Title: OBSERVATION ON THE WEIGHT ENUMERATORS FROM CLASSICAL INVARIANT THEORY

Author(s): Oura, Manabu

Citation: Hokkaido University Preprint Series in Mathematics, 622, 1-14

Issue Date: 2003

DOI: 10.14943/83776

Doc URL: http://hdl.handle.net/2115/69430

Type: bulletin (article)
OBSERVATION ON THE WEIGHT ENUMERATORS FROM CLASSICAL INVARIANT THEORY

By Manabu Oura

The purpose of this paper is to collect computations related to the weight enumerators and to present some relationships among invariant rings. The latter is done by combining two maps, the Broué–Enguehard map and Igusa’s $\rho$ homomorphism. For the completeness of the story, some formulae are given which are not necessarily used in the present manuscript. Sections 1 and 2 contain no new result.

1. Classical Invariant Theory. In this section we recall classical invariant theory. For the detail we refer to [17]. We consider a homogeneous polynomial

$$\sum_{i=0}^{n} \binom{n}{i} u_i x^{n-i} y^i$$

of degree $n$ in 2 variables $x$, $y$. The group $SL(2, \mathbb{C})$ operates on the variable space and, if we require that the above form is invariant, the same group operates on the coefficient space. In this way, we get an irreducible representation of $SL(2, \mathbb{C})$ of degree $n+1$. We consider the graded ring of polynomials in the $u_0, u_1, \ldots, u_n$ with coefficients in $\mathbb{C}$ and operate $SL(2, \mathbb{C})$ on this graded ring using its action on its homogeneous part of degree one defined by the above representation. Then, the invariant subring $S(2,n)$ is a graded, integrally closed, integral domain over $\mathbb{C}$. In the present note we deal with $S(2,4)$ and $S(2,6)$. The structure theorems of those rings are established in the 19th century and we shall describe them. The invariant ring $S(2,4)$ is generated by $P, Q$, which are algebraically independent. The explicit forms are

$$P = u_0 u_4 - 4u_1 u_3 + 3u_2^2,$$

$$Q = \det \begin{pmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{pmatrix}$$

and the dimension formula of $S(2,4)$ is

$$\frac{1}{(1-t^2)(1-t^3)} = 1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + 3t^{12} + \cdots$$

in which the coefficient of $t^k$ denotes the dimension of degree $k$-part of $S(2,4)$. The invariant ring $S(2,6)$ is generated by $J_2, J_4, J_6, J_{10}, J_{15}$. We give the definitions of $J_2, \ldots, J_{15}$ in the appendix, taken from [17]. Among them $J_2, J_4, J_6, J_{10}$ are algebraically independent. The ring $\mathbb{C}[J_2, J_4, J_6, J_{10}]$ contains $J_2^2$ but no $J_{15}$. We also give the explicit formula for $J_2^2$ in the appendix. The dimension formula of $S(2,6)$ is given by

$$\frac{1 + t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})} = 1 + t^2 + 2t^4 + 3t^6 + 4t^8 + 6t^{10} + 8t^{12} + 10t^{14} + t^{15} + 13t^{16} + t^{17} + 16t^{18} + 2t^{19} + 20t^{20} + 3t^{21} + 24t^{22} + 4t^{23} + 29t^{24} + 6t^{25} + 34t^{26} + 8t^{27} + 40t^{28} + 10t^{29} + 47t^{30} + \cdots$$

1This work is supported in part by KAKENHI(No.14740081).
2. Weight Enumerators and Siegel Modular Forms. In this section we recall coding theory and Siegel modular forms. For the details we refer to [1], [14] for coding theory and to [3] for Siegel modular forms. Let $F_2$ be the field of two elements. A linear code (a code for short) of length $n$ is a subspace of $F_2^n$. The vector space $F_2^n$ equips with the inner product $x \cdot y = \sum x_i y_i$. We define the weight $wt(v)$ of a vector $v \in F_2^n$ by the number of the non-zero coordinates of $v$. We shall define special classes of codes. If a code $C$ coincides with its dual code $C^\perp = \{ x \in F_2^n | (x, y) = 0, \forall y \in C \}$, it is called self-dual. We observe that the dimension of the self-dual code is a half of its length. If the weight of any element in $C$ is divisible by 4, it is called doubly-even. In this manuscript we will focus on the self-dual doubly-even codes. We shall next define a homogeneous polynomial of the code which is on the title of this paper. The weight enumerator $W_C^{(g)}$ of the code $C$ in genus $g$ is defined by

$$W_C^{(g)} = \sum_{x \in F_2^g} \prod_{i=1}^{g} x_i^{n_i(v_1, \ldots, v_g)},$$

where $n_i(v_1, \ldots, v_g)$ denotes the number of $i$ such that $a = (v_1, \ldots, v_g)$. If we need the ordering of the elements of $F_2^n$, we fix $F_2^g = \{0 \cdots 00, \, 0 \cdots 01, \, 0 \cdots 10, \ldots, \, 1 \cdots 1\}$. We sometimes use the symbols $x, y, z, \ldots$ instead of the $x_i$’s for simplicity. The weight enumerator in genus 1 is interpreted as

$$W_C^{(1)} = \sum_{v \in C} x^{n-wt(v)} y^{wt(v)},$$

where $n$ denotes the length of the code $C$. In this case the weight enumerator of a self-dual doubly-even code is a symmetric polynomial in the variables $x, y$. The examples are

$$W_{e_8}^{(1)} = x^8 + 14x^4y^4 + y^8,$$

$$W_{g_24}^{(1)} = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24},$$

where $e_8$ denotes the extended Hamming code and $g_24$ the extended Golay code. We omit the definitions of $e_8, \, g_24$ as well as $d_n^+$ appearing below and refer to the references cited above. In the case when $g = 2$ the weight enumerator of a self-dual doubly-even code is also symmetric in the variables $x, y, z, w$. We have

$$W_{e_8}^{(2)} = (8) + 14(4, 4) + 168(2, 2, 2, 2),$$

$$W_{g_24}^{(2)} = (24) + 759(16, 8) + 2576(12, 12) + 212520(12, 4, 4, 4) + 340032(10, 6, 6, 2) + 22770(8, 8, 8) + 1275120(8, 8, 4, 4) + 4080384(6, 6, 6, 6),$$

