+\dot{f}(t)^2, -1) & \text{if} \quad t > 0, \\ \left(1/\sqrt{2}\right)(0, 1, -1) & \text{if} \quad t = 0, \\ \left(1/\sqrt{(1+\dot{f}(t)^2)(2+\dot{f}(t)^2)}\right)(-\dot{f}(t), 1, -1-\dot{f}(t)^2) & \text{if} \quad t < 0, \end{cases}
\end{aligned}$$

where $f(t) = e^{-1/t^2}$ for $t \neq 0$. It is easy to see that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is a framed curve. Since $\boldsymbol{t} : \mathbb{R} \to S^2$;

$$\boldsymbol{t}(t) = \boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t) = \begin{cases} \left(1/\sqrt{1+\dot{f}(t)^2}\right)(1, 0, \dot{f}(t)) & \text{if} \quad t > 0, \\ (1,0,0) & \text{if} \quad t = 0, \\ \left(1/\sqrt{1+\dot{f}(t)^2}\right)(1, \dot{f}(t), 0) & \text{if} \quad t < 0, \end{cases}$$

the curvature of the framed curve $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is given by

$$\begin{aligned}
\ell(t) &= \begin{cases} \dot{f}(t)\ddot{f}(t)/\left((2+\dot{f}(t)^2)\sqrt{1+\dot{f}(t)^2}\right)(1, 0, \dot{f}(t)) & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0, \\ -\dot{f}(t)\ddot{f}(t)/\left((2+\dot{f}(t)^2)\sqrt{1+\dot{f}(t)^2}\right)(1, 0, \dot{f}(t)) & \text{if} \quad t < 0, \end{cases} \\
m(t) &= \begin{cases} \ddot{f}(t)/\sqrt{(1+\dot{f}(t)^2)(2+\dot{f}(t)^2)} & \text{if} \quad t \neq 0, \\ 0 & \text{if} \quad t = 0, \end{cases} \\
n(t) &= \begin{cases} \ddot{f}(t)/\left((1+\dot{f}(t)^2)\sqrt{2+\dot{f}(t)^2}\right) & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0, \\ -\ddot{f}(t)/\left((1+\dot{f}(t)^2)\sqrt{2+\dot{f}(t)^2}\right) & \text{if} \quad t < 0, \end{cases} \\
\alpha(t) &= \begin{cases} \sqrt{1+\dot{f}(t)^2} & \text{if} \quad t \neq 0, \\ 1 & \text{if} \quad t = 0. \end{cases}
\end{aligned}$$

We remark that consider the projection to the direction $\boldsymbol{t}(0) = (1,0,0)$. Then $\boldsymbol{t_t} : \mathbb{R} \to \mathbb{R}^2$ is given by

$$\gamma_{\boldsymbol{t}}(t) = \begin{cases} -(1/\sqrt{2})(e^{-1/t^2}, e^{-1/t^2}) & \text{if} \quad t > 0, \\ (0,0) & \text{if} \quad t = 0, \\ (1/\sqrt{2})(-e^{-1/t^2}, e^{-1/t^2}) & \text{if} \quad t < 0. \end{cases}$$

It follows that $\gamma_{\boldsymbol{t}} : \mathbb{R} \to \mathbb{R}^2$ is not a frontal (cf. [10]). We plot $\gamma_{\boldsymbol{t}}$ in Figure 1.2.

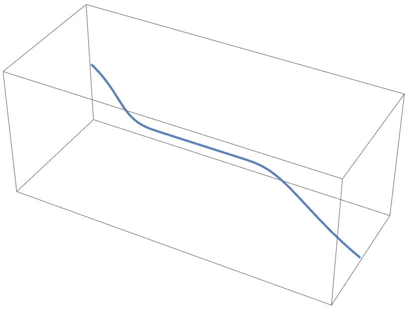

FIGURE 1.1: γ of Example 1.3.12

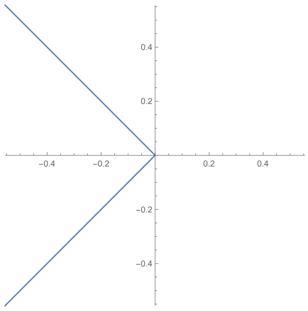

FIGURE 1.2: $\gamma_{\boldsymbol{t}}$ of Example 1.3.12

Chapter 2

Frenet type framed base curves and developable surfaces

There are several articles concerning singularities of the tangent developable surface (i.e. the envelope of osculating planes), the focal developable surface (i.e. the envelope of normal planes) and the rectifying developable surface (i.e. the envelope of rectifying planes) of a space curve (cf. [7, 23–26, 31, 33, 34, 36, 37, 40]). In [23–25] Goo Ishikawa investigated relationships between singularities of the tangent developable surface and the type (n_1, n_2, n_3) of a space curve. An (n_1, n_2, n_3)-type space curve may have singular points. On the other hand, the author can not find any article concerning the focal developable surface and the rectifying developable surface of a space curve with singular points.

In this chapter, we consider the focal developable surface and the rectifying developable surface of a space curve with singular points. In order to define these notions, we apply the theory of framed curves under a certain condition (cf. Chapter 1 and [22]). For them, we give characterizations as developable surfaces (cf. Theorems 2.2.2 and 2.3.2) and characterizations of singularities (cf. Theorems 2.2.3 and 2.3.3). This chapter is based on [19, 21].

2.1 Preliminaries

In this section, in order to investigate the focal developable surface and the rectifying developable surface of a space curve with singular points, we briefly review necessary notions.

2.1.1 Frenet curves and developable surfaces

We briefly review basic concepts on classical differential geometry of regular space curves in $\mathbb{R}^3$. Let I be an interval, and let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve, that is, $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linearly independent for all $t \in I$. Then we have an orthonormal frame

$$\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\} = \left\{ \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}, \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)}{\|(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)\|}, \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|} \right\}$$

which is called the *Frenet frame* along $\gamma(t)$. By standard arguments, we have the following Frenet-Serret formula:

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \|\dot{\gamma}(t)\|\kappa(t) & 0 \\ -\|\dot{\gamma}(t)\|\kappa(t) & 0 & \|\dot{\gamma}(t)\|\tau(t) \\ 0 & \|\dot{\gamma}(t)\|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix},$$

where

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}, \ \tau(t) = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t))}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}.$$

We call $\kappa(t)$ a *curvature* and $\tau(t)$ a *torsion* of the Frenet curve $\gamma(t)$. We remark that the curvature $\kappa(t)$ and the torsion $\tau(t)$ are independent of the choice of parameterizations.

Let γ be a Frenet curve. It is known that developable surfaces associated with the Frenet frame as follows:

(1) The tangent developable surface:

$$TD_\gamma(t,u) = \gamma(t) + u\boldsymbol{t}(t).$$

(2) The focal developable surface:

$$FD_\gamma(t,u) = \gamma(t) + \frac{1}{\kappa(t)} - \frac{\dot{\kappa}(t)}{\|\dot{\gamma}(t)\|\kappa^2(t)\tau(t)}\boldsymbol{b}(t) \text{ (when } \tau(t) \neq 0).$$

(3) The rectifying developable surface:

$$RD_\gamma(t,u) = \gamma(t) + u\frac{\tau(t)\boldsymbol{t}(t) + \kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.$$

2.1.2 Frenet type framed base curves

If γ has a singular point, we cannot construct the Frenet frame along $\gamma(t)$. However, we can define the Frenet type frame along $\gamma(t)$ under a certain condition.

Definition 2.1.1 We say that $\gamma : I \to \mathbb{R}^3$ is a *Frenet type framed base curve* if there exist a regular spherical curve $\boldsymbol{t} : I \to S^2$ and a smooth function $\alpha : I \to \mathbb{R}$ such that $\dot{\gamma}(t) = \alpha(t)\boldsymbol{t}(t)$ for all $t \in I$. Then we call $\boldsymbol{t}(t)$ a *unit tangent vector* and $\alpha(t)$ a *speed function* of $\gamma(t)$.

Clearly, t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$. We define a *unit principal normal vector* $\boldsymbol{n}(t) = \dot{\boldsymbol{t}}(t)/\|\dot{\boldsymbol{t}}(t)\|$ and a *unit binormal vector* $\boldsymbol{b}(t) = \boldsymbol{t}(t) \times \boldsymbol{n}(t)$ of $\gamma(t)$. Then we have an orthonormal frame $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ along $\gamma(t)$ which is called the *Frenet type frame* along $\gamma(t)$. By standard arguments, we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix},$$

where

$$\kappa(t) = \|\dot{\boldsymbol{t}}(t)\|, \ \tau(t) = \frac{\det(\boldsymbol{t}(t), \dot{\boldsymbol{t}}(t), \ddot{\boldsymbol{t}}(t))}{\|\dot{\boldsymbol{t}}(t)\|^2}.$$

We call $\kappa(t)$ a *curvature* and $\tau(t)$ a *torsion* of the Frenet type framed base curve $\gamma(t)$. We can easily check that γ is a framed base curve. More precisely, $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ is a framed curve with the curvature $(\tau(t), -\kappa(t), 0, \alpha(t))$ (cf. Section 1.1).

We define a vector field $\boldsymbol{d}(t)$ along $\gamma(t)$ by

$$\boldsymbol{d}(t) = \tau(t)\boldsymbol{t}(t) + \kappa(t)\boldsymbol{b}(t),$$

which is called a *Darboux type vector* along $\gamma(t)$. By using the Darboux type vector, the Frenet-Serret type formula is written as follows:

$$\begin{cases} \dot{\boldsymbol{t}}(t) &= \boldsymbol{d}(t) \times \boldsymbol{t}(t), \\ \dot{\boldsymbol{n}}(t) &= \boldsymbol{d}(t) \times \boldsymbol{n}(t), \\ \dot{\boldsymbol{b}}(t) &= \boldsymbol{d}(t) \times \boldsymbol{b}(t). \end{cases}$$

Therefore, the Darboux type vector plays an important role for study of the Frenet type framed base curve. Since $\kappa(t) > 0$, we can define a *spherical Darboux type vector* by

$$\overline{\boldsymbol{d}}(t) = \frac{\tau(t)\boldsymbol{t}(t) + \kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.$$

Remark 2.1.2 Since $\boldsymbol{t}(t)$ is a regular curve, we uniquely obtain the unit principal normal vector $\boldsymbol{n}(t)$ and the unit binormal vector $\boldsymbol{n}(t)$. Therefore, $\kappa(t)$, $\tau(t)$ and $\overline{\boldsymbol{d}}(t)$ is uniquely determined with respect to $\boldsymbol{t}(t)$.

2.1.3 Support functions

Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. We define a family of functions $F_{\boldsymbol{t}} : I \times \mathbb{R} \to \mathbb{R}$; $F_{\boldsymbol{t}}(t, \boldsymbol{x}) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{t}(t)$. We call $F_{\boldsymbol{t}}$ the *support function* of γ with respect to the unit tangent vector $\boldsymbol{t}$. We denote $f_{\boldsymbol{t},\boldsymbol{x}_0}(t) = F_{\boldsymbol{t}}(t, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$. Then we have the following proposition:

Proposition 2.1.3 *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following:*

(1) $f_{\boldsymbol{t},\boldsymbol{x}_0}(t_0) = 0$ *if and only if there exist $a, b \in \mathbb{R}$ such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = a\boldsymbol{n}(t_0) + b\boldsymbol{b}(t_0).$$

(2) $f_{\boldsymbol{t},\boldsymbol{x}_0}(t_0) = \dot{f}_{\boldsymbol{t},\boldsymbol{x}_0}(t_0) = 0$ *if and only if there exists $a \in \mathbb{R}$ such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = -\frac{\alpha(t_0)}{\kappa(t_0)}\boldsymbol{n}(t_0) + a\boldsymbol{b}(t_0).$$

(3) $f_{\boldsymbol{t},\boldsymbol{x}_0}(t_0) = \dot{f}_{\boldsymbol{t},\boldsymbol{x}_0}(t_0) = \ddot{f}_{\boldsymbol{t},\boldsymbol{x}_0}(t_0) = 0$ *if and only if there exists $a \in \mathbb{R}$ such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = -\frac{\alpha(t_0)}{\kappa(t_0)}\boldsymbol{n}(t_0) + a\boldsymbol{b}(t_0)$$

and

$$\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0) - a\kappa^2(t_0)\tau(t_0) = 0.$$

Proof. Since $f_{\boldsymbol{t},\boldsymbol{x}_0}(t) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{t}(t)$, we have the following calculations:

(i) $f_{\boldsymbol{t},\boldsymbol{x}_0}(t) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{t}(t)$,

(ii) $\dot{f}_{\boldsymbol{t},\boldsymbol{x}_0}(t) = \alpha(t) + (\gamma(t) - \boldsymbol{x}) \cdot (\kappa(t)\boldsymbol{n}(t))$,

(iii) $\ddot{f}_{\boldsymbol{t},\boldsymbol{x}_0}(t) = \dot{\alpha}(t) + (\gamma(t) - \boldsymbol{x}) \cdot (-\kappa^2(t)\boldsymbol{t}(t) + \dot{\kappa}(t)\boldsymbol{n}(t) + \kappa(t)\tau(t)\boldsymbol{b}(t))$.

By (i), (1) follows.

By (ii), $f_{t,x_0}(t_0) = \dot{f}_{t,x_0}(t_0) = 0$ if and only if there exist $a, b \in \mathbb{R}$ such that $\gamma(t_0) - \boldsymbol{x} = a\boldsymbol{n}(t_0) + b\boldsymbol{b}(t_0)$ and $\alpha(t_0) + a\kappa(t_0) = 0$. Since $\kappa(t_0) > 0$, we have

$$a = -\frac{\alpha(t_0)}{\kappa(t_0)},$$

so that there exists $c \in \mathbb{R}$ such that

$$\gamma(t_0) - \boldsymbol{x}_0 = -\frac{\alpha(t_0)}{\kappa(t_0)}\boldsymbol{n}(t_0) + c\boldsymbol{b}(t_0).$$

Therefore, (2) holds.

By (iii), $f_{t,x_0}(t_0) = \dot{f}_{t,x_0}(t_0) = \ddot{f}_{t,x_0}(t_0) = 0$ if and only if there exists $a \in \mathbb{R}$ such that

$$\gamma(t_0) - \boldsymbol{x}_0 = -\frac{\alpha(t_0)}{\kappa(t_0)}\boldsymbol{n}(t_0) + a\boldsymbol{b}(t_0)$$

and

$$\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0) - a\kappa^2(t_0)\tau(t_0) = 0.$$

This completes the proof. $\square$

For a support function of γ with respect to the unit tangent vector $\boldsymbol{t}$, the discriminant set

$$\mathcal{D}_{F_t} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \text{there exists } t \in I \text{ such that } F_t = \frac{\partial F_t}{\partial t} = 0 \text{ at } (t, \boldsymbol{x}) \right\}$$

corresponds with the focal developable surface of γ (cf. Section 2.2.1). Moreover the secondary discriminant set

$$\mathcal{D}^2_{F_t} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \text{there exists } t \in I \text{ such that } F_t = \frac{\partial F_t}{\partial t} = \frac{\partial^2 F_t}{\partial t^2} = 0 \text{ at } (t, \boldsymbol{x}) \right\}$$

corresponds with the evolute of γ under the condition $\tau(t) \neq 0$ (cf. Section 2.2.2).

On the other hand, we define a family of functions $F_{\boldsymbol{n}} : I \times \mathbb{R} \to \mathbb{R}$; $F_{\boldsymbol{n}}(t, \boldsymbol{x}) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{n}(t)$. We call $F_{\boldsymbol{n}}$ the *support function* of γ with respect to the unit principal normal vector $\boldsymbol{n}$. We denote $f_{n,x_0}(t) = F_{\boldsymbol{n}}(t, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$. Then we have the following proposition:

Proposition 2.1.4 *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following:*

(1) $f_{n,x_0}(t_0) = 0$ *if and only if there exist $a, b \in \mathbb{R}$ such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = a\boldsymbol{t}(t_0) + b\boldsymbol{b}(t_0).$$

(2) $f_{n,x_0}(t_0) = \dot{f}_{n,x_0}(t_0) = 0$ *if and only if there exists $a \in \mathbb{R}$ such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = a\frac{\tau(t_0)\boldsymbol{t}(t_0) + \kappa(t_0)\boldsymbol{b}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

(3) $f_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = \dot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = \ddot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = 0$ *if and only if there exists* $a \in \mathbb{R}$ *such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = a\frac{\tau(t_0)\boldsymbol{t}(t_0) + \kappa(t_0)\boldsymbol{b}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}$$

and

$$\alpha(t_0)\kappa(t_0) + a\frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} = 0.$$

Proof. Since $f_{\boldsymbol{n},\boldsymbol{x}_0}(t) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{n}(t)$, we have the following calculations:

(i) $f_{\boldsymbol{n},\boldsymbol{x}_0}(t) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{n}(t)$,

(ii) $\dot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t) = (\gamma(t) - \boldsymbol{x}) \cdot (-\kappa(t)\boldsymbol{t}(t) + \tau(t)\boldsymbol{b}(t))$,

(iii) $\ddot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t) = \alpha(t)\kappa(t) + (\gamma(t) - \boldsymbol{x}) \cdot (-\dot{\kappa}(t)\boldsymbol{t}(t) - (\kappa^2(t) + \tau^2(t)) + \dot{\tau}(t)\boldsymbol{b}(t))$.

By (i), (1) follows.

By (ii), $f_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = \dot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = 0$ if and only if there exist $a, b \in \mathbb{R}$ such that $\gamma(t_0) - \boldsymbol{x} = a\boldsymbol{t}(t_0) + b\boldsymbol{b}(t_0)$ and $-a\kappa(t_0) + b\tau(t_0) = 0$. Since $\kappa(t_0) > 0$, we have

$$a = b\frac{\tau(t_0)}{\kappa(t_0)},$$

so that there exists $c \in \mathbb{R}$ such that

$$\gamma(t_0) - \boldsymbol{x}_0 = c\frac{\tau(t_0)\boldsymbol{t}(t_0) + \kappa(t_0)\boldsymbol{b}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

Therefore (2) holds.

By (iii), $f_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = \dot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = \ddot{f}_{\boldsymbol{n},\boldsymbol{x}_0}(t_0) = 0$ if and only if there exists $a \in \mathbb{R}$ such that

$$\gamma(t_0) - \boldsymbol{x}_0 = a\frac{\tau(t_0)\boldsymbol{t}(t_0) + \kappa(t_0)\boldsymbol{b}(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

and

$$\alpha(t_0)\kappa(t_0) + a\frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}} = 0.$$

This completes the proof. □

For a support function of γ with respect to the unit principal normal vector $\boldsymbol{n}$, the discriminant set

$$\mathcal{D}_{F_n} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \text{there exists } t \in I \text{ such that } F_n = \frac{\partial F_n}{\partial t} = 0 \text{ at } (t, \boldsymbol{x}) \right\}$$

corresponds with the rectifying developable surface of γ (cf. Section 2.3.1).

2.1.4 Criteria of singularities for wave fronts and frontals

We briefly review some properties of wave fronts and frontals. For details, see [1, 2, 9, 29, 32]. Let $f : V \to \mathbb{R}^3$; $(u, v) \mapsto f(u, v)$ be a smooth mapping, where $V \subset \mathbb{R}^2$ is an open set. We say that f is a *frontal* if there exists a unit vector field $\boldsymbol{\nu}$ along f such that $(f, \boldsymbol{\nu}) : U \to T_1\mathbb{R}^3$ is an isotropic map, where $T_1\mathbb{R}^3$ is the unit tangent bundle of $\mathbb{R}^3$ equipped with the canonical contact structure. If $(f, \boldsymbol{\nu})$ is an immersion, we call

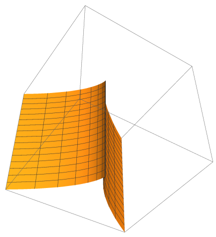

FIGURE 2.1: The cuspidal edge

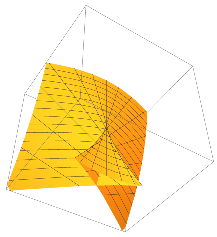

FIGURE 2.2: The swallowtail

f a *wave front*. By definition, $(f, \boldsymbol{\nu})$ is isotropic if and only if

$$\frac{\partial f}{\partial u}(u,v)\cdot\boldsymbol{\nu}(u,v)=0 \text{ and } \frac{\partial f}{\partial v}(u,v)\cdot\boldsymbol{\nu}(u,v)=0.$$

We call $\boldsymbol{\nu}$ the *unit normal vector field* or the *Gauss map* of f. Let $f : V \to \mathbb{R}^3$ be a frontal. The *signed area density function* $\lambda : U \to \mathbb{R}$ is defined to be

$$\lambda(u,v)=\det\left(\frac{\partial f}{\partial u}(u,v),\frac{\partial f}{\partial v}(u,v),\boldsymbol{\nu}(u,v)\right).$$

We call (u_0, v_0) a singular point of f if $\lambda(u_0, v_0) = 0$. A singular point (u_0, v_0) of f is *non-degenerate* if $d\lambda(u_0, v_0) \neq 0$. Let (u_0, v_0) be a non-degenerate singular point of f. Then $S(f)$ is parametrized as a regular curve $\boldsymbol{c} : (\mathbb{R}, 0) \to (V, (u_0, v_0))$ which is called a *singular curve*, where $S(f)$ is the set of singular points of f. Moreover, there exists a unique non-zero vector field $\boldsymbol{\eta}$ up to non-zero scalar multiplications such that $df(\eta(t)) = \mathbf{0}$ on $S(f)$. We call $\boldsymbol{\eta}(t)$ the *null vector field*.

Two map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphisms $s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $t \circ f = g \circ s$. The following criteria of singularities for wave fronts and frontals are known:

Theorem 2.1.5 ([32]) *Let $f : V \to \mathbb{R}^3$ be a wave front, and let $(u_0, v_0) \in V$ be a non-degenerate singular point of f.*

(1) *f is $\mathcal{A}$-equivalent to the cuspidal edge **ce** at (u_0, v_0) if and only if $\boldsymbol{\eta}\lambda(p) \neq 0$.*

(2) *f is $\mathcal{A}$-equivalent to the swallowtail **sw** at (u_0, v_0) if and only if $\boldsymbol{\eta}\lambda(p) = 0$ and $\boldsymbol{\eta}\boldsymbol{\eta}\lambda(p) \neq 0$.*

Here, $\boldsymbol{ce} : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$; $(u, v) \mapsto (u, v^2, v^3)$ is the cuspidal edge (Figure 2.1) and $\boldsymbol{sw} : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$; $(u, v) \mapsto (3u^4 + u^2v, 4u^3 + 2uv, v)$ is the swallowtail (Figure 2.2).

Theorem 2.1.6 ([9]) *Let $f : V \to \mathbb{R}^3$ be a frontal (not front), and let $(u_0, v_0) \in V$ be a non-degenerate singular point of f. f is $\mathcal{A}$-equivalent to the cuspidal cross cap **ccr** if and only if $\boldsymbol{\eta}\lambda(p) \neq 0$, $\Phi(0) = 0$ and $\dot{\Phi}(0) \neq 0$. Here, $\Phi(t) = \det(\dot{f}(\boldsymbol{c}(t)), \boldsymbol{\nu}(\boldsymbol{c}(t)), d\nu(\boldsymbol{\eta}(t)))$* and $\boldsymbol{ccr} : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$; $(u, v) \mapsto (u, uv^3, v^2)$ is the cuspidal cross cap (Figure 2.3).

