Factorizations Of Functions In $H^p(T^n)$

By

Takahiko Nakazi

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan

2000 Mathematics Subject Classification : 32 A 35, 46 J 15

Key words and phrases : Hardy space, polydisc, factorization, extreme point
Abstract. We are interested in extremal functions in a Hardy space $H^p(T^n)$ ($1 \leq p \leq \infty$). For example, we study extreme points of the unit ball of $H^1(T^n)$ and give a factorization theorem. In particular, we show that any rational function can be factorized.
§1. Introduction

Let $D^n$ be the open unit polydisc in $C^n$ and $T^n$ be its distinguished boundary. The normalized Lebesgue measure on $T^n$ is denoted by $dm$. For $0 < p \leq \infty$, $H^p(D^n)$ is the Hardy space and $L^p(T^n)$ is the Lebesgue space on $T^n$. Let $N(D^n)$ denote the Nevanlinna class. Each $f$ in $N(D^n)$ has radial limits $f^*$ defined on $T^n$ a.e.$dm$. Moreover, there is a singular measure $d\sigma_f$ on $T^n$ determined by $f$ such that the least harmonic majorant $u(\log |f|)$ of $\log |f|$ is given by $u(\log |f|)(z) = P(z)(\log |f^*| + d\sigma_f)$ where $P(z)$ denotes Poisson integration and $z = (z_1, z_2, \cdots, z_n) \in D^n$. Put $N_s(D^n) = \{f \in N(D^n) ; \ d\sigma_f \leq 0\}$, then $H^p(D^n) \subset N_s(D^n) \subset N(D^n)$ and $H^p(D^n) = N_s(D^n) \cap L^p(T^n) \cap L^p(T^n)$. These facts are shown in [5, Theorem 3.3.5].

Let $\mathcal{L}$ be a subset of $L^\infty(T^n)$. For a function $f$ in $H^p$, put

$$\mathcal{L}^f_p = \{\phi \in \mathcal{L} ; \ \phi f \in H^p\}.$$

When $\mathcal{L}^f_p \subseteq H^\infty$, $f$ is called an $\mathcal{L}$-extremal function for $H^p$. When $\mathcal{L} = L^\infty(T^n)$, $\mathcal{L} = L_n^\infty(T^n)$ or $\mathcal{L} = L^\infty(T^n)$ is the set of all unimodular functions, such $\mathcal{L}$-extremal functions have been considered in [3]. In [3], the author studied functions which have harmonic properties (A),(B),(C). For example, the property (A) is the following : If $f \in H^p$ and $|f| \geq |g|$ a.e. on $T^n$, then $|f| \geq |g|$ on $D^n$. It is easy to see that $f$ is an $\mathcal{L}$-extremal function for $H^p$ and $\mathcal{L} = L^\infty(T^n)$ if and only if $f$ has the property (A). The properties (B) and (C) are related to $\mathcal{L} = L_n^\infty(T^n)$ and $\mathcal{L} = L_\infty^\infty(T^n)$, respectively. In this paper, as $\mathcal{L}$ we consider only the above three sets.

**Definition.** When $f$ is not $\mathcal{L}$-extremal for $H^p$, if there exists a function $\phi$ in $\mathcal{L}$ such that $\phi f = h$ is an $\mathcal{L}$-extremal function for $H^p$, we say that $f$ is factorized as $f = \phi^{-1}h$.

In this paper, we are interested in when $f$ is factorized for $\mathcal{L} = L^\infty(T^n)$ or $\mathcal{L} = L_n^\infty(T^n)$. The function $h$ in $N(D^n)$ is called outer function if

$$\int_{T^n} \log |h| dm = \log \left| \int_{T^n} h dm \right| > -\infty.$$

The function $q$ in $N_s(D^n)$ is called inner function if $|q| = 1$ a.e.$dm$ on $T^n$. When $\mathcal{L} \subset \mathcal{L}'$, a $\mathcal{L}'$-extremal function is always $\mathcal{L}$-extremal. If $f$ is an outer function, then $f$ is $\mathcal{L}$-extremal for $\mathcal{L} = L^\infty(T^n)$. In fact, if $\phi f$ is in $H^p$ then $\phi$ belongs to $f^{-1}H^p$ and $f^{-1}H^p \subset N_s$. Hence if $\phi$ is bounded then $\phi$ belongs to $H^\infty$ because $N_s \cap L^\infty(T^n) = H^\infty$. When $n = 1$, $f$ is $\mathcal{L}$-extremal if and only if $f$ is an outer function. This is known because $f$ has an inner outer factorization.