where $(8) = x^8 + y^8 + z^8 + w^8$, $(12, 4, 4, 4) = x^{12}y^4z^4w^4 + x^4y^{12}z^4w^4 + x^4y^4z^{12}w^4 + x^4y^4z^4w^{12}$, etc. For an arbitrary positive integer $n$, $n \equiv 0 \pmod{8}$, we have

$$W_{d_n^+}^{(2)} = \sum_{\beta, \gamma \in F_2^g} \left( \sum_{\alpha \in F_2^g} (-1)^{\alpha \beta x_{\alpha + \gamma} x_{\alpha}} \right)^{n/2}.$$
We shall describe the cases for modular forms, we introduce the theta-constants. Theta-constants are defined by \( \theta_m(\tau) = \sum_{n} e^{2\pi i n^2 \tau} \), where \( \tau \) is a complex number with \( \Im(\tau) > 0 \). The theta-constants have a rich algebraic and geometric structure.

The action of \( G \) on \( \mathrm{H} \) and we observe that \( \ker(\Phi) = \mathbb{Z}/4\mathbb{Z} \star 2 \) for every \( M \). The group which directly concerns the weight enumerators is \( G \). Indeed the weight enumerator of any self-dual doubly-even code is invariant under the action of \( G \).

We recall that that the ring \( \mathbb{Z} \) is defined by \( \mathbb{Z}/4\mathbb{Z} \star 2 \) for matrices \( \mathbb{Z} \) and \( 2 \). The group which directly concerns the weight enumerators is \( G \). Indeed the weight enumerator of any self-dual doubly-even code is invariant under the action of \( G \).

Igusa’s homomorphism is, under some condition, one from the ring \( A(\Gamma_g) \) of Siegel modular forms to the ring \( S(2, 2g + 2) \) of projective invariants of a binary form of degree \( 2g + 2 \) (see [7]). We recall that that the ring \( A(\Gamma_g) \) is the graded ring generated by holomorphic functions \( \psi \) on the Siegel upper-half space \( \mathfrak{H} \) in genus \( g \) satisfying the functional equation

\[
\psi(M \cdot \tau) = \det(c\tau + d)^k \cdot \psi(\tau)
\]

for every \( M \) in \( \Gamma_g = Sp(2g, \mathbb{Z}) \) (plus a condition at infinity for \( g = 1 \)). In order to construct Siegel modular forms, we introduce the theta-constants. Theta-constants \( \theta_{m,m'}(\tau) \) are defined by

\[
\theta_{m,m'}(\tau) = \sum_{\nu \in \mathbb{Z}^g} \exp 2\pi i \nu^T \left( \frac{1}{2} \nu \nu' + \frac{1}{2} \nu \nu'' \right),
\]

in which \( \nu \) and \( \nu'' \) are the column vectors in \( \mathbb{R}^g \). If we put \( f_m(\tau) = \theta_{m,0}(2\tau) \). The Broué-Enguehard map \( Th \) is defined by \( x_a \mapsto f_a \). A modular form is called a cusp form if it is in the kernel of Siegel’s \( \Phi \)-operator which maps a modular form of genus \( g \) to a modular form of genus \( g - 1 \).

The structures of the invariant rings and of Siegel modular forms in small genera are known. We shall describe the cases \( g = 1, 2 \). First suppose that \( g = 1 \). The groups \( H_1, G_1 \) are finite unitary reflection groups of order 96, 192, respectively (No.8, No.9 in the list of [18]). We have

\[
C[x, y]^{H_1} = C[W_{e_8}^{(1)}, h_{12}^{(1)}], \quad C[x, y]^{G_1} = C[W_{e_8}^{(1)}, W_{24}^{(1)}],
\]

where

\[
h_{12}^{(1)} = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}.
\]

The dimension formulae of these invariant rings are given by

\[
\frac{1}{(1 - t^8)(1 - t^{12})}, \quad \frac{1}{(1 - t^8)(1 - t^{24})}.
\]

The map \( Th \) induces the isomorphisms

\[
C[x, y]^{H_1} \xrightarrow{\sim} A(\Gamma_1) \quad \text{and} \quad C[x, y]^{G_1} \xrightarrow{\sim} A(\Gamma_1)^{(4)}.
\]

Here we remark that \( A(\Gamma_1) = A(\Gamma_1)^{(2)} \). The invariant ring \( C[x, y]^{G_1} \) is a subring of \( C[x, y]^{H_1} \) and we observe that

\[
W_{24}^{(1)} = 11 \cdot 2^{-1} 3^{-2} (W_{e_8}^{(1)})^3 + 7 \cdot 2^{-1} 3^{-2} (h_{12}^{(1)})^2.
\]

\footnote{If \( S \) is a graded ring composed of homogeneous parts \( S_k \) with \( k \) running over non-negative integers and if \( d \) is a positive integer, then \( S^{(d)} = \oplus_{k \geq d} S_k \).}
The isomorphisms above are given by

\[ Th(W_{e_8}^{(1)}) = \phi_4(\omega), \]
\[ Th(h_{12}^{(1)}) = \phi_6(\omega), \]
\[ Th(W_{g_{24}}^{(1)}) = 11 \cdot 2^{-1}3^{-2}(\phi_4(\omega))^3 + 7 \cdot 2^{-1}3^{-2}(\phi_6(\omega))^2, \]

where \( \phi_k(\omega) \) denotes the normalized Eisenstein series of weight \( k \): \( \phi_k(\omega) = 1 + \cdots \).

