Existence, Nonexistence of Global Solution and Large Time Behavior of Solutions of a Weakly Coupled System of Reaction-Diffusion Equations

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1 Introduction

We consider nonnegative solutions of the initial value problem for a weakly coupled system

$$(1) \qquad \begin{cases} (u_i)_t = \Delta u_i + |x|^{\sigma_i} u_{i+1}^{p_i}, & x \in \mathbf{R}^d, t > 0, i \in N^*, \\ u_i(x,0) = u_{i,0}(x), & x \in \mathbf{R}^d, i \in N^*, \end{cases}$$

where $N \geq 1$, $N^* = \{1, 2, \ldots N\}$, $u_{N+i} = u_i$, $u_{N+i,0} = u_{i,0}$, $p_{N+i} = p_i$, $\sigma_{N+i} = \sigma_i$ $(i \in N^*)$, $u = (u_1, u_2, ..., u_N)$, $u_0 = (u_{1,0}, u_{2,0}, ..., u_{N,0})$, $p = (p_1, p_2, ..., p_N)$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$, $d \geq 1$, $p_i \geq 1$ $(i \in N^*)$, $\prod_{i=1}^N p_i > 1$ and $0 \leq \sigma_i < d(p_i - 1)$ (if $p_i = 1$, we choose $\sigma_i = 0$) $(i \in N^*)$, and $u_{i,0}$ is a nonnegative bounded continuous function satisfying

$$\limsup_{|x| \to \infty} |x|^{\delta_i} u_{i,0}(x) < \infty$$

for any $i \in N^*$, where

$$(2) \qquad \delta_i = \frac{\sigma_i + p_i \sigma_{i+1} + \ldots + p_i p_{i+1} \ldots p_{i+N-2} \sigma_{i+N-1}}{p_1 p_2 \ldots p_N - 1}.$$

Problem (1) has a unique, nonnegative and bounded solution in a suitable weighted space (see Theorem 2.4) at least locally in time. For given an initial value u_0, let $T^* = T^*(u_0)$ be the maximal existence time of the solution. If $T^* = \infty$ the solution is global. On the other hand, if $T^* < \infty$ there exists $i \in N^*$ such that

$$(3) \qquad \limsup_{t \to T^*} \| < x >^{\delta_i} u_i(t)\|_{\infty,} = \infty,$$

where δ_i is defined in (2) and $< x >= (1 + |x|^2)^{1/2}$. When (3) holds we say that the solution blows up in a finite time.

The purpose of the paper is to study systematically the effect of inhomogeneity $|x|^{\sigma_i}$ on the critical blow up exponent to the system (1) and the asymptotic behavior of global solutions for general $N \geq 1$.

In the paper, we present a unified approach to the study of blow up and global existence of solution to the system (1) for the general $N = 1$ and $\sigma_i \leq 1$. Especially, we extend the previous results by Mochizuki-Huang[10] (for the case $N = 2$ and $\sigma_i \geq 0$) and the author[16] (for the case $N \geq 3$ and $\sigma_i = 0$).

Throughout this paper we shall use the following notation. We put some constants:

$$(4) \begin{cases} \alpha_i = \dfrac{2(1 + p_i + p_i p_{i+1} + \ldots + p_i p_{i+1} \ldots p_{i+N-2})}{p_1 p_2 \ldots p_N - 1}, \quad i \in N^*, \\[2ex] \delta_i = \dfrac{\sigma_i + p_i \sigma_{i+1} + \ldots + p_i p_{i+1} \ldots p_{i+N-2} \sigma_{i+N-1}}{p_1 p_2 \ldots p_N - 1}, \quad i \in N^*, \end{cases}$$

which solve

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -p_{N-1} \\ -p_N & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{pmatrix} = - \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \\ 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -p_{N-1} \\ -p_N & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{N-1} \\ \delta_N \end{pmatrix} = - \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{N-1} \\ \sigma_N \end{pmatrix},$$

where δ_i $(i \in N^*)$ are the same constants given by (2). These constants play an important role in our problem. Actually, we show that the number $\max_{i\in N^*}\{\alpha_i + \delta_i\}$ is the "first cutoff" which divides the blow up case and the global existence case. This is a natural existence of the previous result in [10] for the case $N = 2$.

We denote by BC the space of all bounded continuous functions in $\mathbf{R}^d$ and define for $a \geq 0$,

$$\begin{aligned} I^a &= \{\xi \in BC; \xi(x) \geq 0 \text{ and } \limsup_{|x|\to\infty} |x|^a \xi(x) < \infty\}, \\ I_a &= \{\xi \in BC; \xi(x) \geq 0 \text{ and } \liminf_{|x|\to\infty} |x|^a \xi(x) > 0\}. \end{aligned}$$

Let L^∞_a be the Banach space of L^∞-functions such that

$$\|\xi\|_{\infty,a} = \sup_{x\in\mathbf{R}^d} < x >^a |\xi(x)| < \infty.$$

Obviously $I^a \subset L^\infty_a$. The letter C stands for a positive generic constant which may vary from line to line. We use the notation $S(t)\xi$ to represent the solution of the heat equation with an initial value $\xi(x)$:

$$S(t)\xi(x) = (4\pi t)^{-d/2} \int_{\mathbf{R}^d} e^{-|x-y|^2/4t} \xi(y) dy. \tag{5}$$

By using the notation above, throughout paper, we suppose that initial conditions satisfy

$$u_{i,0} \in I^{\delta_i} \qquad (i \in N^*), \tag{6}$$

where δ_i is a nonnegative constant defined by (4).

Now, the results of this paper can be summarized in the following four theorems. First, we state our blow-up result for solutions to (1).

Theorem 1. *Assume* $\max_{i\in N^*}\{\alpha_i+\delta_i\}\geq d$. *Then every nontrivial solution* $u(t)$ *of* (1) *blows up in a finite time.*

When $\max_{i\in N^*}\{\alpha_i+\delta_i\}<d$, we show that there exists both non-global solutions and non-trivial global solution of (1). Precisely, requiring a polynomial decay of initial values u_0:

$$u_{i,0}(x)\sim\lambda^{\mu_i}<x>^{-a_i}\qquad (i\in N^*), \tag{7}$$

where λ, μ_i and a_i are positive constants, we obtain the "second cutoff" $a=(a_1,a_2,...,a_N)$ on the decay rate of initial values, namely $a_i=\alpha_i+\delta_i$ which divides the blow-up case and the global existence case when $\max_{i\in N^*}\{\alpha_i+\delta_i\}<d$.

Theorem 2. *Assume* $\max_{i\in N^*}\{\alpha_i+\delta_i\}<d$.

(i) *Suppose that there exists some* $i\in N^*$ *such that*

$$u_{i,0}\in I_{a_i}\quad \textit{with}\quad a_i<\alpha_i+\delta_i. \tag{8}$$

Then every solution $u(t)$ *of* (1) *is blows up in a finite time.*

(ii) *suppose that for any* $i\in N^*$

$$u_{i,0}\in I^{a_i}\quad \textit{with}\quad a_i>\alpha_i+\delta_i \tag{9}$$

and $\|u_{i,0}\|_{\infty,a_i}$ *is small enough. Then, every solution* $u(t)$ *of* (1) *is global. Moreover, we have a decay estimate:*

$$u_i(x,t)\leq CS(t)<x>^{-\hat{a}_i} \tag{10}$$

in $\boldsymbol{R}^d\times(0,\infty)$, *where* C *is a positive constant and* $\hat{a}_i\leq a_i$ $(i\in N^*)$ *are chosen to satisfy*

$$p_i\min\{\hat{a}_{i+1},d\}-\hat{a}_i>2+\sigma_i. \tag{11}$$

We also obtain the blow up result for large initial dates, even if initial data has an exponential decay.

Theorem 3. *Assume* $\max_{i\in N^*}\{\alpha_i+\delta_i\}<d$. *Suppose that there exists some* $i\in N^*$ *such that* $u_{i,0}(x)\geq Ce^{-\nu_0|x|^2}$ *for some* $\nu_0>0$ *and* $C>0$ *large enough. Then every solution* $u(t)$ *of* (1) *blows up in a finite time.*

Remark. Especially, when $u_{i,0} \in I^{a_i}$ with $a_i > \alpha_i + \delta_i$ for any $i \in N^*$ and $\|u_{i,0}\|_{\infty,a_i}$ is large enough, every solution $u(t)$ of (1) blows up in a finite time.

Furthermore, we obtain the precise asymptotic profile for global solutions for a certain class of vector $a = (a_1, a_2, ..., a_N)$ in the domain $\{a; a_i > \alpha_i + \delta_i, i \in N^*\}$. For these purposes a scaling argument for solutions $u(x,t)$ will play an important role.

Theorem 4 (i) *If we can choose* $\hat{a}_i < d$ $(i \in N^*)$ *in* (11) *and if*

$$\lim_{|x|\to\infty} |x|^{\hat{a}_i} u_{i,0}(x) = A_i > 0, \tag{12}$$

then

$$t^{\hat{a}_i/2}|u(x,t) - A_i S|x|^{-\hat{a}_i}| \to 0 (t \to \infty) \tag{13}$$

as $t \to \infty$ *uniformly in* $\mathbf{R}^d$.

(ii) *If we can choose* $\hat{a}_j > d$ $(j \in N^*)$ *in* (11), *then*

$$t^{d/2}|u_j(x,t) - M_j(4\pi t)^{-d/2}e^{-|x|^2/4t}| \to 0 (t \to \infty) \tag{14}$$

uniformly on the set $\{x \in \boldsymbol{R}^d; |x| \leq Rt^{1/2}\} (R > 0)$, *where* M_j *is a positive constant given by*

$$M_j = \int_{\mathbf{R}^d} u_{j,0}(x)dx + \int_0^\infty \int_{\mathbf{R}^d} |x|^{\sigma_j} u_{j+1}^{p_j}(x,s)dxds < \infty. \tag{15}$$

We briefly recall a history of the study on blow up and global existence of solution to the system (1). First, the blow-up and the global existence of solutions in the case $N = 1$ and $\sigma_1 = 0$,

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^d, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbf{R}^d \end{cases} \tag{16}$$

was studied by Fujita[4]. Fujita proved that when $d(p-1) < 2$ the solution of (16) blow up in a finite time for any $u_0 \not\equiv 0$. On the other hand he also proved that when $d(p-1) > 2$ the solution of (16) exists globally in time if the initial value u_0 is small and has an exponential decay. The number $p = 1 + 2/d$ is called a critical blow-up exponent for (16).