2.1.5 Ruled surfaces and developable surfaces

We briefly review notions and basic properties of ruled surfaces and developable surfaces. For details, see [30]. Let $\gamma : I \to \mathbb{R}^3$ and $\boldsymbol{\xi} : I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ be C^∞-mappings.

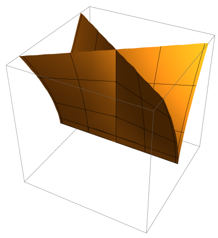

FIGURE 2.3: The cuspidal cross cap

Then we define a mapping $F_{(\gamma,\xi)} : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$F_{(\gamma,\xi)}(t,u) = \gamma(t) + u\boldsymbol{\xi}(t).$$

We call the mapping $F_{(\gamma,\xi)}$ a *ruled surface*, the mapping γ a *base curve* and the mapping $\boldsymbol{\xi}$ a *director curve*. The line defined by $\gamma(t) + u\boldsymbol{\xi}(t)$ for a fixed $t \in I$ is called a *ruling*. We call the ruled surface with vanishing Gaussian curvature on the regular part a *developable surface*. It is known that a ruled surface $F_{(\gamma,\xi)}$ is a developable surface if and only if

$$\det\left(\dot{\gamma}(t), \boldsymbol{\xi}(t), \dot{\boldsymbol{\xi}}(t)\right) = 0$$

(cf. [30]). If the director curve $\boldsymbol{\xi}$ is constant, we call $F_{(\gamma,\xi)}$ a (generalized) *cylinder*. If we define a mapping $\overline{\boldsymbol{\xi}} : I \to S^2$ by $\overline{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(t)/\|\boldsymbol{\xi}(t)\|$, we have $F_{(\gamma,\xi)}(I \times \mathbb{R}) = F_{(\gamma,\overline{\xi})}(I \times \mathbb{R})$. In this case, $F_{(\gamma,\xi)}$ is a cylinder if and only if $\dot{\overline{\boldsymbol{\xi}}}(t) \equiv \mathbf{0}$. We say that $F_{(\gamma,\xi)}$ is *non-cylindrical* if $\dot{\overline{\boldsymbol{\xi}}}(t) \neq \mathbf{0}$. Suppose that $F_{(\gamma,\xi)}$ is non-cylindrical. Then the *striction curve* is defined to be

$$\sigma(t) = \gamma(t) - \frac{\dot{\gamma}(t) \cdot \dot{\overline{\boldsymbol{\xi}}}(t)}{\dot{\overline{\boldsymbol{\xi}}}(t) \cdot \dot{\overline{\boldsymbol{\xi}}}(t)} \overline{\boldsymbol{\xi}}(t).$$

It is known that singularities of a non-cylindrical ruled surface is located on the striction curve (cf. [30]). A non-cylindrical ruled surface $F_{(\gamma,\xi)}$ is a *cone* if the striction curve σ is constant. It is known that a non-cylindrical developable surface $F_{(\gamma,\xi)}$ is a wave front if and only if

$$\det\left(\boldsymbol{\xi}(t), \dot{\boldsymbol{\xi}}(t), \ddot{\boldsymbol{\xi}}(t)\right) \neq 0$$

(cf. [30]). In this case, we call $F_{(\gamma,\xi)}$ a (non-cylindrical) *developable front*.

2.2 Focal developable surfaces and evolutes

For a Frenet curve, the focal developable surface and the evolute are classical objects in differential geometry. There are many articles concerning the focal developable (i.e. the envelope of normal planes) and the evolute (i.e. the locus of center of osculating spheres) of a Frenet curve (For instance [5, 8, 17, 37, 38, 42]).

In this section, we consider the focal developable surface and the evolute of a Frenet type framed base curve.

2.2.1 Focal developable surfaces

We define the focal developable surface of a Frenet type framed base curve as the discriminant set of the support function of the curve with respect to the unit tangent vector (cf. Proposition 2.1.3). Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$.

Definition 2.2.1 We define a mapping $\mathcal{FD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\mathcal{FD}_\gamma(t,u) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) + u\boldsymbol{b}(t).$$

We call $\mathcal{FD}_\gamma$ the *focal developable surface* of γ.

The focal developable surface $\mathcal{FD}_\gamma$ is a ruled surface and we have

$$\begin{aligned}
&\det\left(\frac{d}{dt}\left(\gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t)\right), \boldsymbol{b}(t), \dot{\boldsymbol{b}}(t)\right) \\
&= \det\left(\frac{d}{dt}\left(\frac{\alpha(t)}{\kappa(t)}\right)\boldsymbol{n}(t) + \frac{\alpha(t)\tau(t)}{\kappa(t)}\boldsymbol{b}(t), \boldsymbol{b}(t), -\tau(t)\boldsymbol{n}(t)\right) \\
&= 0.
\end{aligned}$$

This means that $\mathcal{FD}_\gamma$ is a developable surface (cf .Section 2.1.5). Moreover, $\mathcal{FD}_\gamma$ is a developable front (cf. Sections 2.1.4 and 2.1.5). We introduce an invariants $\sigma_f(t)$ as follows:

$$\sigma_f(t) = \frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right), \ (\text{when } \tau(t) \neq 0).$$

We remark that $\tau(t_0) = 0$ if and only if $t_0 \in I$ is a singular point of the unit binormal vector $\boldsymbol{b}(t)$.

Theorem 2.2.2 *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following:*

(1) *The following are equivalent:*

  (a) *$\mathcal{FD}_\gamma$ is a cylinder,*

  (b) *$\tau(t) \equiv 0$.*

(2) *If $\tau(t) \neq 0$, then the following are equivalent:*

  (c) *$\mathcal{FD}_\gamma$ is a conical surface,*

  (d) *$\sigma_f(t) \equiv 0$.*

Proof. (1) By definition, $\mathcal{FD}_\gamma$ is a cylinder if and only if $\boldsymbol{b}(t)$ is constant. Since $\dot{\boldsymbol{b}}(t) = -\tau(t)\boldsymbol{n}(t)$, $\boldsymbol{b}(t)$ is constant if and only if $\tau(t) \equiv 0$.

(2) We consider the striction curve $\boldsymbol{\sigma} : I \to \mathbb{R}$ is defined by

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) - \frac{\frac{d}{dt}\left(\gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t)\right) \cdot \dot{\boldsymbol{b}}(t)}{\dot{\boldsymbol{b}}(t) \cdot \dot{\boldsymbol{b}}(t)}\boldsymbol{b}(t) \\
&= \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) - \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\boldsymbol{b}(t).
\end{aligned}$$

Then (2)-(c) is equivalent to the condition $\dot{\sigma}(t) \equiv 0$. We can calculate that

$$\begin{aligned}
\dot{\sigma}(t) &= \dot{\gamma}(t) + \frac{d}{dt}\left(\frac{\alpha(t)}{\kappa(t)}\right)\boldsymbol{n}(t) + \frac{\alpha(t)}{\kappa(t)}\dot{\boldsymbol{n}}(t) \\
&\quad - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\boldsymbol{b}(t) - \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\dot{\boldsymbol{b}}(t) \\
&= \alpha(t)\boldsymbol{t}(t) - \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)}\boldsymbol{n}(t) + \frac{\alpha(t)}{\kappa(t)}(-\kappa(t)\boldsymbol{t}(t) + \tau(t)\boldsymbol{b}(t)) \\
&\quad - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\boldsymbol{b}(t) - \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}(-\tau(t)\boldsymbol{n}(t)) \\
&= \left(\frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\right)\boldsymbol{b}(t) \\
&= \sigma_f(t)\boldsymbol{b}(t).
\end{aligned}$$

It follows that (2)-(c) and (2)-(d) are equivalent. □

We remark that developable surfaces are classified into cylinders, cones or tangent surfaces of space curves (cf. [43]). By result of Theorem 2.2.2, two invariants $\tau(t)$ and $\sigma_f(t)$ might be related to singularities of the focal developable surface. Actually, we can characterize singularities of the focal developable surface of a Frenet type framed base curve by using these invariants $\tau(t)$ and $\sigma_f(t)$.

Theorem 2.2.3 *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following:*

(1) *(t_0, u_0) is a singular point of $\mathcal{FD}_\gamma$ if and only if*

$$\frac{\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0)}{\kappa^2(t_0)} + u_0\tau(t_0) = 0.$$

(2) *$\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge* ***ce*** *at (t_0, u_0) if and only if*

   (a) *$\tau(t_0) \neq 0$, $\sigma_f(t_0) \neq 0$ and*

$$u_0 = -\frac{\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0)}{\kappa^2(t_0)\tau(t_0)},$$

   or

   (b) *$\tau(t_0) = 0$, $\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0) = 0$ and*

$$\frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)}\right)\bigg|_{t=t_0} + u_0\dot{\tau}(t_0) \neq 0.$$

(3) *$\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail* ***sw*** *at (t_0, u_0) if and only if $\tau(t_0) \neq 0$, $\sigma_f(t_0) = 0$, $\dot{\sigma}_f(t_0) \neq 0$ and*

$$u_0 = -\frac{\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0)}{\kappa^2(t_0)\tau(t_0)}.$$

Proof. By a straightforward calculation, we have

$$\frac{\partial \mathcal{FD}_\gamma}{\partial t}(t,u) \times \frac{\partial \mathcal{FD}_\gamma}{\partial t}(t,u) = -\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)} + u\tau(t)\right)\boldsymbol{t}(t).$$

Therefore (t_0, u_0) is a singular point of $\mathcal{FD}_\gamma$ if and only if

$$\frac{\alpha(t_0)\dot{\kappa}(t_0) - \dot{\alpha}(t_0)\kappa(t_0)}{\kappa^2(t_0)} + u_0\tau(t_0) = 0.$$

This means that (1) holds. Assertions (2) and (3) can be proven by using the criteria for the cuspidal edge $\boldsymbol{ce}$ and the swallowtail $\boldsymbol{sw}$ (cf. [32]). We give the signed density function $\lambda : I \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}\lambda(t,u) &= \det\left(\frac{\partial \mathcal{FD}_\gamma}{\partial t}(t,u), \frac{\partial \mathcal{FD}_\gamma}{\partial u}(t,u), \boldsymbol{t}(t)\right) \\ &= \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)} + u\tau(t).\end{aligned}$$

Suppose that (t_0, u_0) is a non-degenerate singular point of $\mathcal{FD}_\gamma$, that is,

$$\begin{aligned}\frac{\partial\lambda}{\partial t}(t_0,u_0) &= \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)}\right)\Bigg|_{t=t_0} + u_0\dot{\tau}(t_0) \neq 0 \\ &\text{or} \\ \frac{\partial\lambda}{\partial u}(t_0,u_0) &= \tau(t_0) \neq 0.\end{aligned}$$

(2)-(a) When $\tau(t_0) \neq 0$, we have the singular curve $\boldsymbol{c} : (I, t_0) \to (I \times \mathbb{R}, (t_0, u_0))$;

$$\boldsymbol{c}(t) = \left(t, -\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)$$

and the null vector field $\boldsymbol{\eta} : (I, t_0) \to \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$;

$$\boldsymbol{\eta}(t) = \left(1, -\frac{\alpha(t)\tau(t)}{\kappa(t)}\right).$$

By using the criterion for the cuspidal edge $\boldsymbol{ce}$, the focal developable surface $\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at (t_0, u_0) if and only if

$$\det(\dot{\boldsymbol{c}}(t_0), \boldsymbol{\eta}(t_0)) = -\frac{\alpha(t_0)\tau(t_0)}{\kappa(t_0)} + \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\Bigg|_{t=t_0} = -\sigma_f(t_0) \neq 0.$$

Therefore, (2)-(a) holds.

(2)-(b) When $\tau(t_0) = 0$ and

$$\frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)}\right)\Bigg|_{t=t_0} + u_0\dot{\tau}(t_0) \neq 0,$$

there exists a smooth function $\phi : (\mathbb{R}, u_0) \to (I, t_0)$ such that $\lambda(\phi(u), u) = 0$ and

$$\frac{d\phi}{du}(u) = -\left.\frac{\lambda_u(t,u)}{\lambda_t(t,u)}\right|_{t=\phi(u)}.$$

Accordingly, we have the singular curve $\boldsymbol{c} : (\mathbb{R}, u_0) \to (I \times \mathbb{R}, (t_0, u_0))$; $\boldsymbol{c}(u) = (\phi(u), u)$ and the null vector field

$$\boldsymbol{\eta}(u) = \left(1, -\frac{\alpha(\phi(u))\tau(\phi(u))}{\kappa(\phi(u))}\right).$$

Since

$$\det\left(\frac{d\boldsymbol{c}}{du}(u_0), \boldsymbol{\eta}(u_0)\right) = \det\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix} = -1 \neq 0,$$

the focal developable surface $\mathcal{FD}_\gamma$ is always $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at (t_0, u_0) under the assumption. Therefore, (2)-(b) holds.

(3) By the proof of (2)-(b), it is enough to consider in the same assumption as (2)-(a). By using the criterion for the swallowtail $\boldsymbol{sw}$, the focal developable surface $\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail $\boldsymbol{sw}$ at (t_0, u_0) if and only if

$$\det(\dot{\boldsymbol{c}}(t_0), \boldsymbol{\eta}(t_0)) = -\sigma_f(t_0) = 0 \text{ and } \frac{d\det(\dot{\boldsymbol{c}}, \boldsymbol{\eta})}{dt}(t_0) = -\dot{\sigma}_f(t_0) \neq 0.$$

Therefore, (3) holds. This completes the proof. □

2.2.2 Evolutes

We define the evolute of a Frenet type framed base curve as the secondary discriminant set of the support function of the curve with respect to the unit tangent vector under a certain condition (cf. Proposition 2.1.3). In the other words, the evolute is the striction curve of the focal developable surface. Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Throughout this section, we assume that $\tau(t) \neq 0$.

Definition 2.2.4 We define a *mapping* $\mathcal{E}_\gamma : I \to \mathbb{R}^3$ by

$$\mathcal{E}_\gamma(t) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) - \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\boldsymbol{b}(t).$$

We call $\mathcal{E}_\gamma$ the evolute of γ.

By a straightforward calculation, we have

$$\begin{aligned} \dot{\mathcal{E}}_\gamma(t) &= \left(\frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\right)\boldsymbol{b}(t) \\ &= \sigma_f(t)\boldsymbol{b}(t). \end{aligned}$$

Then we have the following proposition:

Proposition 2.2.5 *The evolute $\mathcal{E}_\gamma$ is a Frenet type framed base curve with $(\boldsymbol{b}, \sigma_f)$. The curvature and the torsion is given by $\kappa_{\mathcal{E}_\gamma}(t) = \mathrm{sgn}(\tau)\tau(t)$ and $\tau_{\mathcal{E}_\gamma}(t) = \kappa(t)$, where $\mathrm{sgn}(\tau)$ is the sign of τ.*

Proof. We can easily check that the evolute $\mathcal{E}_\gamma$ is a Frenet type framed base curve with $(\boldsymbol{b}, \sigma_f)$. By straightforward calculations, we have the principal normal vector $\boldsymbol{n}_{\mathcal{E}_\gamma}(t) = \dot{\boldsymbol{b}}(t)/\|\dot{\boldsymbol{b}}(t)\| = -\mathrm{sgn}(\tau)\boldsymbol{n}(t)$ and the binormal vector $\boldsymbol{b}_{\mathcal{E}_\gamma}(t) = \boldsymbol{b}(t) \times (-\mathrm{sgn}(\tau)\boldsymbol{n}(t)) = \mathrm{sgn}(\tau)\boldsymbol{t}(t)$. It follows that

$$\kappa_{\mathcal{E}_\gamma}(t) = \dot{\boldsymbol{b}}(t) \cdot \boldsymbol{n}_{\mathcal{E}_\gamma}(t) = (-\tau(t)\boldsymbol{n}(t)) \cdot (-\mathrm{sgn}(\tau)\boldsymbol{n}(t)) = \mathrm{sgn}(\tau)\tau(t)$$

and

$$\tau_{\mathcal{E}_\gamma}(t) = \dot{\boldsymbol{n}}_{\mathcal{E}_\gamma}(t) \cdot \boldsymbol{b}_{\mathcal{E}_\gamma}(t) = \mathrm{sgn}^2(\tau)((\kappa(t)\boldsymbol{t}(t) - \tau(t)\boldsymbol{b}(t)) \cdot \boldsymbol{t}(t)) = \kappa(t).$$

□

We give relationships between singularities of the evolute $\mathcal{E}_\gamma$ and of the focal developable surface $\mathcal{FD}_\gamma$ as corollary of Theorem 2.2.3.

Corollary 2.2.6 ([21]) *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\tau(t) \neq 0$. Suppose that (t_0, u_0) is a singular point of $\mathcal{FD}_\gamma$. Then we have the following:*

(1) *$\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at (t_0, u_0) if and only if the evolute $\mathcal{E}_\gamma$ is regular at t_0.*

(2) *$\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail $\boldsymbol{sw}$ at (t_0, u_0) if and only if the evolute $\mathcal{E}_\gamma$ is $\mathcal{A}$-equivalent to the $3/2$-cusp $\boldsymbol{c}$ at t_0.*

Here, $\boldsymbol{c} : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0);\ t \mapsto (t^2, t^3, 0)$ is the $3/2$-cusp.

Evolutes and spheres

For a Frenet curve $\gamma : I \to \mathbb{R}^3$ with $\tau(t) \neq 0$, the evolute E_γ is constant if and only if γ is a curve on a sphere $S^2(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x} - \boldsymbol{c}\| = r\}$ (cf. [5, 17]), where $\boldsymbol{c} \in \mathbb{R}^3$ and $\boldsymbol{c} \in \mathbb{R}_{>0}$. We investigate relationships between the evolute of a Frenet type framed base curve and a sphere. Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\tau(t) \neq 0$.

Proposition 2.2.7 ([21]) *If $\mathcal{E}_\gamma$ is constant, then there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a non-negative number $r \in \mathbb{R}$ such taht $\gamma(t) \in S^2(\boldsymbol{c}, r)$.*

Proof. Suppose that $\mathcal{E}_\gamma$ is constant, that is, $\sigma_f(t) \equiv 0$. Then we put $\boldsymbol{c} = \mathcal{E}_\gamma(t)$. By a straightforward calculation, we have

$$
\begin{aligned}
&\frac{d}{dt}\|\gamma(t) - \boldsymbol{c}\|^2 \\
&= 2\left(\frac{\alpha(t)}{\kappa(t)}\right)\frac{d}{dt}\left(\frac{\alpha(t)}{\kappa(t)}\right) + 2\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right) \\
&= -\frac{2(\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t))}{\kappa^2(t)\tau(t)}\left(\frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\right) \\
&= -\frac{2(\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t))}{\kappa^2(t)\tau(t)}\sigma_f(t) \\
&= 0
\end{aligned}
$$

Therefore, there exists a non-negative number $r = \|\gamma(t) - \boldsymbol{c}\|$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$. □

Lemma 2.2.8 ([21]) *$\mathcal{E}_\gamma$ is constant if and only if there exist functions $f, g : I \to \mathbb{R}$ and a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ such that $\gamma(t) - \boldsymbol{c} = f(t)\boldsymbol{n}(t) + g(t)\boldsymbol{b}(t)$.*

Proof. If $\mathcal{E}_\gamma$ is constant, then we can write $\gamma(t) - \boldsymbol{c} = f(t)\boldsymbol{n}(t) + g(t)\boldsymbol{b}(t)$, where

$$\boldsymbol{c} = \mathcal{E}_\gamma(t),\ f(t) = -\frac{\alpha(t)}{\kappa(t)} \text{ and } g(t) = \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}.$$

Conversely, suppose that there exist functions $f, g : I \to \mathbb{R}$ and a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ such that $\gamma(t) - \boldsymbol{c} = f(t)\boldsymbol{n}(t) + g(t)\boldsymbol{b}(t)$. Taking the derivative of the both sides, we have

$$\alpha(t)\boldsymbol{t}(t) = -\kappa(t)f(t)\boldsymbol{t}(t) + (\dot{f}(t) - \tau(t)g(t))\boldsymbol{n}(t) + (\dot{g}(t) + \tau(t)f(t))\boldsymbol{b}(t).$$

It follows that

$$\alpha(t) = -\kappa(t)f(t),\ \dot{f}(t) - \tau(t)g(t) = 0,\ \dot{g}(t) + \tau(t)f(t) = 0$$

and

$$\begin{aligned}\sigma_f(t) &= \frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right)\\ &= -\tau(t)f(t) - \frac{d}{dt}\left(\frac{\dot{f}(t)}{\tau(t)}\right)\\ &= \dot{g}(t) - \dot{g}(t)\\ &= 0.\end{aligned}$$

Therefore, $\mathcal{E}_\gamma$ is constant. □

Proposition 2.2.9 ([21]) *Suppose that the set of regular points of γ is dense in I. Then $\mathcal{E}_\gamma$ is constant if and only if there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a positive number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$.*

Proof. Suppose that $\mathcal{E}_\gamma$ is constant. By Proposition 2.2.7, there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a non-negative number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$. If $r = 0$, $\gamma(t) = \boldsymbol{c}$ and $\dot{\gamma}(t) \equiv \boldsymbol{0}$. This case does not occur because the set of reular points of γ is dense in I. Therefore, r is positive.

Conversely, suppose that there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a positive number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$. By the assumption, we have $\|\gamma(t) - \boldsymbol{c}\| = r^2$. Taking the derivative of the both sides, we have $\alpha(t)\boldsymbol{t}(t) \cdot (\gamma(t) - \boldsymbol{c}) \equiv 0$. Since the set of regular points of γ is dense in I, we have $\boldsymbol{t}(t) \cdot (\gamma(t) - \boldsymbol{c}) \equiv 0$. Then there exist functions $f, g : I \to \mathbb{R}$ such that $\gamma(t) - \boldsymbol{c} = f(t)\boldsymbol{n}(t) + g(t)\boldsymbol{b}(t)$. By Lemma 2.2.8, $\mathcal{E}_\gamma$ is constant. □

Contact between Frenet type framed base curves and evolutes

We investigate contact between Frenet type framed base curves and evolutes.