We shall next discuss the case when \( g = 2 \). The group \( H_2 \) is a finite unitary reflection group of order 46080, No.31 in the list of [18]. The invariant ring of \( H_2 \) is generated by the four elements \( W_{e_8}^{(2)}, W_{g_{24}}^{(2)} \),

\[ h_{12}^{(2)} = (12) - 33(8, 4) + 330(4, 4, 4) + 792(6, 2, 2, 2), \]
\[ F_{20} = (20) - 19(16, 4) - 336(14, 2, 2, 2) - 494(12, 8) + 716(12, 4, 4) \]
\[ + 1038(8, 8, 4) + 7632(10, 6, 2, 2) + 108684(6, 6, 6, 2) + 129012(8, 4, 4, 4). \]

The dimension formula of this ring is

\[
\frac{1}{(1 - t^8)(1 - t^{12})(1 - t^{20})(1 - t^{24})} = 1 + t^8 + t^{12} + t^{16} + 2t^{20} + 3t^{24} + 2t^{28} + 4t^{32} + 4t^{36} + 5t^{40} + \cdots.
\]

The group \( G_2 \) contains \( H_2 \) by index 2 and is not a finite unitary reflection group. The invariant ring is generated by \( W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}}^{(2)}, W_{d_{32}}^{(2)} \) and \( W^{(2)} \). The four elements \( W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}}^{(2)}, W_{d_{32}}^{(2)} \) are algebraically independent and the square of \( W_{d_{32}}^{(2)} \) is written by the polynomial in \( W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}}^{(2)}, W_{d_{32}}^{(2)} \), as follows:

\[
(W_{d_{32}}^{(2)})^2 = -113 \cdot 32621 \cdot 3^{-45} \cdot 7^{-2} \cdot 41^{-1}(W_{e_8}^{(2)})^8
- 2^8 \cdot 60289 \cdot 3^{-45} \cdot 7^{-2} \cdot 11^{-1} \cdot 41^{-1}(W_{e_8}^{(2)})^5 W_{g_{24}}^{(2)}
+ 2^4 \cdot 821477 \cdot 3^{-45} \cdot 7^{-2} \cdot 11^{-1} \cdot 41^{-1}(W_{e_8}^{(2)})^5 W_{d_{24}}^{(2)}
+ 2 \cdot 751 \cdot 3^{-2} \cdot 7^{-1} \cdot 41^{-1}(W_{e_8}^{(2)})^4 W_{d_{32}}^{(2)}
- 2^9 \cdot 11^2 \cdot 3^{-35} \cdot 7^{-1} \cdot 41^{-1}(W_{e_8}^{(2)})^3 W_{d_{40}}^{(2)}
+ 2^4 \cdot 14 \cdot 62 \cdot 3^{-47} \cdot 7^{-2} \cdot 11^{-1} \cdot 41^{-1}(W_{e_8}^{(2)})^2 (W_{e_8}^{(2)})^2
+ 2^{11} \cdot 73 \cdot 79 \cdot 3^{-47} \cdot 7^{-1} \cdot 11^{-1} \cdot 41^{-1}(W_{e_8}^{(2)})^2 W_{g_{24}}^{(2)} W_{d_{24}}^{(2)}
- 2^6 \cdot 10 \cdot 7 \cdot 199 \cdot 3^{-47} \cdot 7^{-2} \cdot 11^{-1} \cdot 41^{-1}(W_{e_8}^{(2)}) W_{d_{24}}^{(2)}
- 2^8 \cdot 389 \cdot 3^{-27} \cdot 7^{-1} \cdot 11^{-1} \cdot 41^{-1} W_{e_8}^{(2)} W_{g_{24}}^{(2)} W_{d_{32}}^{(2)}
+ 2^4 \cdot 5 \cdot 197 \cdot 3^{-2} \cdot 11^{-1} \cdot 41^{-1} W_{e_8}^{(2)} W_{d_{24}}^{(2)} W_{d_{32}}^{(2)}
+ 2^1 \cdot 23 \cdot 9 \cdot 15 \cdot 2 \cdot 7 \cdot 5 \cdot 3^{-15} \cdot 7^{-1} \cdot 11^{-1} \cdot 41^{-1} W_{e_8}^{(2)} W_{g_{24}}^{(2)} W_{d_{40}}^{(2)}
+ 2^9 \cdot 3^{-15} \cdot 41^{-1} W_{e_8}^{(2)} W_{d_{24}}^{(2)} W_{d_{40}}^{(2)}.
\]

The dimension formula of this invariant ring \( \mathbb{C}[x, y, z, w]^{G_2} \) is

\[
\frac{1}{(1 - t^8)(1 - t^{24})^2(1 - t^{40})} = 1 + t^8 + 2t^{24} + 4t^{32} + 5t^{40} + 8t^{48} + 10t^{56} + 12t^{64} + \cdots.
\]
The elements $W_{d_{24}}^{(2)}, W_{d_{32}}^{(2)}, W_{d_{40}}^{(2)}$ can be written by the generators of $\mathbb{C}[x, y, z, w]G$ as follows:

\[
W_{d_{24}}^{(2)} = 11^23^{-2}7^{-1}(W_{e_{8}}^{(2)})_3^4 + 2 \cdot 3^7(h_{12}^{(2)})_7^2 - 237^{-1}W_{g_{24}}^{(2)},
\]
\[
W_{d_{32}}^{(2)} = 43 \cdot 53 \cdot 3^{-4}7^{-1}(W_{e_{8}}^{(2)})_4^4 + 2^3 \cdot 7 \cdot 53^5 \cdot 7^{-3}11^{-1}W_{e_{8}}^{(2)}(h_{12}^{(2)})^2
- 2^63^2 \cdot 2^5 \cdot 7^{-1}W_{g_{24}}^{(2)}W_{e_{8}}^{(2)} + 2^63^2h_{12}^{(2)}F_{20},
\]
\[
W_{d_{40}}^{(2)} = 3 \cdot 19 \cdot 7^{-1}(W_{e_{8}}^{(2)})_5^5 + 2 \cdot 7 \cdot 53^7 \cdot 3^{-7}11^{-1}(W_{e_{8}}^{(2)})^2(h_{12}^{(2)})^2
- 2^35^2 \cdot 19 \cdot 7^{-1}W_{e_{8}}^{(2)}W_{g_{24}}^{(2)} + 2^65^23^{-7}W_{e_{8}}^{(2)}h_{12}^{(2)}F_{20} + 2^25 \cdot 41 \cdot 3^{-7}F_{20}^2.
\]

