Large Time Behavior and Uniqueness of Solutions of a Weakly Coupled System of Reaction-Diffusion Equations

Noriaki Umeda

Division of Mathematics,
Graduate School of Science,
Hokkaido University
Sapporo, 060-0810 Japan

1 Introduction

We consider nonnegative solutions of the initial value problem for a weakly coupled system

$$(1)\quad \begin{cases} \dfrac{\partial u_i(x,t)}{\partial t} = \Delta u_i(x,t) + u_{i+1}^{p_i}(x,t), & x \in \mathbf{R}^d, t > 0, i \in N^*, \\ u_i(x,0) = u_{i,0}(x), & x \in \mathbf{R}^d, i \in N^*, \end{cases}$$

where $N \geq 1$, $N^* = \{1, 2, \ldots N\}$, $d \geq 1$, $p_i > 0 (i \in N^*)$ and $u_{i,0} (i \in N^*)$ are nonnegative bounded and continuous functions. Throughout this paper we mean $u_{N+i} = u_i$, $u_{N+i,0} = u_{i,0}$, $p_{N+i} = p_i$ for each $i \in \mathbf{Z}$ and $u = (u_1, u_2, \cdots, u_N)$, $u_0 = (u_{1,0}, u_{2,0}, \cdots, u_{N,0})$.

Problem (1) has a nonnegative and bounded solution at least locally in time (see Theorem 2.1). For any given initial value u_0, let $T^* = T^*(u_0)$ be the maximal existence time of the solution. If $T^* = \infty$, it is called a global solution. On the other hand, if $T^* < \infty$, there exists $i \in N^*$ such that

$$(2)\qquad \limsup_{t \to T^*} \|u_i(t)\|_\infty = \infty.$$

When (2) holds, we say that the solution blows up in a finite time.

Since the pioneering work of Fujita [6], the blow up and global existence of solutions to weakly coupled semilinear parabolic systems have been studied by several authors ([2], [3], [4] and [7]).

In the previous paper ([8]), we have considered the case $p_i \geq 1 (i \in N^*)$, $p_1 p_2 \dots p_N > 1$ and proved the following results.

(I) If $2 \max_{i\in N^*}\{1+p_i+p_i p_{i+1}+\dots+p_i p_{i+1}\dots p_{i+N-2}\} \geq d(p_1 p_2 \dots p_N - 1)$, then $T^* < \infty$ for every nontrivial solution $u(t)$ of (1);

(II) If $2 \max_{i\in N^*}\{1+p_i+p_i p_{i+1}+\dots+p_i p_{i+1}\dots p_{i+N-2}\} < d(p_1 p_2 \dots p_N - 1)$, then there exist both non-global solutions and non-trivial global solutions of (1).

In [8], we also have considered the large time behavior of global solutions. These results extend the previous results for the case $N = 2$ ([2] and [7]). We also refer [4] and the references therein for the study of the blow-up rate in the case (I) for $N \geq 1$.

In this article, we consider the case $p_1 p_2 \dots p_N \leq 1$ with $p_i > 0$ and we may allow the situation $p_i > 1$ for some i. Our first result is the following global existence of solutions.

Theorem 1. *Assume that $0 < p_1 p_2 \dots p_N \leq 1$. Let $u(t)$ be a nonnegative solution of* (1), *and let $T^* = T^*(u_0)$ be the maximal existence time of the solution. Then $T^* = \infty$, i. e., every solution is global.*

The uniqueness of solutions to (1) is a delicate issue, because the nonlinearity does not satisfy the Lipschitz condition when $p_i < 1$ for some i. Actually, the uniqueness of solutions does not hold in general and we can have the following two theorems.

Theorem 2. *Assume that $0 < p_i < 1$ $(i \in N^*)$ and $u_0 \not\equiv 0$. The problem* (1) *has a unique nonnegative solution.*

Theorem 3. *Assume that $0 < p_1 p_2 \dots p_N < 1$ and $u_0 \equiv 0$. Then any nontrivial nonnegative solution of* (1) *has the form*

$$u_i(x, t; s) = c_i(t - s)_+^{\alpha_i} \qquad (i \in N^*),$$

where $(r)_+ = \max\{r, 0\}$, s *is any nonnegative constant and* $c_i, \alpha_i (i \in N)$ *are positive constants given by*

$$
(3) \qquad \begin{cases} \alpha_i = \dfrac{1 + p_i + p_i p_{i+1} + \ldots + p_i p_{i+1} \ldots p_{i+N-2}}{1 - p_1 p_2 \ldots p_N}, \\ \\ c_i^{p_1 p_2 \ldots p_N - 1} = \alpha_i \alpha_{i+1}^{p_i} \alpha_{i+2}^{p_i p_{i+1}} \ldots \alpha_{i+N-1}^{p_i p_{i+1} \ldots p_{i+N-2}}. \end{cases}
$$

In the case of $N \leq 2$, Theorem 2 has been proved under the weaker condition $0 < p_1 p_2 < 1$ [3;Theorem (a)]. However, it is an open problem whether Theorem 2 is true or not under the weaker condition $p_1 p_2 \ldots p_N < 1$ for the case $N \geq 3$.

Finally, we consider the asymptotic behavior of solutions of (1) as $t \to \infty$.

Theorem 4. *Assume that* $0 < p_1 p_2 \ldots p_N < 1$. *Then, for any nontrivial nonnegative solution* $u(t)$ of (1),

$$
(4) \qquad \lim_{t \to \infty} t^{-\alpha_i} u_i(x, t) = c_i \qquad (i \in N^*)
$$

holds uniformly in $\boldsymbol{R}^d$, *where* c_i, α_i *are positive constants given by* (3).

As for the global existence, the case $N = 1$ is easy ([1]). The case $N = 2$ was studied by M. Escobedo and M. A. Herrero ([2;Theorem 1]). As for the uniqueness of solutions, the same results have been obtained by J. Aguirre and M. Escobedo ([1]) in the case $N = 1$, and by M. A. Herrero and M. Escobedo ([3]) in the case $N = 2$. Theorems 1, 2 and 3 in this paper extend these results to the general case $N \geq 3$. Basically Theorems 1, 2 and 3 can be proved in a similar way to that in [2] and [3]. But for our big system, the procedure to obtain the key differential inequalities (23) and (24) below becomes very complicated. To control this big system properly, we make use of Lemma 2.2 which is a new observation. Moreover, to obtain an important lower bound estimate (Lemma 3.2), we also need to control more complicated iteration process than the one for the case $N = 2$. Theorem 4 is completely new even for the case $N = 1$, 2. We also note that in the proof of Theorem 4 we employ a new comparison argument which yields a simple proof of the global existence under the condition $0 < p_1 p_2 \dots p_N < 1$ even for the case $N = 2$.

In §2 we prove Theorem 1. Theorems 2 and 3 are proved in §3 and finally Theorem 4 is proved in §4.

For simplicity, we use the following notation throughout this paper:

$$p_{i,j} = \begin{cases} p_i p_{i+1} \dots p_j, & (i < j), \\ p_i, & (i = j), \\ 1, & (i > j). \end{cases}$$

2 Proof of Theorem 1

First we note the local existence of solutions of (1).

Theorem 2.1. *Let $p_i > 0 (i \in N^*)$ and assume that u_0 is nonnegative, continuous and bounded. Then there exists $T > 0$ such that (1) admits a nonnegative and bounded classical solution u in $[0, T) \times \mathbf{R}^d$.*

Proof. Although we follow the same argument as in [2;Theorem 2.1] for the case $N = 2$ and [1;Lemma (1.3)] for the case $N = 1$, we give the outline of the proof for reader's convenience. For arbitrary $T > 0$, let

$$E_T = \{u : [0, T] \to (L^\infty)^N ; \|u\|_{E_T} < \infty\}, \tag{5}$$

where

$$\|u\|_{E_T} = \sup_{t \in [0,T]} \{\sum_{i=1}^{N} \|u_i(t)\|_\infty\}.$$

We consider in E_T the related integral system

$$u_i(t) = S(t)u_{i,0} + \int_0^t S(t-s) u_{i+1}^{p_i}(s) ds, \qquad i \in N^*, \tag{6}$$

where $S(t)\xi$ represents the solution of the heat equation with an initial function ξ:

$$S(t)\xi(x) = (4\pi t)^{-d/2} \int_{\mathbf{R}^d} e^{-|x-y|^2/4t} \xi(y) dy.$$

Note that in the closed subset $P_T = \{u \in E_T ; v_i \geq 0 (i \in N^*)\}$ of E_T, (1) is reduced to (6). If $p_j < 1$ for some j, let $\{g_{j,n}\}$ be a sequence of globally

Lipschitz continuous functions such that, for any fixed $n > 0$

$$g_{j,n}(r) = \begin{cases} 0, & r \le 0, \\ c_{j,n} r, & 0 < r \le 1/2n, \\ r^{p_j}, & r > 1/2n, \end{cases}$$

where $c_{j,n} = (2n)^{1-p_j}$. Consider now the approximating problems for (1) as in [2]:

$$(7) \quad \begin{cases} (u_{i,n})_t - \Delta u_{i,n} = u_{i+1,n}^{p_i}, & t > 0, x \in \mathbf{R}^d, \text{ if } p_i \ge 1, \\ (u_{j,n})_t - \Delta u_{j,n} = g_{j,n}(u_{j+1,n}), & t > 0, x \in \mathbf{R}^d, \text{ if } p_j < 1, \\ u_{i+1,n}(0) = u_{i+1,0}, & x \in \mathbf{R}^d, \text{ if } p_i \ge 1, \\ u_{j+1,n}(0) = u_{j+1,0} + 1/n, & x \in \mathbf{R}^d, \text{ if } p_j < 1. \end{cases}$$

