



Title	On the one dimensional Gelfand and Borg-Levinson spectral problems for discontinuous coefficients
Author(s)	Sini, Mourad
Citation	Hokkaido University Preprint Series in Mathematics, 625, 2-16
Issue Date	2003
DOI	10.14943/83779
Doc URL	<a href="http://hdl.handle.net/2115/69433">http://hdl.handle.net/2115/69433</a>
Type	bulletin (article)
File Information	pre625.pdf



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# On the one dimensional Gelfand and Borg-Levinson spectral problems for discontinuous coefficients.

Mourad Sini† §

† Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.

E-mail: [sini@math.sci.hokudai.ac.jp](mailto:sini@math.sci.hokudai.ac.jp)

## Abstract

In this paper, we deal with the inverse spectral problem for the equation  $-(pu')' + qu = \lambda\rho u$  on a finite interval  $(0, h)$ . We give some uniqueness results on  $q$  and  $\rho$  from the Gelfand spectral data, when the coefficients  $p$  and  $\rho$  are piecewise Lipschitz and  $q$  is bounded. We also prove an equivalence result between the Gelfand spectral data and the Borg-Levinson spectral data. As a consequence, we have similar uniqueness results if we consider the Borg-Levinson spectral data. Finally, we consider the inverse problem from the nodes and give uniqueness results on  $\rho$  and in the case where the coefficients  $p, q$  and  $\rho$  are smooth we give a uniqueness results on both  $q$  and  $\rho$ .

## 1. Introduction and statement of the result.

Let  $p$  and  $\rho$ ,  $j = 1, 2$  be bounded and positives measurable functions,  $q$  be bounded measurable function and  $h > 0$  is a constant. We denote by  $\lambda_i$  and  $e_i$ ,  $i \in \mathbb{N}$ , the associated eigenvalues and  $L^2(0, h)$ -orthonormal eigenfunctions of the Sturm-Liouville problem:

$$\begin{cases} -(pu')' + qu - \lambda\rho u = 0 \text{ in } (0, h), \\ u(0) = 0, \\ u(h) = 0. \end{cases} \quad (1)$$

The Gelfand inverse spectral problem consists on the reconstruction of some of the coefficients  $p, q$  and  $\rho$  from the spectral data  $(\lambda_i, |p(e_i)'(0)|)_{i=1}^{\infty}$ . This last sequence is called the Gelfand spectral data. If in (1), we take the Dirichlet boundary condition on  $x_0 = 0$  and the Neuman one on  $x_0 = h$  then we have an other sequence, which we denote by  $(\mu_i)_{i \in \mathbb{N}}$ . The sequence  $(\lambda_i, \mu_i)_{i \in \mathbb{N}}$  is called the Borg-Levinson spectral data. We call Borg-Levinson problem the identification of some of the coefficients  $p, q$  and  $\rho$  from the Borg-Levinson spectral data.

It is well known that when these coefficients are regular ( $p$  and  $\rho$  are  $C^{1,1}(0, h)$  for example) then by reducing the equation  $-(pu')' + qu - \lambda\rho u = 0$  to the normal form, i.e. where  $p = r = 1$ , we can prove uniqueness of one over the three coefficients from the Borg-Levinson spectral data or from the Gelfand spectral data. The aim is to prove similar results for discontinuous coefficients. The case where  $q = 0$  is quite well known. Du to the vast literature on these cases we refer the reader to [Mc1], [Mc2] or [HM] for details and references.

§ [sini@math.sci.hokudai.ac.jp](mailto:sini@math.sci.hokudai.ac.jp)

In the following work, we consider the case where  $q \neq 0$  and prove uniqueness of  $q \in L^\infty(0, h)$  and  $h$  when  $p$  and  $\rho$  are fixed and assumed to be piecewise Lipschitz. We also prove uniqueness of  $\rho \in L^\infty(0, h)$  if  $p$  and  $q$  are fixed in  $L^\infty(0, h)$ . The method of analysis to prove uniqueness of  $q$  is the BC method. One of the main tools of the BC method is the boundary approximate controllability for the corresponding hyperbolic problem. To prove this boundary approximate controllability, for the kind of discontinuities we mentioned above, we approximate the corresponding hyperbolic problem given by irregular coefficients by hyperbolic problems given by regular coefficients and take advantage of the equivalence between the space and the time variables in this one dimensional case.

Let us now give the results:

**Theorem 1** *Let  $p_j$  and  $r_j$ ,  $j = 1, 2$  be bounded and positives and  $q_j$ ,  $j = 1, 2$ , be bounded and measurable functions. We denote by  $\lambda_i^j$  and  $e_i^j$ ,  $i \in \mathbb{N}$ , the associated eigenvalues and orthonormal eigenfunctions on  $\Omega_j := (0, h_j)$ .*

*We suppose that the boundary spectral data are the same, i.e.*

$$\lambda_i^1 = \lambda_i^2 \text{ and } |p_1(e_i^1)'(0)| = |p_2(e_i^2)'(0)| \text{ for all } i \in \mathbb{N}. \quad (2)$$

*A1) If we fix  $p_j$  and  $r_j$  with piecewise Lipschitz regularity, then we have uniqueness of  $q_j$  and  $h_j$ .*

*A2) If in addition  $p_j$  (or  $\rho_j$ ) are piecewise constant, then fixing  $\rho_j$  (or  $p_j$ ) we have uniqueness of both  $p_j$  (or  $\rho_j$ ),  $q_j$  and  $h_j$ .*

*B) If we fix  $p_j$  and  $q_j$  bounded then we have uniqueness of  $\rho_j$  and  $h_j$ .*

The condition  $q_1 = q_2$  means that  $q_j = q$  a.e. in  $(0, h_j)$  where  $q$  is defined on  $(0, H)$  with  $H \geq h_j$ ,  $j = 1, 2$  and similarly for  $p_j$  and  $r_j$ . In the proof of this theorem we assume that  $q_j > 0$  but the results holds also in the general case by shifting the spectrum. Since we can change the sign of  $p_j(e_i^j)'(0)$ , then the absolute value in (2) can be deleted.

The proof of the parts A1), A2) and B) are given in section 2, section 3 and section 4 respectively. This is a continuation of [S] where we proved similar results for piecewise analytic coefficients. In A2), we prove uniqueness of two coefficients, but we assumed that  $p$  (or  $\rho$ ) are piecewise constants. It is interesting to see if we can consider more general discontinuities for  $p$  or  $\rho$ .

In section 5, we use the gauge transformation introduced in [Ku] to the inverse spectral problems and the Alessandrini identity to prove the equivalence between the Gelfand and the Borg-Levinson data. This equivalence enables us to state the results of Theorem 1 considering the Borg-Levinson spectral data. To our knowledge these kind of results are also new with respect to the literature, see [Mc1] and [Mc2].

Section 6 deals with the inverse nodal problem for the equation  $-(pu)'' + qu - \lambda\rho u = 0$  in  $(0, h)$ . This problem has been proposed in [Mc] for the case where  $p = q = 1$  where uniqueness result of  $q$  is given. In [HaMc1], we find reconstruction algorithms for the identification of  $q$ . This case was considered later in [ShTs], [S] and [LSY] where we find other algorithms of reconstructions and informations are given on the smoothness of  $q$ . In [HaMc2], the case  $q = 0$  was considered and uniqueness results on  $p$  or  $\rho$  were established. In our case we are just concerned with the uniqueness, but considering both the three coefficients  $p, q$  and  $\rho$ . We are motivated by the question asked in [HaMc1], see also [HaMc2], on uniqueness of two coefficients from the nodes. This section follows the spirit of this last reference. Fixing  $p$  and  $q$  we prove uniqueness, up to a multiplicative constant, of  $\rho$ . For this last result,  $p$  and  $\rho$  are assumed to be BV-functions and  $q$  bounded. If in addition we take  $p$  and  $\rho$  in

$H^2(0, h)$  and  $q$  continuous almost every where, then fixing  $p$  we prove uniqueness of  $\rho$ , up to a multiplicative constant, and also uniqueness of  $q$ , up to an additive one dimensional subspace generated by the function  $\rho$ . These results complete the work [HaMc2] and give a positive answer for the possibility to prove uniqueness of two coefficients from the nodes.