We give a comment on the paper [9]. In that paper, Maschke determined the invariant ring of some finite group $G$. $G$ is a subgroup of $\text{SL}(4, \mathbb{C})$ and has an order 46080 which is the same as our $H_2$. $G$ is a subgroup of our $G_2$, which is of an order $2 \cdot 46080 = 92160$. $H_2$ is generated by three elements

\[
\left(\frac{1+i}{2}\right)^2 \left((-1)^{a-b}\right)_{a, b \in F_2^2}, \text{ diag}(1, 1, \sqrt{-1}, \sqrt{-1}), \text{ diag}(1, 1, 1, -1),
\]

and $G_2$ by $H_2$ and $\frac{1+i}{\sqrt{2}}$, while $G$ is generated by

\[
\left(\frac{1+i}{2}\right)^2 \left((-1)^{a-b}\right)_{a, b \in F_2^2}, \frac{1+i}{\sqrt{2}} \cdot \text{ diag}(1, 1, \sqrt{-1}, \sqrt{-1}), \frac{1+i}{\sqrt{2}} \cdot \text{ diag}(1, 1, 1, -1).
\]

The dimension formula of $\mathbb{C}[x, y, z, w]^G$ is given by

\[
\frac{1 + t^{32} + t^{60} + t^{92}}{(1 - t^8)(1 - t^{24})(1 - t^{40})}.
\]

From the dimension formulae, for example, we can read off the differences among the invariant rings of the said groups.

We continue our discussion on our case. We shall recall that $A(\Gamma_2)$ is generated over $\mathbb{C}$ by five elements and they are\(^3\)

\[
2^2 \cdot \psi_4 = \sum (\theta_m)^8, \\
2^2 \cdot \psi_6 = \sum_{\text{syzygous}} \pm (\theta_m \theta_{m_2} \theta_{m_3})^4, \\
-2^{14} \cdot \chi_{10} = \prod (\theta_m)^2, \\
2^{17} \cdot \chi_{12} = \sum (\theta_m \theta_{m_2} \cdots \theta_{m_6})^4, \\
2^{39}5^2 \sqrt{-1} \cdot \chi_{35} = \left(\prod \theta_m \left(\sum_{\text{syzygous}} \pm (\theta_m \theta_{m_2} \theta_{m_3})^{20}\right)\right).
\]

In the second symmetrization, the monomial $(\theta_m \theta_{m_2} \theta_{m_3})^4$ with $i_{m_1} = (0, 0, 0, 0), i_{m_2} = (0, 0, 0, 1), i_{m_3} = (0, 0, 1, 0)$ has +1 as its coefficient. In the definition of $\chi_{12}$, the summation is extended over fifteen complements of syzygous quadruples. In the definition of $\chi_{35}$, the symmetrization of $\pm (\theta_m \theta_{m_2} \theta_{m_3})^{20}$ is taken by the stabilizer of $\prod \theta_m$ in $Sp(2, \mathbb{Z})$ modulo

\(^3\chi_{35}\) is not used in Section 3.
the stabilizer of $(\theta_{m_1}\theta_{m_2}\theta_{m_3})^{20}$ with $^4m_1 = (0, 0, 0, 0), ^4m_2 = (0, 0, 0, 1), ^4m_3 = (0, 1, 0, 0)$. The Broué-Enguehard map gives rise the following:

$$Th(W_{cs}^{(2)}) = \psi_4,$$
$$Th(h_{12}^{(2)}) = \psi_6,$$
$$Th(F_{20}) = \psi_4\psi_6 + 21^{12}3^4\chi_{10},$$
$$Th(W_{g_{24}}^{(2)}) = 11 \cdot 2^{-1}3^{-2}\psi_4^3 + 7 \cdot 2^{-1}3^{-2}\psi_6^2 - 210^43^27 \cdot 11\chi_{12}.$$

These can be obtained by comparing the Fourier coefficients (cf. [15], [13], [12]). There have been extensive studies on Fourier coefficients of Siegel modular forms, however, in our case we do not need a deep theory of Fourier coefficients. Since there is a misprint in the definition of $F_4$ in [8], we reproduce the formulae which are useful for our computations of Fourier coefficients. In the case when $g = 1$, we shall use $\omega$ instead of $\tau$. If we put

$$F_0(r) = \sum_{p=1}^{\infty} r^2, \quad F_1(r) = \sum_{p=1}^{\infty} r^{(p-1/2)^2},$$

in which $r = \exp \pi\sqrt{-1}\omega$, then we have

$$\theta_{00}(\omega) = 1 + 2F_0(r), \quad \theta_{01}(\omega) = 1 + 2F_0(-r), \quad \theta_{10}(\omega) = 2F_1(r).$$

In the case when $g = 2$ if we put

$$F_0(r_1, r_2) = F_0(r_1) + F_0(r_2) + \sum_{p_1, p_2=1}^{\infty} A_{p_1, p_2} r_1^{p_1} r_2^{p_2},$$
$$F_1(r_1, r_2) = F_1(r_2) + \sum_{p_1, p_2=1}^{\infty} B_{p_1, p_2} r_1^{p_1} r_2^{(p_2-1/2)^2},$$
$$F_2(r_1, r_2) = F_1(r_2, r_1),$$
$$F_3(r_1, r_2) = \sum_{p_1, p_2=1}^{\infty} C_{p_1, p_2} r_1^{(p_1-1/2)^2} r_2^{(p_2-1/2)^2},$$
$$F_4(r_1, r_2) = \sum_{p_1, p_2=1}^{\infty} D_{p_1, p_2} r_1^{(p_1-1/2)^2} r_2^{(p_2-1/2)^2},$$

in which $r_1 = \exp \pi\sqrt{-1}\tau_1, r_2 = \exp \pi\sqrt{-1}\tau_2, q_{12} = \exp 2\pi\sqrt{-1}\tau_{12}$, and

$$A_{p_1, p_2} = q_{12}^{p_1 p_2} + q_{12}^{-p_1 p_2},$$
$$B_{p_1, p_2} = q_{12}^{(p_2-1/2)} + q_{12}^{-p_1(p_2-1/2)},$$
$$C_{p_1, p_2} = q_{12}^{-(p_1-1/2)(p_2-1/2)} + q_{12}^{-(p_1-1/2)(p_2-1/2)},$$
$$D_{p_1, p_2} = (-1)^{p_1 + p_2 - 1} q_{12}^{-(p_1-1/2)(p_2-1/2)} + (-1)^{p_1 - p_2} q_{12}^{-(p_1-1/2)(p_2-1/2)},$$