In this paper, for a subset $S$ in $L^\infty$ we say that $S$ is of finite dimension if the linear span of $S$ is of finite dimension. We use the following notations.

$$z = (z_j, z_j'), \quad z_j' = (z_1, \cdots, z_{j-1}, \ z_{j+1}, \cdots, z_n).$$
exists a real number \( \lambda \) for \( L \) in Lebesgue measure on \( T \). \( P \) has a \( L^2 \) and similarly. Put \( h \) as a basis.

\[ \int_{-\infty}^{\infty} \]

If \( f = qh \) where \( q \) is inner and \( h \) is \( L^\infty \)-extremal for \( H^p \), then \( f \) has a \( L^\infty \)-factorization in \( H^p \): \( f = \phi^{-1}k \) where \( \phi = q+\bar{q} \) and \( k = (1+q^2)h \) is \( L^\infty \)-extremal for \( H^p \).

\[ \] \[ \] \[ \] \[ \] \[ \] \[ \] Proof. It is enough to show that \((1+q^2)h\) is \( L^\infty \)-extremal for \( H^p \). If \( \psi \in L^\infty(T^n) \) and \( \psi(1+q^2)h \in H^p \) then \( \psi(1+q^2) \) belongs to \( H^\infty \) because \( h \) is \( L^\infty \)-extremal for \( H^p \). Since \( 1+q^2 \) is outer and so \( 1+q^2 \) is \( L^\infty \)-extremal for \( H^p \), \( \psi \) belongs to \( H^\infty \).

Proposition 2. Suppose \( f \) is a nonzero function in \( H^1 \). \( f \) is \( L^\infty \)-extremal for \( H^1 \) if and only if \( f/\|f\|_1 \) is an extreme point of the unit ball of \( H^1 \).

Proof. It is well known.

The degree of a monomial \( z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) (where \( \alpha_i \in \mathbb{Z} \)) is \( \alpha_1 + \cdots + \alpha_n \). The degree of a polynomial \( P \) is the maximum of the degrees of the monomials which occur in \( P \) with non-zero coefficient. The degree of a rational function \( f = P/Q \) is the maximum of deg \( P \), deg \( Q \), provided that all common factors of positive degree have first been cancelled.

Theorem 3. Let \( 0 < p \leq \infty \) and \( \mathcal{L} = L^\infty_R \). If \( f \) is a nonzero function in \( H^p \) and \( \mathcal{L}_p^f \) is of finite dimension then there exists a function \( \phi \) in \( \mathcal{L} \) such that \( f = \phi^{-1}h \) and \( h \) is \( \mathcal{L} \)-extremal for \( H^p \).

Proof. Suppose that \( \mathcal{L}_p^f \) is of finite dimension. Then there exist \( s_1, s_2, \ldots, s_n \) in \( L^\infty_R \) such that \( \{s_j\}_{j=1}^n \) is a basis of \( \mathcal{L}_p^f \), \( s_1 = 1 \) and \( s_n^{-1} \notin L^\infty \). For if \( s_n^{-1} \in L^\infty \) then there exists a real number \( \lambda \) such that \( (s_n - \lambda)^{-1} \notin L^\infty \). Then \( \{s_1, s_2, \ldots, (s_n - \lambda)\} \) is also a basis.

When \( \mathcal{L}_p^{s_n f} = R \), put \( \phi = s_n \) and \( h = s_n f \), then the theorem is proved. Suppose that \( \mathcal{L}_p^{s_n f} \neq R \). If \( \ell_1 \) is a nonconstant function in \( \mathcal{L}_p^{s_n f} \) then \( \ell_1 s_n \) is nonconstant because \( s_n^{-1} \notin L^\infty \). We may assume that \( \ell_1^{-1} \notin L^\infty \). When \( \mathcal{L}_p^{\ell_1 s_n f} = R \), put \( \phi = \ell_1 s_n \) and \( h = \ell_1 s_n f \), then the theorem is proved. Suppose that \( \mathcal{L}_p^{\ell_1 s_n f} \neq R \). Then there exists \( \ell_2 \) in \( \mathcal{L}_p^{\ell_1 s_n f} \) such that \( \ell_2 \ell_1 s_n \) is nonconstant and \( \ell_2^{-1} \notin L^\infty \). When \( \mathcal{L}_p^{\ell_2 \ell_1 s_n f} \neq R \), we can proceed similarly. Put