## 2. Proof of the part A1) of Theorem 1.

Let us define the following hyperbolic problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{\rho} \frac{\partial}{\partial x} (p \frac{\partial u}{\partial x}) + \frac{q}{\rho} u = 0 \text{ in } (0, T) \times (0, h), \\ u(t, 0) = f(t), u(t, h) = 0, t \in (0, T), \\ u(0, x) = \frac{\partial u}{\partial t}(0, x) = 0, x \in (0, h), \end{cases} \quad (3)$$

where  $f \in H^1(0, T)$  such that  $f(0) = 0$  and  $T$  is positive constant.

This problem has one and only one solution in  $C[(0, T), H^1(0, h)] \cap C^1[(0, T), L^2(0, h)]$  satisfying:

$$\max_{0 \leq t \leq T} [\|u(t)\|_{H^1(0, h)} + \|\frac{\partial u}{\partial t}(t)\|_{L^2(0, h)}] \leq c(T)(\|f\|_{H^1(0, h)}), \quad (4)$$

see [KaKuLa] or [LaLiTr].

We set  $u^f(t, x)$  this solution. We recall the spectral representation of  $u^f(t, x)$ :

$$u^f(t, x) = \sum_{i=1}^{\infty} u_i^f(t) e_i(x)$$

where

$$u_i^f(t) := \int_0^h u^f(t, x) e_i(x) \rho(x) dx \quad (5)$$

satisfy the Cauchy problem:

$$\begin{cases} \frac{d^2 u_i^f}{dt^2} = -\lambda_i u_i^f + p \frac{de_i}{dx}(0) f(t), \text{ in } (0, T), \\ u_i^f(0) = \frac{du_i^f}{dt}(0) = 0. \end{cases} \quad (6)$$

Hence

$$u_i^f(t) = \int_0^t f(t') \frac{\sin \sqrt{\lambda_i}(t-t')}{\sqrt{\lambda_i}} dt' (p \frac{de_i}{dx})(0), \quad (7)$$

since  $\lambda_i > 0$  for all  $n \in \mathbb{N}$ .

Since the coefficients  $p, q$  and  $\rho$  are not regular, even if we take  $f$  regular we can not differentiate two times with respect to the time variable in (5) to justify (6). To prove that  $u_i^f(t, x)$  satisfy (6), we start by taking the derivative of (5), using (4), to get  $\frac{du_i^f}{dt} = \int_0^h \frac{du^f}{dt}(t, x) e_i(x) \rho(x) dx$ . We take any function  $\phi \in C_0^\infty(0, T)$  then:

$$\int_0^T \frac{du_i^f}{dt} \frac{d\phi}{dt} dt = \int_0^T \int_0^h \frac{\partial u^f(t, x)}{\partial t} \frac{\partial(\phi(t) e_i(x))}{\partial t} \rho(x) dx dt.$$

Using the definition of the weak solution of (3), we deduce that:

$$\int_0^T \frac{du_i^f}{dt} \frac{d\phi}{dt} dt = \int_0^T \int_0^h p \frac{\partial u^f(t, x)}{\partial x} \frac{(\phi(t) e_i(x))}{\partial x} + q u^f(t, x) e_i(x) \phi(t) dx dt.$$

An integration by parts gives:

$$\int_0^T \frac{du_i^f}{dt}(t) \frac{d\phi}{dt}(t) dt = \lambda_i \int_0^T \int_0^h u^f(t, x) e_i(x) \phi(t) \rho(x) dx dt - \int_0^T p \frac{de_i}{dx}(0) f(t) \phi(t) dt.$$

Hence in the distribution sense we have the first equation of (6). The initial conditions of (6) come from the ones of (3) which is justified by (4).

### 2.1. Domain of influence for the problem (3).

The object of this section is to prove, for general bounded coefficients  $p, q$  and  $\rho$ , that the domain of influence of the solution of the problem (3) is given by  $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt \leq t\}$  where  $v(x) := \sqrt{\frac{\rho}{p}}(x)$  is the velocity of propagation.

Let  $(p_n), (q_n)$  and  $(\rho_n)$  be three sequences of regular coefficients given by:  $p_n = r_n \star \tilde{p}$ ,  $q_n = r_n \star \tilde{q}$  and  $\rho_n = r_n \star \tilde{\rho}$  where  $r_n$  is a mollifier sequence and  $\tilde{p}, \tilde{q}$  and  $\tilde{\rho}$  are given by extending  $p, q$  and  $\rho$  by zero to  $\mathfrak{R} \setminus (0, h)$  respectively. Then  $(p_n, q_n, \rho_n)$  tend to  $(p, q, \rho)$  in  $(L^s(0, h))^3 \forall s, 1 \leq s < \infty$  and the sequences  $(p_n, q_n, \rho_n)$  are bounded from below and above in  $L^\infty(0, h)$ , i.e  $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma$  where  $\delta$  and  $\gamma$  are constants.

Let  $u_n^f$  be the solution of (3) when we replace  $(p, q, \rho)$  by  $(p_n, q_n, \rho_n)$ .

We set  $y = g_n(x) := \frac{1}{h_n} \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt$ ,  $\tilde{u}_n^f(t, y) := (\frac{\tilde{p}_n(y)\tilde{\rho}_n(y)}{\tilde{p}_n(0)\tilde{\rho}_n(0)})^{\frac{1}{4}} u_n^f(t, y)$ ,  $\tilde{p}_n(y) := p_n(g_n^{-1}(y))$ ,  $\tilde{q}_n(y) := q_n(g_n^{-1}(y))$ ,  $\tilde{\rho}_n(y) := \rho_n(g_n^{-1}(y))$ ,  $c_n(y) := \frac{\frac{\partial^2}{\partial y^2}(\tilde{p}_n(y)\tilde{\rho}_n(y))}{\tilde{p}_n(y)\tilde{\rho}_n(y)} + h_n \frac{\tilde{q}_n}{\tilde{\rho}_n}$  and  $h_n := \int_0^h \sqrt{\frac{\rho_n}{p_n}}(t) dt$  then  $\tilde{u}_n^f(y)$  satisfy:

$$\begin{cases} \frac{\partial^2 \tilde{u}_n^f}{\partial t^2} - \frac{1}{h_n^2} \frac{\partial^2 \tilde{u}_n^f}{\partial y^2} + c_n \tilde{u}_n^f = 0 \text{ in } (0, T) \times (0, 1), \\ \tilde{u}_n^f(t, 0) = f(t), \tilde{u}_n^f(t, 1) = 0, \\ \tilde{u}_n^f(0, y) = \frac{\partial \tilde{u}_n^f}{\partial t}(0, y) = 0, y \in (0, 1), \end{cases} \quad (8)$$

Hence by the finiteness of the velocity of the propagation, which is equal here to  $\frac{1}{h_n}$ , we deduce that the support of  $\tilde{u}_n^f$  is included in  $\{(t, y) \in (0, T) \times (0, 1) / y \leq \frac{1}{h_n} t\}$  then the support of  $u_n^f$  is included in  $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt \leq t\}$ .

In the appendix we prove that  $u_n^f$  tends to  $u^f$  in  $L^2[(0, T) \times (0, h)]$ . Hence, since  $(p_n, q_n, \rho_n)$  tend to  $(p, q, \rho)$  in  $L^2(0, h)$  and  $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma$  we deduce that the support of  $u^f$  is included in  $\{(t, y) \in (0, T) \times (0, 1) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt \leq t\}$ .

Indeed, let  $(t_0, x_0) \in \{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt > t\}$ . Then there exists  $\epsilon > 0$ , such that  $\int_0^x \sqrt{\frac{\rho}{p}}(t) dt > t$ , for all  $(t, x)$  satisfying  $|t - t_0| < \epsilon$  and  $|x - x_0| < \epsilon$ . A simple computation shows that  $|\int_0^x \sqrt{\frac{\rho}{p}}(t) dt - \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt| \leq c(\|p_n - p\|_{L^2(0, h)}^2 + \|\rho_n - \rho\|_{L^2(0, h)}^2)$ , where  $c$  is independent of  $n$  and  $x$ .

Hence we can find  $n_0 \in \mathbb{N}$  such that  $\forall n \geq n_0$ ,  $\int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt > t$ ,  $\forall (t, x) / |t - t_0| < \epsilon$  and  $|x - x_0| < \epsilon$  (taking  $\epsilon$  smaller if necessary). Then  $u_n^f(t, x) = 0 \forall (t, x) / |t - t_0| < \epsilon$  and  $|x - x_0| < \epsilon$  for all  $n \geq n_0$ . This implies that  $u^f(t, x) = 0 \forall (t, x) / |t - t_0| < \epsilon$  and  $|x - x_0| < \epsilon$ . Finally,  $u^f(t, x) = 0$  in  $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt > t\}$ .