Fujita's results were extended by Bandle-Levine[1] for the $\sigma \geq 0$:

$$(17) \qquad \begin{cases} u_t = \Delta u + |x|^\sigma u^p, & x \in \mathbf{R}^d, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbf{R}^d, \end{cases}$$

and they showed that when $d(p-1) < 2+\sigma$ the solution of (17) blows up in a finite time for any $u_0 \not\equiv 0$. Hamada[5] proved the same blow-up result for the critical case $d(p-1) = 2+\sigma$ (see also [14]).

Fujita's results were also extended by Escobedo-Herrero[2] and Mochizuki[11] to the system (1) with $N = 2$ and $\sigma_i = 0$ $(i = 1, 2)$, and by Mochizuki-Huang[10] to the system (1) with $N = 2$ and $\sigma_i \geq 0$.

Although the Fujita type critical blowup exponent to the system (1) with $N = 2$ and $\sigma_i = 0$ was established by Escobedo-Herrero[2], their proofs were rather complicated.

Mochizuki[11] and Mochizuki-Huang[10] simplified their proof and also determined the "second cut off" on the decay rate of initial data. The asymptotic behavior of global solutions was also studied in [10] and [11] for the case $N = 2$ and $\sigma_i \geq 0$.

Our result is a natural extension of [10]. We emphasize that our proof gives a unified approach to show blow up results, although the proof in [10] for the case $\sigma_i > 0$ is slightly different from the one for case σ_i.

For a big system (1) with $N \geq 3$ and $\sigma_i = 0$, the auther [16] and Reccławowicz[13] (see also [12]) determined independently the Fujita type critical blow up exponent. See also [3] for large initial data. The methods in [16] and [13] are different, Moreover, in [16] we also determined the "second cutoff" on the decay rate of initial data.

on results extend the results of [16]. The novelty of this paper is the choice of an appropriate weighted function space in which the system (1) is locally well-passed, a unified approach to establish blow up results and a systematic controls of solutions.

Finally, we remark on the problem to estimate the life span $T^*(u_0)$ as λ go to 0 or ∞, when the initial data has the form (7). Such problem was studied by Mochizuki[11], Pinsky[15] and Kobayashi[7],[8], in the case $N = 2$ and $\sigma_i = 0$ $(i = 1, 2)$, the case $N = 1$ and $\sigma > -2$ and the case $N = 2$ and $\sigma_i \geq -2$ $(i = 1, 2)$, respectively. However, it is an open problem to obtain sharp estimate of the life span $T^*(u_0)$ for general $N \geq 3$ even in the case $\sigma_i = 0$.

The rest of the paper is organized as follows. In section 2, we note preliminary results including the local existence for (1) and some useful lemmas. In section 3, we prove the blow up results (Theorems 1, 2(i) and 3). In section 4, we show the result of global existence (Theorem 2 (ii)). In section 5, we prove the result of asymptotic behavior (Theorem 4).

2 Preliminaries

First, we note useful lemmas. The lemmas are well-known and are used throughout this paper. But the proof of Lemma 2.6 is complicated for the case $N \geq 3$. We need to control the precise estimate by induction.

We set for $\gamma > 0$

$$\eta_\gamma(x,t) = S(t) < x >^{-\gamma} . \tag{18}$$

Lemma 2.1. *Let $\gamma > 0$, $0 \leq \delta \leq \min\{d, \gamma\}$. Then we have*

$$\|\eta_\gamma(x,t)\|_{\infty,\delta} \leq \begin{cases} C(1+t)^{(-\min\{d,\gamma\}+\delta)/2} & (\gamma \neq d), \\ C(1+t)^{(-d+\delta)/2}\log(2+t) & (\gamma = d). \end{cases}$$

Proof. See [10;Lemma 2.1] or [9;Lemma 2.12]. □

Lemma 2.2. (i) *The following inequality holds*

$$\eta_\gamma(x,t) \geq C\min\{< x >^{-\gamma}, (1+t)^{-\gamma/2}\}.$$

(ii) *We have in $\boldsymbol{R}^d \times (0,\infty)$*

$$\begin{aligned} &|x|^{\sigma_i}\eta_{a_{i+1}}(x,t)^{p_i} \\ &\leq \begin{cases} C(1+t)^{(\sigma_i+a_i-a_{i+1,d}p_i)/2}\eta_{a_i}(x,t) & (a_{i+1} \neq d), \\ C(1+t)^{(\sigma_i+a_i-dp_i)/2}[\log(2+t)]^{p_i}\eta_{a_i}(x,t) & (a_{i+1} = d). \end{cases} \end{aligned} \tag{19}$$

Proof See [10:Lemmas 4.1 or 4.2]. □

Now, we establish the local solvability of the Cauchy problem (1). Basically, we follow the same argument as in [10].

For arbitrary $T > 0$, let

$$E_T = \{u : [0,T] \to (L^\infty)^N; \|u\|_{E_T} < \infty\}, \tag{20}$$

where

$$\|u\|_{E_T} = \sup_{t\in[0,T]} \{\sum_{i=1}^{N} \|u_i(t)\|_{\infty,\delta_i}\}.$$

We consider in E_T the related integral system

$$u_i(t) = S(t)u_{i,0} + \int_0^t S(t-s)(|x|^{\sigma_i} u_{i+1}^{p_i}(s))ds, \tag{21}$$

where $i \in N^*$. Note that in the closed subset $P_T = \{u \in E_T; u_i \geq 0, i \in N^*\}$ of E_T, (1) is reduced to (21). Define

(22)

$$\Psi(u) = (S(t)u_{1,0} + \Phi_1(u_2), S(t)u_{2,0} + \Phi_2(u_3), ..., S(t)u_{N,0} + \Phi_N(u_1)),$$

where

$$\Phi_i(u_{i+1}) = \int_0^t S(t-s)(|x|^{\sigma_i} u_{i+1}^{p_i}(s))ds \quad (i \in N^*).$$

Then a fixed point u of Ψ corresponds to a solution of (1).

Lemma 2.3 (i) *Let $u_{i,0} \in I^{\delta_i}$ $(i \in N^*)$. Then $(S(\cdot)u_{1,0}, S(\cdot)u_{2,0}, \ldots, S(\cdot)u_{N,0}) \in E_T$ for ant $T > 0$ and we have*

$$\|S(\cdot)u_{1,0}, S(\cdot)u_{2,0}, ..., S(\cdot)u_{N,0}\|_{E_T} \leq C \sum_{i=1}^{N} \|u_{i,0}\|_{\infty,\delta_i}.$$

(ii) *Let $u \in E_T$. Then $(\Phi_1(u_2), \Phi_2(u_3), ..., \Phi_N(u_1)) \in E_T$ and we have*

$$\|\Phi_1(u_2), \Phi_2(u_3), ..., \Phi_N(u_1))\|_{E_T} \leq CT \sum_{i=1}^{N} \|U_i\|_{E_T}^{p_i},$$

where

$$\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{pmatrix} = \begin{pmatrix} 0 & u_2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & u_3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & u_N \\ u_1 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Proof. (i) is obvious from Lemma 2.1 with $\gamma = \delta_i$ $(i \in N^*)$.

(ii) Note That

$$\int_0^t S(t-s)|\cdot|^{\sigma_i} u_{i+1}^{p_i}(s)ds \;\leq\; \int_0^t S(t-s) <\cdot>^{\sigma_i-\delta_{i+1}p_i} ds \sup_{s\in[0,t]} \|u_{i+1}(s)\|_{\infty,\delta_{i+1}}^{p_i}.$$

By a simple calculation (see (4)) $-\sigma_i+\delta_{i+1}p_i=\delta_i<d$. Then it follows from Lemma 2.1 with $\gamma=\delta_i$ $(i\in N^*)$ that

$$\|S(t-s) <\cdot>^{\sigma_i-\delta_{i+1}p_i} \|_{\infty,\delta_i} \leq C.$$

Thus we have

$$\left\|\int_0^t S(t-s)|\cdot|^{\sigma_i} u_{i+1}^{p_i}(s)ds\right\|_{\infty,\delta_i} \leq Ct \sup_{s\in[0,t]} \|u_{i+1}(s)\|_{\infty,\delta_{i+1}}^{p_i}$$

for $i\in N^*$. These inequalities conclude the assertion (ii). □

Now we can prove the following

Theorem 2.4. *Assume that u_0 is a vector of nonnegative bounded continuous functions such that $u_{i,0}\in I^{\delta_i}$ $(i\in N^*)$. Then there exists $0<T\leq\infty$ and a unique vector $u(t)\in P_T$ which solves* (1) *in $\mathbf{R}^d\times[0,T)$.*

Proof. Let $B_R=\{u\in E_T; \|u\|_{E_T}\leq R\}$. We consider two vector-valued fuanctions $v_1(x,t)=(v_{1,1}(x,t),v_{1,2}(x,t),\dots,v_{1,N}(x,t))$ and $v_2(x,t)=(v_{2,1}(x,t),v_{2,2}(x,t),\dots,v_{2,N}(x,t))$. For (22)

$$\begin{aligned}(23)\quad &\|\Psi(v_1)-\Psi(v_2)\|_{E_T}\\ &= \|(\Phi_1(v_{1,2})-\Phi_1(v_{2,2}),\Phi_2(v_{1,3})-\Phi_2(v_{2,3})\\ &\quad ,...,\Phi_{N-1}(v_{1,N})-\Phi_{N-1}(v_{2,N}),\Phi_N(v_{1,1})-\Phi_N(v_{2,1}))\|_{E_T}.\end{aligned}$$