\[ k_j = \ell_j \ell_{j-1} \cdots \ell_1 \ (j = 1, 2, \ldots, n) \]
where $\ell_i^{-1} \notin L^\infty$ ($1 \leq i \leq j$). Suppose that $L_p^{k_j s_n} \neq R$ for $j = 1, 2, \cdots, n$. Hence

$$k_j s_n = \sum_{i=1}^{n} \alpha_{ij} s_i \quad (j = 1, 2, \cdots, n)$$

and so for $j = 1, 2, \cdots, n$

$$\sum_{i=1}^{n-1} \alpha_{ij} s_i + (\alpha_{nj} - k_j) s_n = 0.$$ 

Hence

$$\begin{vmatrix} \alpha_{11} & \cdots & \alpha_{n-1 \, 1} & \alpha_{n1} - k_1 \\ \alpha_{12} & \cdots & \alpha_{n-1 \, 2} & \alpha_{n2} - k_2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{n-1 \, n} & \alpha_{nn} - k_n \end{vmatrix} = 0$$

and so there exist $\gamma_1, \cdots, \gamma_n$ in $\mathcal{C}$ such that

$$\gamma_1 (\alpha_{n1} - k_1) + \gamma_2 (\alpha_{n2} - k_2) + \cdots + \gamma_n (\alpha_{nn} - k_n) = 0$$

where

$$\gamma_j = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{n-1 \, 1} \\ \vdots & \vdots & \vdots \\ \alpha_{1 \, j-1} & \cdots & \alpha_{n-1 \, j-1} \\ \alpha_{1 \, j+1} & \cdots & \alpha_{n-1 \, j+1} \\ \vdots & \vdots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{n-1 \, n} \end{vmatrix} < j$$

Hence $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$. Here we need the following claim.

**Claim** For any $t$ ($1 \leq t \leq n$), if $(\delta_1, \cdots, \delta_t) \neq (0, \cdots, 0)$ then $\delta = \sum_{j=1}^{t} \delta_j k_j$ cannot be constant.

Proof. Let $s$ be the smallest integer such that $\delta_s \neq 0$ and $1 \leq s \leq t$. Then $\delta = \sum_{j=s}^{t} \delta_j k_j$. Hence

$$\delta = \delta_s (\ell_1 \cdots \ell_s) + \cdots + \delta_t (\ell_1 \cdots \ell_s) \ell_{s+1} \cdots \ell_t.$$ 

If $\delta = 0$, then $0 = \delta_s + \delta_{s+1} \ell_{s+1} + \cdots + \delta_t \ell_{s+1} \cdots \ell_t$ and this contradicts that $\ell_{s+1}^{-1} \notin L^\infty$ because $\delta_s \neq 0$. If $\delta$ is a nonzero constant, then this contradicts that $(\ell_1 \cdots \ell_s)^{-1} \notin L^\infty$.

Now we will prove that the equality : $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$ contradicts the definition of $k_j$ ($1 \leq j \leq n$). If $\gamma_n = 0$, then there exist $(\delta_1, \cdots, \delta_{n-1}) \neq (0, \cdots, 0)$ such that
\[ \delta_1(\alpha_{11}, \ldots, \alpha_{n-1, 1}) + \cdots + \delta_{n-1}(\alpha_{1, \ldots, 1}, \ldots, \alpha_{n-1, n-1}) = (0, \ldots, 0). \]

Hence
\[
\left( \sum_{j=1}^{n-1} \delta_j \alpha_{n_j} \right) s_n = \left( \sum_{j=1}^{n-1} \delta_j k_j \right) s_n
\]
because \( \sum_{j=1}^{n} \alpha_{ij} s_i + \alpha_n s_n = k_j s_n \) for \( j = 1, 2, \ldots, n \). Hence \( \sum_{j=1}^{n-1} \delta_j \alpha_{n_j} = \sum_{j=1}^{n-1} \delta_j k_j \) because \( |s_n| > 0 \). This contradicts the claim. Hence \( \gamma_n \neq 0 \). Thus \((\gamma_1, \ldots, \gamma_n) \neq (0, \ldots, 0) \) and \( \sum_{j=1}^{n} \gamma_j k_j \) is constant. This also contradicts the claim. Thus \( \mathcal{L}^{k_n s_n f} = R \) and so the theorem is proved.