We denote by  $\Gamma_T = \{x \in (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt \leq T\}$ .

## 2.2. Boundary controllability

The exact boundary controllability for the problem (3) is to find for every  $z(x) \in L^2(\Gamma_T)$  a function (i.e a control )  $f(t) \in L^2(0, T)$  such that  $u^f(x, T) = z(x)$ . In the case where  $p = \rho = 1$ , this result is known and the proof uses the Green function of the non perturbed part of the equation, given by  $q = 0$ , which is explicitly known in this case (see for example [KaKuLa], chapter 1). An other proof is given in ([Bel2]) (see also ([Bel1])). The authors considered the case where  $p = 1$ ,  $q = 0$  and  $\rho \in C^2[0, h]$ . In both these cases if  $T \leq \int_0^h \sqrt{\frac{\rho}{p}}(s)ds$  then we have uniqueness of the control.

In our general case, we prove the following approximate boundary controllability, which is enough for the inverse spectral problem we are considering. As we said in the introduction, to prove this theorem, we add two arguments to the usual proof in the multidimensional case (see [KaKuLa] Theorem 3.10). The first is to approximate the irregular coefficients by regular ones to justify the integration by parts. The second is to use the equivalence between the time variable and the space variable to consider the unique continuation of the hyperbolic problem as a usual Cauchy problem with time dependent coefficients.

**Theorem 2** *The linear subspace  $\{u^f(x, T), f \in H_0^1(0, T)\}$  is dense in  $L_\rho^2(\Gamma_T)$ .*

**Proof of theorem 2**

Let  $\psi \in L_\rho^2(\Gamma_T)$  such that  $\int_0^h u^f \psi \rho dx = 0$  for all  $f \in H_0^1(0, T)$ . Let us take  $f \in C_0^\infty[0, T]$ . We take also  $\psi_n \in C_0^\infty[0, h]$  such that  $\psi_n$  tends to  $\psi$  in  $L^2(0, h)$ . We recall that  $u_n^f$  is the solution of (3) replacing  $p, q$  and  $\rho$  by  $p_n, q_n$  and  $\rho_n$ .

We define the initial boundary value problem:

$$\begin{cases} \frac{\partial^2 v_n}{\partial t^2} - \frac{1}{\rho_n} \frac{\partial}{\partial x} (p_n \frac{\partial v_n}{\partial x}) + \frac{q_n}{\rho_n} v_n = 0 \text{ in } (0, T) \times (0, h), \\ v_n(t, 0) = v_n(t, h) = 0, \\ v_n(T, x) = 0, \frac{\partial v_n}{\partial t}(T, x) = \psi_n(x) \in (0, h). \end{cases} \quad (9)$$

This problem is called the dual problem of the problem (3). In the appendix, we prove that  $v_n$  tends to  $v$  in  $L^2((0, T) \times (0, h))$  where  $v$  satisfies (9) replacing  $p_n, q_n, \rho_n$  and  $\psi_n$  by  $p, q, \rho$  and  $\psi$  respectively.

Since the coefficients  $p_n, q_n, \rho_n$  are in  $C^\infty(0, h)$  and  $\psi_n \in C_0^\infty(0, h)$ , which guarantee the compatibility conditions, then  $v_n \in C^\infty((0, T) \times (0, h))$ .

By an integration by parts, one finds that:

$$\begin{aligned} 0 &= \int_0^h \int_0^T u_n^f [\rho_n \frac{\partial^2 v_n}{\partial t^2} - \frac{\partial}{\partial x} (p_n \frac{\partial v_n}{\partial x}) + q_n v_n] - v_n^f [\rho_n \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial}{\partial x} (p_n \frac{\partial u_n}{\partial x}) + q_n u_n] dt dx \\ &= \int_0^h u_n^f(T, x) \psi_n \rho_n dx + \int_0^T f p_n \frac{\partial v_n}{\partial x}(0, t) dt. \end{aligned}$$

We have:

$$\begin{aligned} \int_0^h u_n^f(T, x) \psi_n \rho_n dx - \int_0^h u^f(T, x) \psi \rho dx &= \int_0^h u_n^f(T, x) \psi_n \rho_n dx - \int_0^h u_n^f(T, x) \psi_n \rho dx + \\ &\int_0^h u_n^f(T, x) \psi_n \rho dx - \int_0^h u_n^f(T, x) \psi \rho dx + \int_0^h u_n^f(T, x) \psi \rho dx - \int_0^h u^f(T, x) \psi \rho dx \end{aligned}$$

We know that the sequence  $u_n^f(t, x)$  is bounded in the energy norm from the energy inequality (see the appendix), then in particular in  $C([0, T], H^1(0, h))$  hence  $u_n^f(T, x)$  is bounded in

$C[0, h]$ . Since  $\psi_n$  is bounded in  $L^2(0, h)$ , then from the inequality:

$$\left| \int_0^h u_n^f(T, x) \psi_n (\rho_n - \rho) dx \right| \leq \int_0^h |u_n^f(T, x)|^2 |\psi_n|^2 dx \int_0^h (\rho_n - \rho)^2 dx$$

we deduce that  $\int_0^h u_n^f(T, x) \psi_n (\rho_n - \rho) dx$  tends to zero when  $n$  tends to  $\infty$ . Arguing similarly for the other terms we deduce that  $\left| \int_0^h u_n^f(T, x) \psi_n \rho_n dx \right|$  tends to zero since  $\int_0^h u^f(T, x) \psi \rho dx = 0$ .

This means that  $\int_0^T f p_n \frac{\partial v_n}{\partial x}(0, t) dt$  tends to zero as  $n$  tends to  $\infty$ .

We can say also that  $v_n$  satisfies the following problem:

$$\begin{cases} \frac{\partial}{\partial x} (p_n \frac{\partial v_n}{\partial x}) - \rho_n \frac{\partial^2 v_n}{\partial t^2} - q_n v_n = 0 \text{ in } (0, T) \times (0, h), \\ v_n(t, 0) = 0, p_n \frac{\partial v_n}{\partial x}(t, 0) = \xi_n(t), t \in (0, T), \\ v_n(T, x) = 0, v_n(0, x) = \eta_n(x) \in (0, h), \end{cases} \quad (10)$$

where  $\xi_n$  tends weakly to zero in  $L^2(0, T)$  since we took  $f$  arbitrary in a dense set of  $L^2(0, T)$ . We look at this problem as a Cauchy problem where the time variable is  $x$  and the space variable is  $t$ .

Since  $v_n$  is solution of (9), then the boundness of  $v_n$  in  $C([0, T], H^1(0, h))$  and of  $\frac{\partial v_n}{\partial t}$  in  $C([0, T], L^2(0, h))$  are given by the energy inequality applied to (9). Then we take a subsequence, which we denote also by  $v_n$ , and find a function  $\tilde{v}$  such that  $v_n$  tends to  $\tilde{v}$  weakly in  $H^1[(0, T), L^2(0, h)] \cap L^2[(0, T), H^1(0, h)]$ . Hence,  $v_n(0, x)$  and  $v_n(T, x)$  tend to  $\tilde{v}(0, x)$  and  $\tilde{v}(T, x)$  respectively in  $L^2(0, h)$ . As a consequence, we have  $\tilde{v}(T, x) = 0$  since  $v_n(T, x)$  and  $\tilde{v}(0, x) \in H_0^1(0, h)$  since  $v_n(0, x) \in H_0^1(0, h)$ , see (9) and  $v_n(0, x)$  tends weakly to  $v(0, x)$  in  $H^1(0, h)$ .