We put $\tilde{u}_n = (u_{1,n}, u_{2,n}, \dots, u_{N,n})$. Define

$$\begin{aligned} \Psi_n(\tilde{u}_n)(t) &= (S(t)u_{1,n}(0) + \Phi_{1,n}(u_{2,n})(t), S(t)u_{2,n}(0) + \Phi_{2,n}(u_{3,n})(t), \\ &\quad \dots, S(t)u_{N,n}(0) + \Phi_{N,n}(u_{1,n})(t)), \end{aligned}$$

where

$$\begin{aligned} \Phi_{i,n}(u_{i+1,n})(t) &= \int_0^t S(t-s) u_{i+1,n}^{p_i}(s) ds \qquad (p_i \ge 1), \\ \Phi_{j,n}(u_{j+1,n})(t) &= \int_0^t S(t-s) g_{j,n}(u_{j+1,n})(s) ds \quad (p_j < 1). \end{aligned}$$

Then we can easily obtain the following estimates:

$$\|(S(\cdot)u_{1,n}(0), S(\cdot)u_{2,n}(0), ..., S(\cdot)u_{N,n}(0))\|_{E_T} \le C \sum_{i=1}^N \|u_{i,n}(0)\|_\infty,$$

$$\|(\Phi_{1,n}(u_{2,n}), \Phi_{2,n}(u_{3,n}), ..., \Phi_{N,n}(u_{1,n}))\|_{E_T} \le CT \sum_{i=1}^N \|U_{i,n}\|_{E_T}^{p_i},$$

where

$$\begin{pmatrix} U_{1,n} \\ U_{2,n} \\ \vdots \\ U_{N-1,n} \\ U_{N,n} \end{pmatrix} = \begin{pmatrix} 0 & u_{2,n} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & u_{3,n} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & u_{N,n} \\ u_{1,n} & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let $B_R = \{\tilde{u}_n \in E_T; \|\tilde{u}_n\|_{E_T} \le R\}$. If R is large enough and $T > 0$ is small enough, one can easily see from the above inequalities that Ψ_n is a strict contraction from $B_R \cap P_T$ into itself, whence there exists a unique fixed point $\tilde{u}_n \in B_R \cap P_T$ which solves

(8)
$$\begin{cases} u_{i,n}(t) = S(t)u_{i,n}(0) + \displaystyle\int_0^t S(t-s)u_{i+1,n}^{p_i}(s)ds, & i \in N^*, p_i \ge 1, \\ u_{j,n}(t) = S(t)u_{j,n}(0) + \displaystyle\int_0^t S(t-s)g_{j,n}(u_{j+1,n}(s))ds, & j \in N^*, p_j < 1. \end{cases}$$

Thus we obtain a unique nonnegative and bounded solution $\tilde{u}_n(t)$ to (8) in $\mathbf{R}^d \times [0,T)$ for some T. Furthermore, we can show

$$u_{i,n}(t) \le u_{i,m}(t) \qquad \text{if } n \ge m,$$

where we use the argument of [1;Lemma (1.3)]. Therefore, the sequences $\{u_{i,n}(t)\}$ are nonincreasing with respect to n and bounded below. So, we can define $u_i(t) = \lim_{n\to\infty} u_{i,n}(t)$. Then we can conclude that $u_i(t)$ satisfies (6) (see [1]).

To complete the proof of Theorem 2.1, let $u(x,t)$ be the nonnegative and bounded solution of (6) that has been obtained in $[0,T) \times \mathbf{R}^d$ for some $T > 0$. By (6), $u(x,t)$ is continuous in $[0,T) \times \mathbf{R}^d$. Moreover, by considering the difference quotients $(1/h)\{u_i(x_1, x_2, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_d, t) - u_i(x,t)\}$ with $h \to 0$, one easily sees that $\partial u_i(x,t)/\partial x_j$ is locally bounded in $\mathbf{R}^d \times [\tau, T)$ for $j = 1, 2, \ldots, d$ and any τ such that $0 < \tau < T$. Then $u_i^{p_{i+1}} (i \in N^*)$ are locally Hölder continuous functions in space uniformly with respect to time. It then follows from the representation formula (6) that u is a classical solution of (1) in $\mathbf{R}^d \times (0,T)$ (see [5;Chapter 1, Theorem 10]). □

Remark. By the proof of Theorem 2.1, solutions are unique when $p_i = 1$ $(i \in N^*)$. If this assumption is dropped, this result is false in general. For instance, see Theorem 3.

Next, we show an important lemma for the proof of Theorem 1.

Lemma 2.2. *Let $p_1p_2\dots p_N \leq 1$. Then, there exists $m \in N^*$ such that*

$$p_{N+m-k+1,N+m} \leq 1$$

for any $k \in N^*$.

Proof. If $p_i \leq 1$ for any $i \in N^*$, it is obvious. We consider the other cases. We say that $P = p_{k,k+m} = p_kp_{k+1}\dots p_{k+m}$ satisfies the property ($\sharp$), if

$$p_{k+m} \leq 1, p_{k+m-1,k+m} \leq 1, \dots, p_{k+1,k+m} \leq 1, p_{k,k+m} \leq 1$$

hold. We also say that P is a *good* block if $P \leq 1$. We define

$$\begin{cases} i_{1,1} & = & \min\{i; 1 < i \leq N, p_{i-1} \leq 1, p_i > 1\}, \\ i_{1,2} & = & \min\{i; i_{1,1} < i \leq N, p_{i-1} \leq 1, p_i > 1\}, \\ & & \vdots \\ i_{1,k(1)} & = & \min\{i; i_{1,k(1)-1} < i \leq N, p_{i-1} \leq 1, p_i > 1\}, \end{cases}$$

where $k(1)$ is the least number such that $\{i; i_{1,k(1)} < i \leq N, p_{i-1} \leq 1, p_i > 1\}$ becomes empty. We also use the convention $i_{1,k(1)+1} = i_{1,1}$. Then we put

$$\begin{cases} P_1(1) & = & \displaystyle\prod_{l=i_{1,1}}^{i_{1,2}-1} p_l, \\ P_1(2) & = & \displaystyle\prod_{l=i_{1,2}}^{i_{1,3}-1} p_l, \\ \vdots & & \\ P_1(k(1)) & = & \displaystyle\prod_{l=i_{1,k(1)}}^{i_{1,k(1)+1}-1} p_l \\ & & \Big(= \ p_{i_{1,k(1)}}p_{i_{1,k(1)}+1}\dots p_Np_1\dots p_{i_{1,1}-1}\Big). \end{cases}$$

Here we used the convention $p_{N+i} = p_i$ ($i \in \mathbf{Z}$) in the last expression. It is easy to see that if $P_1(m) = p_{i_{1,m}}p_{i_{1,m}+1}\dots p_{i_{1,m+1}-1}$ is a *good* block, then $P_1(m)$ satisfies the property ($\sharp$).

By repeating these procedure, we define

$$\begin{cases} i_{2,1} & = & \min\{i; 1 < i \le k(1), P_1(i-1) \le 1, P_1(i) > 1\}, \\ i_{2,2} & = & \min\{i; i_{2,1} < i \le k(1), P_1(i-1) \le 1, P_1(i) > 1\}, \\ & & \vdots \\ i_{2,k(2)} & = & \min\{i; i_{2,k(2)-1} < i \le k(1), P_1(i-1) \le 1, P_1(i) > 1\}, \end{cases}$$

where $k(2)$ is the least number such that $\{i; i_{2,k(2)} < i \le k(1), P_1(i-1) \le 1, P_1(i) > 1\}$ becomes empty. We also use the convention $i_{2,k(2)+1} = i_{2,1}$. We put

$$\begin{cases} P_2(1) & = \displaystyle\prod_{l=i_{2,1}}^{i_{2,2}-1} P_1(l), \\ P_2(2) & = \displaystyle\prod_{l=i_{2,2}}^{i_{2,3}-1} P_1(l), \\ & \vdots \\ P_2(k(2)) & = \displaystyle\prod_{l=i_{2,k(2)}}^{i_{2,k(2)+1}-1} P_1(l) \\ \Big(= P_1(i_{2,k(2)}) P_1(i_{2,k(2)}+1) \dots P_1(k(1)) P_1(1) \dots P_1(i_{2,1}-1)\Big). \end{cases}$$

Here we used the notation $P_1(k(1)+i) = P_1(i)$ ($i \in \mathbf{Z}$) for the last expression. We also note that if $P_2(l)$ is a *good* block, then $P_2(l)$ satisfies the property ($\sharp$).

Furthermore, by repeating the above procedure inductively and using the assumption $p_{1,N} \le 1$, we arrive at number l such that

$$i_{l,1} \;=\; \min\{i; 1 < i \le k(l+1), P_{l-1}(i-1) \le 1, P_{l-1}(i) > 1\},$$

where $\{i; i_{l,1} < i \le k(l-1), P_{l-1}(i-1) \le 1, P_{l-1}(i) > 1\}$ becomes empty. We put

$$P_l(1) = P_{l-1}(i_{l,1}) P_{l-1}(i_{l,1}+1) \dots P_{l-1}(i_{k(l-1)-1}) P_{l-1}(1) \dots P_{l-1}(i_{l-1}-1).$$

Here, we rewrite $P_l(1)$ in an original form without changing the order of multiplications. Then we obtain

$$P_l(1) = p_{m-N+1} \dots p_{m-1} p_m$$

for some $m \in N^*$. Then $P_l(1)$ satisfies the property ($\sharp$) and here this m is a desired one. $\square$

We also collect the following inequalities which will be frequently used in the proofs of Lemma 2.4 and Theorem 1.