Proposition 2.2.10 ([21]) *Let $\gamma : I \to \mathbb{R}^3;\ t \mapsto \gamma(t)$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^3;\ u \mapsto \widetilde{\gamma}(u)$ be Frenet type framed base curves with $\tau(t) \neq 0$ and $\widetilde{\tau}(t) \neq 0$, respectively. If $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ have at least $(k+2)$-th order contact at $t = t_0$, $u = u_0$ for $k \in \mathbb{N}$, then $(\mathcal{E}_\gamma, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})$ and $(\mathcal{E}_{\widetilde{\gamma}}, -\mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{n}}, \mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{t}})$ have at least k-th order contact $t = t_0$, $u = u_0$.*

Proof. We give the proof by using the induction on k. First suppose that $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ have at least third order contact at $t = t_0$, $u = u_0$. By Theorem 1.3.10, we have

$$(\gamma, \boldsymbol{n}, \boldsymbol{b})(t_0) = (\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})(u_0),\ \frac{d}{dt}(\gamma, \boldsymbol{n}, \boldsymbol{b})(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})(u_0),$$

$$\frac{d^2}{dt^2}(\gamma, \boldsymbol{n}, \boldsymbol{b})(t_0) = \frac{d^2}{du^2}(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})(u_0),\ \mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0),\ \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\widetilde{\mathcal{F}}(u_0),$$

where $\mathcal{F}(t) = (\tau(t), -\kappa(t), 0, \alpha(t))$ and $\widetilde{\mathcal{F}}(u) = (\widetilde{\tau}(u), -\widetilde{\kappa}(u), 0, \widetilde{\alpha}(u))$. Therefore, we have $(\mathcal{E}_\gamma, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})(t_0) = (\mathcal{E}_{\widetilde{\gamma}}, -\mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{n}}, \mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{t}})(u_0)$.

Suppose that the assertion holds for the case of $k+2$. Suppose that $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ have at least $(k+3)$-th order contact at $t=t_0$, $u=u_0$ and

$$\frac{d^i}{dt^i}(\mathcal{E}_\gamma, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})(t_0) = \frac{d^i}{du^i}(\mathcal{E}_{\widetilde{\gamma}}, -\mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{n}}, \mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{t}})(u_0)$$

for $i=0,1,\dots,k-1$. By Theorem 1.3.10, we have

$$\frac{d^i}{dt^i}\mathcal{F}(t_0) = \frac{d^i}{du^i}\widetilde{\mathcal{F}}(u_0)$$

for $i=0,1,\dots,k+1$. We denote by

$$(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)(t) = (\boldsymbol{b}, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})(t),\ (\widetilde{\boldsymbol{v}}_1, \widetilde{\boldsymbol{v}}_2, \widetilde{\boldsymbol{v}}_3)(u) = (\widetilde{\boldsymbol{b}}, -\mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{n}}, \mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{t}})(u),$$

$\mathcal{F}_{\mathcal{E}_\gamma}(t) = (\kappa, -\mathrm{sgn}(\tau)\tau, 0, \sigma_f)(t)$ and $\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(t) = (\widetilde{\kappa}, -\mathrm{sgn}(\widetilde{\tau})\widetilde{\tau}, 0, \widetilde{\sigma}_f)(u)$ for convenience. It follows that

$$\begin{aligned}
\frac{d^k}{dt^k}\mathcal{E}_\gamma(t) &= \left(\frac{d^{k-1}}{dt^{k-1}}\sigma_f(t)\right)\boldsymbol{v}_1(t) + \sum_{i=1}^{3} f_i\left(\mathcal{F}_{\mathcal{E}_\gamma}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}_{\mathcal{E}_\gamma}(t)\right)\boldsymbol{v}_i(t), \\
\frac{d^k}{dt^k}\boldsymbol{v}_2(t) &= -\mathrm{sgn}(\tau)\left(\frac{d^{k-1}}{dt^{k-1}}\tau(t)\right)\boldsymbol{v}_1(t) + \mathrm{sgn}(\tau)\left(\frac{d^{k-1}}{dt^{k-1}}\kappa(t)\right)\boldsymbol{v}_3(t) \\
&\quad + \sum_{i=1}^{3} g_i\left(\mathcal{F}_{\mathcal{E}_\gamma}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}_{\mathcal{E}_\gamma}(t)\right)\boldsymbol{v}_i(t), \\
\frac{d^k}{dt^k}\boldsymbol{v}_3(t) &= -\mathrm{sgn}(\tau)\left(\frac{d^{k-1}}{dt^{k-1}}\kappa(t)\right)\boldsymbol{v}_2(t) + \sum_{i=1}^{3} h_i\left(\mathcal{F}_{\mathcal{E}_\gamma}(t), \dots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}_{\mathcal{E}_\gamma}(t)\right)\boldsymbol{v}_i(t)
\end{aligned}$$

for some smooth functions f_i, g_i and h_i $(i=1,2,3)$. Moreover, we have

$$\frac{d^j}{dt^j}\sigma_f(t) = \sigma_j\left(\mathcal{F}(t), \dots, \frac{d^{j+2}}{dt^{j+2}}\mathcal{F}(t)\right)$$

for some smooth functions σ_j for $j=0,1,\dots,k-1$. By the same calculations, we have

$$\begin{aligned}
\frac{d^k}{du^k}\mathcal{E}_{\widetilde{\gamma}}(u) &= \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\sigma}_f(t)\right)\widetilde{\boldsymbol{v}}_1(u) + \sum_{i=1}^{3} f_i\left(\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(u)\right)\widetilde{\boldsymbol{v}}_i(u), \\
\frac{d^k}{du^k}\widetilde{\boldsymbol{v}}_2(u) &= -\mathrm{sgn}(\widetilde{\tau})\left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\tau}(u)\right)\widetilde{\boldsymbol{v}}_1(u) + \mathrm{sgn}(\widetilde{\tau})\left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\kappa}(u)\right)\widetilde{\boldsymbol{v}}_3(u) \\
&\quad + \sum_{i=1}^{3} g_i\left(\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(u)\right)\widetilde{\boldsymbol{v}}_i(u), \\
\frac{d^k}{du^k}\widetilde{\boldsymbol{v}}_3(u) &= -\mathrm{sgn}(\widetilde{\tau})\left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\kappa}(u)\right)\widetilde{\boldsymbol{v}}_2(u) + \sum_{i=1}^{3} h_i\left(\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\mathcal{F}_{\mathcal{E}_{\widetilde{\gamma}}}(u)\right)\widetilde{\boldsymbol{v}}_i(u)
\end{aligned}$$

and

$$\frac{d^j}{du^j}\widetilde{\sigma}_f(u) = \sigma_j\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{j+2}}{du^{j+2}}\widetilde{\mathcal{F}}(u)\right).$$

It follows that

$$\frac{d^k}{dt^k}(\mathcal{E}_\gamma, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})(t_0) = \frac{d^k}{du^k}(\mathcal{E}_{\widetilde{\gamma}}, -\mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{n}}, \mathrm{sgn}(\widetilde{\tau})\widetilde{\boldsymbol{t}})(u_0).$$

Therefore, $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$. By the induction, we have the result. □

Proposition 2.2.11 ([21]) *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $\tau(t) \neq 0$. $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\mathcal{E}_\gamma, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})$ are congruent as framed curves if and only if $\kappa(t) = \tau(t)$ and $\alpha(t) = \sigma_f(t)$.*

Proof. Suppose that $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\mathcal{E}_\gamma, -\mathrm{sgn}(\tau)\boldsymbol{n}, \mathrm{sgn}(\tau)\boldsymbol{t})$ are congruent as framed curves. By Lemma 1.2.1 and Proposition 2.2.5, we have

$$(\tau(t), -\kappa(t), 0, \alpha(t)) = (\kappa(t), -(\tau)\tau(t), 0, \sigma_f(t))$$

It follows that $\kappa(t) = \tau(t)$ and $\alpha(t) = \sigma_f(t)$. Moreover the converse is true by the above discussion. □

Remark 2.2.12 A function $\tau(t)/\kappa(t)$ is constant if and only if γ is a framed helix (cf. Section 2.3.2 and [19]). Therefore, γ is a framed helix if $\kappa(t) = \tau(t)$.

Example 2.2.13 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = \left((t+1)\sin t + \cos t, -(t+1)\cos t + \sin t, \frac{1}{2}t^2 + t\right)$$

(cf. Figure 2.4). The curve γ has a singular point at $t = -1$, so that it is not a Frenet curve. On the other hand, γ is a Frenet type framed base curve with the mapping $(\boldsymbol{t}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\begin{aligned} \boldsymbol{t}(t) &= \frac{1}{\sqrt{2}}(\cos t, \sin t, 1), \\ \alpha(t) &= \sqrt{2}(t+1). \end{aligned}$$

By straightforward calculations, we have

$$\begin{aligned} \boldsymbol{n}(t) &= \frac{\dot{\boldsymbol{t}}(t)}{\|\dot{\boldsymbol{t}}(t)\|} = (-\sin t, \cos t, 0), \\ \boldsymbol{b}(t) &= \boldsymbol{t}(t) \times \boldsymbol{n}(t) = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 1). \end{aligned}$$

The curvature $\kappa(t)$ and the torsion $\tau(t)$ are $\kappa(t) = 1/\sqrt{2}$, $\tau(t) = 1/\sqrt{2}$, respectively. The evolute $\mathcal{E}_\gamma : I \to \mathbb{R}^3$ is given

$$\mathcal{E}_\gamma(t) = \left(-(t+1)\sin t - \cos t, (t+1)\cos t - \sin t, \frac{1}{2}t^2 + t + 2\right)$$

(cf. Figure 2.5). By Proposition 2.2.11, $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\mathcal{E}_\gamma, -\boldsymbol{b}, \boldsymbol{t})$ are congruent as framed curves. Then γ is a framed helix (cf. 2.3.2 and [19]).

2.2.3 Examples

In order to understand the phenomena for the focal developable surfaces and the evolutes of Frenet type framed base curves, we give examples.

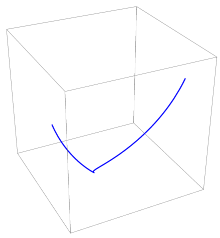

FIGURE 2.4: γ of Example 2.2.13

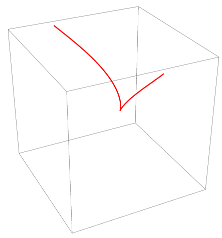

FIGURE 2.5: $\mathcal{E}_\gamma$ of Example 2.2.13

Example 2.2.14 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = (t^2, t^3, 0)$$

(cf. Figure 2.6). We call γ a $3/2$-*cusp*. The curve γ has a singular point at $t = 0$, so that it is not a Frenet curve. On the other hand, γ is a Frenet type framed base curve with the smooth mapping $(\boldsymbol{t}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\begin{aligned} \boldsymbol{t}(t) &= \frac{1}{\sqrt{4+9t^2}}(2, 3t, 0), \\ \alpha(t) &= t\sqrt{4+9t^2}. \end{aligned}$$

By straightforward calculation, we have

$$\begin{aligned} \boldsymbol{n}(t) &= \frac{\dot{\boldsymbol{t}}(t)}{\|\dot{\boldsymbol{t}}(t)\|} = \frac{1}{\sqrt{4+9t^2}}(-3t, 2, 0), \\ \boldsymbol{b}(t) &= \boldsymbol{t}(t) \times \boldsymbol{n}(t) = (0, 0, 1). \end{aligned}$$

The curvature $\kappa(t)$ and the torsion $\tau(t)$ are $\kappa(t) = 6/(4+9t^2)$, $\tau(t) = 0$, respectively. The focal developable surface $\mathcal{FD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\mathcal{FD}_\gamma(t, u) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) + u\boldsymbol{b}(t)$$

(cf. Figure 2.6). The base curve $\gamma(t) + (\alpha(t)/\kappa(t)\boldsymbol{n}(t))$ is the evolute as planer front (cf. [11,12]). By Theorem 2.2.2, the focal developable surface $\mathcal{FD}_\gamma$ is a cylinder. Since $\tau(t) \equiv 0$, we cannot consider the evolute of the curve γ as framed base curve.

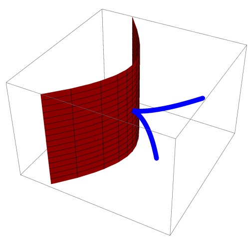

FIGURE 2.6: $(\gamma, \mathcal{FD}_\gamma)$ of Example 2.2.14

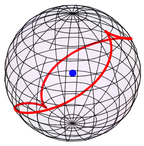

FIGURE 2.7: $(\gamma, \mathcal{E}_\gamma)$ of Example 2.2.15

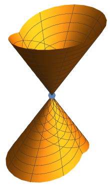

FIGURE 2.8: $\mathcal{FD}_\gamma$ of Example 2.2.15

Example 2.2.15 Let $\gamma : \mathbb{R} \to S^2 \subset \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{\sqrt{3}}{2}\cos t\right)$$

(cf. Figure 2.7). We call the curve γ a *spherical nephroid* (cf. [41]). The curve γ has singular points at $t = 0, \pi$, so that it is not a Frenet curve. On the other hand, γ is a Frenet type framed base curve with the smooth mapping $(\boldsymbol{t}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\begin{aligned} \boldsymbol{t}(t) &= \left(\frac{\sqrt{3}}{2}\cos 2t, \frac{\sqrt{3}}{2}\sin 2t, -\frac{1}{2}\right), \\ \alpha(t) &= \sqrt{3}\sin t. \end{aligned}$$

By straightforward calculations, we have

$$\begin{aligned} \boldsymbol{n}(t) &= \frac{\dot{\boldsymbol{t}}(t)}{\|\dot{\boldsymbol{t}}(t)\|} = (-\sin 2t, \cos 2t, 0), \\ \boldsymbol{b}(t) &= \boldsymbol{t}(t) \times \boldsymbol{n}(t) = \frac{1}{2}(-\cos 2t, -\sin 2t, \sqrt{3}). \end{aligned}$$

The curvature $\kappa(t)$ and the torsion $\tau(t)$ are $\kappa(t) = \sqrt{3}$, $\tau(t) = 1$, respectively. The focal surface $\mathcal{FD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\mathcal{FD}_\gamma(t, u) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) + u\boldsymbol{b}(t).$$

(cf. Figure 2.8). By Theorem 2.2.2 and

$$\sigma_f(t) = \frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right) = 0,$$

the focal developable surface $\mathcal{FD}_\gamma$ is a conical surface. Moreover, the evolute $\mathcal{E}_\gamma$ is a point $\mathcal{E}_\gamma(t) = (0, 0, 0)$ (cf. Figure 2.7).

Example 2.2.16 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = \left(t^2, t^3, t^4\right)$$

(cf. Figure 2.9). The curve γ has a singular point at $t = 0$, so that it is not a Frenet curve. On the other hand, γ is a Frenet type framed base curve with the smooth

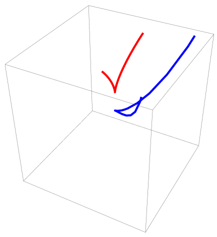

FIGURE 2.9: $(\gamma, \mathcal{E}_\gamma)$ of Example 2.2.16

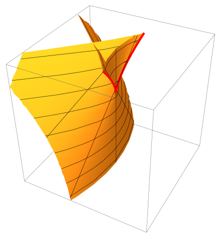

FIGURE 2.10: $\mathcal{FD}_\gamma$ of Example 2.2.16

mapping $(\boldsymbol{t}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\begin{aligned} \boldsymbol{t}(t) &= \frac{1}{\sqrt{4+9t^2+16t^4}}(2, 3t, 4t^2), \\ \alpha(t) &= t\sqrt{4+9t^2+16t^4}. \end{aligned}$$

By straightforward calculations, we have

$$\begin{aligned} \boldsymbol{n}(t) &= \frac{1}{\sqrt{(4+9t^2+16t^4)(9+64t^2+36t^4)}}(-9t-32t^3, 6-24t^4, 16t+18t^3), \\ \boldsymbol{b}(t) &= \frac{1}{\sqrt{9+64t^2+36t^4}}(6t^2, -8t, 3). \end{aligned}$$

The curvature $\kappa(t)$ and the torsion $\tau(t)$ are

$$\begin{aligned} \kappa(t) &= \frac{2\sqrt{9+64t^2+36t^4}}{4+9t^2+16t^4}, \\ \tau(t) &= \frac{12\sqrt{4+9t^2+16t^4}}{9+64t^2+36t^4}, \end{aligned}$$

respectively. The focal surface $\mathcal{FD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\mathcal{FD}_\gamma(t, u) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) + u\boldsymbol{b}(t)$$

(cf. Figure 2.10). We can calculate that

$$\sigma_f(t) = \frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right) = t(3+20t^2)\sqrt{9+64t^2+36t^4}.$$

By Theorem 2.2.3, $\sigma_f(0) = 0$ and $\dot{\sigma}_f(0) = 9$, the focal developable surface $\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail $\boldsymbol{sw}$ at $(0, -1/2)$. The evolute $\mathcal{E}_\gamma : I \to \mathbb{R}^3$ is given by

$$\mathcal{E}_\gamma(t) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) - \frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\boldsymbol{b}(t)$$

(cf. Figure 2.9).

Example 2.2.17 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = (\cos^3 t, \sin^3 t, \cos 2t)$$

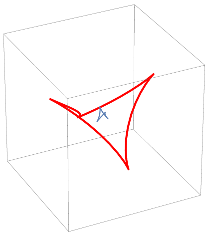

FIGURE 2.11: $(\gamma, \mathcal{E}_\gamma)$ of Example 2.2.17

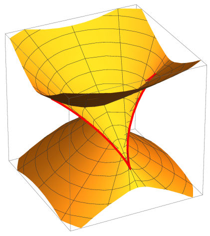

FIGURE 2.12: $\mathcal{FD}_\gamma$ of Example 2.2.17

(cf. Figure 2.12). We call γ an *astroid*. The curve γ has singular points at $t = 0, \pi/2, \pi, 3\pi/2$, so that it is not a Frenet curve. On the other hand, γ is a Frenet type framed base curve with the smooth mapping $(\boldsymbol{t}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\begin{aligned} \boldsymbol{t}(t) &= \frac{1}{5}(-3\cos t, 3\sin t, -4), \\ \alpha(t) &= 5\cos t \sin t. \end{aligned}$$

By straightforward calculations, we have

$$\begin{aligned} \boldsymbol{n}(t) &= (\sin t, \cos t, 0), \\ \boldsymbol{b}(t) &= \frac{1}{5}(4\cos t, -4\sin t, -3). \end{aligned}$$

The curvature $\kappa(t)$ and the torsion $\tau(t)$ are $\kappa(t) = -3/5$, $\tau(t) = -4/5$, respectively. The focal surface $\mathcal{FD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\mathcal{FD}_\gamma(t,u) = \gamma(t) + \frac{\alpha(t)}{\kappa(t)}\boldsymbol{n}(t) + u\boldsymbol{b}(t)$$

(cf. Figure 2.12). By Theorem 2.2.3 and

$$\sigma_f(t) = \frac{\alpha(t)\tau(t)}{\kappa(t)} - \frac{d}{dt}\left(\frac{\alpha(t)\dot{\kappa}(t) - \dot{\alpha}(t)\kappa(t)}{\kappa^2(t)\tau(t)}\right) = 35\cos t \sin t,$$

the focal developable surface $\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at $t = 0, \pi/2, \pi, 3\pi/2$. The evolute $\mathcal{E}_\gamma : I \to \mathbb{R}^3$ is given by

$$\mathcal{E}_\gamma(t) = \left(\frac{28}{3}\cos^3 t, \frac{28}{3}\sin^3 t, -\frac{21}{4}\cos 2t\right)$$

(cf. Figure 2.11). Therefore, the evolute of the astroid is the astroid.

2.3 Rectifying developable surfaces and framed helices

For a Frenet curve, the rectifying developable surface is investigated in [26]. They showed relationships between singularities of the rectifying developable surfaces of a Frenet curve and geometric invariants of the curve. A Frenet curve is always a deodesic of its rectifying developable surface.

In this section, we consider the rectifying developable surface of a Frenet type framed base curve and a framed helix.

2.3.1 Rectifying developable surfaces

We define the rectifying developable surface of a Frenet type framed base curve as the discriminant set of the support function of the curve with respect to the unit principal normal vector (cf. Proposition 2.1.4). Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$.

Definition 2.3.1 We define a mapping $\mathcal{RD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\mathcal{RD}_\gamma(t,u) = \gamma(t) + u\overline{\boldsymbol{d}}(t) = \gamma(t) + u\frac{\tau(t)\boldsymbol{t}(t) + \kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}.$$

We call $\mathcal{RD}_\gamma$ the *rectifying developable surface* of γ, where $\overline{\boldsymbol{d}}$ is the spherical Darboux type vector (cf. Section 2.1.2).

The rectifying developable surface $\mathcal{RD}_\gamma$ is a ruled surface and we have

$$\dot{\overline{\boldsymbol{d}}}(t) = \left(\frac{\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t)}{\kappa^2(t)+\tau^2(t)}\right)\frac{\kappa(t)\boldsymbol{t}(t) - \tau(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}},$$

so that we have

$$\begin{aligned}
&\det\left(\dot{\gamma}(t), \overline{\boldsymbol{d}}(t), \dot{\overline{\boldsymbol{d}}}(t)\right)\\
&= \det\left(\alpha(t)\boldsymbol{t}(t), \frac{\tau(t)\boldsymbol{t}(t)+\kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}, \left(\frac{\kappa(t)\dot{\tau}(t)-\dot{\kappa}(t)\tau(t)}{\kappa^2(t)+\tau^2(t)}\right)\frac{\kappa(t)\boldsymbol{t}(t)-\tau(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}\right)\\
&= 0
\end{aligned}$$

for all $t \in I$. This means that $\mathcal{RD}_\gamma$ is a developable surface (cf .Section 2.1.5). Moreover, $\mathcal{RD}_\gamma$ is a developable front (cf. Sections 2.1.4 and 2.1.5). We introduce two invariants $\delta_r(t)$, $\sigma_r(t)$ as follows:

$$\begin{aligned}
\delta_r(t) &= \frac{\kappa(t)\dot{\tau}(t)-\dot{\kappa}(t)\tau(t)}{\kappa^2(t)+\tau^2(t)},\\
\sigma_r(t) &= \frac{\alpha(t)\tau(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\right) \quad (\text{when } \delta_r(t) \neq 0).
\end{aligned}$$

Theorem 2.3.2 ([19]) *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following:*

(1) *The following are equivalent:*

- (a) *$\mathcal{RD}_\gamma$ is a cylinder,*
- (b) $\delta_r(t) \equiv 0$.

(2) *If $\delta_r(t) \neq 0$, then the following are equivalent:*

- (c) *$\mathcal{RD}_\gamma$ is a conical surface,*
- (d) $\sigma_r(t) \equiv 0$.

Proof. (1) By definition, $\mathcal{RD}_\gamma$ is a cylinder if and only if $\overline{\boldsymbol{d}}(t)$ is constant. Since

$$\dot{\overline{\boldsymbol{d}}}(t) = \left(\frac{\kappa(t)\dot{\tau}(t)-\dot{\kappa}(t)\tau(t)}{\kappa^2(t)+\tau^2(t)}\right)\frac{\kappa(t)\boldsymbol{t}(t)-\tau(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} = \delta_r(t)\frac{\kappa(t)\boldsymbol{t}(t)-\tau(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}},$$

$\overline{\boldsymbol{d}}(t)$ is constant if and only if $\delta_r(t) \equiv 0$.