Now take  $\phi \in H^1((0, T) \times (0, h))$  such that  $\phi(t, h) = 0$ , for  $t \in (0, T)$  and  $\phi(0, x) = \phi(T, x) = 0$ , for  $x \in (0, h)$ . Multiplying in (10) by  $\phi(t, x)$  and integrating by parts, we find:

$$\int_0^T \int_0^h -p_n \frac{\partial v_n}{\partial x} \frac{\partial \phi}{\partial x} + \rho_n \frac{\partial v_n}{\partial t} \frac{\partial \phi}{\partial t} - q_n v_n \phi dx dt = - \int_0^T \xi_n(t) \phi(0, t) dt \quad (11)$$

Now going to the limit in the previous equality using the properties of  $v_n$  and  $\xi_n$ , we have:

$$\int_0^T \int_0^h -p \frac{\partial \tilde{v}}{\partial x} \frac{\partial \phi}{\partial x} + \rho \frac{\partial \tilde{v}}{\partial t} \frac{\partial \phi}{\partial t} - q \tilde{v} \phi dx dt = 0.$$

From the definition of a weak solution of the hyperbolic problem, we deduce that  $\tilde{v}$  satisfy the following problem:

$$\begin{cases} \frac{\partial}{\partial x} (p \frac{\partial \tilde{v}}{\partial x}) - \rho \frac{\partial^2 \tilde{v}}{\partial t^2} - q \tilde{v} = 0 \text{ in } (0, T) \times (0, h), \\ \tilde{v}(t, 0) = 0, p \frac{\partial \tilde{v}}{\partial x}(t, 0) = 0, t \in (0, T), \\ \tilde{v}(T, x) = 0, \tilde{v}(0, x) = \eta(x) \in (0, h), \end{cases} \quad (12)$$

where  $\eta \in H_0^1(0, h)$ .

We write  $v_n := v_n^1 + v_n^2$ , where  $v_n^1$  is the solution of (10) when we take  $\xi_n(t) = 0$  and  $v_n^2$  the solution when  $\eta_n(x) = 0$ . Arguing as in the subsection 2.1, we see that the support of  $v_n^1$  is included in  $\{(t, x) \in (0, T) \times (0, h) / t \leq \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt\}$ . Since  $\xi_n$  tends weakly to zero in  $L^2(0, T)$ , then it is bounded. Arguing as for  $v_n$ , we deduce that  $v_n^2$  tends to zero in  $L^2((0, T) \times (0, h))$ . Since in this case the coefficients depend on the time variable, a little

change is needed since the boundness in the energy norm requires Lipschitz regularity (see [La]). For this we state the problem satisfied by  $v_n^1$  on the parts where the coefficients  $p$  and  $\rho$  are Lipschitz and we proceed by steps.

Since  $v_n$  tends to  $\tilde{v}$  in  $L^2((0, T) \times (0, h))$ , this means that  $v_n^1$  tends to  $\tilde{v}$  in  $L^2((0, T) \times (0, h))$ . Now arguing as in section 2.1, we deduce that the support of  $\tilde{v}$  is included in  $\{(t, x) \in (0, T) \times (0, h) / t \leq \int_0^x \sqrt{\frac{\rho}{p}}(t) dt\}$ .

Hence  $\tilde{v}(t, x) = 0$  in  $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt < t\}$ . Since  $v_n$  tends to  $v$  hence  $v = \tilde{v}$  then  $\psi(x) = \frac{\partial \tilde{v}}{\partial t}(T, x) = 0$ .

**Remark 2.1** *It is in this last step, i.e. to characterize the domain of influence for the problem (12) (where the coefficients are time dependent), where we need the piecewise Lipschitz regularity of the coefficients  $p$  and  $\rho$ .*

*An other possible way to prove this boundary approximate controllability is to use the approach of [AvIv] where this question is related to a "good" distribution of the eigenvalues  $(\lambda_i)_{i \in \mathbb{N}}$ .*

Let us now give the following theorem which gives the projections of the eigenfunctions from the Gelfand spectral data.

**Theorem 3** *Let  $(p_j, q_j, \rho_j)$  be two families of coefficients,  $j = 1, 2$ . Let  $(\lambda_i^j, e_i^j)_{i=1}^\infty$  be the eigenlements of the two Sturm-Liouville operators corresponding to the two families of coefficients.*

*Then we have:*

$$(\lambda_i^1, p_1 \frac{d}{dx}(e_i^1)(0))_{i \in \mathbb{N}} = (\lambda_i^2, p_2 \frac{d}{dx}(e_i^2)(0))_{i \in \mathbb{N}} \Rightarrow \int_{\Gamma_T} (e_i^1)^2 \rho_1 dx = \int_{\Gamma_T} (e_i^2)^2 \rho_2 dx, \forall T > 0.$$

**Proof of Theorem 3**

Let  $\alpha_k(t)$ ,  $k \in \mathbb{N}$ , be a dense set of  $H_0^1(0, T)$ . From the previous theorem and the energy inequality we deduce that the finite combinations of the functions  $u^{\alpha_k}(T, x)$  form a dense set in  $L^2_{\rho_j}(\Gamma_T)$ . Since the Fourier coefficients of  $u^{\alpha_k}$  are determined by  $\lambda_k^j$  and  $p_j \frac{d}{dx}(e_k^j)(0)$ , then the products  $\int_{\Gamma_T} u^{\alpha_i} u^{\alpha_k} \rho_j dx$ ,  $(i, k) \in \mathbb{N}^2$  and  $j = 1, 2$  are also determined by  $(\lambda_i^j, p_j \frac{d}{dx}(e_i^j)(0))_{i \in \mathbb{N}}$ . But may be  $\int_{\Gamma_T} u^{\alpha_i} u^{\alpha_k} \rho_j dx \neq \delta_{i,k}$ ,  $(i, k) \in \mathbb{N}^2$ . By the Gram-Schmidt orthogonalisation procedure we can find an orthonormal basis of  $L^2_{\rho_j}(\Gamma_T)$  given by combinations of  $u^{\alpha_i}$ , i.e.  $v_s = \sum_{i=1}^{n(s)} d_{s,i} u^{\alpha_i}$ . By linearity we have  $v_s = u^{\beta_s}$  where  $\beta_s = \sum_{i=1}^{n(s)} d_{s,i} \alpha_i$ .

For every function  $a \in L^2_{\rho_j}(0, h_j)$ , we set

$$P_{\Gamma_T} a(x) = \begin{cases} a(x), & x \in \Gamma_T, \\ 0, & x \in (0, h_j) \setminus \Gamma_T, \end{cases}$$

then,  $P_{\Gamma_T} a$  can be written as

$$P_{\Gamma_T} a(x) = \sum_{i \in \mathbb{N}} \int_{\Gamma_T} a(x) v_i(T, x) \rho_j(x) dx \cdot v_i(T, x),$$

hence:

$$\int_{\Gamma_T} e_i^j(x) e_l^j(x) \rho_j(x) dx = \sum_{s \in \mathbb{N}} \int_{\Gamma_T} e_i^j(x) v_s(T, x) \rho_j(x) dx \int_{\Gamma_T} e_l^j(x) v_s(T, x) \rho_j(x) dx. \quad (13)$$



But  $\int_{\Gamma_T} e_i^j(x) v_s(T, x) \rho_j(x) dx$  satisfies:

$$\begin{cases} \frac{d^2 u}{dt^2} = -\lambda_i^j u + p_j \frac{\partial e_i^j}{\partial x}(0) \beta_s(t), & \text{in } (0, T), \\ u_i^f(0) = \frac{du_i^f}{dt}(0) = 0, \end{cases} \quad (14)$$

then

$$\int_{\Gamma_T} e_i^j(x) v_s(T, x) \rho_j(x) dx = \int_0^T \beta_s(t') \frac{\sin \sqrt{\lambda_i}(T - t')}{\sqrt{\lambda_i}} dt (p \frac{\partial e_i^j}{\partial x})(0).$$

Hence the sequence  $(\lambda_i^j, |p_j \frac{d}{dx}(e_i^j)(0)|)_{i \in \mathbb{N}}$  determine the infinite matrix given by the entries  $M_{i,l} := \int_{\Gamma_T} e_i^j e_l^j \rho_j dx$ ,  $(i, l) \in \mathbb{N}^2$ ,  $\forall T > 0$ . In particular, taking  $i = l$ , the spectral data  $(\lambda_i^j, |p_j \frac{d}{dx}(e_i^j)(0)|)_{i \in \mathbb{N}}$  determine the diagonal part of the matrix  $M_{i,l}$ , i.e.  $M_{i,i} = \int_{\Gamma_T} (e_i^j)^2 \rho_j dx$ ,  $i \in \mathbb{N}$ .