**Lemma 1.** Let \( p \geq 1 \) and \( f \) be in \( H^p \). If \( f_\zeta(\zeta) = f(\zeta) \) is a rational function (of one variable) of degree \( \leq k_0 < \infty \), for almost all \( z \in T^n \) then \( f \) is a rational function (of \( n \) variables) of degree \( k \) and \( k \leq k_0 \).

Proof. There exist a nonnegative integer \( k \leq k_0 \) and a closed set \( E_k \) such that \( f_\zeta(\zeta) \) is a rational function (of one variable) of degree \( k \) for all \( z \in E_k \) and \( E_k \) is a nonempty interior. We will use [5, Theorem 5.2.2]. In Theorem 5.2.2 in [5], we put \( \Omega = D^n \) and \( E = E_k \). If \( f_\zeta(\zeta) \) is a rational function (of one variable) of degree \( k \) for all \( z \in E \), then \( f \) belongs to \( Y \) in Theorem 5.2.2 in [5]. For \( f_\zeta \) is in \( H^p(D) \), \( p \geq 1 \) and so \( f_\zeta \) is continuous on \( \partial D \). Now Theorem 5.2.2 in [5] implies the lemma.

**Proposition 4.** Suppose \( 1 \leq p \leq \infty \). \( \mathcal{L}^f_p \) is of finite dimension if \( f \) is a rational function.

Proof. Suppose \( f = P/Q \) is a nonzero function in \( H^p \) where \( P \) and \( Q \) are polynomials. If \( s \in \mathcal{L}^f_p \) then \( sP/Q \in H^p \) and so \( sP \in H^p \). Hence \( s \) belongs to \( \mathcal{L}^P_p \) and so \( \mathcal{L}^f_p \subseteq \mathcal{L}^P_p \). It is enough to prove that \( \mathcal{L}^P_p \) is of finite dimension.

**Case** \( n = 1 \). We have the inner outer factorization for \( n = 1 \), that is, \( P = qh \) where \( q \) is a finite Blaschke product and \( h \) is an outer function in \( H^p \). Then it is easy to see that \( \mathcal{L}^P_p = \mathcal{L}^q_p \). Since \( \mathcal{L}^q_p \subset \bar{q}H^p \cap q\bar{H}^p \) and \( \mathcal{L}^q_p \subset \bar{q}H^2 \cap q\bar{H}^2 = \bar{q}(H^2 \cap q^2H^2) = q(H^2 \ominus q^2H_0^2) \). \( H^2 \ominus q^2H_0^2 \) is of finite dimension because \( q \) is a finite Blaschke product.

**Case** \( n \neq 1 \). If \( s \in \mathcal{L}^P_p \) then \( sP \in H^p \) and by Case \( n = 1 \) \((sP)_z(\zeta) \) is a rational function (of one variable) of degree \( \leq k_0 \) for almost all \( z \in T^n \). By Lemma 1, \( sP \) is a rational function (of \( n \) variables) of degree \( k \) and \( k \leq k_0 \). This implies that \( \mathcal{L}^P_p \) is of finite dimension.

When \( f \) is a rational function in \( H^p \), by Theorem 3 and Proposition 4 \( f \) has our factorization. The function \( h \) in \( N(D^n) \) is called \( z_j \)-outer if
\[
\int_{T_j \times T'_j} \log |h(z_j, z'_j)|dm = \int_{T'_j} \left( \int_{T_j} \log |h(z_j, z'_j)|dm_j \right)dm'_j > -\infty.
\]
Proposition 5. Fix $1 \leq j \leq n$. If $f(z_1, \ldots, z_n)$ is $z_i$-outer in $H^p$ for $i \neq j$ and $1 \leq i \leq n$, then $f$ has a factorization in $H^p$.