(2) We consider the striction curve $\boldsymbol{\sigma}(t)$ defined by

$$\boldsymbol{\sigma}(t) = \boldsymbol{\gamma}(t) - \frac{\dot{\boldsymbol{\gamma}}(t) \cdot \dot{\overline{\boldsymbol{d}}}(t)}{\dot{\overline{\boldsymbol{d}}}(t) \cdot \dot{\overline{\boldsymbol{d}}}(t)} \overline{\boldsymbol{d}}(t) = \boldsymbol{\gamma}(t) - \frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}} \overline{\boldsymbol{d}}(t).$$

Then (2)-(c) is equivalent to the condition $\dot{\boldsymbol{\sigma}}(t) \equiv 0$. We can calculate that

$$\begin{aligned}
\dot{\boldsymbol{\sigma}}(t) &= \dot{\boldsymbol{\gamma}}(t) - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\right)\overline{\boldsymbol{d}}(t) - \frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\dot{\overline{\boldsymbol{d}}}(t) \\
&= \alpha(t)\boldsymbol{t}(t) - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\right)\overline{\boldsymbol{d}}(t) - \frac{\alpha(t)\kappa(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} \frac{\kappa(t)\boldsymbol{t}(t) - \tau(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} \\
&= \left(\frac{\alpha(t)\tau(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\right)\right)\frac{\tau(t)\boldsymbol{t}(t)+\kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} \\
&= \sigma_r(t)\overline{\boldsymbol{d}}(t).
\end{aligned}$$

It follows that (2)-(c) and (2)-(d) are equivalent. □

We give characterizations of singularities of the rectifying developable surface $\mathcal{RD}_\gamma$ by using $\delta_r(t)$ and $\sigma_r(t)$.

Theorem 2.3.3 ([19]) *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following:*

(1) *(t_0, u_0) is a singular point of $\mathcal{RD}_\gamma$ if and only if*

$$\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t_0)+\tau^2(t_0)}} + u_0\delta_r(t_0) = 0.$$

(2) *$\mathcal{RD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge **ce** at (t_0, u_0) if and only if*

(a) *$\delta_r(t_0) \neq 0$, $\sigma_r(t_0) \neq 0$ and*

$$u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta_r(t_0)\sqrt{\kappa^2(t_0)+\tau^2(t_0)}},$$

or

(b) *$\delta_r(t_0) = \alpha(t_0) = 0$, $\dot{\delta}_r(t_0) \neq 0$ and*

$$u_0 \neq -\frac{\dot{\alpha}(t_0)\kappa(t_0)\sqrt{\kappa^2(t_0)+\tau^2(t_0)}}{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)},$$

or

(c) *$\delta_r(t_0) = \alpha(t_0) = 0$ and $\dot{\alpha}(t_0) \neq 0$.*

(3) *$\mathcal{RD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail **sw** at (t_0, u_0) if and only if $\delta_r(t_0) \neq 0$, $\sigma_r(t_0) = 0$, $\dot{\sigma}_r(t_0) \neq 0$ and*

$$u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta_r(t_0)\sqrt{\kappa^2(t_0)+\tau^2(t_0)}},$$

Proof. By a straightforward calculation, we have

$$\frac{\partial \mathcal{RD}_\gamma}{\partial t}(t,u) \times \frac{\partial \mathcal{RD}_\gamma}{\partial t}(t,u) = -\left(\frac{\alpha(t)\kappa(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} + u\delta_r(t)\right)\boldsymbol{n}(t).$$

Therefore (t_0,u_0) is a singular point of $\mathcal{RD}_\gamma$ if and only if

$$\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t_0)+\tau^2(t_0)}} + u_0\delta_r(t_0) = 0.$$

This means that (1) holds. Assertions (2) and (3) can be proven by using the criterion for the cuspidal edge $\boldsymbol{ce}$ and the swallowtail $\boldsymbol{sw}$ in [32]. We give the signed density function $\lambda : I \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}\lambda(t,u) &= \det\left(\frac{\partial \mathcal{RD}_\gamma}{\partial t}(t,u), \frac{\partial \mathcal{RD}_\gamma}{\partial u}(t,u), \boldsymbol{n}(t)\right)\\ &= -\frac{\alpha(t)\kappa(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}} - u\delta_r(t).\end{aligned}$$

Suppose that (t_0,u_0) is a non-degenerate singular point of $\mathcal{RD}_\gamma$, that is,

$$\begin{aligned}\frac{\partial\lambda}{\partial t}(t_0,u_0) &= -\frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}\right)\Bigg|_{t=t_0} - u\dot{\delta}_r(t) \neq 0\\ &\quad\text{or}\\ \frac{\partial\lambda}{\partial u}(t_0,u_0) &= \delta_r(t_0) \neq 0.\end{aligned}$$

(2)-(a) When $\delta_r(t_0) \neq 0$, we have the singular curve $\boldsymbol{c} : (I,t_0) \to (I \times \mathbb{R}, (t_0,u_0))$;

$$\boldsymbol{c}(t) = \left(t, -\frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\right)$$

and the null vector field $\boldsymbol{\eta} : (I,t_0) \to \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$;

$$\boldsymbol{\eta}(t) = \left(1, -\frac{\alpha(t)\tau(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}\right).$$

By using the criterion for the cuspidal edge $\boldsymbol{ce}$, the focal developable surface $\mathcal{FD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at (t_0,u_0) if and only if

$$\det(\dot{\boldsymbol{c}}(t_0), \boldsymbol{\eta}(t_0)) = -\frac{\alpha(t_0)\tau(t_0)}{\sqrt{\kappa^2(t_0)+\tau^2(t_0)}} + \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta_r(t)\sqrt{\kappa^2(t)+\tau^2(t)}}\right)\Bigg|_{t=t_0} = -\sigma_r(t_0) \neq 0.$$

Therefore we have (2)-(a).

(2)-(b, c)) When $\delta_r(t_0) = 0$ and

$$-\frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\kappa^2(t)+\tau^2(t)}\right)\Bigg|_{t=t_0} - u\dot{\delta}_r(t) \neq 0,$$

there exists a smooth function $\phi : (\mathbb{R}, u_0) \to (I, t_0)$ such that $\lambda(\phi(u), u) = 0$ and

$$\frac{d\phi}{du}(u) = -\left.\frac{\lambda_u(t,u)}{\lambda_t(t,u)}\right|_{t=\phi(u)}.$$

Accordingly, we have the singular curve $\boldsymbol{c} : (\mathbb{R}, u_0) \to (I \times \mathbb{R}, (t_0, u_0))$; $\boldsymbol{c}(u) = (\phi(u), u)$ and the null vector field

$$\boldsymbol{\eta}(u) = \left(1, -\frac{\alpha(\phi(u))\tau(\phi(u))}{\sqrt{\kappa^2(\phi(u)) + \tau^2(\phi(u))}}\right).$$

Since

$$\det\left(\frac{d\boldsymbol{c}}{du}(u_0), \boldsymbol{\eta}(u_0)\right) = \det\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix} = -1 \neq 0,$$

the rectifying developable surface $\mathcal{FD}_\gamma$ is always $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at (t_0, u_0) under the assumption. Therefore, (2)-(b), (c) holds.

(3) By the proof of (2)-(b), (c), it is enough to consider in the same assumption as (2)-(a). By using the criterion for the swallowtail $\boldsymbol{sw}$, the rectifying developable surface $\mathcal{RD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail $\boldsymbol{sw}$ at (t_0, u_0) if and only if

$$\det(\dot{\boldsymbol{c}}(t_0), \boldsymbol{\eta}(t_0)) = -\sigma_r(t_0) = 0 \text{ and } \frac{d\det(\dot{\boldsymbol{c}}, \boldsymbol{\eta})}{dt}(t_0) = -\dot{\sigma}_r(t_0) \neq 0.$$

Therefore, (3) holds. This completes the proof. □

2.3.2 Framed helices

In this section, we define a (generalized) helix which may have singular points. Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$.

Definition 2.3.4 (Framed helix) We say that $\gamma : I \to \mathbb{R}^3$ is a *framed helix* if there exist a constant vector $\boldsymbol{v} \in S^2$ and a constant number $c \in \mathbb{R}$ such that $\boldsymbol{t}(t) \cdot \boldsymbol{v} \equiv c$.

This means that tangent lines of γ make a constant angle with a fixed direction, so that a framed helix is a natural generalization of a regular helix. The invariant $\delta_r(t)$ characterize a framed helix. We can prove the following proposition.

Proposition 2.3.5 ([19]) *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then the following are equivalent:*

(1) *γ is a framed helix,*

(2) $\delta_r(t) \equiv 0$.

Proof. Suppose that γ is a framed helix. Here, we put $\boldsymbol{v} = a(t)\boldsymbol{t}(t) + b(t)\boldsymbol{n}(t) + c(t)\boldsymbol{b}(t)$, where $a(t)$, $b(t)$ and $c(t)$ are smooth functions. By the assumption,

$$\boldsymbol{v} \cdot \boldsymbol{t}(t) = a(t) = c. \tag{2.1}$$

Moreover, taking the derivative of the both sides of the equation (2.1), we have

$$-b(t)\kappa(t)\boldsymbol{t}(t) + (c\kappa(t) + \dot{b}(t) - c(t)\tau(t))\boldsymbol{n}(t) + (\dot{c}(t) + b(t)\tau(t))\boldsymbol{b}(t) = \boldsymbol{0}.$$

Then we have $b(t) = 0$, $c(t) = c_1$ and $c = c(\tau(t)/\kappa(t))$, where c_1 is a constant number. On the other hand, since

$$1 = \|\boldsymbol{v}\|^2 = c_1^2\left(\frac{\tau^2(t)}{\kappa^2(t)} + 1\right),$$

we have $c_1 \neq 0$. Thus $c/c_1 = \tau(t)/\kappa(t)$. We remark that $\delta_r(t) = 0$ if and only if

$$\frac{d}{dt}\left(\frac{\tau(t)}{\kappa(t)}\right) = \frac{\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t)}{\kappa^2(t)} = 0.$$

Therefore, we have $\delta_r(t) \equiv 0$.

Conversely, suppose that $\delta_r(t) \equiv 0$. We set $\boldsymbol{v} = (\tau(t)/\kappa(t))\boldsymbol{t}(t) + \boldsymbol{b}(t)$ and $\overline{\boldsymbol{v}} = \boldsymbol{v}/\|\boldsymbol{v}\|$. Then

$$\overline{\boldsymbol{v}} \cdot \boldsymbol{t}(t) = \frac{\frac{\tau(t)}{\kappa(t)}}{\sqrt{\frac{\tau^2(t)}{\kappa^2(t)} + 1}},$$

that is, $\overline{\boldsymbol{v}} \cdot \boldsymbol{t}(t)$ is constant. This means that γ is a framed helix. □

Corollary 2.3.6 ([19]) *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then we have the following are equivalent:*

(a) *$\mathcal{RD}_\gamma$ is a cylinder,*

(b) *$\delta_r(t) \equiv 0$,*

(c) *γ is a framed helix.*

By Theorem 1.3.10, we can show the following propositions.

Proposition 2.3.7 ([19]) *If $(\gamma, \boldsymbol{n}, \boldsymbol{b})(t_0)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})(u_0)$ have at least $(k+2)$-th order contact as framed curves, then $\delta_r^{(p)}(t_0) = \widetilde{\delta}_r^{(p)}(u_0)$ for $0 \le p \le k-1$, where*

$$\delta_r^{(p)}(t_0) = \frac{d^p\delta_r}{dt^p}(t_0) \text{ and } \widetilde{\delta}_r^{(p)}(u_0) = \frac{d^p\widetilde{\delta}_r}{du^p}(u_0).$$

Proposition 2.3.8 ([19]) *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve with $(\boldsymbol{t}, \alpha)$. Then there exists a framed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}}) : I \to \mathbb{R}^3 \times V_{3,2}$ such that $\widetilde{\gamma}$ is a framed helix, and $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ have at least second order contact as framed curves at a point $t_0 \in I$.*

Proof. Choose any fixed value $t = t_0$ of the parameter. We consider a new curvature as a framed curve

$$(\widetilde{\tau}(t), -\widetilde{\kappa}(t), 0, \widetilde{\alpha}(t)) = ((\tau(t_0)/\kappa(t_0))\kappa(t), -\kappa(t), 0, \alpha(t)).$$

As an application of Theorems 1.1.4 and 1.1.5, there exists a framed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ with the curvature $(\widetilde{\tau}(t), -\widetilde{\kappa}(t), 0, \widetilde{\alpha}(t))$. By Theorem 1.3.10 and an appropriate Euclid transformation, $(\gamma, \boldsymbol{n}, \boldsymbol{b})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}})$ have at least second order contact as framed curves at $t_0 \in I$. Moreover, by a straightforward calculation, we have $\widetilde{\delta}(t) \equiv 0$. Therefore, $\widetilde{\gamma}$ is a framed helix. □

2.3.3 Examples

In order to understand the phenomena for rectifying the developable surfaces of framed base curves and framed helices, we give examples.

Example 2.3.9 (The astroid) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = (\cos^3 t, \sin^3 t, \cos 2t)$$

(cf. Figure 2.13). The curve γ is a Frenet type framed base curve (cf. Example 2.2.17). By Corollary 2.3.6 and $\delta_r(t) \equiv 0$, γ is a framed helix. The rectifying developable surface is given by $\mathcal{RD}_\gamma(t,u) = \big(\cos^3 t, \sin^3 t, -u + \cos 2t\big)$. By Theorem 2.3.3 (2)-(c), the rectifying developable surface $\mathcal{RD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at $t = 0, \pi/2, \pi, 3\pi/2$ (cf. Figure 2.14).

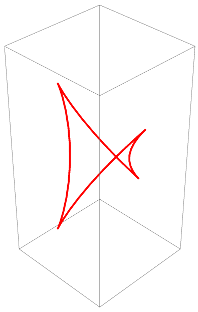

FIGURE 2.13: γ of Example 2.3.9

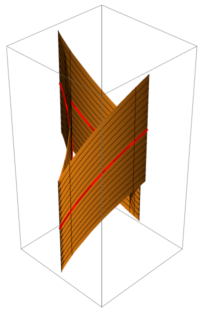

FIGURE 2.14: $(\gamma, \mathcal{RD}_\gamma)$ of Example 2.3.9

Example 2.3.10 (The spherical nephroid (cf. [41])) Let $\gamma : [0, 2\pi) \to S^2 \subset \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{\sqrt{3}}{2}\cos t\right)$$

(cf. Figure 2.15). The curve γ is a Frenet type framed base curve (cf. Example 2.2.15). By Corollary 2.3.6 and $\delta_r(t) \equiv 0$, γ is a framed helix. The rectifying developable surface is given by

$$\mathcal{RD}_\gamma(t,u) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, u + \frac{\sqrt{3}}{2}\cos t\right).$$

By Theorem 2.3.3 (2)-(c), the rectifying developable surface $\mathcal{RD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at $t = 0, \pi$ (cf. Figure 2.16).

Example 2.3.11 ((2, 3, 5)-type) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(t) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{5}t^5\right)$$

(cf. Figure 2.17). The curve γ has a singular point at $t = 0$, so that it is not a Frenet curve. On the other hand, γ is a Frenet type framed base curve with the mapping

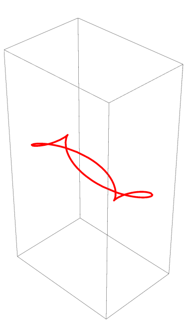

FIGURE 2.15: γ of Example 2.3.10

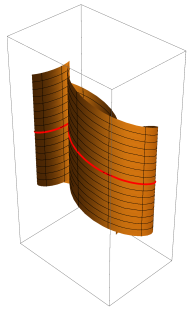

FIGURE 2.16: $(\gamma, \mathcal{RD}_\gamma)$ of Example 2.3.10

$(\boldsymbol{t}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\begin{aligned} \boldsymbol{t}(t) &= \frac{1}{\sqrt{1+t^2+t^6}}\left(1, t, t^3\right), \\ \alpha(t) &= t\sqrt{1+t^2+t^6}. \end{aligned}$$

By straightforward calculations, we have

$$\kappa(t) = \frac{\sqrt{1+9t^4+4t^6}}{1+t^2+t^6}, \ \tau(t) = \frac{6t\sqrt{1+t^2+t^6}}{1+9t^4+4t^6}.$$

By Theorem 2.3.3 (2)-(b), $\delta(0) = 6$, $\sigma(0) = 1/6$ and $\alpha(0) = 0$, the rectifying developable surface $\mathcal{RD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at $(0,0)$ (cf. Figure 2.18).

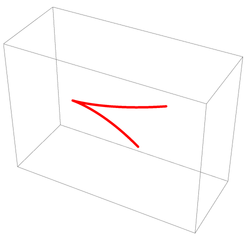

FIGURE 2.17: γ of Example 2.3.11

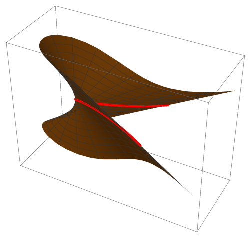

FIGURE 2.18: $(\gamma, \mathcal{RD}_\gamma)$ of Example 2.3.11

Chapter 3

Frontal curves on embedded surfaces and developable surfaces

In this chapter, we consider two types of developable surfaces along a frontal curve on an embedded surface in the Euclidean 3-space. One is called the osculating developable surface, and the other is called the normal developable surface. The frontal curve may have singular points. We give new invariants of the frontal curve which characterize singularities of the developable surfaces. Moreover, a frontal curve is a contour generator with respect to an orthogonal projection or a central projection if and only if one of these invariants constantly equal to zero. This chapter is based on [20].

3.1 Preliminaries

3.1.1 Regular curves on embedded surfaces and developable surfaces

Let $M = \boldsymbol{X}(V)$ be a surface given locally by an embedding $\boldsymbol{X} : V \to \mathbb{R}^3$, $V \subset \mathbb{R}^2$ is an open set. Let $\overline{\gamma} : I \to V$ be a regular plane curve, where $\overline{\gamma}(t) = (u(t), v(t))$ and I is an open interval. Then we have a regular space curve $\gamma = \boldsymbol{X} \circ \overline{\gamma} : I \to M \subset \mathbb{R}^3$ on the surface M. On the surface, we have the unit normal vector field $\boldsymbol{n}$ defined by

$$\boldsymbol{n}(u,v) = \frac{\boldsymbol{X}_u(u,v) \times \boldsymbol{X}_v(u,v)}{\|\boldsymbol{X}_u(u,v) \times \boldsymbol{X}_v(u,v)\|}.$$

Since γ is a regular space curve in $\mathbb{R}^3$, we adopt the arc-length parameter and denote $\gamma(s) = \boldsymbol{X}(u(s), v(s))$. Then we have the unit tangent vector field $\boldsymbol{t}(s) = \gamma'(s)$, where $\gamma'(s) = (d\gamma/ds)(s)$. We have $\boldsymbol{n}_\gamma(s) = \boldsymbol{n} \circ \overline{\gamma}(s)$, which is the unit normal vector field of M along $\gamma(s)$. Moreover, we define $\boldsymbol{b}(s) = \boldsymbol{n}_\gamma(s) \times \boldsymbol{t}(s)$. Then we have an orthonormal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_\gamma(s), \boldsymbol{b}(s)\}$ along $\gamma(s)$, which is called the *Darboux frame* along $\gamma(s)$. Then we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{b}'(s) \\ \boldsymbol{n}_\gamma'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ -\kappa_g(s) & 0 & \tau_g(s) \\ -\kappa_n(s) & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{b}(s) \\ \boldsymbol{n}_\gamma(s) \end{pmatrix}.$$

Here,

$$\begin{aligned}
\kappa_g(s) &= \boldsymbol{t}'(s) \cdot \boldsymbol{b}(s) = \det\left(\gamma'(s), \gamma''(s), \boldsymbol{n}_\gamma(s)\right), \\
\kappa_n(s) &= \boldsymbol{t}'(s) \cdot \boldsymbol{n}_\gamma(s) = \gamma''(s) \cdot \boldsymbol{n}_\gamma(s), \\
\tau_g(s) &= \boldsymbol{b}'(s) \cdot \boldsymbol{n}_\gamma(s) = \det\left(\gamma'(s), \boldsymbol{n}_\gamma(s), \boldsymbol{n}_\gamma'(s)\right).
\end{aligned}$$

We call $\kappa_g(s)$ a *geodesic curvature*, $\kappa_n(s)$ a *normal curvature* and $\tau_g(s)$ a *geodesic torsion* of $\gamma(s)$, respectively. It is known that

(1) γ is an asymptotic curve of M if and only if $\kappa_n(s) \equiv 0$,

(2) γ is a geodesic of M if and only if $\kappa_g(s) \equiv 0$,

(3) γ is a principal curve of M if and only if $\tau_g(s) \equiv 0$.

We define two vector fields $\boldsymbol{d}_o(s), \boldsymbol{d}_r(s)$ along $\gamma(s)$ by

$$\boldsymbol{d}_o(s) = \tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s),$$
$$\boldsymbol{d}_r(s) = \tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_\gamma(s)$$

which are called the *osculating Darboux vector field*, the *rectifying Darboux vector field* along $\gamma(s)$, respectively. We also define spherical Darboux images as follows:

$$\overline{\boldsymbol{d}}_o(s) = \frac{\tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}} \text{ if } \big(\kappa_n(s), \tau_g(s)\big) \neq (0,0),$$
$$\overline{\boldsymbol{d}}_r(s) = \frac{\tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_\gamma(s)}{\sqrt{\tau_g^2(s) + \kappa_g^2(s)}} \text{ if } \big(\tau_g(s), \kappa_g(s)\big) \neq (0,0).$$

These are called the *spherical osculating Darboux image*, the *spherical rectifying Darboux image* along $\gamma(s)$, respectively. The condition $(\kappa_n(s), \tau_g(s)) \neq (0,0)$ means that $\boldsymbol{n}_\gamma(s)$ is a regular spherical curve. Then $\overline{\boldsymbol{d}}_o(s)$ is a spherical dual of $\boldsymbol{n}_\gamma(s)$. On the other hand, the condition $(\kappa_g(s), \tau_g(s)) \neq (0,0)$ means that $\boldsymbol{b}(s)$ is a regular spherical curve. Then $\overline{\boldsymbol{d}}_r(s)$ is a spherical dual of $\boldsymbol{b}(s)$.

Definition 3.1.1 (The osculating developable surface, [28]) Let $\gamma = \boldsymbol{X} \circ \overline{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve on a surface M with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$. We define a mapping $OD_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$OD_\gamma(s,u) = \gamma(s) + u\overline{\boldsymbol{d}}_o(s) = \gamma(s) + u\frac{\tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}.$$

We call OD_γ the *osculating developable surface* along γ on M.

It is easy to show that OD_γ is a developable surface. If $(s_0, 0)$ is a regular point of OD_γ (i.e. $\kappa_n(s_0) \neq 0$), the normal vector of OD_γ at $OD_\gamma(s_0,0) = \gamma(s_0)$ has the same direction of the normal vector of M at $\gamma(s_0)$. This is the reason why we call OD_γ the osculating developable surface of M along γ.

Definition 3.1.2 (The normal developable surface, [18]) Let $\gamma = \boldsymbol{X} \circ \overline{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve on a surface M with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$. We define a mapping $ND_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$ND_\gamma(s,u) = \gamma(s) + u\overline{\boldsymbol{d}}_r(s) = \gamma(s) + u\frac{\tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_\gamma(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}.$$

We call ND_γ the *normal developable surface* along γ on M.