$$\theta_{00}(\omega) = 1 + 2F_0(r), \quad \theta_{01}(\omega) = 1 + 2F_0(-r), \quad \theta_{10}(\omega) = 2F_1(r).$$
then we will have

\[
\begin{align*}
\theta_{0000}(\tau) &= 1 + 2F_0(r_1, r_2), & \theta_{0001}(\tau) &= 1 + 2F_0(r_1, -r_2), \\
\theta_{0010}(\tau) &= 1 + 2F_0(-r_1, r_2), & \theta_{0011}(\tau) &= 1 + 2F_0(-r_1, -r_2), \\
\theta_{0100}(\tau) &= 2F_1(r_1, r_2), & \theta_{0110}(\tau) &= 2F_1(-r_1, r_2), \\
\theta_{1000}(\tau) &= 2F_2(r_1, r_2), & \theta_{1001}(\tau) &= 2F_2(r_1, -r_2), \\
\theta_{1100}(\tau) &= 2F_3(r_1, r_2), & \theta_{1111}(\tau) &= 2F_4(r_1, r_2).
\end{align*}
\]

Cusp forms can be written by Eisenstein series as follows:

\[
\begin{align*}
\chi_{10} &= -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 13^{-1} (\psi_3 \psi_6 - \psi_{10}), \\
\chi_{12} &= 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} (3^2 \cdot 7^2 \cdot \psi_4^3 + 2 \cdot 5^3 \psi_6^2 - 691 \psi_{12}).
\end{align*}
\]

We note that there is a misprint in the formula of \(\chi_{10}\) at p.102 in [15].

3. The Broué-Enguehard map and Igusa’s homomorphism. Before proceeding to Igusa’s homomorphism studied in [7], we go back to the invariant rings \(S(2,4), S(2,6)\). In addition to the generators of them given in Section 1, we give the different generators in irrational forms. If we decompose a binary form into linear factors as

\[
\sum_{i=0}^{n} \binom{n}{i} u_i x^{n-i} y^i = u_0 \prod_{i=1}^{n} (x - \xi_i y),
\]

then we have

\[-\binom{n}{1} \frac{u_1}{u_0} = \sum_{i=1}^{n} \xi_i, \quad \binom{n}{2} \frac{u_2}{u_0} = \sum_{i<j} \xi_i \xi_j, \ldots, \quad (-1)^n \binom{n}{n} \frac{u_n}{u_0} = \prod_{i=1}^{n} \xi_i.\]

Preparing this, we consider each case separately\(^4\). We put \(P_{2g+2}(x) = u_0(x - \xi_1)(x - \xi_2) \cdots (x - \xi_{2g+2})\). Suppose that \(g = 1\). In [7], Igusa takes \(I_2(P_4(x)), I_3(P_4(x))\) as the generators of \(S(2,4)\), given as follows:

\[
\begin{align*}
I_2(P_4(x)) &= u_0^2 \sum_{\text{three}} (12)^2(34)^2 \\
&= u_0^2 \left\{(12)^2(34)^2 + (13)^2(24)^2 + (14)^2(23)^2 \right\}, \\
I_3(P_4(x)) &= u_0^3 \sum_{\text{six}} (12)^2(34)^2(13)(24) \\
&= u_0^3 \left\{(12)^2(34)^2 \{(13)(24) + (14)(23)\} + (13)^2(24)^2 \{(12)(34) - (14)(23)\} + (14)^2(23)^2 \{- (12)(34) - (13)(24)\} \right\}.
\end{align*}
\]

Here \((ij)\) is an abridged notation for \(\xi_i - \xi_j\). We already gave the generators of \(S(2,4)\) in Section 1 and these two sets of the generators are related each other by

\[
I_2(P_4(x)) = 2^3 3P, \quad I_3(P_4(x)) = 2^4 3^3 Q.
\]

\(^4\)We note that the case when \(g = 1\) is roughly sketched in [11].
Igusa’s homomorphism $\rho$ gives the isomorphism $\rho : A(\Gamma_1) \xrightarrow{\cong} S(2, 4)$, where

$$\rho(\psi_4(\tau)) = 2^{-1}I_2(P_4(x)),$$
$$\rho(\psi_6(\tau)) = 2^{-1}I_3(P_4(x)).$$

Combining two maps $Th$ and $\rho$ to denote $\tilde{\rho}$, we have the isomorphism $C[x, y]^{H_1} \xrightarrow{\cong} S(2, 4)$ given by

$$\tilde{\rho}(W_{ea}^{(1)}) = 2^{-1}I_2(P_4(x)),$$
$$\tilde{\rho}(h_{12}^{(1)}) = 2^{-1}I_3(P_4(x)).$$

If we use $P$, $Q$ defined in Section 1, then

$$\tilde{\rho}(W_{ea}^{(1)}) = 2^23P,$$
$$\tilde{\rho}(h_{12}^{(1)}) = 2^33^2Q.$$

We also get

$$C[x, y]^{G_i} \xrightarrow{\cong} S(2, 4)^{(2)}$$

where

$$\tilde{\rho}(W_{ea}^{(1)}) = 11 \cdot 2^{-4}3^{-2}I_2(P_4(x))^3 + 7 \cdot 2^{-3}3^{-2}I_3(P_4(x))^2$$
$$= 2^53(11P^3 + 3^2Q^2).$$

We shall next consider the case when $g = 2$. In [7] the following elements⁵ are used as the generators of $S(2, 6)$.

$$A(P_6(x)) = u_0^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2,$$
$$B(P_6(x)) = u_0^2 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2,$$
$$C(P_6(x)) = u_0^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2,$$
$$D(P_6(x)) = u_0^{10} \sum_{\text{fifteen}} (12)^2(13)^2 \cdots (56)^2,$$
$$E(P_6(x)) = u_0^{15} \prod_{\text{fifteen}} \det \begin{pmatrix} 1 & \xi_1 + \xi_2 & \xi_1\xi_2 \\ 1 & \xi_3 + \xi_4 & \xi_3\xi_4 \\ 1 & \xi_5 + \xi_6 & \xi_5\xi_6 \end{pmatrix}.$$
We do not need the formula of $E^2$ in this paper, however, since it is not contained in [7], we give the explicit formula of $E^2$. This is derived from the formula of $J_{15}^2$ in the appendix, or directly.