Lemma 2.3.

(I) *For all nonnegative numbers a and b, it holds that*

$$\begin{aligned} (a+b)^p &\le 2^{p-1}(a^p+b^p) \qquad && \text{if } p \ge 1, \\ (a+b)^p &\le (a^p+b^p) && \text{if } p \le 1. \end{aligned}$$

(II) (*Jensen's inequality*) *Let $v = v(x,t)$ be any nonnegative function. Then it holds that, for all $t > 0$,*

$$\begin{aligned} (S(t)v(s))^q &\le S(t)v^q(s), \\ \left(\int_0^t S(t-s)v(s)ds\right)^q &\le t^{q-1}\int_0^t S(t-s)v^q(s)ds \end{aligned}$$

if $q \ge 1$, and

$$\begin{aligned} S(t)v^q(s) &\le (S(t)v(s))^q, \\ \int_0^t S(t-s)v^q(s)ds &\le t^{1-q}\left(\int_0^t S(t-s)v(s)ds\right)^q \end{aligned}$$

if $q \le 1$.

We omit the proof of Lemma 2.3, since it is well-known. We also use the semigroup property $S(t-s)(S(s-r)u(r)) = S(t-r)u(r)$ frequently.

First, we establish the basic estimate which will be used frequently in the iteration process of the proof of Theorem 1.

Lemma 2.4. *If $p_i > 1$, $p_{i,i+1} > 1$, $\ldots$, $p_{i,i+j-1} > 1$, and $p_{i,i+j} \le 1$ then there exist $g_h(t)$ $(i \le h \le i+j)$ such that*

$$u_i(t) \le S(t)u_{i0} + \sum_{h=i}^{i+j-1} g_h(t)S(t)u_{h+1,0}^{p_{i,h}} + g_{i+j}(t)u_{i+j+1}^{p_{i,i+j}}(t).$$

and $g_h(t) = O(t^{\gamma_h})$ with some γ_h as $t \to \infty$.

Remark. By taking $\gamma = \max\{\gamma_h; i \leq h \leq i+j\}$, we may write $g_h(t) = O(t^\gamma)$ as $t \to \infty$ for any $i \leq h \leq i+j$. We will use this convention frequently in the proof of Theorem 1.

Proof. It follows from (6) that

(9)

$$u_i(t) = S(t)u_{i,0} + \int_0^t S(t-s)\Big\{S(s)u_{i+1,0} + \int_0^s S(s-r)u_{i+2}^{p_{i+1}}(r)dr\Big\}^{p_i} ds.$$

From Lemma 2.3, we have

$$\begin{aligned}(10)\ u_i(t) &\leq S(t)u_{i,0} + \int_0^t S(t-s)\Big[2^{p_i-1}S(s)u_{i+1,0}^{p_i} \\ &\qquad +2^{p_i-1}\Big(\int_0^s S(s-r)u_{i+2}^{p_{i+1}}(r)dr\Big)^{p_i}\Big]ds \\ &= S(t)u_{i,0} + 2^{p_i-1}\int_0^t S(t)u_{i+1,0}^{p_i}ds \\ &\qquad +2^{p_i-1}\int_0^t S(t-s)\Big(\int_0^s S(s-r)u_{i+2}^{p_{i+1}}(r)dr\Big)^{p_i}ds.\end{aligned}$$

From (II) of Lemma 2.3, it follows that

(11)

$$\begin{aligned}u_i(t) &\leq S(t)u_{i,0} + 2^{p_i-1}tS(t)u_{i+1,0}^{p_i} \\ &\qquad +2^{p_i-1}\int_0^t S(t-s)s^{p_i-1}\int_0^s S(s-r)u_{i+2}^{p_{i,i+1}}(r)drds \\ &\leq S(t)u_{i,0} + 2^{p_i-1}tS(t)u_{i+1,0}^{p_i} + 2^{p_i-1}t^{p_i}\int_0^t S(t-r)u_{i+2}^{p_{i,i+1}}(r)dr.\end{aligned}$$

Substituting $u_{i+2}(r) = S(r)u_{i+2,0} + \int_0^r S(r-s)u_{i+3}^{p_{i+2}}(s)ds$ into the integral of the last inequality of (11), we obtain

(12)

$$\begin{aligned}u_i(t) &\leq S(t)u_{i,0} + g_i(t)S(t)u_{i+1,0}^{p_i} \\ &\quad + g_i(t)t^{p_i-1}\int_0^t \Big(S(t-r)\Big[S(r)u_{i+2,0} + \int_0^r S(r-s)u_{i+3}^{p_{i+2}}(s)ds\Big]^{p_{i,i+1}}\Big)dr,\end{aligned}$$

where $g_i(t) = 2^{p_i-1}t$. Using Jensen's inequality again for $p_{i,i+1} > 1$, we have

$$(13)\qquad u_i(t) \leq S(t)u_{i,0} + g_i(t)S(t)u_{i+1,0}^{p_i} + g_{i+1}(t)S(t)u_{i+2,0}^{p_{i,i+1}}$$

$$+g_{i+1}t^{p_{i,i+1}-1}(t)\int_0^t S(t-s)u_{i+3}^{p_{i,i+2}}(s)ds,$$

where $g_{i+1}(t) = 2^{p_i+p_{i,i+1}-2}t^{1+p_i}$. Since $p_i > 1$, $p_{i,i+1} > 1$, …,$p_{i,i+j-1} > 1$, we repeat the arguments from (9) to (13) to obtain

(14)

$$\begin{aligned} u_i(t) \;\; &\leq \;\; S(t)u_{i,0} + g_i(t)S(t)u_{i+1,0}^{p_i} + g_{i+1}(t)S(t)u_{i+2}^{p_{i,i+1}} + \dots \\ &\quad + g_{i+j-1}(t)S(t)u_{i+j,0} + g_{i+j-1}t^{p_{i,i+j-1}-1}\int_0^t S(t-s)u_{i+j+1}^{p_{1,i+j}}(s)ds, \end{aligned}$$

where $g_{i+k}(t) = 2^{p_i+p_{i,i+1}+\ldots+p_{i,i+k}-k-1}t^{1+p_i+p_{i,i+1}+\ldots+p_{i,i+k-1}}$ $(j > 1)$. Now we substitute $u_{i+j+1}(s) = S(s)u_{i+j+1,0} + \int_0^s S(s-r)u_{i+j+2}^{p_{i+j+1}}(r)dr$ in the last term of (14):

$$\int_0^t S(t-s)u_{i+j+1}^{p_{1,i+j}}(s)ds.$$

Then we have

$$\begin{aligned} &\int_0^t S(t-s)u_{i+j+1}^{p_{1,i+j}}(s)ds \\ &\quad = \int_0^t S(t-s)\left(S(s)u_{i+j+1,0} + \int_0^s S(s-r)u_{i+j+2}^{p_{i+j+1}}(r)dr\right)^{p_{1,i+j}} ds \\ &\quad \leq t^{1-p_{i,i+j}}\Big\{\int_0^t S(t-s) \\ &\qquad \times\Big(S(s)u_{i+j+1,0} + \int_0^s S(s-r)u_{i+j+2}^{p_{i+j+1}}(r)dr\Big)ds\Big\}^{p_{i,i+j}} \\ &\quad \leq t^{1-p_{i,i+j}}\Big(\int_0^t S(t)u_{i+j+1,0}ds \\ &\qquad + \int_0^t S(t-s)\int_0^s S(s-r)u_{i+j+2}^{p_{i+j+1}}(r)drds\Big)^{p_{i,i+j}} \\ &\quad \leq t\Big(S(t)u_{i+j+1,0} + \int_0^t S(t-r)u_{i+j+2}^{p_{i+j+1}}(r)dr\Big)^{p_{i,i+j}} \\ &\quad = tu_{i+j+1}^{p_{i,i+j}}(t). \end{aligned}$$

Substituting this into (14), we conclude

$$u_i(t) \;\leq\; S(t)u_{i,0} + \sum_{h=i}^{i+j-1} g_h(t)S(t)u_{h+1,0}^{p_{i,h}} + g_{i+j}(t)u_{i+j+1}^{p_{i,i+j}}, \tag{15}$$

where $g_{i+j}(t) = t^{p_{i,i+j-1}} g_{i+j-1}(t)$. Thus Lemma 2.4 is proved. $\square$

Proof of Theorem 1. Without loss of generality, we can assume $m = N-1$ in Lemma 2.2. We put

$$\begin{aligned}
j(1) &= \min\{i; 1 \le i \le N-1, p_{1,i} \le 1\}, \\
j(2) &= \min\left\{i; j(1) < i \le N-1, p_{j(1)+1,i} \le 1\right\}, \\
&\vdots \\
j(h) &= \min\left\{i; j(h-1) < i \le N-1, p_{j(h-1)+1,i} \le 1\right\},
\end{aligned}$$

where h is the least number, such that

$$\left\{i; j(h) < i \le N-1, p_{j(h)+1,i} \le 1\right\}$$

becomes empty. Then we have the following general situation which will be divided into h blocks:

$$\begin{aligned}
\text{1-blocks} \ &: \ p_1 > 1, p_{1,2} > 1, \ldots, p_{1,j(1)-1} > 1, p_{1,j(1)} \le 1, \\
\text{2-blocks} \ &: \ p_{j(1)+1} > 1, p_{j(1)+1,j(1)+2} > 1, \\
&\quad , \ldots, p_{j(1)+1,j(2)-1} > 1, p_{j(1)+1,j(2)} \le 1, \\
&\vdots \\
h\text{-blocks} \ &: \ p_{j(h-1)+1} > 1, p_{j(h-1)+1,j(h-1)+2} > 1, \\
&\quad , \ldots, p_{j(h-1)+1,j(h)-1} > 1, p_{j(h-1)+1,j(h)} \le 1,
\end{aligned}$$

where from $j(h) = N-1$, $p_{j(h-1)+1,j(h)} = p_{j(h-1)+1,N-1} \le 1$. Although some k-block may become degenerate so that just $p_{j(k)} \le 1$, if $j(h+1) = j(h)+1$, we consider the general situation above without loss of generality.