We can also show that ND_γ is a developable surface. If $(s_0, 0)$ is a regular point of ND_γ (i.e. $\kappa_g(s_0) \neq 0$), the normal vector of ND_γ at $ND_\gamma(s_0, 0) = \gamma(s_0)$ is tangent to M at $\gamma(s_0)$. This is the reason why we call ND_γ the normal developable surface of M along γ.

We introduce four invariants of (M, γ) as follows:

$$\delta_o(s) = \kappa_g(s) + \frac{\kappa_n(s)\tau_g'(s) - \kappa_n'(s)\tau_g(s)}{\kappa_n^2(s) + \tau_g^2(s)},$$

$$\sigma_o(s) = \frac{\tau_g(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}} - \frac{d}{ds}\left(\frac{\kappa_n(s)}{\delta_o(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}\right), \text{ (when } \delta_o(s) \neq 0\text{)},$$

$$\delta_r(s) = \kappa_n(s) - \frac{\kappa_g(s)\tau_g'(s) - \kappa_g'(s)\tau_g(s)}{\kappa_g^2(s) + \tau_g^2(s)},$$

$$\sigma_r(s) = \frac{\tau_g(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} + \frac{d}{ds}\left(\frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}\right), \text{ (when } \delta_r(s) \neq 0\text{)}.$$

These invariants characterize singularities of OD_γ and ND_γ (cf. [18,28]). Moreover, [28] showed that a regular curve is a contour generator with respect to an orthogonal projection (respectively, a central projection) if and only if $\delta_o(t)$ (respectively, $\sigma_o(t)$) constantly equal to zero.

3.1.2 Contour generators

We now briefly review the notion of contour generators. Let $M \subset \mathbb{R}^3$ be a surface and $\boldsymbol{n}$ be a unit normal vector field on M. For a unit vector $\boldsymbol{k} \in S^2$, the *contour generator of orthogonal projection* with the direction $\boldsymbol{k}$ is defined to be

$$\{p \in M \mid \boldsymbol{k} \cdot \boldsymbol{n}(p) = 0\}.$$

It is actually the singular set of the orthogonal projection with direction $\boldsymbol{k}$. Moreover, for a point $\boldsymbol{c} \in \mathbb{R}^3$, the *contour generator of the central projection* with the center $\boldsymbol{c}$ is defined to be

$$\{p \in M \mid (p - \boldsymbol{c}) \cdot \boldsymbol{n}(p) = 0\}.$$

It is also the singular set of the central projection with the center $\boldsymbol{c}$. The notion of contour generators plays an important role in the computer vision theory [6].

3.1.3 Frontal curves on embedded surfaces

Hereafter, we do not assume that $\overline{\gamma} : I \to V$ is a regular curve. It follows that $\gamma : \boldsymbol{X} \circ \overline{\gamma} : I \to M \subset \mathbb{R}^3$ may have singular points. If γ has a singular point, we can not construct the Darboux frame. However, we can define a moving frame of a frontal curve for a Legendre curve in $T_1M = \{(\boldsymbol{x}, \boldsymbol{v}) \in TM \mid \|\boldsymbol{v}\| = 1\}$, where T_1M is the unit tangent bundle over M equipped with the canonical contact structure.

Definition 3.1.3 We say that $(\gamma, \boldsymbol{b}) : I \to T_1M$ is a *Legendre curve* on the unit tangent bundle T_1M if $\dot{\gamma}(t) \cdot \boldsymbol{b}(t) = 0$ for all $t \in I$. We call γ a *frontal curve*. Moreover, if $(\gamma, \boldsymbol{b})$ is a Legendre immersion, we call γ a *front curve*.

For a surface $M = \boldsymbol{X}(V)$ given locally by an embedding $\boldsymbol{X} : V \to \mathbb{R}^3$, let $\overline{\gamma} : I \to V$ be a plane curve, and let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve, where $\gamma = \boldsymbol{X} \circ \overline{\gamma} :$

$I \to M \subset \mathbb{R}^3$. Then we have the Frenet-Serret type formula of the frontal curve γ as follows. $\boldsymbol{n}_\gamma(t)$ is defined by

$$\boldsymbol{n}_\gamma(t) = \boldsymbol{n} \circ \overline{\gamma}(t).$$

We remark that $\boldsymbol{n}_\gamma(t)$ may have singular points. We put $\boldsymbol{t}(t) = \boldsymbol{n}_\gamma(t) \times \boldsymbol{b}(t)$. We call an orthonormal frame $\{\boldsymbol{n}_\gamma(t), \boldsymbol{b}(t), \boldsymbol{t}(t)\}$ the *Darboux type frame* along the frontal curve $\gamma(t)$ on $M \subset \mathbb{R}^3$ and we have the Frenet-Serret type formula of the frontal curve (or, the Legendre curve) which is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}_\gamma(t) \\ \dot{\boldsymbol{b}}(t) \\ \dot{\boldsymbol{t}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_\gamma(t) \\ \boldsymbol{b}(t) \\ \boldsymbol{t}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\boldsymbol{n}}_\gamma(t) \cdot \boldsymbol{b}(t)$, $m(t) = \dot{\boldsymbol{n}}_\gamma(t) \cdot \boldsymbol{t}(t)$ and $n(t) = \dot{\boldsymbol{b}}(t) \cdot \boldsymbol{t}(t)$. Moreover, there exists a smooth function $\alpha(t)$ such that

$$\dot{\gamma}(t) = \alpha(t)\boldsymbol{t}(t).$$

We call the mapping $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ *the curvature* of the Legendre curve (with respect to the parameter t). By Definition 3.1.3, $(\gamma, \boldsymbol{n}_\gamma, \boldsymbol{b}) : I \to V_{3,2} \times V_{3,2}$ is a framed curve, so that we can apply the existence and the uniqueness theorem for framed curves (cf. Theorems 1.1.4 and 1.1.5). This is the reason why we call (ℓ, m, n, α) the curvature of $(\gamma, \boldsymbol{b})$.

Suppose that $\boldsymbol{n}_\gamma(t)$ is a spherical frontal curve, that is, there exists a smooth mapping $\boldsymbol{d}_o : I \to S^2$ such that $(\boldsymbol{n}_\gamma, \boldsymbol{d}_o) : I \to V_{3,2}$ is a spherical Legendre curve (cf. [41]). Then we call $\boldsymbol{d}_o(t)$ the *spherical osculating Darboux frontal curve*. $\boldsymbol{d}_o(t)$ is a dual of $\boldsymbol{n}_\gamma(t)$ as spherical frontal curve. In this sense, the spherical osculating Darboux frontal curve is a natural generalization of the spherical osculating Darboux image for a regular curve. We put $\boldsymbol{t}_o(t) = \boldsymbol{n}_\gamma(t) \times \boldsymbol{d}_o(t)$ and we have the Frenet-Serret type formula of $(\boldsymbol{n}_\gamma, \boldsymbol{d}_o)$ which is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}_\gamma(t) \\ \dot{\boldsymbol{d}}_o(t) \\ \dot{\boldsymbol{t}}_o(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_o(t) \\ 0 & 0 & n_o(t) \\ -m_o(t) & -n_o(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_\gamma(t) \\ \boldsymbol{d}_o(t) \\ \boldsymbol{t}_o(t) \end{pmatrix},$$

where $m_o(t) = \dot{\boldsymbol{n}}_\gamma(t) \cdot \boldsymbol{d}_o(t)$ and $n_o(t) = \dot{\boldsymbol{n}}_\gamma(t) \cdot \boldsymbol{t}_o(t)$. Moreover, we have the following relationship:

$$\begin{pmatrix} \boldsymbol{d}_o(t) \\ \boldsymbol{t}_o(t) \end{pmatrix} = \begin{pmatrix} \cos\theta_o(t) & -\sin\theta_o(t) \\ \sin\theta_o(t) & \cos\theta_o(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(t) \\ \boldsymbol{t}(t) \end{pmatrix},$$

where $\cos\theta_o(t) = \boldsymbol{d}_o(t) \cdot \boldsymbol{b}(t) = \boldsymbol{t}_o(t) \cdot \boldsymbol{t}(t)$ and $\sin\theta_o(t) = -\boldsymbol{d}_o(t) \cdot \boldsymbol{t}(t) = \boldsymbol{t}_o(t) \cdot \boldsymbol{b}(t)$.

On the other hand, suppose that $\boldsymbol{b}(t)$ is a spherical frontal curve, that is, there exists a smooth mapping $\boldsymbol{d}_r : I \to S^2$ such that $(\boldsymbol{b}, \boldsymbol{d}_r) : I \to V_{3,2}$ is a spherical Legendre curve. Then we call $\boldsymbol{d}_r(t)$ the *spherical rectifying Darboux frontal curve*. We put $\boldsymbol{t}_r(t) = \boldsymbol{b}(t) \times \boldsymbol{d}_r(t)$ and we have the Frenet-Serret type formula of $(\boldsymbol{b}, \boldsymbol{d}_r)$ which is given by

$$\begin{pmatrix} \dot{\boldsymbol{b}}(t) \\ \dot{\boldsymbol{d}}_r(t) \\ \dot{\boldsymbol{t}}_r(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_r(t) \\ 0 & 0 & n_r(t) \\ -m_r(t) & -n_r(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(t) \\ \boldsymbol{d}_r(t) \\ \boldsymbol{t}_r(t) \end{pmatrix},$$

where $m_r(t) = \dot{\boldsymbol{b}}(t) \cdot \boldsymbol{d}_r(t)$ and $n_r(t) = \dot{\boldsymbol{b}}(t) \cdot \boldsymbol{t}_r(t)$. We have the following relationship:

$$\begin{pmatrix} \boldsymbol{d}_r(t) \\ \boldsymbol{t}_r(t) \end{pmatrix} = \begin{pmatrix} \cos\theta_r(t) & -\sin\theta_r(t) \\ \sin\theta_r(t) & \cos\theta_r(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}_\gamma(t) \end{pmatrix},$$

where $\cos\theta_r(t) = \boldsymbol{d}_r(t) \cdot \boldsymbol{t}(t) = \boldsymbol{t}_r(t) \cdot \boldsymbol{n}_\gamma(t)$ and $\sin\theta_r(t) = -\boldsymbol{d}_r(t) \cdot \boldsymbol{n}_\gamma(t) = \boldsymbol{t}_r(t) \cdot \boldsymbol{t}(t)$.

3.2 Osculating developable surfaces

In this section, we introduce the osculating developable surface along a frontal curve on a embedded surface. Let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve with a spherical osculating Darboux frontal curve $\boldsymbol{d}_o(t)$.

Definition 3.2.1 We define a mapping $\mathcal{OD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\mathcal{OD}_\gamma(t, u) = \gamma(t) + u\boldsymbol{d}_o(t).$$

We call $\mathcal{OD}_\gamma$ a (generalized) *osculating developable surface* of M along γ.

The osculating developable surface $\mathcal{OD}_\gamma$ is a ruled surface and we have

$$\begin{aligned} \det\left(\dot{\gamma}, \boldsymbol{d}_o, \dot{\boldsymbol{d}}_o\right) &= \det\left(\alpha \boldsymbol{t}, \boldsymbol{d}_o, n_o \boldsymbol{t}_o\right) \\ &= \det\left(\alpha \boldsymbol{t}, \cos\theta_o \boldsymbol{b} - \sin\theta_o \boldsymbol{t}, n_o\left(\sin\theta_o \boldsymbol{b} + \cos\theta_o \boldsymbol{t}\right)\right) \\ &= 0. \end{aligned}$$

This means that $\mathcal{OD}_\gamma$ is a developable surface. If γ is a regular curve, then we have $\mathcal{OD}_\gamma(I \times \mathbb{R}) = OD_\gamma(I \times \mathbb{R})$. This is the reason why we call $\mathcal{OD}_\gamma$ the (generalized) osculating developable surface of M along γ (cf. Definition 3.1.1).

We introduce an invariant σ_o of (M, γ) as follows:

$$\sigma_o(t) = \alpha(t)\sin\theta_o(t) + \frac{d}{dt}\left(\frac{\alpha(t)\cos\theta_o(t)}{n_o(t)}\right) \text{ (when } n_o(t) \neq 0).$$

Then $n_o(t)$ and $\sigma_o(t)$ characterize contour generators of M as follows:

Theorem 3.2.2 *Let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve with a spherical osculating Darboux frontal curve $\boldsymbol{d}_o(t)$. Suppose that the set of regular points of $\boldsymbol{n}_\gamma$ is dense in I.*

(1) *The following are equivalent:*

   (a) *$\mathcal{OD}_\gamma$ is a cylinder,*

   (b) *$n_o(t) \equiv 0$,*

   (c) *γ is a contour generator with respect to an orthogonal projection.*

(2) *If $n_o(t) \neq 0$, then the following are equivalent:*

   (d) *$\mathcal{OD}_\gamma$ is a conical surface,*

   (e) *$\sigma_o(t) \equiv 0$,*

   (f) *γ is a contour generator with respect to a central projection.*

Proof. (1) By definition, $\mathcal{OD}_\gamma$ is a cylinder if and only if $\boldsymbol{d}_o(t)$ is constant. Since $\dot{\boldsymbol{d}}_o(t) = n_o(t)\boldsymbol{t}_o(t)$, $\boldsymbol{d}_o(t)$ is constant if and only if $n_o(t) \equiv 0$. Therefore, (a) is equivalent to (b). Suppose that (c) holds. Then there exists a vector $\boldsymbol{k} \in S^2$ such that $\boldsymbol{n}_\gamma(t) \cdot \boldsymbol{k} \equiv 0$. Then there exist functions $\lambda, \mu : I \to \mathbb{R}$ such that $\boldsymbol{k} = \lambda(t)\boldsymbol{d}_o(t) + \mu(r)\boldsymbol{t}_o(t)$. Since $\dot{\boldsymbol{n}}_\gamma(t) \cdot \boldsymbol{k} \equiv 0$ and the assumption, we have $\mu(t) \equiv 0$, so that we have $\boldsymbol{k} = \pm\boldsymbol{d}_o(t)$. Therefore, (a) holds. Conversely, suppose that (a) holds. Then we choose $\boldsymbol{k} = \boldsymbol{d}_o(t) \in S^2$. By definition of $\boldsymbol{d}_o(t)$, we have $\boldsymbol{n}_\gamma(t) \cdot \boldsymbol{k} = \boldsymbol{n}_\gamma(t) \cdot \boldsymbol{d}_o(t) \equiv 0$. Therefore, (a) implies (c).

(2) We consider a striction curve $\boldsymbol{\sigma}(t)$ given by

$$\boldsymbol{\sigma}(t) = \boldsymbol{\gamma}(t) - \frac{\alpha(t)\cos\theta_o(t)}{n_o(t)}\boldsymbol{d}_o(t).$$

Then (d) is equivalent to the condition that $\dot{\boldsymbol{\sigma}}(t) \equiv \boldsymbol{0}$. We can calculate that

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \alpha\boldsymbol{t} - \frac{d}{dt}\left(\frac{\alpha\cos\theta_o}{n_o}\right)\boldsymbol{d}_o - \left(\frac{\alpha\cos\theta_o}{n_o}\right)n_o\boldsymbol{t}_o \\
&= \alpha\left(-\sin\theta_o\boldsymbol{d}_o + \cos\theta_o\boldsymbol{t}_o\right) - \frac{d}{dt}\left(\frac{\alpha\cos\theta_o}{n_o}\right)\boldsymbol{d}_o - \alpha\cos\theta_o\boldsymbol{t}_o \\
&= -\left(\alpha\sin\theta_o + \frac{d}{dt}\left(\frac{\alpha\cos\theta_o}{n_o}\right)\right)\boldsymbol{d}_o \\
&= -\sigma\boldsymbol{d}_o.
\end{aligned}$$

It follows that (d) and (e) are equivalent. By the definition of the contour generator with respect to a central projection, (f) means that there exists $\boldsymbol{c} \in \mathbb{R}^3$ such that $(\boldsymbol{\gamma}(t) - \boldsymbol{c}) \cdot \boldsymbol{n}_\gamma(t) \equiv 0$. If (d) holds, then $\boldsymbol{\sigma}(t)$ is constant. For the constant point $\boldsymbol{c} = \boldsymbol{\sigma}(t) \in \mathbb{R}^3$, we have

$$\begin{aligned}
(\boldsymbol{\gamma}(t) - \boldsymbol{c}) \cdot \boldsymbol{n}_\gamma(t) &= (\boldsymbol{\gamma}(t) - \boldsymbol{\sigma}(t)) \cdot \boldsymbol{n}_\gamma(t) \\
&= \left(\frac{\alpha(t)\cos\theta_o(t)}{n_o(t)}\boldsymbol{d}_o(t)\right) \cdot \boldsymbol{n}_\gamma(t) \\
&= 0.
\end{aligned}$$

This means that (f) holds. For the converse, by (f), there exists a point $\boldsymbol{c} \in \mathbb{R}^3$ such that $(\boldsymbol{\gamma}(t) - \boldsymbol{c}) \cdot \boldsymbol{n}_\gamma(t) = 0$. Taking the derivative of the both side, we have $0 = (\boldsymbol{\gamma}(t) - \boldsymbol{c}) \cdot (m_o(t)\boldsymbol{t}_o(t))$. Since the set of regular points of $\boldsymbol{n}_\gamma$ is dense, there exists a function $\lambda : I \to \mathbb{R}$ such that $\boldsymbol{\gamma}(t) - \boldsymbol{c} = \lambda(t)\boldsymbol{d}_o(t)$. Taking the derivative again, we have

$$\begin{aligned}
0 &= \alpha m_o\cos\theta_o + (\boldsymbol{\gamma} - \boldsymbol{c}) \cdot \left(-m_o^2\boldsymbol{n}_\gamma - m_o n_o\boldsymbol{d}_o + \dot{m}_o\boldsymbol{t}_o\right) \\
&= \alpha m_o\cos\theta_o + \lambda\boldsymbol{d}_o \cdot \left(-m_o^2\boldsymbol{n}_\gamma - m_o n_o\boldsymbol{d}_o + \dot{m}_o\boldsymbol{t}_o\right) \\
&= m_o\left(\alpha\cos\theta_o - \lambda n_o\right).
\end{aligned}$$

Since the set of regular points of $\boldsymbol{n}_\gamma$ is dense, $\alpha(t)\cos\theta_o(t) - \lambda(t)n_o(t) = 0$. It follows that

$$\boldsymbol{c} = \boldsymbol{\gamma}(t) - \lambda(t)\boldsymbol{d}_o(t) = \boldsymbol{\gamma}(t) - \frac{\alpha(t)\cos\theta_o(t)}{n_o(t)}\boldsymbol{d}_o(t) = \boldsymbol{\sigma}(t).$$

Therefore, $\boldsymbol{\sigma}(t)$ is constant, so that (iv) holds. This completes the proof. □

Corollary 3.2.3 *The osculating developable surface $\mathcal{OD}_\gamma$ is non-cylindrical if and only if $n_o(t) \neq 0$.*

By the result of Theorem 3.2.2, two invariants $n_o(t)$ and $\sigma_o(t)$ might be related to the singularities of osculating developable surfaces. Actually, we can characterize the singularities of osculating developable surfaces of M along curves by using theses two invariants $n_o(t)$ and $\sigma_o(t)$.

Theorem 3.2.4 *Let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve a with spherical osculating Darboux frontal curve $\boldsymbol{d}_o(t)$. Then we have the following:*

(1) *(t_0, u_0) is a singular point of the osculating developable surface $\mathcal{OD}_\gamma$ if and only if*

$$\alpha(t_0)\cos\theta_o(t_0) + u_0 n_o(t_0) = 0.$$

(2) *Suppose that (t_0, u_0) is a singular point of $\mathcal{OD}_\gamma$ and $m_o(t_0) \neq 0$.*

(a) *The osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge* ***ce*** *at (t_0, u_0) if and only if*

(a-i) *$n_o(t_0) \neq 0$ and $\sigma_o(t_0) \neq 0$,*

or

(a-ii) *$n_o(t_0) = 0$ and $\dot\alpha(t_0)\cos\theta_o(t_0) - \alpha(t_0)n(t_0)\sin\theta_o(t) + u_0\dot n_o(t_0) \neq 0$.*

(b) *The osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail* ***sw*** *at (t_0, u_0) if and only if $\sigma_o(t_0) = 0$ and $n_o(t_0)\dot\sigma_o(t_0) \neq 0$.*

(3) *Suppose that (t_0, u_0) is a singular point of $\mathcal{OD}_\gamma$ and $m_o(t_0) = 0$.*

(c) *The osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal cross cap* ***ccr*** *at (t_0, u_0) if and only if $n_o(t_0) \neq 0$ and $\dot m_o(t_0)\sigma_o(t_0) \neq 0$.*

Proof. $\mathcal{OD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ is a frontal with a unit normal vector field $\boldsymbol{n}_\gamma$. We remark that $\mathcal{OD}_\gamma : (I \times \mathbb{R}, (t_0, u_0)) \to \$^3$ is a wave front (not frontal) if and only if $m(t_0) \neq 0$. We give a signed area density function $\lambda : I \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}\lambda(t,u) &= \det\left(\frac{\partial\mathcal{OD}_\gamma}{\partial t}(t,u), \frac{\partial\mathcal{OD}_\gamma}{\partial u}(t,u), \boldsymbol{n}_\gamma(t)\right)\\ &= -\alpha(t)\cos\theta_o(t) - u n_o(t).\end{aligned}$$

This means that (1) holds.

Suppose that (t_0, u_0) is a non-degenerate singular point of $\mathcal{OD}_\gamma$, that is,

$$\frac{\partial\lambda}{\partial t}(t_0,u_0) = -\dot\alpha(t_0)\cos\theta_o(t_0) + \alpha(t_0)\dot\theta_o(t_0)\sin\theta_o(t_0) - u_0\dot n_o(t_0) \neq 0$$

or

$$\frac{\partial\lambda}{\partial u}(t_0,u_0) = -n_o(t_0) \neq 0.$$

(a-i) We consider the case when $m_o(t_0) \neq 0$ and $n_o(t_0) \neq 0$. Then a singular curve $\boldsymbol{c} : (I, t_0) \to (I \times \mathbb{R}, (t_0, u_0))$ is given by

$$\boldsymbol{c}(t) = \left(t, -\frac{\alpha(t)\cos\theta_o(t)}{n_o(t)}\right)$$

and a null vector field $\boldsymbol{\eta} : (I, t_0) \to \mathbb{R}^2$ is given by

$$\boldsymbol{\eta}(t) = (1, \alpha(t) \sin \theta_o(t)).$$

By using the criterion for the cuspidal edge, the osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge $\boldsymbol{ce}$ at (t_0, u_0) if and only if

$$\det(\dot{\boldsymbol{c}}(t_0), \boldsymbol{\eta}(t_0)) = \alpha(t_0) \sin \theta_o(t_0) + \frac{d}{dt}\left(\frac{\alpha \cos \theta_o}{n_o}\right)(t_0) = \sigma_o(t_0) \neq 0.$$

Therefore, (a-i) holds.