**End of the proof of the part A1) of Theorem 1**

Let us set  $\Gamma_T^j$ ,  $j = 1, 2$ , the domain of influence related to  $p_j, q_j, \rho_j$ . Since  $p_1 = p_2 =: p$  and  $\rho_1 = \rho_2 =: \rho$  then  $\Gamma_T^1 = \Gamma_T^2 =: \Gamma_T$ . The asymptotic of the eigenvalues give the equality  $h_1 = h_2 =: h$ . From the previous theorem, we deduce that  $\int_{\Gamma_T} (e_i^1)^2 \rho dx = \int_{\Gamma_T} (e_i^2)^2 \rho dx$  for every  $T > 0$ . This implies that  $\sqrt{p(g^{-1})(y)\rho(g^{-1})(y)}(e_i^1(g^{-1}))^2 = \sqrt{p(g^{-1})(y)\rho(g^{-1})(y)}(e_i^2(g^{-1}))^2$  for all  $y \leq \int_0^h g(x) dx$ ,  $j = 1, 2$ . Hence  $|e_i^1(x)| = |e_i^2(x)|$  in  $(0, h)$ . We recall that the eigenfunction of the first eigenvalue has one sign. Replacing in the equations satisfied by this first eigenvalue and its associated eigenfunction, we deduce that  $q_1 = q_2$ .

**A procedure to reconstruct  $q$ .**

The proof of this previous part gives a procedure to compute  $q$  from the spectral data. The steps can be given as follows:

(1a). Choose any basis  $\alpha_i$  of  $H_0^1(0, T)$  and for  $T \leq \int_0^h \sqrt{\frac{\rho}{p}} ds$  fixed, compute the sum:

$$r_{i,j}(T) := \int_{\Gamma_T} u^{\alpha_i}(T, x) u^{\alpha_j}(T, x) \rho(x) dx = \sum_{l \in \mathbb{N}} (u^{\alpha_i}(T, x), e_l) \times (u^{\alpha_j}(T, x), e_l)$$

where  $(u^{\alpha_i}(T, x), e_l) = \int_0^T \alpha_i(t') \frac{\sin \sqrt{\lambda_l}(T - t')}{\sqrt{\lambda_l}} dt (p \frac{de_l}{dx})(0)$ .

(1b). By the Gram Schmidt orthogonalisation procedure, construct  $\beta_s(T)$ ,  $s = 1, 2, \dots$  from  $r_{i,j}(T)$ .

(2). Compute

$$\int_{\Gamma_T} (e_1(x))^2 \rho(x) dx := \sum_{s \in \mathbb{N}} [\int_0^T \beta_s(t') \frac{\sin \sqrt{\lambda_1}(t - t')}{\sqrt{\lambda_1}} dt (p \frac{de_1}{dx})(0)]^2.$$

(3). Compute the values of  $e_1(x)$  as a derivative of the integral in the step 2.

(4). Compute  $q(x) := \frac{(p(e_1)')'(x) + \lambda_1 \rho e_1(x)}{e_1(x)}$ .

**Remark 2.2** 1) The step 4) can be replaced by:

(4)' Compute  $\int_0^h q(x) e_1(x) \xi_n(x) dx = \int_0^h e_1(x) \xi_n(x) \rho(x) dx (\lambda_1 + \omega_n)$ , where  $(\xi_n, \omega_n)$  are the eigenelements of the problem  $-(pu')' = \lambda pu$  in  $(0, h)$  with Dirichlet boundary conditions.

In the case where  $p = 1$ , we can choose an explicit basis given by  $\sin n(\frac{x}{h})$  to compute the Fourier coefficients of  $q(x)e_1(x)$ .

2) The advantage of this procedure is that we take into account the three coefficients where  $p$  and  $\rho$  can be discontinuous. For the case  $q = 0$  or  $p = \rho = 1$ , one can find, among others, procedures based on the Gelfand-Levitan representation, see [CCPR] chapter 3 for a review of these procedures.

### 3. Proof of the part A2). Uniqueness of two coefficients

In this section we prove uniqueness of two coefficients among  $p, q$  and  $\rho$ . In [S], we proved uniqueness of  $p$  (or  $\rho$ ) and  $q$  if  $p$  (or  $\rho$ ) is piecewise constant,  $q$  is piecewise analytic and  $\rho$  (or  $p$ ) is piecewise indefinitely differentiable. Here we show how to prove the same result for  $q$  in  $L^\infty(0, h)$ ,  $p$  (or  $\rho$ ) piecewise constant and  $\rho$  (or  $p$ ) piecewise Lipschitz.

Indeed, let  $\rho$  be piecewise Lipschitz and fixed. As in [S], we prove that the Guelfand spectral data implies the values of  $p$  on  $x = 0$ . Since  $p$  is piecewise constant then we know  $p$  on  $(0, c_1)$ , where  $c_1$  is the first discontinuity point. Let us now prove that  $q$  is known on  $(0, c_1)$ . To do this, we prove as in the previous section that  $\int_{\Gamma_t} (e_i)^2(x)\rho(x)dx$  is known for every  $t$ . This implies that  $|e_i(x)|$  is known on  $(0, c_1)$ . From the equations satisfied by these eigenfunctions we deduce that  $q(x)$  is known on  $(0, c_1)$ .

Now, proceeding by Layer Stripping, since  $p, q$  and  $\rho$  are known on  $(0, c_1)$ , then we deduce that for the Sturm-Liouville problems stated on  $(0, c_1)$ , we have the Gelfand data. Then step by step, we prove uniqueness of  $p$  (or  $\rho$ ) and  $q$  and hence of  $h$ .

### 4. Proof of the part B).

Let  $p_j$  and  $q_j$  be fixed in  $L^\infty(0, h_j)$ , i.e.  $p_1 = p_2$  and  $q_1 = q_2$ . We set  $Au := \frac{1}{\rho}((-pu')' + \frac{q}{\rho}u)$ . Let  $k > 0$  be a continuous function. We set  $\phi_i := ke_i$ , then  $A_k\phi_i := kAk^{-1}\phi_i = \lambda_i\phi_i$  and  $A_k u = -\frac{k^2}{\rho}(k^{-2}pu')' + kA(k^{-1}) \cdot u$ . We also have  $[k^{-2}p\phi_i'](0) = k^{-1}(0)p(e_i)'(0)$ . Now for  $j = 1, 2$ , we take  $k := k_j$  such that  $k_j^{-1}$  is an  $H^1(0, h)$ -solution of the elliptic problem:

$$\begin{cases} A_j(k_j^{-1}) = 0, & \text{in } (0, h_j), \\ k_j^{-1}(0) = 1, k_j^{-1}(h) = 0. \end{cases} \quad (15)$$

then  $k_j^{-1}$  is positive by the maximum principle, see ([PW], Chapter 1, Theorem 3).

Since  $p$  and  $q$  are fixed then  $k_1 = k_2$ . Using this transformation we have the spectral data of the operators given by taking  $p_j := k^{-2}p$ ,  $q_j := 0$ ,  $\rho_j := k^{-2}\rho_j$  and  $h_j := h_j$ . Now we use the transformation  $g(x) := \int_0^x k^2(t)p^{-1}(t)dt$  to transform the last operators to the case where  $p_j := 1$ ,  $q_j := 0$ ,  $\rho_j := k^{-4}(g^{-1})\rho_j(g^{-1})p(g^{-1})$  and  $h_j := \int_0^{h_j} k^2(t)p^{-1}(t)dt$ . This operator has been studied in [Bell] where it is proved that

$$k^{-4}(g^{-1})\rho_1(g^{-1})p(g^{-1}) = k^{-4}(g^{-1})\rho_2(g^{-1})p(g^{-1}) \quad (16)$$

Hence we deduce that  $\rho_1 = \rho_2$  and  $h_1 = h_2$ .

**Remark 4.1** In [BK], the case where  $p = 1$  and  $q = 0$  is studied and a procedure to reconstruct  $\rho$  from the spectral data is given.

Since in our case, we used the gauge function  $k$ , which can be constructed by solving a direct elliptic problem, and the Liouville type transformation, which is explicit, to transform this general case to the case studied in [BK], then a procedure to reconstruct  $\rho$  from the spectral

data can be obtained. In [BK],  $\rho$  is assumed to be  $C^2[0,h]$ . This condition is assumed to prove the boundary controllability and not to give the reconstruction procedure. Hence, this procedure is applicable in our case.