By using Lemma 2.4 for 1-block, there exist $0 < \gamma_1 < \infty$ and $f_i(t)$ $(i = 1, 2, \ldots, j(1)-1)$ and $\bar{f}_{j(1)}(t)$ behaving like $O(t^{\gamma_1})$ at $t \to \infty$ such that

$$u_1 \le S(t)u_{1,0} + \sum_{h=1}^{j(1)-1} f_h(t) S(t) u_{h+1,0}^{p_{1,h}} + f_{j(1)}(t) u_{j(1)+1}^{p_{1,j(1)}}. \tag{16}$$

By using Lemma 2.4 again for 2-block, there exist $0 < \gamma_2 < \infty$ and $\bar{f}_i(t)$ $(j(1)+1 \le i \le j(2))$ behaving like $O(t^{\gamma_2})$ at $t \to \infty$ such that

(17)

$$u_{j(1)+1} \leq S(t)u_{j(1)+1,0} + \sum_{h=j(1)+1}^{j(2)-1} \bar{f}_h(t)S(t)u_{h+1,0}^{p_{j(1)+1,h}} + \bar{f}_{j(2)}(t)u_{j(2)+1}^{p_{j(1)+1,j(2)}}.$$

Combining (16) with (17), and using (I) of Lemma 2.3 for $p_{1,j(1)} \leq 1$,

(18)

$$\begin{aligned} u_1 &\leq S(t)u_{1,0} + \sum_{k=1}^{j(1)-1} f_k(t)S(t)u_{k+1,0}^{p_{1,k}} + \bar{f}_{j(1)}(t)\Big\{S(t)u_{j(1)+1,0} \\ &\quad + \sum_{k=j(1)+1}^{j(2)-1} \bar{f}_k(t)S(t)u_{k+1,0}^{p_{j(1)+1,k}} + \bar{f}_{j(2)}(t)u_{j(2)+1}^{p_{j(1)+1,j(2)}}\Big\}^{p_{1,j(1)}} \\ &\leq S(t)u_{1,0} + \sum_{k=1}^{j(2)-1} f_k(t)U_k(t) + f_{j(2)}(t)u_{j(2)+1}^{p_{1,j(2)}}, \end{aligned}$$

where

$$U_k(t) = \begin{cases} S(t)u_{k+1,0}^{p_{1,k}} & (1 \leq k \leq j(1)-1), \\ [S(t)u_{j(1)+1,0}]^{p_{1,j(1)}} & (k = j(1)), \\ [S(t)u_{k+1,0}^{p_{j(1)+1,k}}]^{p_{1,j(1)}} & (j(1)+1 \leq k \leq j(2)-1), \end{cases}$$

and

$$\begin{aligned} f_{j(1)}(t) &= \bar{f}_{j(1)}, \\ f_k(t) &= \bar{f}_{j(1)}\bar{f}_k^{p_{1,j(1)}} \qquad (j(1)+1 \leq k \leq j(2)). \end{aligned}$$

Iterating this process for m-blocks ($m = 1, 2, \ldots, h$), we can obtain that there exist $0 < \gamma_h < \infty$ and $f_{j(2)+1}, f_{j(2)+2}, \ldots, f_{j(h)}$ behaving like $O(t^{\gamma_h})$ at $t \to \infty$ such that

$$u_1 \leq S(t)u_{1,0} + \sum_{k=1}^{j(h)-1} f_k(t)U_k(t) + f_{j(h)}(t)u_{j(h)+1}^{p_{1,j(h)}}, \tag{19}$$

where

$$U_k(t) = \begin{cases} S(t)u_{k+1,0}^{p_{1,k}} & (1 \le k \le j(1)-1), \\ [S(t)u_{j(1)+1,0}]^{p_{1,j(1)}} & (k = j(1)), \\ [S(t)u_{k+1,0}^{p_{j(1)+1,k}}]^{p_{1,j(1)}} & (j(1)+1 \le k \le j(2)-1), \\ [S(t)u_{j(2)+1,0}]^{p_{1,j(2)}} & (k = j(2)), \\ [S(t)u_{k+1,0}^{p_{j(2)+1,k}}]^{p_{1,j(2)}} & (j(2)+1 \le k \le j(3)-1), \\ [S(t)u_{j(3)+1,0}]^{p_{1,j(3)}} & (k = j(3)), \\ \qquad\vdots & \\ [S(t)u_{j(h-1)+1,0}]^{p_{1,j(h-1)}} & (k = j(h-1)), \\ [S(t)u_{k+1,0}^{p_{j(h-1)+1,k}}]^{p_{1,j(h-1)}} & (j(h-1)+1 \le k \le j(h)-1). \end{cases}$$

Put $f(t) = 1 + \sum_{k=1}^{j(h)} f_k(t)$. Since $j(h) = N-1$, if follows from (19) that

$$u_1 \le f(t)\Big\{S(t)u_{1,0} + \sum_{k=1}^{N-2} U_k(t) + u_N^{p_{1,N-1}}\Big\}. \tag{20}$$

Note that $a_1 + a_2 + \ldots + a_l \le l(a_1^\alpha + a_2^\alpha + \ldots + a_l^\alpha)^{1/\alpha}$ holds for $\alpha \ge 1$ and $a_j \ge 0$. Applying this inequality as $\alpha = 1/p_{1,N-1}$ (note $p_{1,N-1} \le 1$ by our assumption), we obtain from (20) that

(21)

$$u_1 \le Nf(t)\Big\{(S(t)u_{1,0})^{1/p_{1,N-1}} + \sum_{k=1}^{N-2} U_k^{1/p_{1,N-1}}(t) + u_N(t)\Big\}^{p_{1,N-1}}.$$

Since all p_{N-1}, $p_{N-2,N-1}$, $\ldots$, $p_{1,N-1}$ are not more than 1 by our assumption, Jensen's inequality yields

$$U_k^{1/p_{1,N-1}}(t) \le S(t)u_{k+1,0}^{1/p_{k+1,N-1}} \tag{22}$$

for $1 \le k \le N-2$. Therefore substituting (22) in (21), we have

$$u_1 \;\le\; Nf(t)\Big\{\sum_{k=1}^{N-1} S(t)u_{k,0}^{1/p_{k,N-1}} + u_N(t)\Big\}^{p_{1,N-1}}. \tag{23}$$

Substituting this in the N-th equality of (1), we have

(24)

$$(u_N)_t - \Delta u_N \;\le\; (Nf(t))^{p_N}\Big\{\sum_{k=1}^{N-1} S(t)u_{k,0}^{1/p_{k,N-1}} + u_N(t)\Big\}^{p_{1,N}}.$$

Put

$$\begin{cases} w(x,t) = \displaystyle\sum_{k=1}^{N-1} S(t) u_{k,0}^{1/p_{k,N-1}} + u_N(t), \\ \\ w(x,0) = \displaystyle\sum_{k=1}^{N-1} u_{k,0}^{1/p_{k,N-1}} + u_{N,0}. \end{cases}$$

Then we find $w_t - \Delta w = u_{Nt} - \Delta u_{Nt}$. Since $p_{1,N} \leq 1$, we have $w^{p_{1,N}} \leq 1 + w$, and hence it follows that

$$w_t - \Delta w \leq (Nf(t))^{p_N} w^{p_{1,N}} \leq (Nf(t))^{p_N}(1+w).$$

Let v be the solution of

$$\begin{cases} v_t = (Nf(t))^{p_N}(1+v), \\ v(0) = \sup_{x \in \mathbf{R}^d} w(x,0) = v_0. \end{cases}$$

Then we have $w(t) \leq v(t)$ by the standard comparison theorem and

$$v(t) = (1+v_0)\exp\left(\int_0^t (Nf(s))^{p_N} ds\right) - 1.$$

Since v is global, w is also global. By the definition of w this means u_N is global. From (6) we see that all u_{N-1}, $u_{N-2}, \ldots, u_1$ are global. $\square$

3 Proof of Theorem 2 and 3

Lemma 3.1. *Let $u_0 \not\equiv 0$, and let $u(x,t)$ be a solution of* (1). *Then for any $\tau > 0$ there exist constants $c > 0$* and *$\alpha > 0$ such that*

$$u_i(x,\tau) \geq c\exp(-\alpha|x|^2) \qquad (i \in N^*). \tag{25}$$

Proof. We employ the same argument as in [2;Theorem 2.4] and [3;Lemma 1]. Assume for instance that $u_{1,0} \not\equiv 0$. By shifting the origin if necessary, we may assume that there exists $R > 0$ such that $\nu = \inf\{u_{1,0}(\xi); |\xi| \leq R\} > 0$. Since $u_1(t) \geq S(t)u_{1,0}$, it holds that

$$u_1(t) \geq \nu \exp(-|x|^2/2t)(4\pi t)^{-d/2} \int_{|y|\leq R} \exp(-|y|^2/2t)dy.$$