We consider i-th term of $\|\Psi(v_1)-\Psi(v_2)\|_{E_T}$,

$$\begin{aligned}&|\Phi_i(v_{1,i+1})-\Phi_i(v_{2,i+1})| <x>^{\delta_i}\\ &\leq \int_0^t S(t-s)|x|^{\sigma_i}\Big||v_{1,i+1}(s)|^{p_i}-|v_{2,i+1}(s)|^{p_i}\Big|ds <x>^{\delta_i}.\end{aligned}$$

We consider this expression in $B_R\cap P_T$ for R sufficient large. From proof of Lemma 2.3 (ii),

$$(24)\qquad |\Phi_i(u_{1,i+1})-\Phi_i(u_{2,i+1})| <x>^{\delta_i}$$

$$
\begin{aligned}
&\le CT \sup_{s\in[0,t]} \|v_{1,i+1}^{p_i}(s) - v_{2,i+1}^{p_i}(s)\|_{\infty,delta_{i+1}}\\
&\le CT \sup_{s\in[0,t]} \|R^{p_i-1}p_i v_{1,i+1}(s) - v_{2,i+1}(s)\|_{\infty,delta_{i+1}}\\
&\le CTR^{p_i-1}p_i \sup_{s\in[0,t]} \|v_{1,i+1}(s) - v_{2,i+1}(s)\|_{\infty,delta_{i+1}}.
\end{aligned}
$$

Substitute (24) into (23). From we can put T is small enough for R, we obtain

$$
\begin{aligned}
&\|\Psi(v_1) - \Psi(v_2)\|_{E_T}\\
&\le CT\|(R^{p_1-1}p_1(v_{1,2} - v_{2,2}), R^{p_2-1}p_2(v_{1,3} - v_{2,3}),\\
&\qquad\qquad \dots, R^{p_N-1}p_N(v_{1,1} - v_{2,1})\|_{E_T}\\
&\le CTR^{\max_i\{p_i\}-1} \max_i\{p_i\}\|v_1 - v_2\|_{E_T}\\
&\le \rho\|v_1 - v_2\|_{E_T}
\end{aligned}
$$

for some $\rho < 1$. Then Ψ is a strict contraction of $B_R \cap P_T$ into itself, whence there exists a unique fixed point $u \in B_R \cap P_T$ which solves (4). □

Next, we establish key estimate of solutions which will be used show blow up results.

Lemma 2.5. *Let $u_0 \not\equiv 0$ and u be the solutions of* (1) *with initial data u_0. Then there exist $\tau = \tau(u_0) \ge 0$ and constants $C > 0, \nu > 0$ such that*

$$
u_i(x,\tau) \ge Ce^{-\nu|x|^2} \quad (i \in N^*). \tag{25}
$$

Proof. (cf. [2:Lemma 2.4]) Assume for instance that $u_{1,0} \not\equiv 0$. By shifting the origin if necessary, we may assume that there exists $R > 0$ such that $\nu = \inf\{u_{1,0}(\xi) : |\xi| \le R\} > 0$. Since $u(t) \ge S(t)u_{1,0}$, it follows that

$$
u_1 \ge \nu \exp(-\frac{|x|^2}{2t})(4\pi t)^{-d/2} \int_{|y|\le R} \exp(-|y|^2/2t)dy.
$$

Define $\bar{u}_1(t) = u_1(t + \tau_1)$ for some $\tau_1 > 0$. Then, we obtain

$$
\bar{u}_1(0) = u_1(\tau_1) > c_1 \exp(-\alpha_1|x|^2) \tag{26}
$$

with

$$
\alpha_1 = \frac{1}{2\tau_1}, \qquad c_1 = \nu(4\pi\tau_1)^{-d/2} \int_{|y|<R} \exp\left(-\frac{|y|^2}{2\tau_1}\right) dy. \tag{27}
$$

Substituting (26) in N-th equation of (21), we obtain

$$\begin{aligned} u_N &\geq \int_0^t S(t-s)|x|^{\sigma_N} u_1^{p_N}(s)ds \\ &\geq c_1^{p_N} \int_{\tau_1}^t S(t-s)|x|^{\sigma_N} \exp(p_N \alpha_1 |x|^2) ds. \end{aligned}$$

From for $\nu > 0$ and $\sigma \geq 0$,

$$S(t)(|x|^\sigma e^{-\nu|x|^2}) \geq C_\sigma (2t)^{\sigma/2}(2\nu t+1)^{-(d+\sigma)/2} e^{-|x|^2/2t}, \tag{28}$$

where

$$C_\sigma = (2\pi)^{-d/2} \int_{\mathbf{R}^d} |x|^\sigma e^{-|x|^2} dx. \tag{29}$$

(See [10:Lemma 3.2]), we obtain

$$\begin{aligned} u_N &\geq c_1 C_{\sigma_N} \int_{\tau_1}^t \{2(t-s)\}^{\sigma_N/2} \{2\alpha_1(t-s)+1\}^{-(\sigma_N+d)/2} \\ &\qquad\qquad \times \exp\left(-\frac{|x|^2}{2(t-s)}\right) ds \\ &\geq c_1 C_{\sigma_N} \int_{\tau_1}^{(t+\tau_1)/2} \{2(t-s)\}^{\sigma_N/2} \{2\alpha_1(t-s)+1\}^{-(\sigma_N+d)/2} \\ &\qquad\qquad \times \exp\left(-\frac{|x|^2}{2(t-s)}\right) ds \\ &\geq c_1 C_{\sigma_N} \int_{\tau_1}^{(t+\tau_1)/2} \{(t-\tau_1)\}^{\sigma_N/2} \{2\alpha_1(t-\tau_1)+1\}^{-(\sigma_N+d)/2} \\ &\qquad\qquad \times \exp\left(-\frac{|x|^2}{2(t-\tau_1)}\right) ds \\ &\geq \frac{1}{2} c_1 C_{\sigma_N} \{(t-\tau_1)\}^{1+\sigma_N/2} \{2\alpha_1(t-\tau_1)+1\}^{-(\sigma_N+d)/2} \\ &\qquad\qquad \times \exp\left(-\frac{|x|^2}{2(t-\tau_1)}\right). \end{aligned}$$

Where C_{σ_N} is defined in (29). Define $\bar{u}_N(t) = u_N(t+\tau_N)$ for some $\tau_N > \tau_1$. Then, we obtain

$$\bar{u}_N(0) = u_N(\tau_N) > c_N \exp(-\alpha_N |x|^2) \tag{30}$$

with

$$
(31)\ \begin{cases} \alpha_1 = \dfrac{1}{2(\tau_N - \tau_1)}, \\ c_N = c_1 C_{\sigma_N} (\tau_N - \tau_1)^{1+\sigma_N/2} \{2\alpha_1(\tau_N - \tau_1) + 1\}^{-(\sigma_N + d)/2}. \end{cases}
$$

By repeating this argument, we obtain same results for $u_N, u_{N-1}, \dots, u_2$. This completes the proof. $\square$

We suppose $\alpha_1 + \delta_1 = d$. Let $u(t) \in E_T$ be a nontrivial solution of (1). By Lemma 2.4, we may assume

$$
u_{1,0} \geq C e^{-\mu |x|^2}
$$

for some $C > 0$ and $\mu > 0$.

Lemma 2.6. *We assume $\alpha_1 + \delta_1 = d$. Then we have*

$$
u_1(x,t) \geq C t^{-d/2} e^{-|x|^2/t} \log(t/(2a)) \quad (a \leq t < T),
$$

where $a > 0$ is a small constant.

Proof

$$
\begin{aligned}
u_N(x,t) &\geq \int_0^t S(t-s)|x|^{\sigma_N} u_1(x,s)^{p_N} ds \\
&\geq \int_0^t (4s + 1/\mu)^{-dp_N/2} S(t-s)|x|^{\sigma_N} e^{-p_N |x|^2/(4s+1/\mu)} ds.
\end{aligned}
$$

Since

$$
S(t)|x|^{\sigma_N} e^{-p_N|x|^2/(4s+1/\mu)} \geq C t^{\sigma_N} \left\{ \frac{2p_N t}{4s + 1/\mu} + 1 \right\}^{-(d+\sigma_N)/2} e^{-|x|^2/2t},
$$

we obtain

$$
\begin{aligned}
u_N(x,t) &\geq C \int_{t/4}^{t/2} (4s + 1/\mu)^{-dp_N/2} (t-s)^{\sigma_N/2} e^{-|x|^2/2(t-s)} ds \\
&\geq C t^{\sigma_N/2} (t+1)^{dp_N/2} e^{-|x|^2/t}.
\end{aligned}
$$