$$E^2 = (1/2^{11}3^3)(A^7B^4 - 2^33A^6B^3C - 2^23^5A^6B^2D$$

$$+ 2 \cdot 3 \cdot 13A^3B^5 + 2 \cdot 3^3A^5B^2C^2 + 2^33^6A^3BCD + 2^23^{10}A^5D^2$$

$$- 2^23^237A^4B^4C - 2^33^239A^4B^3D - 2^33^3A^4BC^3 - 2^23^7A^4C^2D$$

$$- 3 \cdot 53A^3B^6 + 2 \cdot 3^45 \cdot 11A^3B^3C^2 + 2^23^57A^3B^2CD + 2^33^75^211A^3BD^2$$

$$+ 3^3A^3C^4 + 2^35^3A^2B^5C + 2^33^4457A^2B^4D - 2^33^317A^2B^3C^3$$

$$- 2^43^5 \cdot 53A^2BC^2D - 2^73^85^3A^2CD^2 + 2^55AB^7 - 2^33^7AB^4C^2$$

$$- 2^63^5 \cdot 61AB^3CD - 2^63^75^329AB^2D^2 + 2^33^437ABC^4 + 2^63^75^3AC^3D$$

$$- 2^33^3B^6C - 2^33^14B^5D + 2^83^3B^3C^3 + 2^93^65^2B^2C^2D$$

$$+ 2^33^85^4BCD^2 - 2^73^5C^5 + 2^{11}3^5D^3).$$

This induces the expression for $\chi_{35}^2$ in [7]. Igusa’s $\rho$-homomorphism is given by

$$\rho(\psi_4) = 2^{-2}B,$$

$$\rho(\psi_5) = 2^{-3}(AB - 3C),$$

$$\rho(\chi_{10}) = -2^{-14}D,$$

$$\rho(\chi_{12}) = 2^{-17}3^{-1}AD$$

$$= 3^25 \cdot 2^{-10}(-2^35^1J_2^6 - 2^53^25^3J_2^4J_4 - 2^23^4J_2^3J_6 + 2^63^45^2J_2^2J_4)$$

$$- 2^33^25^5J_2J_4J_6 + 3^35^2J_2J_{10}),$$

$$\rho(\chi_{35}) = -2^{-39}\sqrt{-1}D^2E.$$
Using these formulae, we know the $\overline{\rho}$ image of the generators (except $W_{c_6}^{(2)}$, $W_{g_{24}}^{(2)}$) of $C[x, y, z, w]^{G_2}$ as follows:

$$
\overline{\rho}(W_{c_6}^{(2)}) = \overline{\rho}(11^23^{-2}7^{-1}(W_{c_6}^{(2)})^3 + 2 \cdot 3^{-2}(h_{12}^{(2)})^2 - 2^37^{-1}W_{g_{24}}^{(2)})
$$

$$
= 2^{-8}3^{-2}(-2A^2B^2 + 2^23ABC + 2^23^311AD + 11B^3 - 2 \cdot 3^2C^2)
$$

$$
= 3^5(409 \cdot 1549J_2^6 - 2^33^559J_2^2J_4 - 2^{-7}5^413J_2^4J_6 + 2^43^55^5463J_2^2J_4)
$$

$$
- 2^63^55^2J_2J_4J_6 + 2^33^55^511JD_{20} - 2^63^65^811J_4^2 - 2 \cdot 5^7J_2^2)
$$

$$
\overline{\rho}(W_{g_{24}}^{(2)}) = \overline{\rho}(43 \cdot 53 \cdot 3^{-4}7^{-1}(W_{c_6}^{(2)})^4 + 2^{45} \cdot 23 \cdot 3^{-5}11^{-1}W_{c_6}^{(2)}(h_{12}^{(2)})^2
$$

$$
- 2^{643} \cdot 3^{-2}7^{-1}11^{-1}W_{c_6}^{(2)}W_{g_{24}}^{(2)} + 2^63^{-5}h_{12}^{(2)}F_{20}
$$

$$
= 2^{-8}3^{-3}(-2^4A^2B^3 + 2^53ABB + 2^53^5ABD + 3BD^2 - 4^33^2BC^2 + 2^83^2CD)
$$

$$
= 3^5(-286322081J_2^8 + 2^43^55^31 \cdot 59 \cdot 5477J_2^6J_4 - 2^{11}5^317 \cdot 83J_2^6J_6
$$

$$
+ 2^{65}67 \cdot 43J_2^2J_4^2 + 2^63^55^313 \cdot 61J_2^6J_4 + 2^63^55^57 \cdot 233J_2^6J_{10}
$$

$$
- 2^{85}6^55^73 \cdot 293J_2^6J_4^2 - 2^55^7J_2^2J_4J_6 - 2^95^57^167J_2J_4J_{10}
$$

$$
+ 2^83^51143J_4^4 + 2^63^25^841J_4J_6^2 - 2^73^35^8J_6J_{10}).
$$

We observe that the $\rho$ images of $A(\Gamma_2)^{(2)}$, $A(\Gamma_2)^{(4)}$ are strictly smaller than $S(2, 6)^{(2)}$, $S(2, 6)^{(4)}$, respectively. On the other hand, it is known that the Broué-Enguehard map gives the isomorphisms $C[x, y, z, w]^{H_2} \cong A(\Gamma_2)^{(2)}$ and $C[x, y, z, w]^{G_2} \cong A(\Gamma_2)^{(4)}$. Therefore the $\overline{\rho}$ images of the rings $C[x, y, z, w]^{H_2}$, $C[x, y, z, w]^{G_2}$ are strictly smaller than $S(2, 6)^{(2)}$, $S(2, 6)^{(4)}$, respectively.