Defining $\bar{u}_1(t) = u_1(t+\tau_0)$ for some $\tau_0 > 0$, we obtain

$$\bar{u}_1(0) = u_1(\tau_0) > c\exp(-\alpha|x|^2)$$

with

$$(26)\qquad \alpha = \frac{1}{2\tau_0}, \qquad c = \nu(4\pi\tau)^{-d/2}\int_{|y|<R}\exp(-\frac{|y|^2}{2\tau})dy.$$

To obtain the corresponding result for $u_N(t)$, we note that, if $p_N \geq 1$, Jensen's inequality yields

$$(27)\qquad \begin{aligned} u_N(t) &\geq \int_0^t S(t-s)(S(s)u_{1,0})^{p_N}ds \\ &\geq \int_0^t (S(t-s)S(s)u_{1,0})^{p_N}ds = \int_0^t (S(t)u_{1,0})^{p_N}ds \\ &\geq t(S(t)u_{1,0})^{p_N} \end{aligned}$$

and, if $p_N < 1$,

$$(28)\qquad \begin{aligned} u_N(t) &\geq \int_0^t S(t-s)(S(s)u_{1,0})^{p_N}ds \\ &\geq \int_0^t S(t-s)S(s)u_{1,0}^{p_N}ds \\ &\geq tS(t)u_{1,0}^{p_N}. \end{aligned}$$

From (27) and (28), the estimate for u_N in (25) holds with a different choice of α and c from the one previously made in (26). By repeating this procedure, we obtain the desired results for $u_{N-1}, u_{N-2}, \ldots, u_2$. This completes the proof. □

Lemma 3.2. *Let $u(x,t)$ be any nontrivial solution of* (1) *with* $0 < p_1p_2 \ldots p_N < 1$ and $u_0 \not\equiv 0$. *Then*

$$(29)\qquad u_i(x,t) \geq c_i t^{\alpha_i}, \qquad (i \in N^*),$$

where c_i and α_i *are positive constants given by* (3).

Proof. We take the same strategy as in [3;Lemma 2]. Assume first that $u_{1,0}(x) \geq c\exp(-\alpha|x|^2)$ for some $c > 0$ and $\alpha > 0$. For simplicity, we consider the case $0 < p_i < 1$ ($i \in N^*$). We will mention the proof for the general case $0 < p_{1,N} < 1$ at the end of the proof. Since

$$(30) \qquad S(t)\exp(-\alpha|x|^2) = (1+4\alpha t)^{-d/2}\exp\left(\frac{-\alpha|x|^2}{1+4\alpha t}\right),$$

it follows that

$$(31) \qquad u_1(x,t) \geq S(t)u_{1,0} \geq c(1+4\alpha t)^{-d/2}\exp\left(\frac{-\alpha|x|^2}{1+4\alpha t}\right).$$

Moreover, by (6)

$$(32) \qquad u_N(x,t) \geq \int_0^t S(t-s)(u_1(x,s))^{p_N}ds.$$

Substituting (31) into (32) and using (30), we have

(33)

$$\begin{aligned}
u_N(x,t) \;&\geq\; c^{p_N}\int_0^t (1+4\alpha s)^{-d(p_N-1)/2}(1+4\alpha s+4\alpha p_N(t-s))^{-d/2} \\
&\qquad \times \exp\left(-\frac{\alpha p_N|x|^2}{1+4\alpha s+4\alpha p_N(t-s)}\right)ds \\
&\geq\; c^{p_N}(1+4\alpha t)^{-d/2}\exp\left(-\frac{\alpha p_N|x|^2}{1+4\alpha p_N t}\right)t.
\end{aligned}$$

Here we used $p_N \leq 1$. We substitute this inequality into (6) and use (30) to find

$$\begin{aligned}
u_{N-1}(x,t) \;&\geq\; c^{p_{N-1,N}}\int_0^t (1+4\alpha s)^{-dp_{N-1}/2} \\
&\qquad\qquad \times(1+4\alpha p_N s+4\alpha p_{N-1,N}(t-s))^{-d/2} \\
&\qquad \times(1+4\alpha p_N s)^{d/2} \\
&\qquad\qquad \times\exp\left(-\frac{\alpha p_{N-1,N}|x|^2}{1+4\alpha p_N s+4\alpha p_{N-1,N}(t-s)}\right)s^{p_{N-1}}ds \\
&\geq\; c^{p_{N-1,N}}(1+4\alpha t)^{-dp_{N-1}/2}(1+4\alpha p_N t)^{-d/2} \\
&\qquad \times\exp\left(-\frac{\alpha p_{N-1,N}|x|^2}{1+4\alpha p_{N-1,N}t}\right)\frac{t^{p_{N-1}}}{p_{N-1}+1}.
\end{aligned}$$

Now we claim that

$$(34) \quad u_{1-k}(x,t) \geq c^{p_{N-k+1,N}}g_k(t)\exp\left(-\frac{\alpha p_{N-k+1,N}|x|^2}{1+4\alpha p_{N-k+1,N}t}\right)\frac{t^{P_k}}{A_k}$$

for any $k \geq 1$, where P_k and A_k are positive constants and $g_k(t)$ is a positive monotone decreasing function of t which will be determined inductively. Here we use the convention $p_{N+j} = p_j$, $u_{N+j} = u_j$ ($j \in \mathbf{Z}$). The inductive relations are obtained as follows. From (34) and

$$u_{-k}(x,t) \geq \int_0^t S(t-s)(u_{1-k}(x,s))^{p_{-k}} ds, \tag{35}$$

we get

$$\begin{aligned} u_{-k}(x,t) &\geq c^{p_{N-k,N}} g_k^{p_{-k}}(t)(1+4\alpha p_{N-k+1,N}t)^{-d/2} \\ &\quad \times \exp\left(-\frac{\alpha p_{N-k,N}|x|^2}{1+4\alpha p_{N-k,N}t}\right) \frac{t^{P_k p_{-k}+1}}{A_k^{p_{-k}}(P_k p_{-k}+1)} \\ &= c^{p_{N-k,N}} g_{k+1}(t) \exp\left(-\frac{\alpha p_{N-k,N}|x|^2}{1+4\alpha p_{N-k,N}t}\right) \frac{t^{P_{k+1}}}{A_{k+1}}. \end{aligned} \tag{36}$$

Then from (33) and (36), we get the following relation:

$$\left\{ \begin{array}{lcll} P_1 &=& 1, & P_{k+1} = P_k p_{-k} + 1 \\ A_1 &=& 1, & A_{k+1} = A_k^{p_{-k}} P_{k+1} \\ g_1(t) &=& (1+4\alpha t)^{-d/2}, & g_{k+1}(t) = g_k^{p_{-k}}(t)(1+4\alpha p_{N-k+1,N}t)^{-d/2} \end{array} \right. \tag{37}$$

for $k \geq 1$. Now, by using (37), we easily see that

$$\left\{ \begin{array}{lcl} P_k &=& 1 + p_{-k+1} + p_{-k+1,-k+2} + \ldots + p_{-k+1,-1}, \\ A_k &=& \prod_{l=2}^{k} P_l^{p_{N-k+1,N-l}}, \\ g_k(t) &=& \prod_{l=1}^{k} (1+4\alpha p_{N-k+l+1,N}t)^{-dp_{-k+1,-k+l-1}/2} \end{array} \right. \tag{38}$$

for $k \geq 2$. From (34) with $k = jN$, noting $p_{N-jN+1,N} = p_{1,jN} = p_{1,N}^j$, we obtain

$$\begin{aligned} u_1(x,t) &= u_{1-jN}(x,t) \\ &\geq c^{(p_{1,N})^j} \bar{g}_j(t) \exp\left(-\frac{\alpha (p_{1,N})^j |x|^2}{1+4\alpha p_{1,N}^j t}\right) \frac{t^{\bar{P}_j}}{\bar{A}_j}, \end{aligned} \tag{39}$$

where $\bar{P}_j = P_{jN}$, $\bar{A}_j = A_{jN}$ and $\bar{g}_j(t) = g_{jN}(t)$. We rewrite $\bar{P}_j$, $\bar{A}_j$ and $\bar{g}_j(t)$ to see their asymptotic behaviors as $j \to \infty$. First it is easy to see that

(40)

$$\begin{aligned}
\bar{P}_j &= P_{jN} = 1 + p_1 + p_{1,2} + \ldots + p_{1,jN-1} \\
&= (1 + p_1 + p_{1,2} + \ldots + p_{1,N-1})(1 + p_{1,N} + p_{1,N}^2 + \ldots + p_{1,N}^{j-1}) \\
&= \frac{(1 + p_1 + p_{1,2} + \ldots + p_{1,N-1})(1 - (p_{1,N})^j)}{1 - p_{1,N}}.
\end{aligned}$$

Furthermore, we need the following expression P_{jN+k} $(1 \le k \le N)$ by using $\bar{P}_j$:

(41)

$$\begin{aligned}
P_{jN+k} &= 1 + p_{-(jN+k-1)} P_{jN+k-1} = 1 + p_{1-k} P_{jN+k-1} \\
&= 1 + p_{1-k}(1 + p_{-(jN+k-2)} P_{jN+k-2}) \\
&= 1 + p_{1-k} + p_{1-k} p_{2-k} P_{jN+k-2} \\
&= \ldots \\
&= 1 + p_{1-k} + p_{1-k,2-k} + \ldots + p_{1-k,-1} + p_{1-k,0} P_{jN} \\
&= 1 + p_{N+1-k} + p_{N+1-k,N+2-k} + \ldots + p_{N+1-k,N-1} + p_{N+1-k,N} \bar{P}_j.
\end{aligned}$$