Substitute this into $u_{N-1}(x,t) \geq \int_0^t S(t-s)|x|^{\sigma_{N-1}} u_N(x,s)^{p_{N-1}}$. Then

$$
\begin{aligned}
u_{N-1}(x,t) \geq C &\int_0^t s^{p_{N-1}(1+\sigma_N/2)+\sigma_{N-1}/2} (s+1)^{-dp_{N-1}p_N/2} \\
&\left\{ \frac{2p_{N-1}(t-s)}{s} + 1 \right\}^{-(d+\sigma_{N-1})/2} e^{-|x|^2/(t-s)} ds \\
&\geq C e^{-|x|^2/t} \int_{t/4}^{t/2} s^{-dp_{N-1}p_N/2 + p_{N-1}(1+\sigma_N/2)+\sigma_{N-1}/2} ds \\
&\geq C t^{-dp_{N-1}p_N/2 + p_{N-1}(1+\sigma_N/2)+(1+\sigma_{N-1}/2)} e^{-|x|^2/t}
\end{aligned}
$$

by (28) again. By repeating this argument,

$$u_2 \geq Ct^{(-dp_2p_3\dots p_N+p_2p_3\dots p_{N-1}(2+\sigma_N)+\dots+p_2p_3(2+\sigma_4)+p_2(2+\sigma_3)+(2+\sigma_2))/2}e^{-|x|^2/t}$$

by using (28) again. Thus we obtain

$$\begin{aligned} u_1(x,t) &\geq C\int_0^t s^{(-dp_1p_2\dots p_N+p_1p_2\dots p_{N-1}(2+\sigma_N)+\dots+p_1p_2(2+\sigma_3)+p_1(2+\sigma_2)+\sigma_1)/2} \\ &\qquad\qquad \times \left\{\frac{2p_1(t-s)}{s}+1\right\}^{-(d+\sigma_1)/2} e^{-|x|^2/(t-s)}ds \\ &\geq C(t+1)^{-d/2}e^{-|x|^2/t} \\ &\quad \times \int_a^{t/2} s^{(-d(p_1p_2\dots p_N-1)+p_1p_2\dots p_{N-1}(2+\sigma_N)+\dots+p_1p_2(2+\sigma_3)+p_1(2+\sigma_2)+\sigma_1)/2}ds \end{aligned}$$

for small $a>0$. Since $\alpha_1+\delta_1 = \dfrac{2(p_1p_2\dots p_{N-1}+\dots+p_1p_2+p_1+1)}{p_1p_2\dots p_N-1}$ $+\dfrac{\sigma_1+p_1\sigma_2+\dots+p_1p_2\dots p_{N-1}\sigma_N}{p_1p_2\dots p_N-1} = d$ and $\{-d(p_1p_2\dots p_N-1)+p_1p_2\dots p_{N-1}(2+\sigma_N)+\dots+p_1p_2(2+\sigma_3)+p_1(2+\sigma_2)+\sigma_1\}/2=-1$, we have

$$u_1(x,t) \geq Ct^{-d/2}e^{-|x|^2/t}\log(t/2a). \qquad \square$$

3 Proof of blow up results

In this section we summarize several blow-up conditions which follow from Theorem 3.2. Here, we take the same strategy as in [10] and [11]. Actually, we can deduce our blow up problem to the one for the systems of ordinary differential equations with a parameter $\epsilon>0$. We fined a nice scaling to reduce the problem furthermore to the one for a simpler (ϵ-independent) system of ordinary differential equations. This gives us a uniform treatment of our blow up results.

Let $\rho_\epsilon(x) = (\epsilon/\pi)^{d/2}e^{-\epsilon|x|^2}$, $\epsilon>0$. For a solution $u(t)\in E_T$ of (1) we put

$$(32) \qquad F_{i\epsilon}(t) = \int_{\mathbf{R}^d} u_i(x,t)\rho_\epsilon(x)dx \quad (i\in N^*).$$

Since $-\Delta\rho_\epsilon(x) \leq 2d\epsilon\rho_\epsilon(x)$, the pair $\{2N\epsilon, \rho_\epsilon(x)\}$ is regarded as an approximate principal eigensolution of $-\Delta$ in $\mathbf{R}^d$. With this fact and

Jensen's inequality we easily have

$$F'_{i,\epsilon}(t) \geq -2d\epsilon F_{i,\epsilon}(t) + C_{p_i}\epsilon^{-\delta_i/2}F_{i+1,\epsilon}(t)^{p_i} \quad (i \in N^*), \tag{33}$$

where

$$C_{p_i} = \left(\pi^{-d/2}\int_{\mathbf{R}^d}|x|^{-\sigma_i/(p_i-1)}e^{-|x|^2}dx\right)^{-p_i+1}$$

for $p_i > 1$ and $C_{p_i} = 1$ for $p_i = 1$.

Let us consider the system of ordinary differential equations

$$\begin{cases} f'_{i,\epsilon}(t) = -2d\epsilon f_{i,\epsilon}(t) + C_{p_i}\epsilon^{-\delta_i/2}f_{i+1,\epsilon}(t)^{p_i} \quad (i \in N^*), \\ f_{i,\epsilon}(0) = F_{i,\epsilon}(0), \quad (i \in N^*). \end{cases} \tag{34}$$

By the scaling with (4)

$$f_i(t) = \frac{(C_{p_i}C_{p_{i+1}}^{p_i}C_{p_{i+2}}^{p_ip_{i+1}}\dots C_{p_{i+N-1}}^{p_ip_{i+1}\cdots p_{i+N-2}})^{1/(p_1p_2\dots p_N-1)}}{2d^{\alpha_i/2}\epsilon^{(\alpha_i+\delta_i)/2}}f_{i\epsilon}\left(\frac{t}{2d\epsilon}\right)$$

for $i \in N^*$, we obtain the simpler system of equations

$$f'_i(t) = -f_i(t) + f_{i+1}(t)^{p_i}(i \in N^*). \tag{35}$$

Lemma 3.1. *Let $f(t) = (f_1(t), f_2(t), ..., f_N(t))$ be the solution to* (35) *with the initial data*

$$f_1(0) = f_0 > 1, f_j(0) = 0 \quad (j \in N^*\backslash\{1\}).$$

If f_0 is sufficiently large, then $f(t)$ blows up in a finite time. Moreover, the life span T_0 of $f(t)$ is estimated from above by

$$T_0 \leq t_0 + \int_{\prod_{i=1}^N f_i(t_0)}^{\infty}\{C_1(p)\xi^{C_2(p)+1} - N\xi\}^{-1}d\xi, \tag{36}$$

where

$$\begin{aligned} C_1(p) &= \prod_{i=1}^N \frac{1}{\beta_i^{\beta_i}} \qquad \left(\beta_i = \frac{\alpha_{i+1}}{\sum_{j=1}^N \alpha_j} \quad (i \in N^*)\right), \\ C_2(p) &= \frac{2}{\sum_{i=1}^N \alpha_i}, \end{aligned}$$

and $0 < t_0 < T_0$ *is chosen to satisfy* $\{\prod_{i=1}^N f_i(t_0)\}^{C_2(p)} > N$.

Proof We take the same strategy as in [11:Lemma2.2]. Multiplying e^t on the both sides of (35) and integrating it, we obtain

$$
(37)\begin{cases}
f_N(t) = e^{-t}\displaystyle\int_0^t e^{s_1} f_1(s_1)^{p_N} ds_1, \\
f_{N-1}(t) = e^{-t}\displaystyle\int_0^t e^{(1-p_2)s_1}\left\{\int_0^{s_1} e^{s_2} f(s_2)^{p_N} ds_2\right\}^{p_{N-1}} ds_1, \\
\qquad\qquad\qquad\qquad\vdots \\
f_2(t) = e^{-t}\displaystyle\int_0^t e^{(1-p_2)s_1}\Big[\int_0^{s_1} e^{(1-p_3)s_2} \times \dots \\
\quad \times\Big(\displaystyle\int_0^{s_{N-3}} e^{(1-p_{N-1})s_{N-2}}\Big\{\int_0^{s_{N-2}} e^{s_{N-1}} f_1(s_{N-1})^{p_N} ds_{N-1}\Big\}^{p_{N-1}} \\
\qquad\qquad \times ds_{N-2}\Big)^{p_{N-2}} \dots ds_2\Big]^{p_2} ds_1,
\end{cases}
$$

$$
\begin{aligned}
f_1(t) &= e^t f_0 + e^{-t}\int_0^t e^{(1-p_1)s_1}\Big[\int_0^{s_1} e^{(1-p_2)s_2} \times \dots \\
&\qquad \times\Big(\int_0^{s_{N-2}} e^{(1-p_{N-1})s_{N-1}}\Big\{\int_0^{s_{N-1}} e^{s_N} f_1(s_N)^{p_N} ds_N\Big\}^{p_{N-1}} \\
&\qquad\qquad \times ds_{N-1}\Big)^{p_{N-2}} \dots ds_2\Big]^{p_2} ds_1.
\end{aligned} \tag{38}
$$

Let $f_0 > 1$ be chosen large enough to satisfy

$$
\begin{aligned}
&\inf_{t_0>0}\Big\{e^{t_0} f_0 + 2^{p_1 p_2 \dots p_N} e^{-t_0}\int_0^{t_0} e^{(1-p_1)s_1}\Big[\int_0^{s_1} e^{(1-p_2)s_2} \times \dots \\
&\quad \times\Big(\int_0^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}}\Big\{\int_0^{s_{N-2}} e^{s_{N-1}} (1-e^{-s_{N-1}})^{p_{N-1}} ds_{N-1}\Big\}^{p_{N-2}} \\
&\qquad \times ds_{N-2}\Big)^{p_{N-3}} \dots ds_2\Big]^{p_2} ds_1\Big\} \ge 2^{p_1 p_2 \dots p_N} - \delta,
\end{aligned} \tag{39}
$$

where $\delta > 0$ is a small constant satisfying $\delta < 2^{p_1 p_2 \dots p_N} - 2$.