(a-ii) We consider the case when $m_o(t_0) \neq 0$, $n_o(t_0) = 0$ and

$$-\dot{\alpha}(t_0) \cos \theta_o(t_0) + \alpha(t_0)\dot{\theta}_o(t_0) \sin \theta_o(t_0) - u_0 \dot{n}_o(t_0) \neq 0.$$

Then there exists $\phi : (\mathbb{R}, u_0) \to (I, t_0)$ such that $\lambda(\phi(u), u) = 0$ and

$$\frac{d\phi}{du}(u) = -\left.\frac{\lambda_u(t,u)}{\lambda_t(t,u)}\right|_{t=\phi(u)}.$$

Accordingly, we have a singular curve $\boldsymbol{c}(u) = (\phi(u), u)$ and a null vector field

$$\boldsymbol{\eta}(u) = (1, \alpha(\phi(u)) \sin \theta_o(\phi(u))).$$

Since

$$\det\left(\frac{d\boldsymbol{c}}{du}(u_0), \boldsymbol{\eta}(u_0)\right) = \det\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix} = -1 \neq 0,$$

the osculating developable surface $\mathcal{OD}_\gamma$ is always $\mathcal{A}$-equivalent to the cuspidal edge **ce** at (t_0, u_0) under the assumption.

(2)-(b) By the proof of (a-ii), it is enough to consider in the same assumption as (a-i). By using the criterion of the swallowtail, the osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the swallowtail $\boldsymbol{sw}$ at (t_0, u_0) if and only if

$$\det(\dot{\boldsymbol{c}}(t_0), \boldsymbol{\eta}(t_0)) = \sigma_o(t_0) = 0 \text{ and } \frac{d \det(\dot{\boldsymbol{c}}, \boldsymbol{\eta})}{dt}(t_0) = \dot{\sigma}_o(t_0) \neq 0.$$

Therefore, (2)-(b) holds.

(3) We consider the case when $m_o(t_0) = 0$ and $n_o(t_0) \neq 0$. (3) can be proven using the criterion for the cuspidal cross cap $\boldsymbol{ccr}$ (cf. Theorem 2.1.6). For $\hat{\boldsymbol{c}}(t) = \mathcal{OD}_\gamma(\boldsymbol{c}(t))$, we consider a function

$$\Phi(t) = \det\left(\dot{\hat{\boldsymbol{c}}}(t), \boldsymbol{n}_\gamma(t), d\boldsymbol{n}_\gamma(\boldsymbol{\eta}(t))\right) = m_o(t)\sigma_o(t).$$

By using the criterion of the cuspidal cross cap, the osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal cross cap $\boldsymbol{ccr}$ at (t_0, u_0) if and only if $\sigma_o(t_0) \neq 0$ and $\dot{m}_o(t) \neq 0$. On the other hand, in the case when $n_o(t_0) = 0$, the osculating developable surface $\mathcal{OD}_\gamma$ is never $\mathcal{A}$-equivalent to the cuspidal cross cap $\boldsymbol{ccr}$ at (t_0, u_0). This completes the proof. □

Theorem 3.2.4, (3)-(c) occurs when we consider frontal curves on embedded surfaces. In other words, we have the following corollary.

Corollary 3.2.5 *For a regular curve $\gamma : I \to M \subset \mathbb{R}^3$, the osculating developable surface $\mathcal{OD}_\gamma$ is never $\mathcal{A}$-equivalent to the cuspidal cross cap $\boldsymbol{ccr}$.*

We consider the case when the surface itself is a developable surface where the curve is located on.

Proposition 3.2.6 *Suppose that the developable surface M is the image of*

$$F_{(\boldsymbol{c},\boldsymbol{\xi})}(s,u) = \boldsymbol{c}(s) + u\boldsymbol{\xi}(s)$$

and $\gamma : I \to M$ is a frontal curve in the regular part of M, where $\boldsymbol{c}(s)$ is the base curve and $\boldsymbol{\xi}(s)$ is the director curve. Then $(\boldsymbol{n}_\gamma, \overline{\boldsymbol{\xi}}) : I \to V_{3,2}$ is a spherical Legendre curve and $\mathcal{OD}_\gamma(I \times \mathbb{R}) \subset M$, where $\overline{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(s(t))/|\boldsymbol{\xi}(s(t))|$.

Proof. By the assumption, we have $\det(\boldsymbol{c}'(s), \boldsymbol{\xi}(s), \boldsymbol{\xi}'(s)) = 0$. We now consider a curve on M parametrized by

$$\gamma(t) = \boldsymbol{c}(s(t)) + u(t)\boldsymbol{\xi}(s(t)),$$

where t is a parameter of γ. Since

$$\frac{\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}}{\partial s}(s,u) = \boldsymbol{c}'(s) + u\boldsymbol{\xi}'(s), \; \frac{\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}}{\partial u}(s,u) = \boldsymbol{\xi}(s),$$

the unit normal vector along γ is

$$\begin{aligned}\boldsymbol{n}_\gamma(t) &= \frac{1}{l(t)}\left((\boldsymbol{c}'(s(t)) + u(t)\boldsymbol{\xi}'(s(t))) \times \boldsymbol{\xi}(s(t))\right) \\ &= \frac{1}{l(t)}\left((\boldsymbol{c}'(s(t)) \times \boldsymbol{\xi}(s(t))) + u(t)\left(\boldsymbol{\xi}'(s(t)) \times \boldsymbol{\xi}(s(t))\right)\right),\end{aligned}$$

where

$$l(t) = \left|\frac{\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}}{\partial s}(s(t),u(t)) \times \frac{\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}}{\partial u}(s(t),u(t))\right|.$$

We can also calculate that

$$\dot{\boldsymbol{n}}_\gamma = \frac{\dot{s}}{l}\left(\boldsymbol{c}'' \times \boldsymbol{\xi} + \boldsymbol{c}' \times \boldsymbol{\xi}'\right) - \frac{\dot{l}}{l^2}\left(\boldsymbol{c}' \times \boldsymbol{\xi}\right) + \frac{\dot{s}u}{l}\left(\boldsymbol{\xi}'' \times \boldsymbol{\xi}\right) + \frac{\dot{u}l - u\dot{l}}{l^2}\left(\boldsymbol{\xi}' \times \boldsymbol{\xi}\right),$$

so that we have $\boldsymbol{n}_\gamma(t) \cdot \overline{\boldsymbol{\xi}}(t) = 0$ and $\dot{\boldsymbol{n}}_\gamma(t) \cdot \overline{\boldsymbol{\xi}}(t) = 0$, where $\overline{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(s(t))/|\boldsymbol{\xi}(s(t))|$. Therefore, $(\boldsymbol{n}_\gamma, \overline{\boldsymbol{\xi}}) : I \to V_{3,2}$ is a spherical Legendre curve and we have $\mathcal{OD}_\gamma(t,v) = \gamma(t) + v\overline{\boldsymbol{\xi}}(t)$. This means that $\mathcal{OD}_\gamma(I \times \mathbb{R}) \subset M$. □

3.3 Normal developable surfaces

In this section, we introduce the normal developable surface along a frontal curve on an embedded surface. Let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve with a spherical rectifying Darboux frontal curve $\boldsymbol{d}_r(t)$.

Definition 3.3.1 We define a mapping $\mathcal{ND}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\mathcal{ND}_\gamma(t,u) = \gamma(t) + u\boldsymbol{d}_r(t).$$

We call $\mathcal{ND}_\gamma$ a (generalized) *normal developable surface* of M along γ.

The normal developable surface $\mathcal{ND}_\gamma$ is a ruled surface and we have

$$\begin{aligned}\det\left(\dot{\gamma}, \boldsymbol{d}_r, \dot{\boldsymbol{d}}_r\right) &= \det\left(\alpha \boldsymbol{t}, \boldsymbol{d}_r, n_r \boldsymbol{t}_r\right) \\ &= \det\left(\alpha \boldsymbol{t}, \cos\theta_r \boldsymbol{t} - \sin\theta_r \boldsymbol{n}_\gamma, n_r\left(\sin\theta_r \boldsymbol{t} + \cos\theta_r \boldsymbol{n}_\gamma\right)\right) \\ &= 0.\end{aligned}$$

This means that $\mathcal{ND}_\gamma(I \times \mathbb{R})$ is a developable surface. If γ is a regular curve, then we have $\mathcal{ND}_\gamma(I \times \mathbb{R}) = ND_\gamma(I \times \mathbb{R})$. This is the reason why we call $\mathcal{ND}_\gamma$ the (generalized) normal developable surface of M along γ.

We introduce an invariant σ_r of (M, γ) as follows:

$$\sigma_r(t) = \alpha(t)\cos\theta_r(t) - \frac{d}{dt}\left(\frac{\alpha(t)\sin\theta_r(t)}{n_r(t)}\right) \text{ (when } n_r(t) \neq 0).$$

Then $n_r(t)$ and $\sigma_r(t)$ characterize cylindrical surfaces and conical surfaces. By the same method of Theorem 3.2.2, we have the following proposition.

Proposition 3.3.2 *Let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve with spherical rectifying Darboux frontal curve $\boldsymbol{d}_r(t)$. Then we have the following:*

(A) *The following are equivalent:*

- (i) *$\mathcal{ND}_\gamma$ is a cylinder,*
- (ii) $n_r(t) \equiv 0$.

(B) *If $n_r(t) \neq 0$, then the following are equivalent:*

- (iii) *$\mathcal{ND}_\gamma$ is a conical surface,*
- (iv) $\sigma_r(t) \equiv 0$.

Corollary 3.3.3 *The normal developable surface $\mathcal{ND}_\gamma$ is non-cylindrical if and only if $n_r(t) \neq 0$.*

By the result of Proposition 3.3.2, two invariants $n_r(t)$ and $\sigma_r(t)$ might be related to the singularities of normal developable surfaces. Actually, we can characterize the singularities of normal developable surfaces of M along curves by using theses two invariants $n_r(t)$ and $\sigma_r(t)$.

Theorem 3.3.4 *Let $(\gamma, \boldsymbol{b}) : I \to T_1M$ be a Legendre curve with spherical rectifying Darboux frontal curve $\boldsymbol{d}_r(t)$. Then we have the following:*

(1) *(t_0, u_0) is a singular point of the normal developable surface $\mathcal{ND}_\gamma$ if and only if*

$$\alpha(t_0)\sin\theta_r(t_0) + u_0 n_r(t_0) = 0.$$

(2) *Suppose that (t_0, u_0) is a singular point of $\mathcal{ND}_\gamma$ and $m_r(t_0) \neq 0$.*

(a) *The normal developable surface $\mathcal{ND}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge* ***ce*** *at (t_0, u_0) if and only if*

(a-i) *$n_r(t_0) \neq 0$ and $\sigma_r(t_0) \neq 0$,*

or

(a-ii) *$n_r(t_0) = 0$ and $\dot{\alpha}(t_0)\sin\theta_r(t_0) - \alpha(t_0)m(t_0)\cos\theta(t_0) + u\dot{n}_r(t_0) \neq 0$.*

(b) *The normal developable surface* $\mathcal{ND}_\gamma$ *is* $\mathcal{A}$*-equivalent to the swallowtail* $\boldsymbol{sw}$ *at* (t_0, u_0) *if and only if* $n_r(t_0) \neq 0$, $\sigma_r(t_0) = 0$ *and* $\dot{\sigma}_r(t_0) \neq 0$.

(3) *Suppose that* (t_0, u_0) *is a singular point of* $\mathcal{ND}_\gamma$ *and* $m_r(t_0) = 0$.

(c) *The normal developable surface* $\mathcal{ND}_\gamma$ *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap* $\boldsymbol{ccr}$ *at* (t_0, u_0) *if and only if* $n_r(t_0) \neq 0$ *and* $\dot{m}_r(t_0)\sigma_r(t_0) \neq 0$.

Remark 3.3.5 In order the normal developable surface $\mathcal{ND}_\gamma$ to be $\mathcal{A}$-equivalent to the cuspidal cross cap $\boldsymbol{ccr}$, $\boldsymbol{b}$ needs to have a singular point. Therefore, the generalization to spherical frontal is necessary different from the regular case (cf. [18]).

3.4 Examples

We give some examples of osculating developable surfaces and normal developable surfaces along frontal curves on embedded surfaces.

Example 3.4.1 Let $\boldsymbol{X} : \mathbb{R}^2 \to \mathbb{R}^3$ be $\boldsymbol{X}(u,v) = (u, v, u^2 + (u^2 - v^3)^2)$, and let $\overline{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be $\overline{\gamma}(t) = (t^3, t^2)$. The unit normal vector field $\boldsymbol{n}$ is given by

$$\boldsymbol{n}(u,v) = \frac{1}{f(u,v)}(-2u - 4u(u^2 - v^3), 6v^2(u^2 - v^3), 1),$$

where $f(u,v) = \sqrt{1 + (2u + 4u(u^2 - v^3))^2 + (6v^2(u^2 - v^3))^2}$. Then $(\gamma, \boldsymbol{b}) : I \to T_1 M$;

$$\gamma(t) = \boldsymbol{X} \circ \overline{\gamma}(t) = (t^3, t^2, t^6),$$
$$\boldsymbol{\nu}(t) = \frac{1}{\sqrt{(1 + 4t^6)(4 + 9t^2 + 36t^8)}}(2, -3t - 12t^7, 4t^3).$$

is a Legendre immersion. By straightforward calculations, we have

$$\boldsymbol{n}_\gamma(t) = \frac{1}{\sqrt{1 + 4t^6}}(-2t^3, 0, 1), \qquad \boldsymbol{t}(t) = \frac{1}{\sqrt{4 + 9t^2 + 36t^8}}(3t, 2, 6t^4),$$
$$\ell(t) = \frac{-12t^3}{(1 + 4t^6)\sqrt{4 + 9t^2 + 36t^8}}, \qquad m(t) = \frac{-18t^3}{\sqrt{(1 + 4t^6)(4 + 9t^2 + 36t^8)}},$$
$$n(t) = \frac{-6 - 96t^6}{(4 + 9t^2 + 36t^8)\sqrt{1 + 4t^6}}.$$

We can easily check that $\boldsymbol{d}_o(t) = (0, 1, 0)$ is the spherical Darboux vector field, so that the osculating developable surface $\mathcal{OD}_\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$\mathcal{OD}_\gamma(t, u) = \gamma(t) + u\boldsymbol{d}_o(t) = (t^3, t^2 + u, t^6).$$

Then we can also calculate that

$$\boldsymbol{t}_o(t) = \frac{1}{\sqrt{1 + 4t^6}}(-1, 0, -2t^3), \quad m_o(t) = \frac{-6t^2}{1 + 4t^6}, \quad n_o(t) = 0.$$

By Theorem 3.2.2-(1) and $n_o(t) \equiv 0$, $\mathcal{OD}_\gamma$ is a cylinder and γ is the contour generator with respect to the orthogonal projection with direction $\boldsymbol{d}_o(t)$. We remark that the apparent contour has regular parameterizations. However, the contour generator γ

has 3/2-cusp at $t = 0$. We cannot recognize singularities of the contour generator from the image of the apparent contour. We draw the pictures of $\boldsymbol{X}$, γ and $\mathcal{OD}_\gamma$ in Figure 3.1.

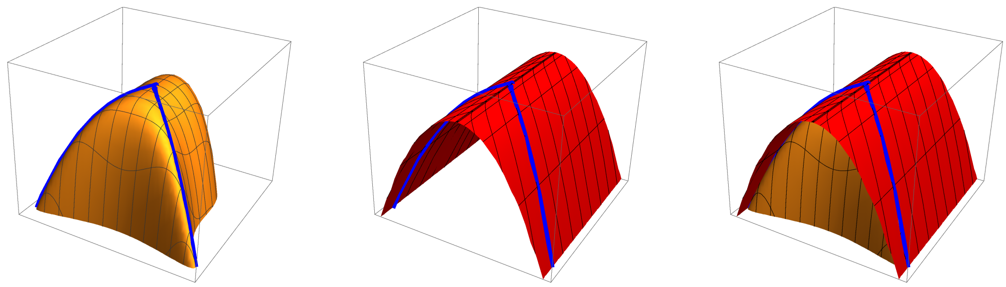

FIGURE 3.1: Figures of Example 3.4.1 (Left to right: $(\boldsymbol{X}, \gamma)$, $(\mathcal{OD}_\gamma, \gamma)$ and $(\boldsymbol{X}, \mathcal{OD}_\gamma, \gamma)$).

Example 3.4.2 Let $\boldsymbol{X} : \mathbb{R}^2 \backslash \{\boldsymbol{0}\} \to \mathbb{R}^3$ be $\boldsymbol{X}(u, v) = (u, v, \sqrt{u^2 + v^2} + (u^2 + (v+1)^3)^2)$ and let $\overline{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be $\overline{\gamma}(t) = (t^3, -1 - t^2)$. Then we have

$$\begin{aligned}
\gamma(t) &= (t^3, -1 - t^2, \sqrt{1 + 2t^2 + t^4 + t^6}),\\
\boldsymbol{n}_\gamma(t) &= \frac{1}{\sqrt{2}} \left(\frac{-t^3}{\sqrt{1 + 2t^2 + t^4 + t^6}}, \frac{1 + t^2}{\sqrt{1 + 2t^2 + t^4 + t^6}}, 1 \right)
\end{aligned}$$

and $(\gamma, \boldsymbol{b}) : I \to T_1 M$ is a Legendre immersion, where

$$\boldsymbol{b}(t) = \frac{1}{f(t)} \left(-\frac{4 + 8t^2 + 7t^4 + 5t^6}{\sqrt{1 + 2t^2 + t^4 + t^6}}, -\frac{3t + 8t^3 + 5t^5 + 6t^7}{\sqrt{1 + 2t^2 + t^4 + t^6}}, 3t + t^3 \right),$$

and $f(t) = \sqrt{16 + 50t^2 + 76t^4 + 50t^6 + 36t^8}$. By straightforward calculations, we have

$$\begin{aligned}
\ell(t) &= \frac{6t^2 + 8t^4 + 11t^6 + 3t^8}{(1 + 2t^2 + t^4 + t^6)\sqrt{8 + 25t^2 + 38t^4 + 25t^6 + 18t^8}},\\
m(t) &= -\frac{9t^3 + 6t^5 + t^7}{(1 + 2t^2 + t^4 + t^6)\sqrt{16 + 50t^2 + 76t^4 + 50t^6 + 36t^8}},\\
n(t) &= \frac{12 + 48t^2 + 45t^4 + 45t^6 + 51t^8 + 23t^{10} + 12t^{12}}{(1 + 2t^2 + t^4 + t^6)(8 + 25t^2 + 38t^4 + 25t^6 + 18t^8)\sqrt{2}}
\end{aligned}$$

and

$$\alpha(t) = t\sqrt{\frac{8 + 25t^2 + 38t^4 + 25t^6 + 18t^8}{1 + 2t^2 + t^4 + t^6}}.$$

We can easily check that the spherical Darboux vector field is given by

$$\boldsymbol{d}_o(t) = \frac{1}{\sqrt{2}} \left(\frac{-t^3}{\sqrt{1 + 2t^2 + t^4 + t^6}}, \frac{1 + t^2}{\sqrt{1 + 2t^2 + t^4 + t^6}}, -1 \right),$$

so that we have the osculating developable surface $\mathcal{OD}_\gamma(t, u) = \gamma(t) + u\boldsymbol{d}_o(t)$. Then we can also calculate that

$$m_o(t) = \frac{3t^2 + t^4}{(1 + 2t^2 + t^4 + t^6)\sqrt{2}}, \quad n_o(t) = \frac{3t^2 + t^4}{(1 + 2t^2 + t^4 + t^6)\sqrt{2}}$$

and $\sigma_o(t) = 0$ (when $t \neq 0$). By Theorem 3.2.2-(2), $\mathcal{OD}_\gamma$ is a conical surface away from $t = 0$. On the other hand, $\mathcal{OD}_\gamma$ is characterized as cylinder at $t = 0$. However, $\mathcal{OD}_\gamma(\mathbb{R} \times \mathbb{R})$ has a parametrization as conical surface. Properties of developable surfaces are intricate when base curves have singular points. We draw pictures of $\boldsymbol{X}$, γ and $\mathcal{OD}_\gamma$ in Figure 3.2.

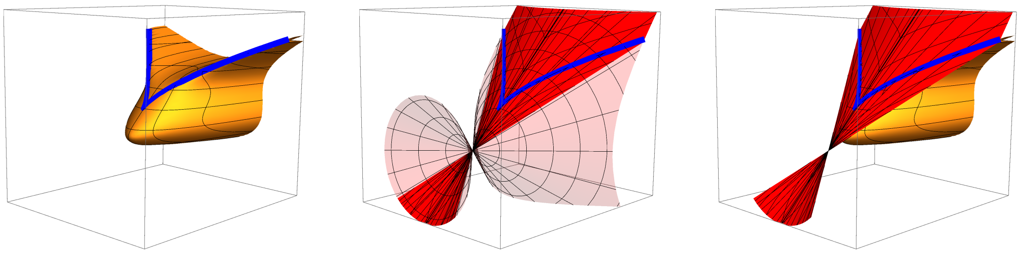

FIGURE 3.2: Figures of Example 3.4.2 (Left to right: $(\boldsymbol{X}, \gamma)$, $(\mathcal{OD}_\gamma, \gamma)$ and $(\boldsymbol{X}, \mathcal{OD}_\gamma, \gamma)$).

Example 3.4.3 Let $\boldsymbol{X} : \mathbb{R}^2 \to \mathbb{R}^3$ be $\boldsymbol{X}(u, v) = (u, v, u^2 + (3/2)v^2)$, and let $\overline{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be $\gamma(t) = (t^2/2, t^3/2)$. Then we have

$$\gamma(t) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{4}t^4 + \frac{1}{6}t^6\right), \ \boldsymbol{n}_\gamma(t) = \frac{1}{\sqrt{1 + t^4 + t^6}}(-t^2, -t^3, 1)$$

and $(\gamma, \boldsymbol{b}) : I \to T_1M$ is a Legendre immersion, where

$$\boldsymbol{b}(t) = \frac{1}{\sqrt{1 + t^2}}(t, -1, 0).$$

By straightforward calculation, we have

$$\ell(t) = \frac{t^2}{\sqrt{(1 + t^2)(1 + t^4 + t^6)}}, \qquad m(t) = -\frac{t(2 + 3t^2)}{(1 + t^4 + t^6)\sqrt{1 + t^2}},$$

$$n(t) = \frac{1}{(1 + t^2)\sqrt{1 + t^4 + t^6}}, \qquad \alpha(t) = t\sqrt{(1 + t^2)(1 + t^4 + t^6)}.$$

We can easily check that the spherical Darboux vector field is given by

$$\boldsymbol{d}_o(t) = \frac{1}{\sqrt{4 + 9t^2 + t^6}}(-3t, 2, -t^3),$$

so that we have the osculating developable surface $\mathcal{OD}_\gamma(t, u) = \gamma(t) + u\boldsymbol{d}_o(t)$. Then we can also calculate that

$$m_o(t) = \frac{t\sqrt{4 + 9t^2 + t^6}}{1 + t^4 + t^6}, \quad n_o(t) = \frac{6\sqrt{1 + t^4 + t^6}}{4 + 9t^2 + t^6}$$

and $\sigma_o(t) \neq 0$. By Theorem 3.2.4 (3)-(c), the osculating developable surface $\mathcal{OD}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal cross cap $\boldsymbol{ccr}$ at $(0, 0)$. We draw pictures of $\boldsymbol{X}, \gamma$ and $\mathcal{OD}_\gamma$ in Figure 3.3.