Substituting (40) into (41), we obtain

$$\begin{aligned}
(42) \quad P_{jN+k} &= 1 + p_{N-k+1} + p_{N-k+1,N-k+2} + \ldots + p_{N-k+1,N-1} \\
&\quad + p_{N-k+1,N}(1 + p_1 + p_{1,2} + \ldots + p_{1,N-1}) \frac{1 - p_{1,N}^j}{1 - p_{1,N}}
\end{aligned}$$

for $j \ge 1$ and $1 \le k \le N$. We also note that this formula is also true even for $j = 0$. Now, we express $\bar{A}_j$ as follows:

$$\begin{aligned}
(43)\ \bar{A}_j &= \prod_{l=1}^{jN} P_l^{p_{-jN+1,-l}} \\
&= \prod_{k=1}^{N} P_k^{p_{-jN+1,-k}} \times P_{N+k}^{p_{-jN+1,-(N+k)}} \times \ldots \times P_{(j-1)N+k}^{p_{-jN+1,-((j-1)N+k)}} \\
&\equiv \prod_{k=1}^{N} A_{k,j} \qquad (j \ge 2),
\end{aligned}$$

where

$$A_{k,j} = \prod_{l=0}^{j-1} P_{lN+k}^{p_{-jN+1,-(lN+k)}}.$$

Note that for $0 \le l \le j-1$

$$(44)\qquad \begin{aligned} p_{-jN+1,-(lN+k)} &= p_{jN-jN+1,jN-(lN+k)} \\ &= p_{1,(j-l)N-k} = p_{1,(j-l-1)N+(N-k)} \\ &= (p_{1,N})^{j-l-1} p_{1,N-k}. \end{aligned}$$

Then using (42) and (44), we have

$$(45)\qquad \begin{aligned} A_{k,j} &= \prod_{l=0}^{j-1} P_{lN+k}^{(p_{1,N}^{j-l-1} p_{1,N-k})} \\ &= \prod_{l=0}^{j-1} \Big[\beta_k + \sigma_k(1-(p_{1,N})^l)\Big]^{(p_{1,N}^{j-l-1} p_{1,N-k})}, \end{aligned}$$

where we used the notation

$$(46)\qquad \begin{cases} \beta_k = 1 + p_{N+1-k} + p_{N+1-k,N+2-k} + \ldots + p_{N+1-k,N-1}, \\ \sigma_k = p_{N+1-k,N}(1 + p_1 + p_{1,2} + \ldots + p_{1,N-1}) \dfrac{1}{1-p_{1,N}} \end{cases}$$

for $1 \le k \le N$. Noting $p_{i,j}p_{1,N} = p_{i,j}p_{j+1,N+j} = p_{i,N+j}$, we see

(47)

$$\begin{aligned} \beta_k + \sigma_k = &\Big\{(1-p_{1,N})(1 + p_{N-k+1} + p_{N-k+1,N-k+2} + \ldots + p_{N-k+1,N-1}) \\ &+ p_{N-k+1,N}(1 + p_1 + p_{1,2} + \ldots + p_{1,N-1})\Big\} \frac{1}{1-p_{1,N}} \\ = &\Big\{(1 + p_{N-k+1} + p_{N-k+1,N-k+2} + \ldots + p_{N-k+1,2N-1}) \\ &- (1 + p_{N-k+1} + p_{N-k+1,N-k+2} + \ldots + p_{N-k+1,2N-1})p_{1,N}\Big\} \frac{1}{1-p_{1,N}} \\ = &\Big\{(1 + p_{N-k+1} + p_{N-k+1,N-k+2} + \ldots + p_{N-k+1,2N-1}) \\ &- (p_{N-k+1,2N-k} + p_{N-k+1,2N-k+1} + \ldots + p_{N-k+1,2N-1})\Big\} \frac{1}{1-p_{1,N}} \\ = &\frac{1 + p_{N-k+1} + \ldots + p_{N-k+1,2N-k-1}}{1-p_{1,N}}. \end{aligned}$$

It follows from (45) and (47) that

$$(48)\qquad A_{k,j} \le \prod_{l=0}^{j-1} \Big(\frac{1 + p_{N-k+1} + \ldots + p_{N+1-k,2N-k-1}}{1-p_{1,N}}\Big)^{p_{1,N-k}(p_{1,N})^{j-l-1}}$$

$$
\begin{aligned}
&= \alpha_{N+1-k}^{p_{1,N-k}\sum_{l=0}^{j-1}(p_{1,N})^{j-l-1}} \\
&= \alpha_{N+1-k}^{p_{1,N-k}(1-(p_{1,N})^{j})/(1-p_{1,N})} \\
&\leq \alpha_{N+1-k}^{p_{1,N-k}/(1-p_{1,N})}
\end{aligned}
$$

for $1 \leq k \leq N$ and any $j \geq 1$.

Next, we show

$$
\bar{g}_{j+1}(t) = \prod_{l=1}^{N}(1 + 4\alpha p_{l+1,N}(p_{1,N})^j t)^{-dp_{1,l-1}/2}\bar{g}_j^{p_{1,N}}(t) \tag{49}
$$

for $j \geq 0$. Here $\bar{g}_0 \equiv 1$.

Proof of (49): By (37), we have

$$
\begin{aligned}
\bar{g}_{j+1}(t) &= g_{jN+N}(t) = g_{jN+N-1}^{p_{-(jN+N-1)}}(1 + 4\alpha p_{N-(jN+N-1)+1,N}t)^{-d/2} \\
&= g_{jN+N-1}^{p_1}(1 + 4\alpha p_{2,N}(p_{1,N})^j t)^{-d/2} \\
&= \left[g_{jN+N-2}^{p_{-(jN+N-2)}}(1 + 4\alpha p_{N-(jN+N-2)+1,N}t)^{-d/2}\right]^{p_1} \\
&\quad \times(1 + 4\alpha p_{2,N}(p_{1,N})^j t)^{-d/2} \\
&= (1 + 4\alpha p_{2,N}(p_{1,N})^j t)^{-d/2}(1 + 4\alpha p_{N-(jN+N-2)+1,N}t)^{-dp_1/2}g_{jN+N-2}^{p_1 p_2}.
\end{aligned}
$$

Here we used $p_{-(jN+N-l)} = p_l$ and

$$
\begin{aligned}
p_{N-(jN+N-l)+1,N} &= p_{l-jN+1,N} = p_{l+1,N+jN} \\
&= p_{l+1,N} \cdot (p_{1,N})^j.
\end{aligned} \tag{50}
$$

Repeating this procedure by using (50), we obtain

$$
\begin{aligned}
\bar{g}_{j+1}(t) &= (1 + 4\alpha p_{2,N}(p_{1,N})^j t)^{-d/2} \\
&\quad \times(1 + 4\alpha p_{3,N}(p_{1,N})^j t)^{-dp_1/2} \\
&\quad \times(1 + 4\alpha p_{4,N}(p_{1,N})^j t)^{-dp_{1,2}/2} \\
&\quad \times \dots \\
&\quad \times(1 + 4\alpha p_{N+1,N}(p_{1,N})^j t)^{-dp_{1,N-1}/2} \\
&\quad \times g_{jN}^{p_{1,N}}(t).
\end{aligned}
$$

This implies the formula (49). $\square$

Using the formula (49) inductively, we have

$$
\begin{aligned}
(51)\quad \bar{g}_j(t) &= \prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})^{j-1}t)^{-dp_{1,l-1}/2}\bar{g}_{j-1}^{p_{1,N}}(t) \\
&= \prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})^{j-1}t)^{-dp_{1,l-1}/2} \\
&\quad \times\prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})^{j-2}t)^{-dp_{1,l-1}p_{1,N}/2}\bar{g}_{j-2}^{(p_{1,N})^2}(t) \\
&= \prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})^{j-1}t)^{-dp_{1,l-1}/2} \\
&\quad \times\prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})^{j-2}t)^{-dp_{1,l-1}p_{1,N}/2} \\
&\quad \times\prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})^{j-3}t)^{-dp_{1,l-1}(p_{1,N})^2/2} \\
&\quad \times\dots \\
&\quad \times\prod_{l=1}^{N}(1+4\alpha p_{l+1,N}(p_{1,N})t)^{-dp_{1,l-1}(p_{1,N})^{j-2}/2} \\
&\quad \times\bar{g}_1^{(p_{1,N})^{j-1}}(t) \\
&= \prod_{l=1}^{N}\left[\prod_{k=0}^{j-1}(1+4\alpha p_{l+1,N}(p_{1,N})^{k}t)^{-dp_{1,l-1}(p_{1,N})^{j-k-1}/2}\right] \\
&\equiv \prod_{l=1}^{N} g_{lj}(t) \qquad (j\geq 1).
\end{aligned}
$$

Now we claim that

$$
(52)\qquad \underline{\lim}_{j\to\infty} g_{lj}(t)\geq 1 \qquad (1\leq l\leq N)).
$$

Proof of (52): First, we note that

$$
\begin{aligned}
(53)\ 4\alpha p_{l+1,N}(p_{1,N})^{j-1}jt &= \sum_{k=0}^{j-1}(p_{1,N})^{j-1-k}4\alpha p_{l+1,N}(p_{1,N})^{k}t \\
&\geq \sum_{k=0}^{j-1}(p_{1,N})^{j-k-1}\log(1+4\alpha p_{l+1,N}t(p_{1,N})^{k})
\end{aligned}
$$

$$= \log \left[\prod_{k=0}^{j-1} (1 + 4\alpha p_{l+1,N}(p_{1,N})^k t)^{(p_{1,N})^{j-1-k}} \right].$$