We shall first show that under this condition $f_1(t) > 2$ for any $0 < t < T_0$. Assume to the contrary that there exist $0 < t_1 < T_0$ such that $f_1(t) > 2$ in $0 \le t < t_1$ and $f_1(t_1) = 2$. Then it follows from (38) and (39) that

$$
\begin{aligned}
2 &= f_1(t_1) \\
&\geq e^{t_1} f_0 + 2^{p_1 p_2 \dots p_N} e^{-t_1} \int_0^{t_1} e^{(1-p_1)s_1} \Big[\int_0^{s_1} e^{(1-p_2)s_2} \times \dots \\
&\qquad \times \Big(\int_0^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \Big\{ \int_0^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})^{p_{N-1}} d_{s_{N-1}} \Big\}^{p_{N-2}} \\
&\qquad\qquad \times ds_{N-2} \Big)^{p_{N-3}} \dots ds_2 \Big]^{p_2} ds_1 \Big\} \\
&\geq 2^{p_1 p_2 \dots p_N} - \delta > 2,
\end{aligned}
$$

and a contradiction occurs. Next, we shall show that $\lim_{t \to T_0} f_1(t) = \infty$ $(T_0 \leq \infty)$. Assume to the contrary that there exist a sequence $\{t_j\}$ such that

$$\lim_{t_j \to \infty} f_1(t_j) = M \text{ for some } 2 \leq M < \infty.$$

We choose $\epsilon > 0$ and $t_* > 0$ to satisfy $M < (M - \epsilon)^{p_1 p_2 \dots p_N}$ and $f_1(t) > M - \epsilon$ in $t_* < t < T$. It then follows from (38) that

$$
\begin{aligned}
f_1(t_j) \quad &\geq \quad e^{t_j} f_0 + 2^{p_1 p_2 \dots p_N} e^{-t_j} \int_0^{t_*} e^{(1-p_1)s_1} \Big[\int_0^{s_1} e^{(1-p_2)s_2} \times \dots \\
&\qquad \times \Big(\int_0^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \Big\{ \int_0^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})^{p_{N-1}} ds_{N-1} \Big\}^{p_{N-2}} \\
&\qquad\qquad \times ds_{N-2} \Big)^{p_{N-3}} \dots ds_2 \Big]^{p_2} ds_1 \\
&\quad + (M - \epsilon)^{p_1 p_2 \dots p_N} e^{-t_j} \int_{t_*}^{t_j} e^{(1-p_1)s_1} \Big[\int_{t_*}^{s_1} e^{(1-p_2)s_2} \times \dots \\
&\qquad \times \Big(\int_{t_*}^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \Big\{ \int_{t_*}^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})^{p_{N-1}} ds_{N-1} \Big\}^{p_{N-2}} \\
&\qquad\qquad \times ds_{N-2} \Big)^{p_{N-3}} \dots ds_2 \Big]^{p_2} ds_1 \\
&\to \quad (M - \epsilon)^{p_1 p_2 \dots p_N} > M \quad (t_j \to \infty).
\end{aligned}
$$

Noting (37), we now conclude

(40) $$\lim_{t \to T_0} f_1(t) = \lim_{t \to T_0} f_2(t) = \dots = \lim_{t \to T_0} f_N(t) = \infty \quad (T_0 \leq \infty).$$

To complete the assertion we put $h(t) = f_1(t) f_2(t) \dots f_N(t)$ .Then by (35) and Young's inequality,

(41) $$h'(t) \geq -3h(t) + C_1(p) h(t)^{C_2(p)+1}.$$

Integrating this, we obtain

$$t - t_0 \leq \int_{h(t_0)}^{h(t)} \left\{ C_1(p)\xi^{C_2(p)+1} - N\xi \right\}^{-1} d\xi.$$

Since $p_1 p_2 \dots p_N > 1$, this and (40) show that $h(t)$ blows up in a finite time and the life span T_0 is estimated by (**??**). □

Let us consider the solution $f_\epsilon(t) = (f_{1,\epsilon}(t), f_{2,\epsilon}(t), ..., f_{N,\epsilon}(t))$ of (**??**). As is shown in Lemma 1.2.2, there exist $A_i > 0$ $(i \in N^*)$ such that if

$$F_{i,\epsilon}(0) > A_i(2d\epsilon)^{\alpha_i/2} \quad (i \in N^*), \tag{42}$$

then F_ϵ blows up in a finite time. Moreover, its life span is estimated from above by $(2d\epsilon)^{-1}T_0$.

Let us consider the solution $f_\epsilon(t) = (f_{1\epsilon}(t), f_{2\epsilon}(t), ..., f_{N\epsilon}(t))$ of (34). As is shown in Lemma 3.1, there exists $A_i > 0$ $(i \in N^*)$ such that if

$$F_{i\epsilon}(0) > A_i(2d\epsilon)^{(\alpha_i+\delta_i)/2} \quad (i \in N^*), \tag{43}$$

then F_ϵ blows up in a finite time. Moreover, its life span is estimated from above by $(2d\epsilon)^{-1}T_0$.

Theorem 3.2. *Let $F_\epsilon(t) = (F_{1\epsilon}(t), F_{2\epsilon}(t), ..., F_{N\epsilon}(t))$ satisfy differential inequalities* (33). *If* (43) *is satisfied for some $\epsilon > 0$, then $F_\epsilon(t)$ blows up in finite time. Moreover, its life span is estimated from above by $(2d\epsilon)^{-1}T_0$. Then, we obtain*

$$T^*(u_0) \leq (2d\epsilon)^{-1}T_0. \tag{44}$$

Proof of Theorem 1. First, we consider the noncritical case as $\max_{i\in N^*}$ $\{\alpha_i+\delta_i\} > d$. Without loss of generality, we can let $\alpha_2+\delta_2 > d$. By means of a comparison principle and Lemma 2.5, we can assume $u_{2,0} \in L^1(\mathbf{R}^d)$ and

$$\int_{\mathbf{R}^d} u_{2,0}(x)dx > 0.$$

The Lebesgue's dominated convergence theorem then shows the existence of ϵ_0 such that

$$F_{2,\epsilon}(0) = \left(\frac{\epsilon}{\pi}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} u_{2,0}(x)e^{-\epsilon|x|^2}dx \geq \frac{1}{2}\left(\frac{\epsilon}{\pi}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} u_{2,0}(x)dx$$

for any $0 < \epsilon \leq \epsilon_0$. Since $\alpha_2 + \delta_2 > d$ by the assumption, this implies that the condition (43) of Theorem 3.2 is satisfied if ϵ is sufficiently small. Thus, $F_\epsilon(t)$ blow up in a finite time.

Next, we consider the critical case as $\max_{i \in N^*}\{\alpha_i + \delta_i\} = d$. For each nontrivial solution $u(t) \in E_T$ of (1), it follows from Lemma 2.6 that

$$\begin{aligned} S(t)u_1(0,t) &\geq Ct^{-d/2}\log(t/2a)\int_{\mathbf{R}^d} e^{-5|x|^2/4t}dx \\ &\geq Ct^{-d/2}\log(t/2a) \end{aligned} \tag{45}$$

in $a < t < T^*$. Contrary to the conclusion, assume that u is global. Then by Theorem 3.2

$$F_{1,\epsilon}(t) = (\epsilon/\pi)^{d/2}\int_{\mathbf{R}^d} u_1(x,t)e^{-\epsilon|x|^2}dx \leq A_1\epsilon^{(\alpha_1+\delta_1)/2}$$

holds for any $t \geq 0$ and $\epsilon > 0$. Thus, choosing $\epsilon = (4t)^{-1}$, we obtain

$$F_{1,1/4t}(t) = S(t)u_1(0,t) \leq A_1(4t)^{-(\alpha_1+\delta_1)/2} = A_1(4t)^{-d/2}.$$

This and (45) contradict to each other if $T^* = \infty$.

The proof of Theorem 1 is thus complete. □

Proof of Theorem 2 (i). If $u_{1,0} \in I_{a_1}$ with $a_1 < \alpha_1 + \delta_1 < d$, we have

$$\begin{aligned} F_{1,\epsilon}(0) &= (\epsilon/\pi)^{d/2}\int_{\mathbf{R}^d} u_{1,0}(x)e^{-\epsilon|x|^2}dx \\ &= \pi^{-d/2}\int_{\mathbf{R}^d} u_{1,0}(\epsilon^{-1/2}x)e^{-|x|^2}dx. \end{aligned}$$

Then it follows that

$$\epsilon^{-\alpha_1+\delta_1/2}F_{1,\epsilon}(0) \geq C\epsilon^{-(\alpha_1+\delta_i-a)/2}\pi^{-d/2}\int_{\mathbf{R}^d}|x|^{-a_1}e^{-|x|^2}dx > A_1$$

for sufficiently small $\epsilon > 0$. If $i \in N^*\backslash\{1\}$, we can obtain a similar estimate for $F_{i,\epsilon}$. Thus $F_\epsilon(t)$ blows up in a finite time by Theorem 3.2.

Proof of Theorem 3. We then have for any $i \in N^*$,

$$F_{i,\epsilon} \geq C(\epsilon/\pi)^{d/2}\int_{\mathbf{R}^d} e^{-(\epsilon+\nu_0)|x|^2}dx = C\left(\frac{\epsilon}{\epsilon+\nu_0}\right)^{d/2}.$$

So, if we choose $\epsilon = 1$ and $C > (2\pi)^{d/2}\max_{i\in N^*}\{A_i\}(1+\nu_0)^{d/2}$, the condition of Theorem 3.2 is also satisfied in this case. □

4 Proof of global existence

In this and next section we require $\max_{i\in n^*}\{\alpha_i+\delta_i\}<d$, and treat the existence and large time behavior of global solutions of (1). Tn this section we show Theorem 2 (ii). Note that our condition implies that there exists $i\in N^*$ such that $p_i>1+2/d$. We may assume that $p_2>1+2/d$ for simplicity.

First note that condition (9) can be replaced by $u_{i,0}\in I^{\hat{a}_i}$ $(i\in N^*)$ since we have $I^{a_i}\subset I^{\hat{a}_i}$ $(i\in N^*)$. Then, to establish Theorem 2 (ii), we have only to consider the special case $\hat{a}_i=a_i$ $(i\in N^*)$. As is easily seen, in this case condition (11) is equivalent to

$$p_i a_{i+1,d}-a_i>2+\sigma_i \qquad (i\in N^*), \tag{46}$$

where $a_{j,d}=\min\{a_j,d\}$.