## 5. Equivalence of the Gelfand data and the Borg-Levinson data.

In the introduction we called the Borg-Levinson data the following sequence  $(\lambda_n, \mu_n)_{n \in \mathbb{N}}$ , where  $(\lambda_n)_{n \in \mathbb{N}}$  is the sequence of the eigenvalues of the Sturm-Liouville problem where we take as boundary conditions the Dirichlet one on  $x_0 = 0$  and  $x_0 = h$  while  $(\mu_n)_{n \in \mathbb{N}}$  is the sequence of the eigenvalues of the Sturm-Liouville problem where we take as boundary conditions the Dirichlet one on  $x_0 = 0$  and the Neuman one on  $x_0 = h$ .

The aim of this section is to prove the equivalence of the Gelfand data and the Borg-Levinson data. More precisely in section 4.1, we prove that the Gelfand spectral data imply the Borg-Levinson data for general bounded coefficients  $p, q$  and  $\rho$ . In section 4.2, we prove that the Borg-Levinson data imply the Gelfand data assuming that  $p$  and  $\rho$  are BV-functions and  $p(0)\rho(0)$  is known.

### 5.1. A) Gelfand data imply Borg-Levinson data.

Suppose that for two families of coefficients  $p_j, q_j, \rho_j, j = 1, 2$ , we have  $(\lambda_i^1, |p_1(e_i^1)'(0)|)_{i \in \mathbb{N}} = (\lambda_i^1, |p_1(e_i^1)'(0)|)_{i \in \mathbb{N}}$ . Now for  $j = 1, 2$ , we take  $k_j$  such that  $k_j^{-1}$  is the  $H^1(0, h)$ -solution of the elliptic problem:

$$\begin{cases} A_j(k_j^{-1}) = 0, & \text{in } (0, h), \\ k_j^{-1}(0) = 1, p_j(k_j^{-1})'(h) = 0. \end{cases} \quad (17)$$

Then  $k_j^{-1}$  is positive by the maximum principle, see ([PW]).

With this choice of  $k_j$ , we have  $A_{k_j}^j u = -\frac{k_j^2}{\rho_j}(k_j^{-2}p_j u)'$ . We set  $\phi_i^j := k_j e_i^j$ , hence:

$$\begin{cases} -(k_j^{-2}p_j(\phi_i^j))' = \lambda_i^j \rho_j k_j^{-2} \phi_i^j, & \text{in } (0, h), \\ \phi_i^j(0) = \phi_i^j(h) = 0, \\ |k_1^{-2}p_1(\phi_i^1)(0)| = |k_2^{-2}p_2(\phi_i^2)(0)|, \\ \int_0^h k_j^{-2} \rho_j (\phi_i^j)^2 dx = 1. \end{cases} \quad (18)$$

In ([S], Remark 4), it is proved that  $|k_1^{-2}p_1(\phi_i^1)'(h)| = |k_1^{-2}p_1(\phi_i^1)'(h)|, i \in \mathbb{N}$ . We set  $\psi_i^j = k_j^{-2}p_j(\phi_i^j)'$ , then  $\psi_i^j$  satisfy the following Neumann problem:

$$\begin{cases} -(k_j^2 \rho_j^{-1}(\psi_i^j))' = \lambda_i^j p_j^{-1} k_j^2 \psi_i^j, & \text{in } (0, h), \\ k_j^2 \rho_j^{-1}(\psi_i^j)'(0) = k_j^2 \rho_j^{-1}(\psi_i^j)'(h) = 0, \\ |\psi_i^1(0)| = |\psi_i^2(0)|, |\psi_i^1(h)| = |\psi_i^2(h)|, \\ \int_0^h k_j^2 p_j^{-1}(\psi_i^j)^2 dx = \lambda_i^1 = \lambda_i^2. \end{cases} \quad (19)$$

Let us now define the following elliptic problems:

$$\begin{cases} \text{Find } u_j \in H^1(0, h) \text{ such that:} \\ (k_j^2 \rho_j^{-1}(u_j))' + \lambda_i^j p_j^{-1} k_j^2 u_j = 0, & \text{in } (0, h), \\ k_j^2 \rho_j^{-1}(u_j)'(0) = a, k_j^2 \rho_j^{-1}(u_j)'(h) = b. \end{cases} \quad (20)$$

The functionals  $(a, b) \longrightarrow \Lambda_j^N(\lambda) := (u^j(0), u^j(h)), j = 1, 2$ , are called the Neumann to Dirichlet operators.

In ([S], section 2.3), It is proved that these two Neuman to Dirichlet operators are equal for all  $\lambda \in \mathfrak{R} \setminus \{\lambda_i^j; j = 1, 2; i \in \mathfrak{N}\}$ .

**Remark 5.1** For general coefficients  $(p, q, \rho)$  and for the Dirichlet problem, we have no proof of equality of the Dirichlet to Neuman map if we have equality of the Gelfand spectral data. This is the reason why we transformed this Dirichlet spectral problem to the Neuman spectral problem for which we have a proof, see [CK] or [S]. In the previous part, we proved equality of the Neuman-Dirichlet operators from the Dirichlet Guelfand spectral data. Using the Alessandrini identity we deduce the equality of the Dirichlet-Neuman operators.

Let us now use the Alessandrini identity [Al]. Let  $u_j$  be an  $H^1(0, h)$ -solution of (20), then we have the following equality:

$$\int_0^h (k_1^2 \rho_1^{-1} - k_2^2 \rho_2^{-1}) u_1' u_2' - \lambda (k_1^2 p_1^{-1} - k_2^2 p_2^{-1}) u_1 u_2 dx = ((\Lambda_1^N - \Lambda_2^N)(a, b)) \cdot (c, d),$$

where  $(k_1^2 \rho_1^{-1} u_1'(0), k_1^2 \rho_1^{-1} u_1'(h)) = (a, b)$  and  $(k_2^2 \rho_2^{-1} u_2'(0), k_2^2 \rho_2^{-1} u_2'(h)) = (c, d)$ .

Since  $\Lambda_1^N(\lambda) = \Lambda_2^N(\lambda)$  for all  $\lambda \in \mathfrak{R} \setminus \{\lambda_i^j; j = 1, 2; i \in \mathfrak{N}\}$ , then:

$$\int_0^h (k_1^2 \rho_1^{-1} - k_2^2 \rho_2^{-1}) u_1' u_2' - \lambda (k_1^2 p_1^{-1} - k_2^2 p_2^{-1}) u_1 u_2 dx = 0 \quad (21)$$

for every  $H^1(0, h)$ -solution of (20).

We define the mixed-mixed operators:

$$(a, b) \longrightarrow \Lambda_j^M := (u_j(0), k_j^2 \rho_j^{-1} u_j'(h)), j = 1, 2$$

We use also the Alessandrini identity for the mixed problems:

$$\int_0^h (k_1^2 \rho_1^{-1} - k_2^2 \rho_2^{-1}) u_1' u_2' - \lambda (k_1^2 p_1^{-1} - k_2^2 p_2^{-1}) u_1 u_2 dx = ((\Lambda_1^M - \Lambda_2^M)(a, b)) \cdot (c, d),$$

for all  $H^1(0, h)$ -solution of (20), where  $(u_1(0), k_2^2 p_2^{-1} u_1'(0)) = (a, b)$  and  $(u_2(0), k_2^2 p_2^{-1} u_2'(0)) = (a, b)$ .

From (21), we deduce that  $(\Lambda_1^M - \Lambda_2^M)(a, b) = 0$  for all  $(a, b) \in \mathfrak{R}^2$  and  $\lambda \in \mathfrak{R} \setminus (S_1 \cup S_2)$  where  $S_j$  are the set of the mixed eigenvalues of the operators given by  $-\frac{p_j}{k_j^2} (\frac{k_j^2}{\rho_j} u')'$  with the mixed homogeneous boundary conditions. We denote these eigenvalues by  $\bar{\mu}_i^j$ .

It is known that if  $\Lambda_1^M = \Lambda_2^M$ ,  $\forall \lambda \in \mathfrak{R} \setminus (S_1 \cup S_2)$ , then  $S_1 = S_2$  (see [S], lemma 2.5). This means that  $\bar{\mu}_i^1 = \bar{\mu}_i^2$ ,  $i \in \mathfrak{N}$ . Set  $\psi_i^{j,M}$  the corresponding eigenfunctions.