Here we used $x \geq \log(1+x)$ for $x \geq 0$. Hence, it follows that

$$\begin{aligned} g_{l,j}(t) &= \prod_{k=0}^{j-1} (1 + 4\alpha p_{l+1,N}(p_{1,N})^k t)^{-dp_{1,l-1}(p_{1,N})^{j-k-1}/2} \\ &\geq \exp\left(-dp_{1,l-1}/2 \times 4\alpha t p_{l+1,N}(p_{1,N})^{j-1} j\right) \\ &\to 1 \qquad (j \to +\infty), \end{aligned}$$

since $p_{1,N} < 1$. Therefore we obtain the desired estimate. $\square$

Now we obtain

$$\underline{\lim}_{j\to\infty} \bar{g}_j(t) \geq 1. \tag{54}$$

So, letting $j \to \infty$ in (39), we can conclude by (40) and (54) that

$$\begin{aligned} u_1(x,t) &\geq \frac{1}{(\alpha_1 \alpha_2^{p_1} \alpha_3^{p_{1,2}} \dots \alpha_N^{p_{1,N-1}})^{1/(1-p_{1,N})}} t^{(1+p_1+\ldots+p_{1,N-1})/(1-p_{1,N})} \\ &= c_1 t^{\alpha_1}. \end{aligned}$$

As to the general case, we take arbitrary $\varepsilon > 0$, and set $u_{i,\varepsilon}(t) \equiv u_i(t+\varepsilon)$. One then has

$$u_{i,\varepsilon}(t) = S(t)u_{i,\varepsilon}(0) + \int_0^t S(t-s) u_{i+1,\varepsilon}^{p_i}(s)ds,$$

where by Lemma 3.1, $u_{1,\varepsilon}(0) \geq c\exp(-\alpha|x|^2)$ with some c and α. Therefore, the preceding argument shows $u_{1,\varepsilon}(t) \geq c_1 t^{\alpha_1}$, and accordingly

$$u_1(t) = u_1(\varepsilon + (t-\varepsilon)) \geq c_1(t-\varepsilon)^{\alpha_1},$$

whence follows the result, because $\varepsilon > 0$ is arbitrary.

This completes the proof for $i = 1$. The estimate for $i \geq 2$ can be obtained in the same way.

Finally, we remark on the proof for the general case $0 < p_{1,N} < 1$. Even for this general case, we can slightly modify the above computations to get similar estimates as before. Precisely, we can obtain the estimate (39) with a slightly different $\bar{g}_j(t)$. Here $\bar{g}_j(t)$ satisfies (49), where only the number $p_{l+1,N}$ should be replaced by a certain number. Thus, we have the estimate (52) even for the general case. So, we can conclude the same result under the general assumption $p_{1,N} < 1$. $\square$

We prepare the following lemma which will be used in the proofs of Theorems 2 and 3.

Lemma 3.3. *Let $\{v_i(t)\}$ $(i \in N^*)$ be the nonnegative continuous function satisfying*

$$v_i(t) \le \int_0^t v_{i+1}^{p_i}(s)ds \qquad (i \in N^*). \tag{55}$$

Assume $p_{1,N} < 1$. Then

$$v_i(t) \le c_i t^{\alpha_i}$$

with c_i and α_i defined by (3).

Proof. From (55)

$$v_i(t) \le \int_0^t \left(\int_0^{s_1} \dots \left(\int_0^{s_{N-1}} v_i^{p_{N+i-1}}(s_N)ds_N \right)^{p_{N+i-2}} \dots ds_2 \right)^{p_i} ds_1. \tag{56}$$

We put

$$V_i(t) = \sup_{s\in[0,t]} \|v_i(s)\|_\infty.$$

Then

$$\begin{aligned} V_i(t) &\le \int_0^t \left(\int_0^{s_1} \dots \left(\int_0^{s_{N-1}} V_i^{p_{N+i-1}}(t)ds_N \right)^{p_{N+i-2}} \dots ds_2 \right)^{p_i} ds_1 \\ &\le V_i^{p_{1,N}}(t) \int_0^t \left(\int_0^{s_1} \dots \left(\int_0^{s_{N-1}} ds_N \right)^{p_{N+i-2}} \dots ds_2 \right)^{p_i} ds_1 \\ &\le V_i^{p_{1,N}}(t) t^{1+p_i+p_{i,i+1}+\dots p_{i,i+N-2}} \Big\{ (p_{N+i-2}+1)^{p_{i,i+N-3}} \\ &\quad \times (p_{N+i-3,N+i-2} + p_{N+i-2} + 1)^{p_{i,i+N-4}} \\ &\quad \times \dots \times (p_{i,i+N-2} + \dots + p_{i,i+1} + p_i + 1) \Big\}^{-1}, \end{aligned}$$

thus

$$\begin{aligned} V_i(t) &\le t^{\alpha_i} \Big\{ (p_{N+i-2}+1)^{p_{i,i+N-3}} \\ &\quad \times (p_{N+i-3,N+i-2} + p_{N+i-2} + 1)^{p_{i,i+N-4}} \\ &\quad \times \dots \times (p_{i,i+N-2} + \dots + p_{i,i+1} + p_i + 1) \Big\}^{-1/(1-p_{1,N})} \\ &\equiv c_i^+ t^{\alpha_i}. \end{aligned}$$

Thus we obtain

$$v_i(t) \le c_i^+ t^{\alpha_i}. \tag{57}$$

To obtain the stronger estimate, put $\theta_i = \sup_{t\in[0,\infty]} v_i(t)/t^{\alpha_i} \le c_i^+$ so that $v_i(t) \le \theta_i t^{\alpha_i}$. Substituting $v_i(t) \le \theta_i t^{\alpha_i}$ into (56), we have

$$\begin{aligned} v_i(t) &\le \int_0^t \left(\int_0^{s_1} \cdots \left(\int_0^{s_{N-1}} (\theta_i s_N^{\alpha_i})^{p_{N+i-1}} ds_N \right)^{p_{N+i-2}} \cdots ds_2 \right)^{p_i} ds_1 \\ &\le \theta_i^{p_{1,N}} \frac{t^{\alpha_i}}{\alpha_i \alpha_{i+1}^{p_i} \alpha_{i+2}^{p_{i,i+1}} \cdots \alpha_{i+N-1}^{p_{i,i+N-2}}}. \end{aligned}$$

Therefore by the definition of θ_i, we obtain

$$\theta_i \le \theta_i^{p_{1,N}} \frac{1}{\alpha_i \alpha_{i+1}^{p_i} \alpha_{i+2}^{p_{i,i+1}} \cdots \alpha_{i+N-1}^{p_{i,i+N-2}}}$$

and hence

$$\theta_i \le \left(\frac{1}{\alpha_i \alpha_{i+1}^{p_i} \alpha_{i+2}^{p_{i,i+1}} \cdots \alpha_{i+N-1}^{p_{i,i+N-2}}} \right)^{1/(1-p_{1,N})} = c_i.$$

Thus $v_i \le c_i t^{\alpha_i}$. $\square$

Proof of Theorem 2. We take the same strategy as in [3;Lemma 3]. Suppose that for some $u_0 \not\equiv 0$ there exist two different solution $(u_1(t), u_2(t), \ldots, u_N(t))$ and $(\bar{u}_1(t), \bar{u}_2(t), \ldots, \bar{u}_N(t))$ defined in some strip $S_T = (0,T) \times \mathbf{R}^d$. Then it follows from (6), Lemma 3.2 and the mean value theorem that

$$\begin{aligned} (u_1(t) - \bar{u}_1(t))_+ &\le \int_0^t S(t-s)(u_2^{p_1}(s) - \bar{u}_2^{p_1}(s))_+ ds \\ &\le p_1 c_2^{p_1-1} \int_0^t S(t-s)(u_2(s) - \bar{u}_2(s))_+ s^{(p_1-1)\alpha_2} ds. \end{aligned}$$

In a similar way, we have

$$\begin{aligned} (u_2(t) - \bar{u}_2(t))_+ &\le p_2 c_3^{p_2-1} \int_0^t S(t-s)(u_3(s) - \bar{u}_3(s))_+ s^{(p_2-1)\alpha_3} ds, \\ &\vdots \\ (u_N(t) - \bar{u}_N(t))_+ &\le p_N c_1^{p_N-1} \int_0^t S(t-s)(u_1(s) - \bar{u}_1(s))_+ s^{(p_N-1)\alpha_1} ds. \end{aligned}$$

From these, it follows that

(58)

$$
\begin{aligned}
&\|(u_1-\bar{u}_1)_+(t)\|_\infty \\
&\quad\le\; p_{1,N}c_1^{p_N-1}c_2^{p_1-1}\dots c_N^{p_{N-1}-1}\int_0^t s_1^{(p_1-1)\alpha_2}\Big[\int_0^{s_1} s_2^{(p_2-1)\alpha_3}\dots \\
&\qquad\Big(\int_0^{s_{N-1}} s_N^{(p_N-1)\alpha_1}\|(u_1-\bar{u}_1)_+(s_N)\|_\infty ds_N\Big)\dots ds_2\Big]ds_1 \\
&\quad=\; p_{1,N}\alpha_1\alpha_2\dots\alpha_N\int_0^t s_1^{(p_1-1)\alpha_2}\Big[\int_0^{s_1} s_2^{(p_2-1)\alpha_3}\dots \\
&\qquad\Big(\int_0^{s_{N-1}} s_N^{(p_N-1)\alpha_1}\|(u_1-\bar{u}_1)_+(s_N)\|_\infty ds_N\Big)\dots ds_2\Big]ds_1.
\end{aligned}
$$

Here we used the relation $c_1^{p_N-1}c_2^{p_1-1}\dots c_N^{p_{N-1}-1}=\alpha_1\alpha_2\dots\alpha_N$ which can be seen from the definition of c_j and α_j.