Using η defined in (18), we define the Banach spaces E_η and X as

$$E_\eta=\Big\{u;\|u\|_{E_\eta}\equiv\sum_{i=1}^{N}(|||u_i/\eta_{a_i}|||_\infty)<\infty\Big\},$$

and

$$X=\Big\{v;|||v/\eta_{a_N}|||_\infty<\infty\Big\},$$

where

$$|||w|||_\infty=\sup_{(x,t)\in\mathbf{R}^d\times(0,\infty)}|w(x,t)|.$$

(21) is reduced to

$$u_N(t)=V(t)(u_0,u_N), \tag{47}$$

where

$$\begin{aligned}V(t)(u_0,v) \;\;=\;\; & S(t)u_{N,0}+\int_0^t S(t-s_1)|x|^{\sigma_N}\Big(S(s_1)u_{1,0}\\
&+\int_0^{s_1}S(s_1-s_2)|x|^{\sigma_1}\Big\{S(s_2)u_{2,0}+\int_0^{s_2}S(s_2-s_3)\\
&\times\ldots\times|x|^{\sigma_{N-2}}\Big[S(s_{N-1})u_{N-1,0}\\
&+\int_0^{s_{N-1}}S(s_{N-1}-s_N)v^{p_{N-1}}(s_N)ds_N\Big]^{p_{N-2}}\times\ldots\times ds_3\Big\}^{p_1}ds_2\Big)^{p_N}ds_1.\end{aligned}$$

Here, if V is a strict contraction, its fixed point yields a solution of (1). Moreover, using that $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $a>0, b>0, p>1$,

$$V(t)(u_0,v) \leq T(t)(u_0) + \Gamma(t)(v),$$

where

$$\begin{aligned}
&T(t)(u_0)\\
&= S(t)u_{N,0} + 2^{p_N-1}\int_0^t S(t-s_1)|x|^{\sigma_N}(S(s_1)u_{1,0})^{p_N}ds\\
&\quad +2^{(p_N-1)(p_1-1)}\int_0^t S(t-s_1)|x|^{\sigma_N}\\
&\quad \times\left(\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\left\{S(s_2)u_{2,0}\right\}^{p_1}dr\right)^{p_N} ds+\ldots\\
&\quad +2^{(p_N-1)(p_1-1)\ldots(p_{N-2}-1)}\int_0^t S(t-s_1)|x|^{\sigma_N}\\
&\quad \times\Big[\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\Big\{\int_0^{s_2} S(s_2-s_3)\times\ldots\times|x|^{\sigma_{N-3}}\\
&\quad \times\Big(\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})|x|^{\sigma_{N-2}}[S(s_{N-1})u_{N-1,0}]^{p_{N-2}}ds_{N-1}\Big)^{\sigma_{N-3}}\\
&\qquad\qquad\qquad\qquad \times\ldots\times ds_3\Big\}^{p_1}ds_2\Big]^{p_N}ds_1,
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma(t)(v) = 2^{(p_N-1)(p_1-1)\ldots(p_{N-2}-1)}\\
&\quad\times\int_0^t S(t-s_1)|x|^{\sigma_N}\Big(\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\Big\{\int_0^{s_2} S(s_2-s_3)\times\ldots\\
&\quad\times|x|^{\sigma_{N-3}}\Big[\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})\\
&\quad\times|x|^{\sigma_{N-2}}\Big\{\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}v^{p_{N-1}}(s_N)ds_N\Big\}^{p_{N-2}}\\
&\quad\times\quad ds_{N-1}\Big]^{p_{N-3}}\times\ldots\times ds_3\Big\}^{p_1}ds_2\Big)^{p_N}ds_1.
\end{aligned}$$

Lemma 4.1.
(i) *Let* u_0 *satisfy* (9). *Then* $T(\cdot)(u_0)\in X$ *and*

$$\begin{aligned}
&|||T(\cdot)(u_0)/\eta_{a_N}(\cdot)|||_\infty\\
&\leq C\left(\|u_{N,0}\|_{\infty,a_N} + \|u_{1,0}\|^{p_N}_{\infty,a_1} + \|u_{2,0}\|^{p_Np_1}_{\infty,a_2} + \ldots + \|u_{N-1,0}\|^{p_Np_1\ldots p_{N-2}}_{\infty,a_{N-1}}\right).
\end{aligned}$$

(ii) Γ *maps* X *into itself and*

$$|||\Gamma(v)/\eta_{a_N}|||_\infty \leq C|||v/\eta_{a_N}|||_\infty^{p_1p_2p_3\dots p_N}.$$

Proof. (i) (cd. [14:Lemma 4.3].) By (18) and (19) in Lemma 2.2, we obtain $T(t)(u_0) = I_1 + I_2 + \ldots + I_N$, where

$$\begin{aligned} I_1 &\leq \|u_{N0}\|_{\infty,a_N}\eta_{a_N}(t), \\ I_2 &\leq 2^{p_N-1}\int_0^t S(t-s)|x|^{\sigma_N}(\eta_1\|u_{1,0}\|_{\infty,a_1})^{p_N}ds \\ &\leq C\|u_{10}\|_{\infty,a_1}^{p_N}\eta_{a_N}(t), \end{aligned}$$

and by same argument, we have

$$\begin{aligned} I_3 &\leq C\|u_{20}\|_{\infty,a_1}^{p_1p_N}\eta_{a_N}(t), \\ &\vdots \\ I_N &\leq C\|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_Np_1\dots p_{N-2}}. \end{aligned}$$

(ii) By (18) and (19)

$$\begin{aligned} \Gamma(v) \leq\,& C|||v/\eta_{a_N}|||_\infty^{p_1p_2\dots p_N}\int_0^t S(t-s_1)|x|^{\sigma_2}\Big(\int_0^s S(s_1-s_2) \\ &\times|x|^{\sigma_3}\Big\{\int_0^{s_2} S(s_2-s_3)\times\ldots|x|^{\sigma_{N-1}}\Big[\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1}) \\ &\times|x|^{\sigma_N}\Big\{\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_1}\eta_{a_N}(s_N)^{p_1}ds_N\Big\}^{p_N} \\ &\times ds_{N-1}\Big]^{p_{N-1}}\times\ldots\times ds_3\Big\}^{p_3}ds_2\Big)^{p_2}ds_1 \\ \leq\,& C2^{(p_N-1)(p_1-1)(p_2-1)\dots(p_{N-2}-1)}|||v/\eta_{a_N}|||_\infty^{p_1p_2\dots p_N}\int_0^t \eta_{a_1}(s)^{p_N}ds \\ \leq\,& C|||v/\eta_{a_N}|||_\infty^{p_1p_2\dots p_N}\eta_{a_N}. \end{aligned}$$

Proof of Theorem 2 (ii). Let

$$C\left(\|u_{N,0}\|_{\infty,a_N} + \|u_{1,0}\|_{\infty,a_1}^{p_N} + \|u_{2,0}\|_{\infty,a_2}^{p_Np_1} + \ldots + \|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_Np_1\dots p_{N-2}}\right) \leq m,$$

$\|u_i\|_{\infty,a_i} \leq m$ ($i \in N^*$), $B_m = \{v \in X : |||v/\eta_{a_3}|||_\infty \leq 2m\}$ and $P = \{u \in X; u \geq 0\}$. Here the constant C is the one appeared in Lemma 4.1. Then we shall show that $V(u_0, v)$ is a strict contraction of $B_m \cap P$ into itself provided m is small enough.

It is trivial that V maps P into P. We shall show that V maps $B_m \to B_m$. If m is small enough, then

$$V(t)(u_0, v)/\eta_{a_N} \leq m + C(2m)^{p_1 p_2 \ldots p_N} \leq 2m.$$

This proves that V maps $B_m \to B_m$.

Now, we show that $V(u_0, v)$ is a strict contraction on $B_m \cap P$. Using $|a^p - b^p| \leq p(a+b)^{p-1}|a-b|$ for $a > 0$, $b > 0$ and $p \geq 1$, with $v = \max\{v_1, v_2\}$, we can estimate as follows.

$$\begin{aligned}
&|V(t)(u_0, v_1) - V(t)(u_0, v_2)| \\
\leq\ & C \int_0^t S(t-s_1)|x|^{\sigma_N} \Big(2S(s_1)u_{1,0} \\
&+2 \int_0^{s_1} S(s_1 - s_2)|x|^{\sigma_1} \Big\{ S(s_2)u_{2,0} + \int_0^{s_2} S(s_2 - s_3) \times \ldots \\
&\times |x|^{\sigma_{N-3}} \Big\{ S(s_{N-2})u_{N-1,0} + \int_0^{s_{N-2}} S(s_{N-2} - s_{N-1})|x|^{\sigma_{N-2}} \\
&\times \Big[S(s_{N-1})u_{N,0} + \int_0^{s_{N-1}} S(s_{N-1} - s_N)|x|^{\sigma_{N-1}} v^{p_{N-1}}(s_N)ds_N \Big]^{p_{N-2}} \\
&ds_{N-1} \Big\}^{p_{N-3}} \times \ldots \times ds_3 \Big\}^{p_1} ds_2 \Big)^{p_N - 1} \\
\times\ & \int_0^{s_1} S(s_1 - s_2)|x|^{\sigma_1} \Big(2S(s_2)u_{2,0} \\
&+2 \int_0^{s_2} S(s_2 - s_3)|x|^{\sigma_2} \Big\{ S(s_3)u_{3,0} + \int_0^{s_3} S(s_3 - s_4) \times \ldots \\
&\times |x|^{\sigma_{N-3}} \Big\{ S(s_{N-2})u_{N-1,0} + \int_0^{s_{N-2}} S(s_{N-2} - s_{N-1})|x|^{\sigma_{N-2}} \\
&\times \Big[S(s_{N-1})u_{N,0} + \int_0^{s_{N-1}} S(s_{N-1} - s_N)|x|^{\sigma_{N-1}} v^{p_{N-1}}(s_N)ds_N \Big]^{p_{N-2}} \\
&ds_{N-1} \Big\}^{p_{N-3}} \times \ldots \times ds_4 \Big\}^{p_2} ds_3 \Big)^{p_1 - 1} \\
\times\ & \ldots \\
\times\ & \int_0^{s_{N-2}} S(s_{N-2} - s_{N-1})|x|^{\sigma_{N-2}} \Big\{ 2S(s_{N-1})u_{N-1,0} \\
&+2 \int_0^{s_{N-1}} S(s_{N-1} - s_N)|x|^{\sigma_{N-1}} v^{p_{N-1}}(s_N)ds_N \Big\}^{p_{N-2}-1}
\end{aligned}$$

$$
\begin{aligned}
\times & \quad \Big(\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}|v_1^{p_{N-1}}(s_N)-v_2^{p_{N-1}}(s_N)|^{p_{N-1}}ds_N\Big) \\
& \times ds_{N-1}\dots ds_2 ds_1.
\end{aligned}
$$