Summing up,

**Theorem.** Let $\overline{\rho}$ be the combination of the Broué-Enguehard map and Igusa’s $\rho$-homomorphism.

(1) In the case when $g = 1$, $\overline{\rho}$ gives rise to the isomorphisms from $C[x, y]^{H_1}$ onto $S(2, 4)$, and from $C[x, y]^{G_1}$ onto $S(2, 4)^{(2)}$.

(2) In the case when $g = 2$, $\overline{\rho}$ transforms injectively $C[x, y, z, w]^{H_2}$ and $C[x, y, z, w]^{G_2}$ into $S(2, 6)$. The $\overline{\rho}$ images of these invariant rings are strictly smaller than $S(2, 6)^{(2)}$, $S(2, 6)^{(4)}$, respectively.

The explicit $\overline{\rho}$-images of the generators of each invariant ring are given above in two ways each.
We give remarks.

(1) Since Igusa’s \( \rho \) homomorphism increases the weight or the degree by a \( \frac{1}{2}g \) ratio, our \( \tilde{\rho} \) increases the degree by a \( \frac{1}{4}g \) ratio. This remark holds in the arbitrary genus \( g \). Here we note that Siegel modular forms we are considering are always of even weights.

(2) The weight enumerator of a code has non-negative integers as its coefficients (in arbitrary genus). We shall consider the case when \( g = 1 \). The \( \tilde{\rho} \) image of the weight enumerator has negative coefficients as the polynomials in \( \mathbb{C}[u_0, u_1, u_2, u_3, u_4] \) in general. For example, we have

\[
\tilde{\rho}(W^{(1)}_{e_8}) = 2^3 3(u_0u_4 - 4u_1u_3 + 3u_2^2), \\
\tilde{\rho}(W^{(1)}_{g_{24}}) = 2^5 3 (11u_0^3u_4^3 - 132u_0^2u_1u_3u_4^2 + 288u_0^2u_2^2u_4^2 - 378u_0u_2u_3u_4 + 189u_0^2u_3^2 \\
- 378u_0u_1^2u_2u_4^2 + 906u_0u_4u_3^2u_4 - 36u_0u_1u_2u_3u_4 - 756u_0u_1u_2u_4^3 \\
- 81u_0u_2^4u_4 + 378u_0u_2^3u_3^2 + 189u_1^4u_4^2 - 756u_1^3u_2u_3u_4 - 704u_1^3u_3^2 \\
+ 378u_1^2u_2^3u_4 + 2340u_1^2u_2^2u_3^2 - 1944u_1u_2^4u_3 + 486u_2^6). 
\]

It would be interesting if we interpret this from coding theoretical or combinatorial point of view.

(3) We mention the paper [19] in which Shioda discussed the close relationship of the ring \( S(3, 4) \) of projective invariants to the invariant theory for the Weyl groups \( W(E_7) \) and \( W(E_8) \). We omit the details.

Sapporo Medical University

REFERENCES.


[11] Oura, M., Note on the weight enumerators, J. of Liberal Arts and Sciences, Sapporo Medical University, School of Medicine.


Appendix. We give the generators of $S(2, 6)$ from [17]. We also give the expression for $J_{15}^2$.

\[
J_2 = u_0 u_6 - 6u_1 u_5 + 15u_2 u_4 - 10u_3^2,
\]

\[
J_4 = \det \begin{pmatrix}
 u_0 & u_1 & u_2 & u_3 \\
 u_1 & u_2 & u_3 & u_4 \\
 u_2 & u_3 & u_4 & u_5 \\
 u_3 & u_4 & u_5 & u_6
\end{pmatrix},
\]

\[
J_6 = \det \begin{pmatrix}
 b_0 & b_1 & b_2 \\
 b_1 & b_2 & b_3 \\
 b_2 & b_3 & b_4
\end{pmatrix},
\]

\[
J_{10} = u_0 c^3 - 6u_1 b c^2 + 3u_2 (ac + 4b^2)c - 4u_3 (3abc + 2b^3) + 3u_4 a (ac + 4b^2) - 6u_5 a^2 b + u_6 a^3,
\]

\[
J_{15} = \det \begin{pmatrix}
 c_0 & c_1 & c_2 & c_3 & c_4 \\
 c_1 & c_2 & c_3 & c_4 & c_5 \\
 c_2 & c_3 & c_4 & c_5 & c_6 \\
 c_3 & c_4 & c_5 & c_6 & c_7 \\
 c_4 & c_5 & c_6 & c_7 & c_8
\end{pmatrix},
\]

where

\[
b_0 = 6(u_0 u_4 - 4u_1 u_3 + 3u_2^2),
\]

\[
b_1 = 3(u_0 u_5 - 3u_1 u_4 + 2u_2 u_3),
\]

\[
b_2 = u_0 u_6 - 9u_2 u_4 + 8u_3^2,
\]

\[
b_3 = 3(u_1 u_6 - 3u_2 u_5 + 2u_3 u_4),
\]

\[
b_4 = 6(u_2 u_6 - 4u_3 u_5 + 3u_4^2),
\]

\[
a = 2(u_0 u_2 u_6 - 3u_0 u_3 u_5 + 2u_0 u_4^2 - u_1^2 u_6 + 3u_1 u_2 u_5 - u_1 u_3 u_4 - 3u_2^2 u_4 + 2u_2 u_3^2),
\]

\[
b = u_0 u_4 u_6 - u_0 u_5 u_5 - u_1 u_2 u_6 - 8u_1 u_3 u_5 + 9u_1 u_4^2 + 9u_2^2 u_5 - 17u_2 u_3 u_4 + 8u_3^3,
\]

\[
c = 2(u_0 u_4 u_6 - u_0 u_5 u_5 - 3u_1 u_3 u_6 + 3u_1 u_4 u_5 + 2u_2 u_6 - u_2 u_3 u_5 - 3u_2 u_4^2 + 2u_3 u_4),
\]

\[
c_0 = 8(u_0^2 u_5 - 5u_0 u_4 u_4 + 2u_0 u_2 u_3 + 8u_0^2 u_3 - 6u_1 u_2^2),
\]

\[
c_1 = u_0^2 u_6 + 2u_0 u_1 u_5 - 19u_0 u_2 u_4 + 8u_0 u_3^2 - 6u_1^2 u_4 + 44u_1 u_2 u_3 - 30u_2^3,
\]