We next show that the integrand above is indeed locally integrable. To this end, we first notice that

$$\|(u_i-\bar{u}_i)_+(t)\|_\infty\le\int_0^t\|(u_{i+1}-\bar{u}_{i+1})_+(s)\|_\infty^{p_i}ds\qquad(i\in N^*)$$

holds, since all p_1, p_2, …, p_N are less than one and $(a^p-b^p)_+\le(a-b)_+^p$ holds for any nonnegative constants a and b if $0<p<1$. Then Lemma 3.3 yields

$$\|(u_1-\bar{u}_1)_+(t)\|_\infty\;\le\;c_1t^{\alpha_1}$$

with α_1 as in (3). We also note the relation $p_i\alpha_{i+1}+1=\alpha_i$ $(i\in N^*)$. This implies that the right-hand side in (58) is convergent. Moreover, substituting this in (58), from $p_i\alpha_{i+1}+1=\alpha_i$ $(i\in N^*)$, we find

$$\|(u_1-\bar{u}_1)_+(t)\|_\infty\le p_{1,N}c_1t^{\alpha_1}.$$

We may now use this to obtain a new bound for $\|(u_1-\bar{u}_1)_+(t)\|_\infty$ via (58). Iterating this procedure k times, we obtain

$$\|(u_1-\bar{u}_1)_+(t)\|_\infty\le p_{1,N}^kc_1t^{\alpha_1}.$$

Now by letting $k\to\infty$, it follows from $p_{1,N}<1$ that $u_1\equiv\bar{u}_1$, whence $u_2\equiv\bar{u}_2$, …, $u_N\equiv\bar{u}_N$ holds. □

Proof of Theorem 3. Although we follow the same argument as in [3;Lemma 4], we give the proof for the sake of completeness. Let u be a nontrivial solution of (1) satisfying the assumption in Theorem 3. Then by (6), for $i \in N^*$

$$\|u_i(t)\|_\infty \leq \int_0^t \|u_{i+1}(s)\|_\infty^{p_i} ds$$

holds and Lemma 3.3 yields

$$\|u_i(t)\|_\infty \leq c_i t^{\alpha_i} \tag{59}$$

with positive constants c_i, α_i defined by (3). By the hypothesis, there exist $i \in N^*$, $t > 0$ and $x \in \mathbf{R}^d$ such that $u_i(x,t) > 0$. Without loss of generality, assume that $u_1(x,t) > 0$, and define τ by

$$\tau = \inf\{t > 0 : u_1(x,t) > 0\}.$$

From Lemma3.1, $u_i(x,t) > 0$ for any $i \in N^*$, $x \in \mathbf{R}^d$ and $t > \tau$. Now take $\bar{t} > \tau$ and set

$$\bar{u}_i(x,t) = u_i(x, t + \bar{t}).$$

Then $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$ solves (1) and $\bar{u}_i > 0$. Therefore, by Lemma 3.2,

$$u_i(x, t + \bar{t}) \geq c_i t^{\alpha_i}$$

for any $t \geq 0$. This implies that

$$u_i(x,t) \geq c_i (t - \tau)_+^{\alpha_i} \tag{60}$$

for $x \in \mathbf{R}^d$, $t \geq 0$. Now choose any $\underline{t} < \tau$ and define

$$\underline{u}_i(x,t) = u_i(x, t + \underline{t}).$$

By our choice of τ, $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_N)$ solves (1), with $\underline{u}_i(0) = 0$. Therefore, by (58)

$$u_i(x, t + \underline{t}) \leq c_i t^{\alpha_i}$$

for any $t \geq 0$ and $x \in \mathbf{R}^N$. Letting $\underline{\mathrm{t}} \to \tau$, we obtain

$$u_i(x,t) \leq c_1 (t - \tau)_+^{\alpha_i} \tag{61}$$

for $x \in \mathbf{R}^d$, $t \geq 0$, and the conclusion follows from (60) and (61). □

4 Proof of Theorem 4

Lemma 4.1. *Assume* $0 < p_{1,N} < 1$ *and consider the problem*

$$\begin{cases} v_{it} = \Delta v_i + v_{i+1}^{p_i}, & x \in \mathbf{R}^d, t > 0, i \in N^* \\ v_i(x,0) = v_{i,0}, & x \in \mathbf{R}^d, i \in N^* \end{cases} \tag{62}$$

with a positive constant $v_{i,0}$ *satisfying*

$$c_i^{1/\alpha_i} \alpha_i v_{i,0}^{p_i \alpha_{i+1}/\alpha_i} = v_{i+1,0}^{p_i} \qquad (i \in N^*). \tag{63}$$

If v_i *is defined by*

$$v_i(x,t) = c_i \Big\{ t + \Big(\frac{v_{i,0}}{c_i} \Big)^{1/\alpha_i} \Big\}^{\alpha_i} \tag{64}$$

for each $i \in N^*$*, then* $v = (v_1, v_2, \ldots, v_N)$ *satisfyes* (62)*, where* c_i *and* α_i *are positive constants given by* (3).

Proof. It is easy to see that (64) is a solution of (62). □

Proof of Theorem 4. It is obvious for the case $u_0 \equiv 0$ from Theorem 3. We show Theorem 4 for the case $u_0 \not\equiv 0$. From Lemma 3.2

$$u_i(x,t) \geq c_i t^{\alpha_i}. \tag{65}$$

Now choose $v_{i,0}$ which satisfies (63) and $v_{i,0} \geq 2\|u_{i,0}\|_\infty$. Define v_i by (64). Then we claim the following estimate:

$$v_i(x,t) > u_i(x,t) \qquad ((x,t) \in \mathbf{R}^d \times [0,\infty), i \in N^*).$$

If not, $v_i(x,t) \leq u_i(x,t)$ holds at some point $(x,t) \in \mathbf{R}^d \times [0,\infty)$ for some i. We put

$$t_i = \inf\{t > 0; v_i(x,t) \leq u_i(x,t) \text{ for some } x \in \mathbf{R}^d\} \qquad (i \in N^*).$$

From (6),

$$v_i(t) - u_i(t) = S(t)(v_{i,0} - u_{i,0}) + \int_0^t S(t-s)(v_{i+1}^{p_i}(s) - u_{i+1}^{p_i}(s))ds,$$

and from $v_{i,0}(x) > u_{i,0}(x)$,

$$S(t)(v_{i,0} - u_{i,0}) > 0.$$

Therefore, we have

$$0 = v_i(x,t_i) - u_i(x,t_i) > \int_0^{t_i} \Big[S(t_i - s)(v_{i+1}^{p_i}(s) - u_{i+1}^{p_i}(s))\Big](x)ds$$

for some $x \in \mathbf{R}^d$. Then we obtain

$$t_i > t_{i+1}.$$

Inductively, we have

$$t_i > t_{i+1} > \ldots > t_{N+i-2} > t_{N+i-1} > t_{N+i} = t_i$$

and we get the contradiction. Therefore it follows from Lemma 4.1 that

$$u_i(x,t) < c_i\Big\{t + \Big(\frac{v_{i,0}}{c_i}\Big)^{1/\alpha_i}\Big\}^{\alpha_i}. \tag{66}$$

From (65) and (66), we obtain

$$c_i t^{\alpha_i} \le u_i(x,t) < c_i\Big\{t + \Big(\frac{v_{i,0}}{c_i}\Big)^{1/\alpha_i}\Big\}^{\alpha_i}. \tag{67}$$

Multiplying (67) by $t^{-\alpha_i}$ and letting $t \to \infty$, we have

$$\lim_{t\to\infty} t^{-\alpha_i} u_i(x,t) = c_i \qquad (i \in N^*).$$

uniformly in $\mathbf{R}^d$. □

Acknowledgement. The author expresses his sincere gratitude to Professor K. Mochizuki and Professor K. Kurata for their constant encouragement and valuable advices, and to the referee for his valuable comments for improving the paper.

References

[1] J. Aguirre and M. Escobedo, *A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$. Asymptotic behavior of solutions.,* Ann. Fac. Sci. Toulouse Math. **8** (1986-87), 175-203.

[2] M. Escobedo and M. A. Herrero, *Boundedness and blow up for a semilinear reaction-diffusion system,* J. Differential Equations **89** (1991), 176-202.

[3] M. Escobedo and M. A. Herrero, *A uniqueness result for a semilinear reaction-diffusion system,* Proc. Amer. Math. Soc. **112** (1991), 175-185.

[4] M. Fila and P. Quittner, *The blow-up rate for a semilinear parabolic system,* J. Math. Anal. Appl. **238** (1999), 468-476.

[5] A. Friedman, *Partial differential equations of parabolic type*, Robert E. Krieger publishing company, Malabar, Florida, 1983.

[6] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t - \Delta u = u^{1+\alpha}$,* J. Fac. Sci. Univ. of Tokyo, Sect. I **13** (1960), 65-79.

[7] K. Mochizuki, *Blow-up,life span and large time behavior of solution of weakly coupled system of reaction-diffusion equation,* Advance in Nonlinear Partial Differential Equation and Stochastics, 175-197, World Scientific Publishing, River Edge, NJ, 1998.

[8] N. Umeda, *Blow-up and large time behavior of solutions of a weakly coupled system of reaction-diffusion equations*, Tsukuba J. Math., **27** (2003), 31-46.