Using the reverse transformation:  $\phi_i^{j,M} = k_j^2 \rho_j^{-1} (\psi_i^{j,M})'$ , we find that the mixed eigenvalues given by the boundary conditions,  $\phi_i^{j,M}(0) = 0$ ,  $k_j^{-2} p_j (\phi_i^{j,M})'(0) = 0$ , are also the same  $\bar{\mu}_i^1 = \bar{\mu}_i^2$ . Now using the reverse gauge transformation defining  $e_i^{j,M} := k_j^{-1} \phi_i^{j,M}$ , then we find that  $\mu_i^j = \bar{\mu}_i^j$ . Hence we have equality of the mixed eigenvalues for the operators given by  $(p_j, q_j, \rho_j)$ .

## 5.2. B) Borg-Levinson data imply Gelfand data.

In this part we suppose that  $p$  and  $\rho$  are  $BV$ -functions and that  $p(0)\rho(0)$  is known.

Suppose that  $\mu_i^1 = \mu_i^2$  and  $\lambda_i^1 = \lambda_i^2$ , for the mixed and Dirichlet operators given by the differential expressions  $-\frac{1}{\rho}(pu')' + \frac{q}{\rho}$ . We are going to prove that the Borg-Levinson spectral data imply the Gelfand spectral data.

First we transform this problem to the case where  $q = 0$  as in the case A) by introducing the gauge function  $k$  solution of the problem  $-(p(k^{-1})')' + qk^{-1} = 0$  in  $(0, h)$  and  $k^{-1}(0) = 1$  and  $p(k^{-1})'(h) = 0$ . Hence  $\mu_i^j$  and  $\lambda_i^j$  are the mixed and Dirichlet spectrum for the operator  $-\frac{1}{\rho k^{-2}}(pk^{-2}u)'$ . Secondly, we transform this problem to the case where the equations are  $-\frac{1}{\tilde{p}}(\tilde{p}u)'$ , where  $\tilde{p} = p(g^{-1}(y))\rho(g^{-1}(y))k^{-4}(g^{-1}(y))$  and  $g(x) = \int_0^x \sqrt{\frac{\rho}{p}}(t)dt$ . From [An], we deduce uniqueness of  $\tilde{p}$  since  $k^{-4}(0)\rho(0)p(0) =: \tilde{p}(0)$  is known and  $\tilde{p}$  is a  $BV$ -function. In particular, one has equality of the Gelfand data. Now, using the reverse transformations we deduce the equality of the Gelfand data for the expression  $-\frac{1}{\rho}(pu')' + \frac{q}{\rho}u$ .

**Remark 5.2** *The additional condition on  $p(0)\rho(0)$  is natural since in the case  $q = 0$ , we have uniqueness just up to a multiplicative constant.*

## 6. Uniqueness from the nodes.

In this section we suppose that  $p$  and  $\rho$  are  $BV$ -functions continuous from the right and at  $x = h$ .

Let  $N$  be the set of the nodes of the eigenfunctions  $e_i$ . As in the section 4 we introduce the gauge function  $k(x)$  solution of the problem  $-(p(k^{-1})')' + qk^{-1} = 0$  in  $(0, h)$  with the boundary conditions  $k^{-1}(0) = k^{-1}(h) = 1$ . The function  $k(x)$  is in  $H^1(0, h)$  and  $k > 0$ . We set  $\phi_i := ke_i$ . We know that  $\phi_i$ ,  $i \in \mathbb{N}$ , are the eigenfunctions of the operators  $A_k u := \rho^{-1}k^{-2}(pk^{-2}u)'$  on  $(0, h)$ . Hence the set of nodes of the functions  $\phi_i$ ,  $i \in \mathbb{N}$ , is exactly  $N$  since  $k > 0$ . We recall the following result which gives the asymptotic of the eigenvalues (see [HaMc2]):

$$|\sqrt{\lambda_n} \int_0^h \sqrt{\frac{\rho}{p}}(x)dx - n\pi| \leq \frac{1}{4}V(\ln(\frac{p\rho}{k^4})), \quad (22)$$

where  $V(f)$  is the total variations of  $f$ .

Since  $p$  and  $q$  are fixed then  $k$  is also known. Hence the only unknown is  $\rho$ .

In [HaMc2], the authors proved that for the equation  $-(pu')' = \lambda\rho u$  with  $p$  and  $\rho$  are  $BV$ -functions continuous from the right and at  $x = h$ , uniqueness of  $\rho$ , up to a multiplicative constant, holds from a dense subset of the nodes if  $p$  is fixed. This set of nodes has the following property: for  $n \in \mathbb{N}$ , if this set contains the  $j^{\text{th}}$  node  $x_j^n$  of the  $n^{\text{th}}$  eigenvalue then it also contains the  $(j+1)^{\text{th}}$  or the  $(j-1)^{\text{th}}$  node of this eigenfunction.

Hence using this result we deduce that  $\rho k^2$  and hence  $\rho$  is unique up to a multiplicative constant from this dense subset of nodes.

Let us now consider the uniqueness of both  $q$  and  $\rho$ . We suppose in addition to the previous part that the coefficient  $p$  and  $\rho$  are in  $H^2(0, h)$  and  $q$  is continuous almost every where. In this case we have the following asymptotic of the eigenvalues:

$$|\sqrt{\lambda_n} \int_0^h \sqrt{\frac{\rho}{p}}(x)dx - n\pi| = O(\frac{1}{n}), \quad (23)$$

see ([CCPR], page 82) and [HaMc2] for more references about the asymptotic of the eigenvalues.

We prove uniqueness of the coefficients on the points of continuity.

The method of the proof is the one given [HaMc2].

Let  $x$  be a point of continuity of the coefficients  $q$  and  $\rho$ . Set  $x_{l(n)}^n$  a sequence of this dense set tending to the point  $x$ .

Consider the equations  $-(pu'_j)' + q_j u_j = \lambda_j^n \rho_j u_j$  in  $(x_{l(n)}^n, x_{l(n)+1}^n)$ . We can consider  $\lambda_j^n$  as the first eigenvalue for the operators replacing  $(0, h)$  by  $(x_{l(n)}^n, x_{l(n)+1}^n)$ . Multiplying the  $(j = 1)$ -equation by  $u_2$  and the  $(j = 2)$ -equation by  $u_1$ , integrating over  $I_n := (x_{l(n)}^n, x_{l(n)+1}^n)$  and subtracting the results we find:

$$\int_{I_n} [(q_1 - q_2) + (\lambda_n^2 \rho_2 - \lambda_n^1 \rho_1)] u_1^n u_2^n dx = 0.$$

Hence there exist  $x'_n$  and  $x''_n$  in  $I_n$  such that:

$$[(q_1 - q_2) + (\lambda_n^2 \rho_2 - \lambda_n^1 \rho_1)](x'_n) \geq 0 \quad (24)$$

and

$$[(q_1 - q_2) + (\lambda_n^2 \rho_2 - \lambda_n^1 \rho_1)](x''_n) \leq 0 \quad (25)$$

since  $u_1^n u_2^n$  can be chosen to be a positive function in  $(x_{l(n)}^n, x_{l(n)+1}^n)$ .

When  $n$  tends to infinity,  $x'_n$  and  $x''_n$  tend to  $x$ . Dividing the inequalities (24) and (25) by  $(\lambda_n^2)$ , going to the limit when  $n$  tends to infinity and using the fact that  $\sqrt{\frac{\lambda_n^1}{\lambda_n^2}}$  tends to a constant  $R$ , which is given by (22) or (23), we deduce that  $\rho_2(x) = R^2 \rho_1(x)$ . Since  $x$  is arbitrary we deduce that  $\rho_2(x) = R^2 \rho_1(x)$  in every point of continuity of the coefficients.

From (22), we deduce that  $R = \frac{\int_0^h \sqrt{\frac{\rho_2}{p}}(x) dx}{\int_0^h \sqrt{\frac{\rho_1}{p}}(x) dx}$ .

Now (24) and (25) can be rewritten as :

$$[(q_1 - q_2) + (\lambda_n^2 R^2 - \lambda_n^1) \rho_1](x'_n) \geq 0 \quad (26)$$

and

$$[(q_1 - q_2) + (\lambda_n^2 R^2 - \lambda_n^1) \rho_1](x''_n) \leq 0. \quad (27)$$

But  $(\lambda_n^2 R^2 - \lambda_n^1) = \frac{1}{R_1^2} [R_2^2 \lambda_n^2 - R_1^2 \lambda_n^1]$  where  $R_j := \int_0^h \sqrt{\frac{\rho_j}{p}}(x) dx$ .