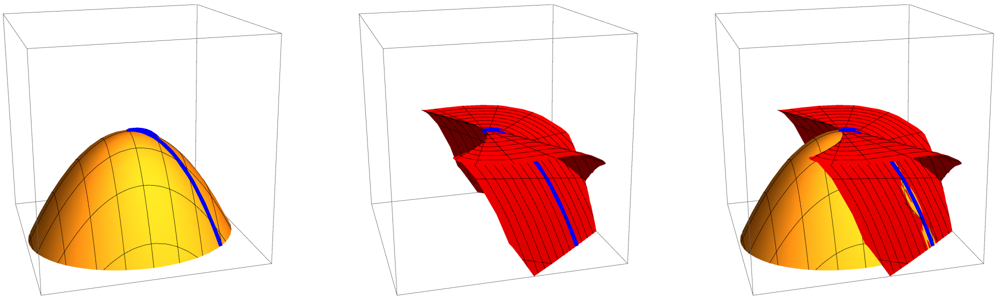

FIGURE 3.3: Figures of Example 3.4.4 (Left to right: $(\boldsymbol{X}, \gamma)$, $(\mathcal{OD}_\gamma, \gamma)$ and $(\boldsymbol{X}, \mathcal{OD}_\gamma, \gamma)$).

Example 3.4.4 Let $\boldsymbol{X} : \mathbb{R}^2 \to \mathbb{R}^3$ be $\boldsymbol{X}(u,v) = (u/2, v/3, u^2/4)$ and let $\overline{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be $\overline{\gamma}(t) = (t^2, t^3)$. Then we have

$$\gamma(t) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{4}t^4\right), \ \boldsymbol{n}_\gamma(t) = \frac{1}{\sqrt{1+t^4}}\left(-t^2, 0, 1\right)$$

and $(\gamma, \boldsymbol{b}) : I \to T_1M$ is a Legendre immersion, where

$$\boldsymbol{b}(t) = \frac{1}{\sqrt{(1+t^4)(1+t^2+t^4)}}\left(t, -1-t^4, t^3\right).$$

By straightforward calculations, we have

$$\ell(t) = \frac{-2t^2}{(1+t^4)\sqrt{1+t^2+t^4}}, \qquad m(t) = \frac{-2t}{\sqrt{(1+t^4)(1+t^2+t^4)}},$$
$$n(t) = \frac{1-t^4}{(1+t^2+t^4)\sqrt{1+t^4}}, \qquad \alpha(t) = t\sqrt{1+t^2+t^4}.$$

We can easily check that the spherical Darboux vector field is given by $\boldsymbol{d}_o(t) = (0,1,0)$. By Theorem 3.2.6, $\mathcal{OD}_\gamma(\mathbb{R}\times\mathbb{R}) \subset \boldsymbol{X}(\mathbb{R}\times\mathbb{R})$. On the other hand, the spherical rectifying Darboux vector field is given by

$$\begin{aligned}\boldsymbol{d}_r(t) = \frac{1}{f(t)}\Big(&3t^2+3t^4+5t^6+2t^8+t^{10}-t^{12}-t^{14},\\ &2t^2+2t^5+4t^7+2t^9+3t^{11},\\ &-1-t^2+t^4+2t^6+5t^8+3t^{10}+3t^{12}\Big),\end{aligned}$$

where $f(t) = (1+t^2+2t^4+t^6+t^8)\sqrt{1+3t^4+4t^6+3t^8+t^{12}}$. We can also calculate that $m_r(t) =$

$$\frac{1+3t^4+4t^6+3t^8+t^{12}}{\sqrt{(1+t^2+2t^4+t^6+t^8)(1+t^2+5t^4+8t^6+14t^8+14t^{10}+14t^{12}+8t^{14}+5t^{16}+t^{18}+t^{20})}}$$

and $n_r(t) =$

$$\frac{6t+6t^3+18t^5+12t^7+18t^9+6t^{11}+6t^{13}}{\sqrt{(1+3t^4+4t^6+3t^8+t^{12})(1+t^2+5t^4+8t^6+14t^8+14t^{10}+14t^{12}+8t^{14}+5t^{16}+t^{18}+t^{20})}}.$$

By Theorem 3.3.4, (2)-(b) and $n_r(0) = 0$, $\lambda_t(0,u) = 1+6u_0$, the normal developable surface $\mathcal{ND}_\gamma$ is $\mathcal{A}$-equivalent to the cuspidal edge **ce** at $(0, u)$ away from $u \neq 1/6$. We draw pictures of $\boldsymbol{X}$, γ, $\mathcal{OD}_\gamma$ and $\mathcal{ND}_\gamma$ in Figure 3.4.

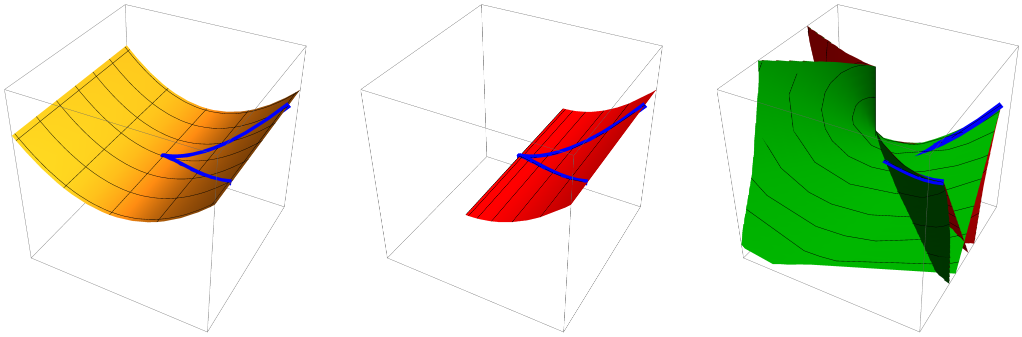

FIGURE 3.4: Figures of Example 3.4.4 (Left to right: $(\boldsymbol{X}, \gamma)$, $(\mathcal{OD}_\gamma, \gamma)$ and $(\mathcal{ND}_\gamma, \gamma)$).

Chapter 4

Spherical framed curves and extrinsic flat great circular surfaces

In this chapter we investigate a special class of surfaces in the 3-sphere which are called extrinsic flat great circular surfaces.

4.1 Preliminaries

We briefly review notions and basic properties of Great circular surfaces and Extrinsic flat great circular surfaces in [27]. Let $(\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) : I \to SO(4)$ be a C^∞-mapping. By standard arguments, we have the following fundamental differential equations:

$$\begin{pmatrix} \dot{\boldsymbol{a}}_0(t) \\ \dot{\boldsymbol{a}}_1(t) \\ \dot{\boldsymbol{a}}_2(t) \\ \dot{\boldsymbol{a}}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ -c_1(t) & 0 & c_4(t) & c_5(t) \\ -c_2(t) & -c_4(t) & 0 & c_6(t) \\ -c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix},$$

where

$$\begin{aligned} c_1(t) &= \dot{\boldsymbol{a}}_0(t) \cdot \boldsymbol{a}_1(t) = -\boldsymbol{a}_0(t) \cdot \dot{\boldsymbol{a}}_1(t), \quad & c_2(t) &= \dot{\boldsymbol{a}}_0(t) \cdot \boldsymbol{a}_2(t) = -\boldsymbol{a}_0(t) \cdot \dot{\boldsymbol{a}}_2(t), \\ c_3(t) &= \dot{\boldsymbol{a}}_0(t) \cdot \boldsymbol{a}_3(t) = -\boldsymbol{a}_0(t) \cdot \dot{\boldsymbol{a}}_3(t), \quad & c_4(t) &= \dot{\boldsymbol{a}}_1(t) \cdot \boldsymbol{a}_2(t) = -\boldsymbol{a}_1(t) \cdot \dot{\boldsymbol{a}}_2(t), \\ c_5(t) &= \dot{\boldsymbol{a}}_1(t) \cdot \boldsymbol{a}_3(t) = -\boldsymbol{a}_1(t) \cdot \dot{\boldsymbol{a}}_3(t), \quad & c_6(t) &= \dot{\boldsymbol{a}}_2(t) \cdot \boldsymbol{a}_3(t) = -\boldsymbol{a}_2(t) \cdot \dot{\boldsymbol{a}}_3(t). \end{aligned}$$

We define a mapping $F_A : I \times [0, 2\pi) \to S^3$ by

$$F_A(t, \theta) = \cos\theta \boldsymbol{a}_1(t) + \sin\theta \boldsymbol{a}_3(t).$$

We call F_A a *great circular surface*, the mapping $\boldsymbol{a}_1$ a *base curve* and the mapping $\boldsymbol{a}_3$ a *directrix*. The great circle defined by $\cos\theta \boldsymbol{a}_1(t_0) + \sin\theta \boldsymbol{a}_3(t_0)$ for a fixed $t_0 \in I$ is called a *generating great circle*. We call the great circular surface with vanishing the extrinsic Gaussian curvature on the regular part an *extrinsic flat great circular surface* (briefly, we call an *E-flat great circular surface*). It is known that a great circular surface $F_A(t, \theta)$ is E-flat if and only if

$$c_1(t)c_6(t) + c_3(t)c_4(t) = 0. \tag{4.1}$$

If $(c_1(t_0), c_3(t_0), c_4(t_0), c_6(t_0)) = (0, 0, 0, 0)$, then all points on the great circle $F_A(\theta, t_0)$ are the singularities. We say that F_A is *non-cyclic* if

$$(c_1(t), c_3(t), c_4(t), c_6(t)) \neq (0, 0, 0, 0) \text{ for all } t \in I. \tag{4.2}$$

In [27], they showed the following criteria:

Theorem 4.1.1 ([27]) *Suppose that $c_1 \equiv c_3 \equiv c_4 \equiv 0$, $c_6(t_0) \neq 0$ and $p_0 = (\theta_0, t_0) \in S(F_A)$.*

(1) *F_A at p_0 is $\mathcal{A}$-equivalent to the cudpidal edge if and only if $c_2(t_0)c_5(t_0) \neq 0$.*

(2) *F_A at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_5(t_0) = 0$ and $c_2(t_0)\dot{c}_5(t_0) \neq 0$.*

(3) *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_2(t_0) = 0$ and $c_5(t_0)\dot{c}_2(t_0) \neq 0$.*

On the other hand, we briefly review the spherical duality from the view point of contact geometry. We now consider the following double fibrations over S^3:

$$\begin{aligned}
&\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in S^3 \times S^3 \mid \boldsymbol{v} \cdot \boldsymbol{w} = 0\}, \\
&\pi_1 : \Delta \ni (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \in S^3, \quad \pi_2 : \Delta \ni (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{w} \in S^3, \\
&\theta_1 = d\boldsymbol{v} \cdot \boldsymbol{w}|_\Delta, \quad \theta_2 = \boldsymbol{v} \cdot d\boldsymbol{w}|_\Delta.
\end{aligned}$$

Here, $d\boldsymbol{v} \cdot \boldsymbol{w} = \sum_{i=4}^4 w_i dv_i$ and $\boldsymbol{v} \cdot d\boldsymbol{w} = \sum_{i=4}^4 v_i dw_i$. By $d(\boldsymbol{v} \cdot \boldsymbol{w}) = d\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot d\boldsymbol{w}$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ on Δ, $\theta_1^{-1}(0)$ and $\theta_2^{-1}(0)$ define the same tangent hyperplane field over Δ which is denoted by K. Then (Δ, K) is a contact manifold and both of π_i are Legendrian fibrations (cf. [27]).

We say that a C^∞-mapping $\mathcal{L} : U \to \Delta$ is an isotropic mapping if $\mathcal{L}^*\theta_i = 0$ $(i = 1$ or $2)$. We remark that the isotropic mapping is Legendrian immersion if it is an immersion. If we have an isotropic mapping $\mathcal{L} : U \to \Delta$, then we say that $\pi_1 \circ \mathcal{L}(U)$ and $\pi_2 \circ \mathcal{L}(U)$ are Δ-dual to each other.

We consider the Δ-dual surface to the locus of singular values of F_A under the assumption that $c_1 \equiv c_3 \equiv c_4 = 0$. By straightforward calculations, the singular value of F_A is $\boldsymbol{a}_1(t)$. We consider a great circular surface defined by

$$F_A^\sharp(t, \theta) = \cos\theta\, \boldsymbol{a}_0(t) + \sin\theta\, \boldsymbol{a}_2(t).$$

Then we have the following diagram and criteria:

$$\begin{tikzcd}
F_A \arrow[r, "\text{taking singular value}"] & \boldsymbol{a}_1 \arrow[d, "\Delta-\text{dual}"] \\
\boldsymbol{a}_0 \arrow[u, "\Delta-\text{dual}"] & F_A^\sharp \arrow[l, "\text{taking singular value}"']
\end{tikzcd}$$

Theorem 4.1.2 ([27]) *Suppose that $c_1 \equiv c_3 \equiv c_4 \equiv 0$, $c_6(t_0) \neq 0$ and $p_0 = (\theta_0, t_0) \in S(F_A)$.*

(1) *F_A at p_0 is $\mathcal{A}$-equivalent to the cudpidal edge if and only if $c_2(t_0)c_5(t_0) \neq 0$.*

(2) *F_A at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_2(t_0) = 0$ and $c_5(t_0)\dot{c}_2(t_0) \neq 0$.*

(3) *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_5(t_0) = 0$ and $c_2(t_0)\dot{c}_5(t_0) \neq 0$.*

We now briefly review the differential geometry on regular curves in S^3. Let $\gamma : I \to S^3$ be a regular curve. Then we can reparameterize γ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So we have the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s)$. In the case when $\boldsymbol{t}'(s) \cdot \boldsymbol{t}'(s) \neq 1$, we have a unit vector $\boldsymbol{n}(s) = (\boldsymbol{t}'(s) + \gamma(s))/(\|\boldsymbol{t}'(s) + \gamma(s)\|)$. Moreover, define $\boldsymbol{e}(t) = \gamma(t) \times \boldsymbol{t}(t) \times \boldsymbol{n}(t)$, then we have an orthonormal frame $\{\gamma(t), \boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{e}(t)\}$ of $\mathbb{R}^4$ along $\gamma(t)$, which called the *Frenet frame* along $\gamma(t)$. By standard arguments, we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{e}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g(s) & 0 \\ 0 & -\kappa_g(s) & 0 & \tau_g(s) \\ 0 & 0 & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{e}(t) \end{pmatrix},$$

where

$$\kappa_g(t) = \|\boldsymbol{t}'(s) + \gamma(s)\| \quad \text{and} \quad \tau_g(t) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa_g^2(s)}.$$

The condition $\dot{\boldsymbol{t}}(t) \cdot \dot{\boldsymbol{t}}(t) \neq 1$ is equivalent to the condition $\kappa_g(t) \neq 0$. It is known that $\gamma(t)$ is a great circle (i.e., the geodesic) if and only if $\kappa_g(t) \equiv 0$.

Let $\gamma : I \to S^3$ be a unit speed curve with $\kappa_g(s) \neq 0$. We review 2-types of E-flat great circular surfaces associated with the Frenet frame as follows:

(1) $F_T(s, \theta) = \cos\theta\gamma(s) + \sin\theta\boldsymbol{t}(s)$: the tangent E-flat great circular surface (cf. [27]),

(2) $DS_\gamma(s, \theta) = \cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{b}(s)$: the dual E-flat great circular surface.

We cannot construct the Frenet frame at singular points of $\gamma : I \to S^3$. In the following section we would like to consider spherical curves which may have singular points.

4.2 Spherical framed curves

Definition 4.2.1 We say that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to V_{4,3}$ is a *spherical framed curve* if $\dot{\gamma}(t) \cdot \boldsymbol{n}_1(t) = 0$ and $\dot{\gamma}(t) \cdot \boldsymbol{n}_2(t) = 0$ for all $t \in I$. Moreover, if $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is an immersion, we call a spherical framed immersion.

Definition 4.2.2 We say that $\gamma : I \to S^3$ is a *spherical framed base curve* if there exists $(\boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \Delta$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to V_{4,3}$ is a spherical framed curve.

We define $\boldsymbol{t}(t) = \gamma(t) \times \boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t)$. Then we have an orthonormal frame $\{\gamma(t), \boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{t}(t)\}$ of $\mathbb{R}^4$ along $\gamma(t)$. By standard arguments, we have the following Frenet-Serret type formula:

Proposition 4.2.3 *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to V_{4,3}$ be a spherical framed curve. Then we have*

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\boldsymbol{n}}_1(t) \\ \dot{\boldsymbol{n}}_2(t) \\ \dot{\boldsymbol{t}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \alpha(t) \\ 0 & 0 & \ell(t) & m(t) \\ 0 & -\ell(t) & 0 & n(t) \\ -\alpha(t) & -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \\ \boldsymbol{t}(t) \end{pmatrix},$$

where $\alpha(t) = \dot{\gamma}(t) \cdot \boldsymbol{t}(t)$, $\ell(t) = \dot{\boldsymbol{n}}_1(t) \cdot \boldsymbol{n}_2(t)$, $m(t) = \dot{\boldsymbol{n}}_1(t) \cdot \boldsymbol{t}(t)$ and $n(t) = \dot{\boldsymbol{n}}_2(t) \cdot \boldsymbol{t}(t)$.

We call the mapping $(\alpha, \ell, m, n) : I \to \mathbb{R}^4$ a *curvature* of the spherical framed curve $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$. Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 4.2.4 Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2), (\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2) : I \to V_{4,3}$ be spherical framed curves. We say that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ are *congruent as spherical framed curves* if there exists a special orthogonal matrix $A \in SO(4)$ such that

$$\widetilde{\gamma}(t) = A(\gamma(t)), \quad \widetilde{\boldsymbol{n}}_1(t) = A(\boldsymbol{n}_1(t)) \quad \text{and} \quad \widetilde{\boldsymbol{n}}_2(t) = A(\boldsymbol{n}_2(t))$$

for all $t \in I$.

As a special case of framed curves, we have the following existence theorem and the uniqueness theorem (cf. [22]).

Theorem 4.2.5 (The existence theorem) *For a smooth mapping $(\alpha, \ell, m, n) : I \to \mathbb{R}^4$, there exists a spherical framed curve $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ whose associated curvature is (α, ℓ, m, n).*

Theorem 4.2.6 (The uniqueness theorem) *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ be spherical framed curves whose curvatures (α, ℓ, m, n) and $(\widetilde{\alpha}, \widetilde{\ell}, \widetilde{m}, \widetilde{n})$ coincide. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{n}}_1, \widetilde{\boldsymbol{n}}_2)$ are congruent as spherical framed curves.*

Example 4.2.7 A regular curve is a typical example of a spherical framed curve. Let $\gamma : I \to S^3$ be a unit speed curve with $\kappa_g(s) \neq 0$. If we take $\boldsymbol{n}_1(s) = \boldsymbol{n}(s)$ and $\boldsymbol{n}_1(s) = \boldsymbol{e}(s)$, then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to V_{4,3}$ is a spherical framed curve. By a straightforward calculation, we have

$$\boldsymbol{t}(s) = \boldsymbol{t}(s), \quad \alpha(s) = 1, \quad \ell(s) = \tau_g(s), \quad m(s) = -\kappa_g(s) \quad \text{and} \quad n(t) = 0.$$

Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to V_{4,3}$ be a spherical framed curve. We define $(\boldsymbol{b}_1(t), \boldsymbol{b}_2(t)) \in \Delta$ by

$$\boldsymbol{b}_1(t) = \cos\theta(t)\boldsymbol{\nu}_1(t) - \sin\theta(t)\boldsymbol{\nu}_2(t),\ \boldsymbol{b}_2(t) = \sin\theta(t)\boldsymbol{\nu}_1(t) + \cos\theta(t)\boldsymbol{\nu}_2(t),$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \boldsymbol{b}_1, \boldsymbol{b}_2) : I \to V_{4,3}$ is also a spherical framed curve and

$$\boldsymbol{t_b}(t) = \gamma(t) \times \boldsymbol{b}_1(t) \times \boldsymbol{b}_2(t) = \boldsymbol{t}(t).$$

We call $\{\gamma(t), \boldsymbol{b}_1(t), \boldsymbol{b}_2(t), \boldsymbol{t_b}(t)\}$ a *rotated frame* along $\gamma(t)$ by $\theta(t)$. If we take a smooth function $\theta(t)$ which satisfies $\dot{\theta}(t) = \ell(t)$, then we call $\{\gamma(t), \boldsymbol{b}_1(t), \boldsymbol{b}_2(t), \boldsymbol{t_b}(t)\}$ a *Bishop type frame* along $\gamma(t)$ (cf. [3]). It follows that the Frenet-Serret type formula is given by

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\boldsymbol{b}}_1(t) \\ \dot{\boldsymbol{b}}_2(t) \\ \dot{\boldsymbol{t}}_{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \alpha(t) \\ 0 & 0 & 0 & \widetilde{m}(t) \\ 0 & 0 & 0 & \widetilde{n}(t) \\ -\alpha(t) & -\widetilde{m}(t) & -\widetilde{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{b}_1(t) \\ \boldsymbol{b}_2(t) \\ \boldsymbol{t_b}(t) \end{pmatrix},$$

where

$$\widetilde{m}(t) = \cos\theta(t)m(t) - \sin\theta(t)n(t),\ \widetilde{n}(t) = \sin\theta(t)m(t) + \cos\theta(t)n(t).$$

On the other hand, we define $(\boldsymbol{f}_1, \boldsymbol{f}_2) \in \Delta$ by

$$\boldsymbol{f}_1(t) = \frac{n(t)\boldsymbol{n}_1(t) - m(t)\boldsymbol{n}_2(t)}{\sqrt{m^2(t) + n^2(t)}},\ \boldsymbol{f}_2(t) = \frac{m(t)\boldsymbol{n}_1(t) + n(t)\boldsymbol{n}_2(t)}{\sqrt{m^2(t) + n^2(t)}}$$

under the condition $m^2(t)+n^2(t)\neq 0$ for all $t\in I$. Then $(\gamma,f_1,f_2):I\to V_{4,3}$ is a spherical framed immersion and

$$\boldsymbol{t}_f(t)=\gamma(t)\times f_1(t)\times f_2(t)=\boldsymbol{t}(t).$$

We call $\{\gamma(t),f_1(t),f_2(t),\boldsymbol{t}_f(t)\}$ a *Frenet type frame* along $\gamma(t)$. Then the Frenet-Serret type formula is given by

$$\begin{pmatrix}\dot\gamma(t)\\ \dot f_1(t)\\ \dot f_2(t)\\ \dot{\boldsymbol{t}}_n(t)\end{pmatrix}=\begin{pmatrix}0&0&0&\alpha(t)\\ 0&0&\overline{\ell}(t)&0\\ 0&-\overline{\ell}(t)&0&\overline{n}(t)\\ -\alpha(t)&0&-\overline{n}(t)&0\end{pmatrix}\begin{pmatrix}\gamma(t)\\ f_1(t)\\ f_2(t)\\ \boldsymbol{t}_n(t)\end{pmatrix},$$

where

$$\overline{\ell}(t)=\ell(t)+\frac{m(t)\dot n(t)-\dot m(t)n(t)}{m^2(t)+n^2(t)},\ \overline{n}(t)=\sqrt{m^2(t)+n^2(t)}.$$

4.3 Dual surfaces and tangent great circular surfaces

In this section, we consider dual surface of spherical framed curve. Let $(\gamma,\boldsymbol{n}_1,\boldsymbol{n}_2):I\to V_{4,3}$ be a spherical framed curve. We define a map $\mathcal{DS}_\gamma:I\times[0,2\pi)\to S^3$ by

$$\mathcal{DS}_\gamma(t,\theta)=\cos\theta\boldsymbol{n}_1(t)+\sin\theta\boldsymbol{n}_2(t).$$

We call $\mathcal{DS}_\gamma$ a *dual surface* of $\gamma(t)$. This is a great circular surface of $(\gamma,\boldsymbol{n}_1,-\boldsymbol{t},\boldsymbol{n}_2):I\to SO(4)$ and we have

$$\begin{pmatrix}\dot\gamma(t)\\ \dot{\boldsymbol{n}}_1(t)\\ -\dot{\boldsymbol{t}}(t)\\ \dot{\boldsymbol{n}}_2(t)\end{pmatrix}=\begin{pmatrix}0&0&-\alpha(t)&0\\ 0&0&-m(t)&\ell(t)\\ \alpha(t)&m(t)&0&n(t)\\ 0&-\ell(t)&-n(t)&0\end{pmatrix}\begin{pmatrix}\gamma(t)\\ \boldsymbol{n}_1(t)\\ -\boldsymbol{t}(t)\\ \boldsymbol{n}_2(t)\end{pmatrix}.$$

By the condition (4.1), $\mathcal{DS}_\gamma$ is an E-flat great circular surface.