\[
c_2 = 2(u_0 u_1 u_6 - 2u_0 u_2 u_5 - 2u_0 u_3 u_4 - 3u_1 u_2 u_4 + 16u_1 u_3^2 - 10u_2^2 u_3),
\]

\[
c_3 = u_0 u_2 u_6 - 4u_0 u_3 u_5 - 2u_0 u_4^2 + 2u_1^2 u_6 - 6u_1 u_2 u_5 + 24u_1 u_3 u_4 - 15u_2^2 u_4,
\]

\[
c_4 = 4(-u_0 u_4 u_5 + u_1 u_2 u_6 + 3u_1 u_4^2 - 3u_3^2 u_5),
\]

\[
c_5 = -u_0 u_4 u_6 - 2u_0 u_5^2 + 4u_1 u_3 u_6 + 6u_1 u_4 u_5 + 2u_2^2 u_6 - 24u_2 u_3 u_5 + 15u_2 u_4^2,
\]

\[
c_6 = 2(-u_0 u_5 u_6 + 2u_1 u_4 u_6 + 2u_2 u_3 u_6 + 3u_2 u_4 u_5 - 16u_3^2 u_5 + 10u_3 u_4^2),
\]

\[
c_7 = -u_0 u_6^2 - 2u_1 u_5 u_6 + 19u_2 u_4 u_6 + 6u_2 u_5^2 - 8u_3^2 u_6 - 44u_3 u_4 u_5 + 30u_4^3,
\]

\[
c_8 = 8(-u_1 u_6^2 + 5u_2 u_5 u_6 - 2u_3 u_4 u_6 - 8u_3 u_5^2 + 6u_4^2 u_5).
\]
\[ J_{15}^2 = -2^{73 - 10} J_{14}^{15} + 2^{97} \cdot 3^{-8} J_{13}^{14} J_{4} - 2^{73} \cdot 3^{-12} J_{12}^{13} J_{6} \\
= -2^{117} \cdot 3^{-5} J_{1}^{13} J_{6}^{2} + 2^{153} \cdot 3^{-9} J_{1}^{10} J_{4} J_{6} + 2^{73} \cdot 7 J_{10}^{10} J_{6} \\
+ 2^{135} \cdot 7 \cdot 3^{-4} J_{2}^{13} J_{4}^{2} - 2^{98} \cdot 29 \cdot 3^{-12} J_{2}^{9} J_{6}^{2} - 2^{115} \cdot 3^{-4} J_{2}^{8} J_{6}^{2} J_{6} \\
- 2^{95} \cdot 3^{-5} J_{2}^{8} J_{4} J_{10} - 2^{155} \cdot 7 \cdot 3^{-2} J_{2}^{7} J_{4}^{1} + 2^{101} \cdot 11 \cdot 3^{-8} J_{2}^{7} J_{4} J_{6}^{2} \\
+ 2^{73} \cdot 6 J_{2}^{7} J_{6} J_{10} + 2^{165} \cdot 11 \cdot 3^{-6} J_{2}^{6} J_{4} J_{6} + 2^{125} \cdot 3^{-3} J_{2}^{5} J_{4}^{2} J_{10} \\
- 2^{87} \cdot 3^{-11} J_{2}^{5} J_{6}^{2} + 2^{173} \cdot 7 J_{2}^{3} J_{4}^{2} - 2^{113} \cdot 3^{-3} J_{2}^{3} J_{4}^{2} J_{6}^{2} \\
- 2^{105} \cdot 3^{-5} J_{2}^{5} J_{4} J_{6} J_{10} + 2^{53} \cdot 3^{-3} J_{2}^{5} J_{10} - 2^{155} \cdot 17 \cdot 3^{-3} J_{2}^{4} J_{4}^{1} J_{6} \\
- 2^{145} \cdot 3^{-1} J_{2}^{4} J_{4}^{4} J_{10} + 2^{123} \cdot 8 J_{2}^{3} J_{4} J_{6}^{2} + 2^{73} \cdot 6 J_{2}^{6} J_{6} J_{10} \\
- 2^{19} \cdot 3^{2} J_{2}^{3} J_{6}^{4} + 2^{15} \cdot 31 \cdot 3^{-6} J_{2}^{3} J_{4} J_{6}^{2} + 2^{11} \cdot 11 \cdot 3^{-3} J_{2}^{3} J_{4}^{2} J_{6} J_{10} \\
- 2^{8} \cdot 3^{-1} J_{2}^{3} J_{4}^{2} J_{10} - 2^{7} \cdot 13 \cdot 3^{-12} J_{2}^{3} J_{6}^{4} + 2^{20} J_{2}^{5} J_{4} J_{6} \\
+ 2^{153} \cdot 5 J_{2}^{3} J_{4}^{3} J_{10} - 2^{113} \cdot 3^{-5} J_{2}^{3} J_{4} J_{6}^{2} - 2^{95} \cdot 3^{-5} J_{2}^{2} J_{4} J_{6} J_{10} \\
+ 2^{7} \cdot 3 \cdot 3 J_{2}^{2} J_{6} J_{10} + 2^{21} \cdot 3 J_{2}^{3} J_{4}^{1} - 2^{157} \cdot 3^{-3} J_{2}^{3} J_{4} J_{6}^{2} \\
- 2^{153} \cdot 3 J_{2}^{3} J_{4} J_{6} J_{10} + 2^{93} J_{2}^{3} J_{4} J_{10} + 2^{93} \cdot 3 J_{2}^{2} J_{4} J_{6}^{2} \\
+ 2^{7} \cdot 3 \cdot 7 J_{2}^{3} J_{6} J_{10} - 2^{19} \cdot 7 J_{4}^{6} J_{6} - 2^{17} \cdot 3 \cdot 3 J_{4}^{2} J_{6}^{2} \\
- 2^{13} \cdot 3 \cdot 6 J_{4}^{3} J_{6}^{2} + 2^{113} \cdot 3^{-3} J_{4}^{3} J_{6} J_{10} + 2^{7} \cdot 3 \cdot 1 J_{4} J_{6} J_{10} \\
- 2^{7} \cdot 3^{-12} J_{6}^{2} - 2^{5} J_{10}^{3}.\]