Or from the bounds of the eigenvalues (23), we deduce that the sequence  $\frac{1}{R_1^2} [R_2^2 \lambda_n^2 - R_1^2 \lambda_n^1]$  is bounded. Hence we can take a converging subsequence. We set  $\alpha$  the limit of this subsequence. Hence from (26) and (27), we deduce using the relative subsequence for  $\lambda_n^j$ , that  $(q_1 - q_2 - \alpha \rho_1)(x) = 0$ . Finally we have  $q_1 - q_2$  is in the one dimensional subspace generated by  $\rho_1$ .

**Remark 6.1** 1) Using this gauge transformation, we deduce similar asymptotic for the eigenvalues for the equation  $-(pu')' + qu = \lambda \rho u$  as those given in [HaMc2].

2) If we take  $p$  and  $\rho$  discontinuous the asymptotic (23) are not valid (see [HaMc2]). Hence we can not deduce the boundness of the sequence  $\frac{1}{R_1^2} [R_2^2 \lambda_n^2 - R_1^2 \lambda_n^1]$ .

## 7. Appendix.

Let  $p_n, q_n$  and  $\rho_n$  be three sequences of functions in  $L^\infty(0, h)$  such that  $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma$   $\forall n$  in  $\mathbb{N}$ . Let  $u_n$  be the solution of the following hyperbolic problem:

$$\begin{cases} \rho_n \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial}{\partial x} (p_n \frac{\partial u_n}{\partial x}) + q_n u_n = 0 \text{ in } (0, T) \times (0, h), \\ u_n(t, 0) = f_1(t), u_n(t, h) = f_2(t), t \in (0, T), \\ u_n(0, x) = \psi_0(x), \frac{\partial u_n}{\partial t}(0, x) = \psi_1(x), x \in (0, h), \end{cases} \quad (28)$$

where  $\psi_0, \psi_1$  are in  $C_0^\infty[0, h]$  and  $f$  is in  $C_0^\infty[0, T]$ .

The aim of this part of the appendix is to prove that  $u_n$  is bounded in the energy norm.

Let  $y(x) := g_n(x) = \frac{1}{h_n} \int_0^x p_n^{-1}(t) dt$ . We set  $v_n(t, y) = u_n(t, g_n^{-1}(y))$ ,  $\tilde{\rho}_n(y) := \rho_n(g_n^{-1}(y))$ ,  $\tilde{p}_n(y) = p_n(g_n^{-1}(y))$ ,  $\tilde{q}_n(y) = q_n(g_n^{-1}(y))$  and  $h_n = \int_0^h p_n(t) dt$ . Hence  $v_n$  satisfies the following problem:

$$\begin{cases} \frac{\partial^2 v_n}{\partial t^2} - \frac{h_n^2}{\tilde{p}_n(y)\tilde{\rho}_n(y)} \frac{\partial^2 v_n}{\partial y^2} + \frac{\tilde{q}_n(y)}{\tilde{\rho}_n(y)} v_n = 0 \text{ in } (0, T) \times (0, 1), \\ v_n(t, 0) = f_1(t), u_n(t, 1) = f_2(t), t \in (0, T), \\ v_n(0, x) = \psi_0(g_n^{-1}(y)), \frac{\partial v_n}{\partial t}(0, x) = \psi_1(g_n^{-1}(y)) y \in (0, 1). \end{cases} \quad (29)$$

Now let  $F(t, y)$  be a regular function on  $(0, T) \times (0, 1)$  such that  $F(t, 0) = f_1(t)$  and  $F(t, 1) = f_2(t)$ ,  $t \in (0, T)$ .

We set  $\bar{v}_n = F(t, y) - v_n(t, y)$ , then  $\bar{v}_n$  satisfies the following problem:

$$\begin{cases} E_n \frac{\partial^2 \bar{v}_n}{\partial t^2} - \frac{\partial^2 \bar{v}_n}{\partial y^2} + B_n \bar{v}_n = G_n(t, y) \text{ in } (0, T) \times (0, 1), \\ \bar{v}_n(t, 0) = \bar{v}_n(t, 1) = 0, t \in (0, T), \\ v_n(0, x) = -\psi_0(g_n^{-1}(y)) + F(0, y), y \in (0, 1), \\ \frac{\partial v_n}{\partial t}(0, x) = -\psi_1(g_n^{-1}(y)) + \frac{\partial F}{\partial t}(0, y), y \in (0, 1), \end{cases} \quad (30)$$

where  $G_n(t, y) := E_n \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial y^2} + B_n F$  and  $E_n = \frac{1}{h_n^2} \tilde{p}_n(y) \tilde{\rho}_n(y)$ ,  $B_n = \frac{1}{h_n^2} \tilde{p}_n(y) \tilde{\rho}_n(y)$ .

It is clear that  $G_n(t, y) \in L^2((0, T) \times (0, 1))$  and  $\|G_n\|_{L^2((0, T) \times (0, 1))} \leq c$ ,  $\forall n \in \mathbb{N}$ . Since  $p_n \geq \gamma > 0$ ,  $\forall n \in \mathbb{N}$ , then  $\psi_0(g_n^{-1}(y))$  and  $\psi_1(g_n^{-1}(y))$  are bounded in  $H^1(0, 1)$  and  $L^2(0, 1)$  respectively. Mimiking the proof given in [[La], section IV.2, (2.15) and section IV, (3.17)] we find that  $\bar{v}_n$  is in  $C([0, T], H^1(0, 1)) \cap C^1([0, T], L_{E_n}^2(0, 1))$  with a bounded norm. Note that the sequence  $E_n$  is bounded from below and above. Since  $F(t, y)$  is regular then we have  $v_n \in C([0, T], H^1(0, 1)) \cap C^1([0, T], L_{E_n}^2(0, 1))$  with a bounded norm. Now using the reverse transformation we deduce that  $v_n \in C([0, T], H^1(0, h)) \cap C^1([0, T], L^2(0, h))$  with a bounded norm.

**Proposition 7.1** *Let  $(p_n, q_n, \rho_n)$  be three sequences of functions in  $L^\infty(0, h)$  satisfying  $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma \forall n$  in  $\mathbb{N}$  and  $(p, q, \rho)$  in  $L^\infty(0, h)$  such that  $(p_n, q_n, \rho_n)$  tends to  $(p, q, \rho)$  in  $(L^2(0, h))^3$ . Let  $u_n$  and  $u$  be the solutions (28) with the coefficients  $(p_n, q_n, \rho_n)$  and  $(p, q, \rho)$  respectively. Then  $u_n$  tends to  $u$  in  $L^2((0, T) \times (0, h))$ .*

### Proof of Proposition 7.1

From the energy inequality, we deduce that  $u_n$  is bounded in  $L^2((0, T), H^1(0, h)) \cap H^1((0, T), L^2(0, h))$  as in the previous part of this appendix. Then we can take a subsequence, denoted also by  $u_n$ , and a function  $\bar{u}$  such that  $u_n$  tends to  $\bar{u}$  in  $H^1((0, T), L^2(0, h))$  weakly. Hence  $\|u_n - \bar{u}\|_{L^2(0, h)}(t)$  is bounded in  $H^1(0, T)$  then, up to a

subsequence,  $\int_0^T \|u_n - \bar{u}\|_{L^2(0,h)}(t)dt$  is convergent. This means that  $u_n - \bar{u}$  is convergent in  $L^2((0, T) \times (0, h))$  and the limit is zero.

Using the fact that  $u_n$  tends weakly to  $\bar{u}$  in  $L^2([0, T], H^1(0, h)) \cap H^1([0, T], L^2(0, h))$  and the definition of the weak solution of the hyperbolic problem, we deduce that  $\bar{u}$  is solution of (28) replacing  $p_n, q_n$  and  $\rho_n$  respectively by  $p, q$  and  $\rho$ . Hence  $\bar{u} = u$  by uniqueness.

## Acknowledgments

The author thanks Prof. Y. Kurylev for proposing him to consider this one dimensional inverse spectral problem via the BC method. He also thanks Prof. G. Nakamura for many helpful and stimulating discussions. This work is supported by the Japanese Society for Promotion of Sciences, JSPS.

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