Remark 4.3.1 In the same situation as in Example 4.2.7, we consider $\mathcal{DS}_\gamma$ of $(\gamma,\boldsymbol{v}_1,\boldsymbol{v}_2)$. Then $\mathcal{DS}_\gamma(s,\theta)=\cos\theta\boldsymbol{n}(s)+\sin\theta\boldsymbol{b}(s)=DS_\gamma(s,\theta)$. Thus, $\mathcal{DS}_\gamma$ is a generalization of DS_γ.

Remark 4.3.2 $\mathcal{DS}_\gamma$ is non-cyclic if and only if $m^2(t)+n^2(t)\neq 0$ for all $t\in I$.

Hereinafter, we consider a non-cyclic dual surface of $\gamma(t)$ only, that is, we assume that $m^2(t)+n^2(t)\neq 0$ for all $t\in I$. For simplicity, we consider a spherical framed immersion by the Frenet type frame. Let $(\gamma,f_1,f_2):I\to V_{4,3}$ be a spherical framed immersion with the Frenet type frame $\{\gamma(t),f_1(t),f_2(t),\boldsymbol{t}_f(t)\}$. Then the singular points of $\mathcal{DS}_\gamma$ are $(t,0)$ and (t,π) so that the singular value of $\mathcal{DS}_\gamma$ is $\pm\boldsymbol{n}_1(t)$.

Proposition 4.3.3 *$(\pm f_1,\gamma,\boldsymbol{t}_f):I\to V_{4,3}$ is a spherical framed immersion.*

Proof. We can easily check that $(\pm f_1(t),\gamma(t),\boldsymbol{t}_f(t))\in V_{4,3}$. By the Frenet-Serret type formula, we have

$$\pm\dot f_1(t)\cdot\gamma(t)=\overline{\ell}(t)f_2(t)\cdot\gamma(t)=0$$

and

$$\pm\dot f_1(t)\cdot\boldsymbol{t}_n(t)=\overline{\ell}(t)f_2(t)\cdot\boldsymbol{t}_f(t)=0.$$

Therefore $(\pm \boldsymbol{f}_1, \gamma, \boldsymbol{t}_f)$ is a spherical framed immersion. □

Since Proposition 4.3.3, we can take the dual surface of $\pm \boldsymbol{f}_1(t)$. It is given by

$$\mathcal{F}_{\boldsymbol{t}}(t,\theta) = \mathcal{DS}_{f_1}(t,\theta) = \cos\theta\gamma(t) + \sin\theta\boldsymbol{t}(t).$$

$\mathcal{F}_{\boldsymbol{t}}$ corresponds to the E-flat tangent great circular surface in [27], so that we call $\mathcal{F}_{\boldsymbol{t}}$ the *E-flat tangent great circular surface* of $\gamma(t)$. Since Δ-duality, we have the following diagram:

$$\begin{array}{ccc} \mathcal{DS}_{\gamma} & \xrightarrow{\text{taking singular value}} & \pm \boldsymbol{f}_1 \\ {\scriptstyle \Delta-\text{dual}}\uparrow & & \downarrow{\scriptstyle \Delta-\text{dual}} \\ (\pm)\gamma & \xleftarrow[\text{taking singular value}]{} & \mathcal{F}_{\boldsymbol{t}} \end{array}$$

By using the criterion in [27], we have the following propositions.

Proposition 4.3.4 *Suppose that $(t_0,\theta_0) \in S(\mathcal{DS}_\gamma)$. Then we have the following:*

(i) *$\mathcal{DS}_\gamma$ at (t_0,θ_0) is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\alpha(t_0)\overline{\ell}(t_0) \neq 0$.*

(ii) *$\mathcal{DS}_\gamma$ at (t_0,θ_0) is $\mathcal{A}$-equivalent to the swallowtail if and only if $\overline{\ell}(t_0) \neq 0$ and $\alpha(t_0)\dot{\overline{\ell}}(t_0) \neq 0$.*

(iii) *$\mathcal{DS}_\gamma$ at (t_0,θ_0) is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\alpha(t_0) = 0$ and $\dot{\alpha}(t_0)\overline{\ell}(t_0) \neq 0$.*

Proof. In this case, we consider the orthonormal frame $\{\gamma(t), \boldsymbol{f}_1(t), -\boldsymbol{t}(t), \boldsymbol{f}_2(t)\} \in SO(4)$. By the Frenet-Serret type formula, we have

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\boldsymbol{f}}_1(t) \\ -\dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{f}}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha(t) & 0 \\ 0 & 0 & 0 & \overline{\ell}(t) \\ \alpha(t) & 0 & 0 & \overline{n}(t) \\ 0 & -\overline{\ell}(t) & -\overline{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{f}_1(t) \\ -\boldsymbol{t}(t) \\ \boldsymbol{f}_2(t) \end{pmatrix}.$$

By criteria (cf. Theorem 4.1.1 and [27]), we have the assertion. □

Proposition 4.3.5 *Suppose that $(t_0,\theta_0) \in S(\mathcal{F}_{\boldsymbol{t}})$. Then we have the following:*

(i) *$\mathcal{F}_{\boldsymbol{t}}$ at (t_0,θ_0) is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\alpha(t_0)\overline{\ell}(t_0) \neq 0$.*

(ii) *$\mathcal{F}_{\boldsymbol{t}}$ at (t_0,θ_0) is $\mathcal{A}$-equivalent to the swallowtail if and only if $\alpha(t_0) = 0$ and $\dot{\alpha}(t_0)\overline{\ell}(t_0) \neq 0$.*

(iii) *$\mathcal{F}_{\boldsymbol{t}}$ at (t_0,θ_0) is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\overline{\ell}(t_0) \neq 0$ and $\alpha(t_0)\dot{\overline{\ell}}(t_0) \neq 0$.*

Remark 4.3.6 For a regular spherical curve $\gamma : I \to S^3$, Proposition 4.3.4-(iii) and 4.3.5-(ii) doesn't occur.

4.4 Focal great circular surfaces and evolutes

In this section, we consider E-flat focal great circular surfaces and evolutes of spherical framed immersion. Let $(\gamma, \boldsymbol{f}_1, \boldsymbol{f}_2) : I \to V_{4,3}$ be a spherical framed immersion with $\overline{\ell}(t) \neq 0$, where $\{\gamma(t), \boldsymbol{f}_1(t), \boldsymbol{f}_2(t), \boldsymbol{t}_f(t)\}$ is the Frenet type frame.

Proposition 4.4.1 $(\boldsymbol{t}, \boldsymbol{\zeta}, \boldsymbol{f}_1) : I \to V_{4,3}$ *is a framed curve, where*

$$\boldsymbol{\zeta}(t) = \frac{\overline{n}(t)\boldsymbol{\gamma}(t) - \alpha(t)\boldsymbol{n}_2(t)}{\sqrt{\alpha^2(t) + \overline{n}^2(t)}}.$$

Proof. By a direct calculation, we have $\boldsymbol{t}(t) \cdot \boldsymbol{t}(t) = \boldsymbol{\zeta}(t) \cdot \boldsymbol{\zeta}(t) = \boldsymbol{f}_1(t) \cdot \boldsymbol{f}_1(t) = 1$ and $\boldsymbol{t} \cdot \boldsymbol{\zeta}(t) = \boldsymbol{t} \cdot \boldsymbol{f}_1(t) = \boldsymbol{\zeta}(t) \cdot \boldsymbol{f}_1(t)$. We can also calculate that

$$\dot{\boldsymbol{t}}(t) \cdot \boldsymbol{\zeta}(t) = (-\alpha(t)\boldsymbol{\gamma}(t) - \overline{n}(t)\boldsymbol{f}_2(t)) \cdot \frac{\overline{n}(t)\boldsymbol{\gamma}(t) - \alpha(t)\boldsymbol{n}_2(t)}{\sqrt{\alpha^2(t) + \overline{f}^2(t)}} = 0$$

and

$$\dot{\boldsymbol{t}}(t) \cdot \boldsymbol{f}_1(t) = (-\alpha(t)\boldsymbol{\gamma}(t) - \overline{n}(t)\boldsymbol{f}_2(t)) \cdot \boldsymbol{f}_1(t) = 0.$$

Therefore $(\boldsymbol{t}, \boldsymbol{\zeta}, \boldsymbol{f}_1) : I \to V_{4,3}$ is a framed curve. $\square$

Remark 4.4.2 $\boldsymbol{\mu}_{\boldsymbol{n}}(t)$ is a regular spherical curve.

We define a mapping $\mathcal{FS}_\gamma : I \times [0, 2\pi) \to S^3$ by

$$\mathcal{FS}_\gamma(t, \theta) = \mathcal{DS}_{\boldsymbol{\mu}}(t, \theta) = \cos\theta \boldsymbol{\zeta}(t) + \sin\theta \boldsymbol{f}_1(t).$$

We call $\mathcal{FS}_\gamma$ the *E-flat focal great circular surface*. By a direct calculation, (t_0, θ_0) is a singular point of $\mathcal{FS}_\gamma$ if and only if

$$\frac{\alpha(t_0)\dot{\overline{n}}(t_0) - \dot{\alpha}(t_0)\overline{n}(t_0)}{\sqrt{\alpha^2(t_0) + \overline{n}^2(t_0)}} \cos\theta_0 + \overline{\ell}(t_0)\overline{n}(t_0) \sin\theta_0 = 0,$$

so that the singular value of $\mathcal{FS}_\gamma$ is

$$\mathcal{E}_\gamma^\pm(t) = \frac{\pm\overline{\ell}(t)\overline{n}^2(t)\boldsymbol{\gamma}(t) \mp \left(\alpha(t)\dot{\overline{n}}(t) - \dot{\alpha}(t)\overline{n}(t)\right) \boldsymbol{f}_1(t) \mp \alpha(t)\overline{\ell}(t)\overline{n}(t)\boldsymbol{f}_2(t)}{\sqrt{\overline{\ell}^2(t)\overline{n}^4(t) + \left(\alpha(t)\dot{\overline{n}}(t) - \dot{\alpha}(t)\overline{n}(t)\right)^2 + \alpha^2(t)\overline{\ell}^2(t)\overline{n}^2(t)}}.$$

We call $\mathcal{E}_\gamma^\pm$ an evolute of γ.

Proposition 4.4.3 $(\mathcal{E}_\gamma^\pm, \boldsymbol{\eta}, \boldsymbol{\mu}) : I \to V_{4,3}$ *is a spherical framed curve, where*

$$\boldsymbol{\eta}(t) = \frac{\alpha(t)\boldsymbol{\gamma}(t) + \overline{n}(t)\boldsymbol{f}_2(t)}{\sqrt{\alpha^2(t) + \overline{n}^2(t)}}$$

The focal E-frat great circular surface and the evolute of γ is given by $\boldsymbol{t}(t)$, so that we have the following proposition.

Proposition 4.4.4 *We have the following:*

(i) $\mathcal{FS}_\gamma(t, \theta) = \mathcal{FS}_{\boldsymbol{b}_1}(t, \theta) = \mathcal{FS}_{\boldsymbol{b}_2}(t, \theta)$,

(ii) $\mathcal{E}_\gamma(t) = \mathcal{E}_{\boldsymbol{b}_1}(t) = \mathcal{E}_{\boldsymbol{b}_2}(t)$,

where $\{\boldsymbol{\gamma}(t), \boldsymbol{b}_1(t), \boldsymbol{b}_2(t), \boldsymbol{\mu}_{\boldsymbol{b}}(t)\}$ *is the Bishop type frame.*

Since Proposition 4.4.3, we can take the dual surface of $\mathcal{E}^{\pm}_{\gamma}(t)$. It is given by

$$\mathcal{DS}_{\mathcal{E}^{\pm}_{\gamma}}(t,\theta) = \cos\theta\boldsymbol{\eta}(t) + \sin\theta\boldsymbol{t}(t).$$

Since Δ-duality, we have the following diagram:

$$\begin{tikzcd}
\mathcal{FS}_{\gamma} \arrow[r, "\text{taking singular value}"] & \mathcal{E}^{\pm}_{\gamma} \arrow[d, "\Delta-\text{dual}"] \\
(\pm)\boldsymbol{t} \arrow[u, "\Delta-\text{dual}"] & \mathcal{DS}_{\mathcal{E}^{\pm}_{\gamma}} \arrow[l, "\text{taking singular value}"]
\end{tikzcd}$$

We introduce two functions $\theta(t), \sigma(t)$ as follow:

$$\theta(t) = \arctan\left(\frac{\alpha(t)\dot{\overline{n}}(t) - \dot{\alpha}(t)\overline{n}(t)}{\overline{\ell}(t)\overline{n}(t)\sqrt{\alpha^2(t)+\overline{n}^2(t)}}\right) \text{ and } \sigma(t) = -\left(\frac{\alpha(t)\overline{\ell}(t)}{\sqrt{\alpha^2(t)+\overline{n}^2(t)}} + \dot{\theta}(t)\right).$$

Theorem 4.4.5 *Suppose that $(t_0, \theta_0) \in S(\mathcal{FS}_{\gamma})$. Then we have the following:*

(i) *$\mathcal{FS}_{\gamma}$ at (t_0, θ_0) is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\sigma(t_0) \neq 0$,*

(ii) *$\mathcal{FS}_{\gamma}$ at (t_0, θ_0) is $\mathcal{A}$-equivalent to the swallowtail if and only if $\sigma(t_0) = 0$ and $\dot{\sigma}(t_0) \neq 0$,*

(iii) *The cuspidal cross cap doesn't appear.*

Theorem 4.4.6 *Suppose that $(t_0, \theta_0) \in S(\mathcal{DS}_{\mathcal{E}^{\pm}_{\gamma}})$. Then we have the following:*

(i) *$\mathcal{DS}_{\mathcal{E}^{\pm}_{\gamma}}$ at (t_0, θ_0) is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\sigma(t_0) \neq 0$,*

(ii) *$\mathcal{DS}_{\mathcal{E}^{\pm}_{\gamma}}$ at (t_0, θ_0) is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\sigma(t_0) = 0$ and $\dot{\sigma}(t_0) \neq 0$,*

(iii) *The swallowtail doesn't appear,*

Bibliography

[1] V. I. Arnol'd. *Singularities of caustics and wave fronts*, Vol. 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[2] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Volume 1*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2012. Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition.

[3] R. L. Bishop. There is more than one way to frame a curve. *Amer. Math. Monthly*, Vol. 82, pp. 246–251, 1975.

[4] M. Bôcher. Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence. *Trans. Amer. Math. Soc.*, Vol. 2, No. 2, pp. 139–149, 1901.

[5] J. W. Bruce and P. J. Giblin. *Curves and singularities*. Cambridge University Press, Cambridge, second edition, 1992. A geometrical introduction to singularity theory.

[6] R. Cipolla and P. Giblin. *Visual motion of curves and surfaces*. Cambridge University Press, Cambridge, 2000.

[7] J. P. Cleave. The form of the tangent-developable at points of zero torsion on space curves. *Math. Proc. Cambridge Philos. Soc.*, Vol. 88, No. 3, pp. 403–407, 1980.

[8] D. Fuchs. Evolutes and involutes of spatial curves. *Amer. Math. Monthly*, Vol. 120, No. 3, pp. 217–231, 2013.

[9] S. Fujimori, K. Saji, M. Umehara, and K. Yamada. Singularities of maximal surfaces. *Math. Z.*, Vol. 259, No. 4, pp. 827–848, 2008.

[10] T. Fukunaga and M. Takahashi. Existence and uniqueness for Legendre curves. *J. Geom.*, Vol. 104, No. 2, pp. 297–307, 2013.

[11] T. Fukunaga and M. Takahashi. Evolutes of fronts in the Euclidean plane. *J. Singul.*, Vol. 10, pp. 92–107, 2014.

[12] T. Fukunaga and M. Takahashi. Evolutes and involutes of frontals in the Euclidean plane. *Demonstr. Math.*, Vol. 48, No. 2, pp. 147–166, 2015.

[13] T. Fukunaga and M. Takahashi. Involutes of fronts in the Euclidean plane. *Beitr. Algebra Geom.*, Vol. 57, No. 3, pp. 637–653, 2016.

[14] T. Fukunaga and M. Takahashi. On convexity of simple closed frontals. *Kodai Math. J.*, Vol. 39, No. 2, pp. 389–398, 2016.

[15] T. Fukunaga and M. Takahashi. Existence conditions of framed curves for smooth curves. *J. Geom.*, Vol. 108, No. 2, pp. 763–774, 2017.

[16] C. G. Gibson. *Elementary geometry of differentiable curves*. Cambridge University Press, Cambridge, 2001. An undergraduate introduction.

[17] A. Gray, E. Abbena, and S. Salamon. *Modern differential geometry of curves and surfaces with Mathematica®*. Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL, third edition, 2006.

[18] S. Hananoi and S. Izumiya. Normal developable surfaces of surfaces along curves. *Proc. Roy. Soc. Edinburgh Sect. A*, Vol. 147, No. 1, pp. 177–203, 2017.

[19] S. Honda. Rectifying developable surfaces of framed base curves and framed helices. to appear in Advance Studies in Pure Mathematics.

[20] S. Honda, S. Izumiya, and M. Takahashi. Developable surfaces along frontal curves on embedded surfaces. Preprint.

[21] S. Honda and M. Takahashi. Evolutes and focal surfaces of framed immersions in the euclidean space. to appear in Proc. Roy. Soc. Edinburgh Sect. A.

[22] S. Honda and M. Takahashi. Framed curves in the Euclidean space. *Adv. Geom.*, Vol. 16, No. 3, pp. 265–276, 2016.

[23] G. Ishikawa. Determinacy of the envelope of the osculating hyperplanes to a curve. *Bull. London Math. Soc.*, Vol. 25, No. 6, pp. 603–610, 1993.

[24] G. Ishikawa. Developable of a curve and determinacy relative to osculation-type. *Quart. J. Math. Oxford Ser. (2)*, Vol. 46, No. 184, pp. 437–451, 1995.

[25] G. Ishikawa. Topological classification of the tangent developables of space curves. *J. London Math. Soc. (2)*, Vol. 62, No. 2, pp. 583–598, 2000.

[26] S. Izumiya, H. Katsumi, and T. Yamasaki. The rectifying developable and the spherical Darboux image of a space curve. In *Geometry and topology of caustics—CAUSTICS '98 (Warsaw)*, Vol. 50 of *Banach Center Publ.*, pp. 137–149. Polish Acad. Sci. Inst. Math., Warsaw, 1999.

[27] S. Izumiya, T. Nagai, and K. Saji. Great circular surfaces in the three-sphere. *Differential Geom. Appl.*, Vol. 29, No. 3, pp. 409–425, 2011.

[28] S. Izumiya and S. Otani. Flat approximations of surfaces along curves. *Demonstr. Math.*, Vol. 48, No. 2, pp. 217–241, 2015.

[29] S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas, and F. Tari. *Differential geometry from a singularity theory viewpoint*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[30] S. Izumiya and N. Takeuchi. Geometry of ruled surface. In *Applicable Mathematics in the Golden Age*, pp. 305–338. Narosa Publishing House, 2003.

[31] J. Koenderink. *Solid shape*. MIT Press Series in Artificial Intelligence. MIT Press, Cambridge, MA, 1990.

[32] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada. Singularities of flat fronts in hyperbolic space. *Pacific J. Math.*, Vol. 221, No. 2, pp. 303–351, 2005.

[33] D. Mond. On the tangent developable of a space curve. *Math. Proc. Cambridge Philos. Soc.*, Vol. 91, No. 3, pp. 351–355, 1982.

[34] D. Mond. Singularities of the tangent developable surface of a space curve. *Quart. J. Math. Oxford Ser. (2)*, Vol. 40, No. 157, pp. 79–91, 1989.

[35] G. Peano. Sur le déterminant wronskien. *Mathesis*, Vol. 9, pp. 75–76, 110–112, 1889.

[36] I. R. Porteous. The normal singularities of a submanifold. *J. Differential Geometry*, Vol. 5, pp. 543–564, 1971.

[37] I. R. Porteous. *Geometric differentiation for the intelligence of curves and surfaces*. Cambridge University Press, Cambridge, 1994.

[38] M. C. Romero-Fuster and E. Sanabria-Codesal. Generalized evolutes, vertices and conformal invariants of curves in $\mathbb{R}^{n+1}$. *Indag. Math. (N.S.)*, Vol. 10, No. 2, pp. 297–305, 1999.

[39] Lawrence S. Developable surfaces: Their history and application. *Nexus Network Journal*, Vol. 13, No. 3, pp. 701–714, 2011.

[40] O. P. Shcherbak. Projectively dual space curves and Legendre singularities. *Trudy Tbiliss. Univ.*, Vol. 232/233, pp. 280–336, 1982.

[41] M. Takahashi. Legendre curves in the unit spherical bundle over the unit sphere and evolutes. In *Real and complex singularities*, Vol. 675 of *Contemp. Math.*, pp. 337–355. Amer. Math. Soc., Providence, RI, 2016.

[42] R. Uribe-Vargas. On vertices, focal curvatures and differential geometry of space curves. *Bull. Braz. Math. Soc. (N.S.)*, Vol. 36, No. 3, pp. 285–307, 2005.

[43] I. Vaisman. *A first course in differential geometry*, Vol. 80 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1984.

[44] K. Wolsson. A condition equivalent to linear dependence for functions with vanishing Wronskian. *Linear Algebra Appl.*, Vol. 116, pp. 1–8, 1989.