We put

$$
\begin{aligned}
& |V(t)(u_0,v_1)-V(t)(u_0,v_2)| \\
\leq & \ C\int_0^t S(t-s_1)\times J_1\times\int_0^{s_1} S(s_1-s_2)\times J_2\times \\
& \quad \dots\times\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})\times J_{N-1}\times J_N ds_{N-1}\dots ds_2 ds_1
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 = & |x|^{\sigma_N}\Big(2S(s_1)u_{1,0} \\
& +2\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\Big\{S(s_2)u_{2,0}+\int_0^{s_2} S(s_2-s_3)\times\dots \\
& \times|x|^{\sigma_{N-3}}\Big\{S(s_{N-2})u_{N-1,0}+\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})|x|^{\sigma_{N-2}} \\
& \times\Big[S(s_{N-1})u_{N,0}+\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}v^{p_{N-1}}(s_N)ds_N\Big]^{p_{N-2}} \\
& ds_{N-1}\Big\}^{p_{N-3}}\times\dots\times ds_3\Big\}^{p_1} ds_2\Big)^{p_N-1}, \\
J_2 = & |x|^{\sigma_1}\Big(2S(s_2)u_{2,0} \\
& +2\int_0^{s_2} S(s_2-s_3)|x|^{\sigma_2}\Big\{S(s_3)u_{3,0}+\int_0^{s_3} S(s_3-s_4)\times\dots \\
& \times|x|^{\sigma_{N-3}}\Big\{S(s_{N-2})u_{N-1,0}+\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})|x|^{\sigma_{N-2}} \\
& \times\Big[S(s_{N-1})u_{N,0}+\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}v^{p_{N-1}}(s_N)ds_N\Big]^{p_{N-2}} \\
& ds_{N-1}\Big\}^{p_{N-3}}\times\dots\times ds_4\Big\}^{p_2} ds_3\Big)^{p_1-1} \\
& \qquad \vdots \\
J_{N-1} = & |x|^{\sigma_{N-2}}\Big\{2S(s_{N-1})u_{N-1,0} \\
& +2\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}v^{p_{N-1}}(s_N)ds_N\Big\}^{p_{N-2}-1}
\end{aligned}
$$

$$J_N = \int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}|v_1^{p_{N-1}}(s_N) - v_2^{p_{N-1}}(s_N)|^{p_{N-1}} ds_N.$$

Noting $(a+b)^p \le 2^{\max\{p-1,0\}}(a^p+b^p)$ for $a>0$, $b>0$ and $p\ge 0$, we find

$$\begin{aligned}
J_{N-1} &\le C|x|^{\sigma_{N-2}}\Big\{\|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_{N-2}-1}\eta_{a_{N-1}}^{p_{N-2}-1}(s_{N-1}) \\
&\quad + \left(\int_0^r S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}|v/\eta_{a_N}|^{p_{N-1}}\eta_{a_N}^{p_{N-1}}(s_N)ds_N\right)^{p_{N-2}-1}\Big\} \\
&\le C|x|^{\sigma_{N-2}}\left\{\left(m^{p_{N-2}-1} + C(2m)^{(p_{N-2}-1)p_{N-1}}\right)\eta_{a_{N-1}}^{p_{N-2}-1}(s_{N-1})\right\} \\
&\le Cm^{p_{N-2}-1}|x|^{\sigma_{N-2}}\eta_{a_{N-1}}^{p_{N-2}-1}(s_{N-1}).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
J_1 &\le Cm^{p_N-1}|x|^{\sigma_N}\eta_{a_1}^{p_N-1}(s_1), \\
J_2 &\le Cm^{p_1-1}|x|^{\sigma_1}\eta_{a_2}^{p_1-1}(s_2), \\
J_3 &\le Cm^{p_2-1}|x|^{\sigma_2}\eta_{a_3}^{p_2-1}(s_3), \\
&\vdots \\
J_{N-2} &\le Cm^{p_{N-2}-1}|x|^{\sigma_{N-3}}\eta_{a_{N-2}}^{p_{N-3}-1}(s_{N-2}), \\
J_N &\le Cm^{p_{N-1}-1}(|v_1-v_2|/\eta_{a_N})\eta_{a_{N-1}}.
\end{aligned}$$

Thus, we obtain

$$|V(t)(u_0,v_1) - V(t)(u_0,v_2)| \le Cm^{p_1+p_2+\ldots+p_N-N}|v_1-v_2|.$$

Since $p_2>1$, $V(t)$ is a strict contraction of $B_m \cap P$ into itself provided m is small enough. Hence, there exists a unique fixed point $v=(u_N)\in X$ which solves (47). We substitute $v=u_N$ into (21). Then the vector u solves (21). Moreover, since $u_N \in B_m$, we find

$$u_N \le CS(t) < x >^{-a_N}.$$

By the same reason as in the proof of Lemma 4.3, we have

$$\begin{aligned}
|u_{N-1}(t)| &\le \eta_{a_{N-1}}(x,t)\left\{\|u_{N-1,0}\|_{\infty,a_{N-1}} + C|\|u_N/\eta_{a_N}|\|_\infty\right\}, \\
&\vdots \\
|u_2(t)| &\le \eta_{a_2}(x,t)\left\{\|u_{20}\|_{\infty,a_2} + C|\|u_3/\eta_{a_3}|\|_\infty\right\}, \\
|u_1(t)| &\le \eta_{a_1}(x,t)\left\{\|u_{10}\|_{\infty,a_1} + C|\|u_2/\eta_{a_2}|\|_\infty\right\}.
\end{aligned}$$

Then $u_i \in B_m (i\in N^*)$ and the proof of Theorem 3 is completed. □

5 Proof of asymptotic behavior

In this section we shall prove Theorem 4 for the global solution $u(t)$ of (1) constructed in the previous section. We use the same strategy as in [11;Theorem 6.1] (see [6] and [10;Teorem 4]).

Note again that, to establish Theorem 4, we have only to consider the special case $\hat{a}_i = a_i$ $(i \in N^*)$ from same reason in Section 4. As is easily seen, in this case condition (11) is equivalent to

$$p_i a_{i+1,d} - a_i > 2 + \sigma_i \qquad (i \in N^*). \tag{48}$$

We put

$$u_{i,k}(x,t) = k^{a_{id}} u_i(kx, k^2 t) \quad (i \in N^*)$$

for $k > 0$. Then $u_{i,k}$ solves

$$\begin{cases} (u_{i,k})_t = \Delta u_{i,k} + k^{a_i+2-a_{i+1,d}p_i} u_{i+1,k}^{p_i} & (i \in N^*), \\ u_{i,k}(x,0) = k^{a_{i,d}} u_{i,0}(kx) & (i \in N^*). \end{cases} \tag{49}$$

Note that we have assumed $a_i \neq d$ in Theorem 4. Then it follows from (48) that

$$\|u_{i,k}(t)\|_\infty \leq k^{a_{id}} C (k^2 t)^{-a_{id}/2} = C t^{-a_{i,d}/2} \quad (i \in N^*).$$

Thus, $\{u_{i,k}(x,t)\}$ are uniformly bounded in $\mathbf{R}^d \times [\delta, \infty)$ for any $\delta > 0$. As is easily seen from the integral equation (21), the uniform boundedness implies the equicontinuity of $\{u_{i,k}(x,t)\}$ in any bounded set of $\mathbf{R}^d \times [\delta, \infty)$. Then using the Ascoli-Arzela theorem and a diagonal sequence method in δ, we see that for any sequence $\{k_j\} \to \infty$, there exists a subsequence $\{k'_j\}$ and continuous functions $w_i(x,t)$ such that

$$u_{ik'_j}(x,t) \to w_i(x,t) \quad (k'_j \to \infty, i \in N^*)$$

locally uniformly in $\mathbf{R}^d \times (0, \infty)$.

Proof of Theorem 4 (i). We shall first show

$$w_i(x,t) = A_i S |x|^{a_i}. \tag{50}$$

It follows from the first equation of (14) that

$$\begin{aligned} &\int_{\mathbf{R}^d} u_{i,k}(x,t)\zeta(x,t)dx - \int_{\mathbf{R}^d} u_{,k}(x,0)\zeta(x,0)dx \\ &= \int_0^t \int_{\mathbf{R}^d} \{u_{i,k}\zeta_t + u_{i,k}\Delta\zeta + k^{a_i+2+\sigma_i-a_{i+1,d}p_i}|x|^{\sigma_i} u_{i+1,,k}^{p_i}\zeta\} dxdt \end{aligned} \tag{51}$$

for any $t > 0$ and nonnegative test function $\zeta \in C_0^\infty(\mathbf{R}^d \times [0,\infty))$. By assumption (12) for the initial value $u_{i,0}$,

$$\int_{\mathbf{R}^d} u_{i,k}(x,0)\zeta(x,0)dx = \int_{\mathbf{R}^d} k^{a_i} u_{i,0}(kx)\zeta(x,0)dx$$

$$\to \int_{\mathbf{R}^d} A_i|x|^{-a_i}\zeta(x,0)dx \qquad (k = k_j' \to \infty).$$

On the other hand,

$$\int_0^t \int_{\mathbf{R}^d} k^{a_i+2-a_{i+1,d}p_i}|x|^{\sigma_i} u_{i+1,k}^{p_i}\zeta dxdt$$
$$= \int_0^{k^2t} \int_{\mathbf{R}^d} k^{a_i+\sigma_i}|x|^{\sigma_i} u_{i+1},(kx,\tau)^{p_i}\zeta(x,k^{-2}\tau)dxd\tau.$$

Here

$$k^{a_i}|x|^{\sigma_i}u_{i+1}(kx,\tau)^{p_i}$$
$$= \left[(k|x|)^{a_{i+1,d}}u_{i+1}(kx,\tau)\right]^{(a_i+\sigma_i)\rho/a_{i+1,d}}$$
$$u_{i+1}^{p_{i+1}-a\rho/a_{i+1,d}}k^{(a_i+\sigma_i)(1-\rho)}|x|^{\sigma_i-(a_i+\sigma_i)\rho}$$

for $\rho > 1$. As is easily seen from (48) and Lemma 2.1 , $(k|x|)^{a_{i+1,d}}u_{i+1}(kx,\tau)$ is bounded in $\mathbf{R}^d \times (0,\infty)$ and

$$u_{i+1}(kx,\tau)^{p_i-a_i\rho/a_{i+1,d}} \le C(1+t)^{-a_{i+1,,d}p_i+a_i)/2}.$$

By assumption (18) and the condition of this theorem, there exists a number $\rho > 1$ such that

$$a_i\rho + \sigma_i(\rho-1) < d, \qquad a_{i+1,d}\rho - (a_i+\sigma_i)\rho < 2.$$

Then since $(a_i+\sigma_i)(1-\rho) < 0$, these imply

$$\int_0^{k^2t}\int_{\mathbf{R}^d} k^{a_i}u_{i+1},(kx,\tau)^{p_i}\zeta(x,k^{-2}\tau)dxd\tau \to 0 \qquad (k = k_j' \to \infty).$$

Thus, letting $k = k_j' \to \infty$ in (49), we obtain

$$\int_{\mathbf{R}^d} w_i(x,t)\zeta(x,t)dx - \int_{\mathbf{R}^d} A_i|x|^{-a_i}\zeta(x,0)dx$$

$$
\begin{aligned}
&= \int_0^t \int_{\mathbf{R}^d} \Big\{ w_i(x,s)\zeta_t(x,s) + w_i(x,s)\Delta\zeta(x,s) \Big\} dxds \\
&= \int_{\mathbf{R}^d} \Big\{ w_i(x,t)\zeta(x,t) - w_i(x,0)\zeta(x,0) \Big\} dx \\
&\quad - \int_0^t \int_{\mathbf{R}^d} \Big\{ w_{it}(x,s) - \Delta w_i(x,s) \Big\} \zeta(x,s) dxds.
\end{aligned}
$$

Using Green theorem,

$$
\begin{aligned}
&\int_{\mathbf{R}^d} \Big\{ w_i(x,0) - A_i|x|^{-a_i} \Big\} \zeta(x,0) dx \\
&= -\int_0^t \int_{\mathbf{R}^d} \Big\{ w_{it}(x,s) - \Delta w_i(x,s) \Big\} \zeta(x,s) dxds.
\end{aligned}
$$

By using the uniqueness of solutions of

$$
u_t = \Delta u, \qquad u(x,0) = A_i|x|^{-a_i},
$$

we will show (13).

The uniqueness result asserts more:

$$
u_{i,k}(x,t) \to A_i S(t)|x|^{-a_i} \quad (k \to \infty) \tag{52}
$$

uniformly in compact sets of $\mathbf{R}^d \times (0,\infty)$.

Note again (48), that is,

$$
u_{i,k} \le C k^{a_i} S(k^2 t)|kx|^{-a_i}.
$$

Let $t = 1$ in this inequality. Then by the self-similarity of $S(t)|x|^{-a_i}$, we have

$$
u_{i,k}(x,1) \le C S(1)|x|^{-a_i}.
$$

This inequality implies that for any $\epsilon > 0$ there exists an $R > 0$ independent of $k > 1$ such that $\{u_{i,k}(x,t)\}$ are uniformly less than ϵ in $|x| > R$. Therefore, it follows from (50) that

$$
u_k(x,1) - AS(1)|x|^{a_i} \to 0 \quad (k \to \infty)
$$

uniformly in $\mathbf{R}^d$. We let $y = kx, s = k^2$ in this relation. Then noting again the self-similarity of $S(t)|x|^{-a_i}$, we conclude that

$$s^{a_i/2}\Big|u(y,s)-AS(s)|y|^{-a_i}\Big|\to 0 \quad (s\to\infty)$$

uniformly in $\mathbf{R}^d$.

Relation (13) is now proved for $u_i(x,t)$. $\square$

Proof of Theorem 4 (ii) As in the above case, we shall show

$$u_{j,k}(x,t)=w_j(x,t)=M_j(4\pi t)e^{-|x|^2/4t} \quad (k\to\infty) \tag{53}$$

locally uniformly in $\mathbf{R}^d\times(0,\infty)$, where M_j is given as in (15). It follows from the second equation of (49) that

$$\begin{aligned}\int_{\mathbf{R}^d} u_{j,k}(x,t)\zeta(x,t)dx &- \int_{\mathbf{R}^d} u_{j,k}(x,0)\zeta(x,0)dx \\ = \int_0^t\int_{\mathbf{R}^d}&\{u_{j,k}\zeta_t+u_{j,k}\Delta\zeta+k^{d+2-a_{j+1,d}p_j}|\cdot|^{\sigma_j}u_{j+1,k}^{p_j}\zeta\}dxdt\end{aligned} \tag{54}$$

for any $t>0$ and nonnegative $\zeta\in C_0^\infty(\mathbf{R}^d\times[0,\infty))$.

Since $a_j>d$, condition (45) implies that $u_{j,0}\in L^1$. Then we have

$$\begin{aligned}\int_{\mathbf{R}^d} u_{j,k}(x,0)\zeta(x,0)dx &= \int_{\mathbf{R}^d} u_{j,k}(x)\zeta(k^{-1}x,0)dx \\ &\to \int_{\mathbf{R}^d} u_{j,0}dx\zeta(0,0) \qquad (k=k_j'\to\infty).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^d} k^{d+2-a_{j+1,d}p_j}u_{j+1,k}^{p_j}\zeta dxdt \\ &= \int_0^{k^2t}\int_{\mathbf{R}^d}|x|^{\sigma_j}u_{j+1}(x,\tau)^{p_j}\zeta(k^{-1}x,k^{-2}\tau)dxd\tau.\end{aligned}$$

Here (11) and Lemma 2.1 show

$$u_{j+1}(x,t)^{p_{j+1}}\le C(1+\tau)^{-a_{j+1,d}(p_j-r)/2}u_{j+1}(x,\tau)^r$$

for some r satisfying $a_{j+1}(p_j-r)>2$. We put $r=d\rho/a_{j+1,d}$. Then by assumption (11), we can choose $\rho>1$ to satisfy

$$a_{j+1}(p_j-r)=a_{j+1}p_j-d\rho>0.$$

Then since

$$u_{j+1}(x,t)^r \leq C[S(t) < x >^{-a_{j+1}}] \leq CS(t) < x >^{-d\rho}$$

it follows that

$$\int_0^\infty \int_{\mathbf{R}^d} |x|^{\sigma_j} u_{j+1}(x,\tau)^{p_j} dx d\tau$$

and we have

$$\begin{aligned} &\int_0^{k^2 t} \int_{\mathbf{R}^d} |x|^{\sigma_j} u_{j+1}(x,\tau)^{p_j} \zeta(k^{-1}x, k^{-2}\tau) dx d\tau \\ \to &\int_0^\infty \int_{\mathbf{R}^d} |x|^{\sigma_j} u_{j+1},(x,\tau)^{p_j} dx d\tau \zeta(0,0) \leq \infty \qquad (k = k_j' \to \infty). \end{aligned}$$

Thus, letting $k = k_j' \to \infty$ in (54), we obtain

$$\begin{aligned} &\int_{\mathbf{R}^d} w_j(x,t)\zeta(x,t)dx - M_j\zeta(0,0) \\ = &\int_0^t \int_{\mathbf{R}^d} \Big\{ w_j(x,s)\zeta_t(x,s) + w_j(x,s)\Delta\zeta(x,s) \Big\} dx dt. \\ = &\int_{\mathbf{R}^d} \Big\{ w_j(x,t)\zeta(x,t) - w_j(x,o)\zeta(x,0) \Big\} dx \\ &-\int_0^t \int_{\mathbf{R}^d} \Big\{ w_{jt}(x,s) - \Delta w_j(x,s) \Big\} \zeta(x,s) dx ds. \end{aligned}$$

Using Green's theorem,

$$\begin{aligned} &\int_{\mathbf{R}^d} (w_j(x,0) - M_j\delta(x))\zeta(x,0)dx \\ = \; &-\int_0^t \int_{\mathbf{R}^d} \{ w_{jt}(x,s) - \Delta w_j(x,s))\zeta(x,s) dx ds. \end{aligned}$$

The uniqueness of solution of

$$u_t = \Delta u, \qquad u(x,0) = M_j\delta(x)$$

where $\delta(x)$ is the Dirac δ -function, then implies (54).

We put $t = 1$ in (54). Then letting $y = kx$ and $s = k^2$, we conclude

$$s^{d/2}\Big|u_j(y,s) - M_j(4\pi s)^{-d/2}e^{-|y|^2/4s}| \to 0 \qquad (s \to \infty)$$

uniformly in $\{y \in \mathbf{R}^d; |y| \leq Rs^{1/2}\}$ for any $R > 0$.

Theorem 4 (ii) is thus proved. $\square$

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