Proof. We will generalize Theorem 2 in [3]. That is, when $h$ is $z_i$-outer in $H^p$ for $i \neq j$, $L^h_p = R$ if and only if the common inner divisor of $\{h_\alpha(z_j)\}_\alpha$ is constant, where

$$h_\alpha(z_j) = \int_{T_j^j} h(z_j, z'_j) \overline{z}_j^\alpha dm_j,$$

$\alpha = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n)$ and $\overline{z}_j^\alpha = \overline{z}_j^{\alpha_1} \cdots \overline{z}_{j-1}^{\alpha_{j-1}} \overline{z}_{j+1}^{\alpha_{j+1}} \cdots \overline{z}_n^{\alpha_n}$. Note that $h_\alpha(z_j)$ belongs to $H^p(T_j)$. For the proof, use the following notation: $H^p_{(j)} = \{f \in L^p(T^n) : \hat{f}(m_1, \ldots, m_n) = 0 \text{ if } m_i < 0 \text{ for all } i \neq j\}$ and $H^p_{(j)} \cap \overline{H^p_{(j)}} = L^p$ is the Lebesgue space on $T_j$.

If $\phi \in L^h_p$ then $g = \phi h$ and $\phi$ belongs to $H^p_{(j)}$ because $h$ is $z_i$-outer in $H^p$ for $i \neq j$. Since $\phi$ is real-valued, $\phi \in L^p$ and so $\phi = \phi(z_j)$. If the common inner divisor of $\{h_\alpha(z_j)\}_\alpha$ is constant, then for each $\alpha$

$$\phi(z_j) h_\alpha(z_j) = \int_{T_j^j} \phi(z_j) h(z_j, z'_j) \overline{z}_j^\alpha dm_j = \int_{T_j^j} g(z_j, z'_j) \overline{z}_j^\alpha dm_j$$

belongs to $H^p(T_j)$ and hence $\phi \in H^\infty(T_j)$. Therefore $\phi$ is constant. This implies that $L^h_p = R$. Conversely suppose that $\overline{L^h_p} = R$. If $\{h_\alpha(z_j)\}_\alpha$ has a non-constant common inner divisor $q(z_j)$, put $\phi(z_j, z'_j) = q(z_j) + q(z_j)$, then $g = \phi h$ belongs to $H^p$. This contradiction shows the ‘only if’ part.

Now we will prove that $f$ has a factorization in $H^p$. If $\{f_\alpha(z_j)\}_\alpha$ does not have common inner divisors, then by what was just proved $L^f_p = R$ and so we need not prove. If $\{f_\alpha(z_j)\}_\alpha$ have common inner divisors, let $q(z_j)$ be the greatest common inner divisor. Put $\phi = q(z_j) + q(z_j)$ and $h = \phi f$, then $h$ belongs to $H^p$ and $L^h_p = R$. This completes the proof.

§3 $L = L^\infty$

In this section, we assume that $L = L^\infty$. If $f$ is a $L^\infty$-extremal function for $H^p$ then $f$ is also a $L^\infty_R$-extremal function for $H^p$. When $n = 1$, the converse is true. However this is not true for $n \neq 1$. For example, $z - 2w$ is a $L^\infty_R$-extremal function but not a $L^\infty$-extremal. We can prove an analog of Proposition 1 for $L^\infty$. Let $M_f$ be an invariant closed subspace generated by $f$ in $H^p$ and $M(M_f)$ the set of multipliers of $M_f$ (see [1]). Then $M(M_f) = L^f_p$ for $L = L^\infty$. It is easy to see that

$$(L^\infty)^f_p \cap \overline{(L^\infty)^f_p} = (L^R)^f_p + i(L^\infty)^f_p.$$ 

It is easy to see that $(L^\infty)^f_p$ is a weak * closed invariant subspace which contains $H^\infty$. Then $(L^\infty)^f_p / H^\infty$ is of infinite dimension (see [4, Theorem 1]). Thus we can not expect the analogy of Theorem 3.
Proposition 6. Let $1 \leq p \leq \infty$ and $f$ a nonzero function in $H^p$. Suppose $\phi$ is in $L^p_f$.

1. $L^p_f \supseteq \phi L^p_f \supseteq \phi H^\infty$.
2. $\phi^{-1}$ is in $L^\infty$ if and only if $L^p_f = \phi L^p_f$.
3. If $L^p_f \supseteq L^p_\phi$ then $\phi$ belongs to $H^\infty$. If $L^p_f \supseteq \phi L^p_f$ and $\phi^{-1}$ is in $L^\infty$ then $\phi^{-1}$ belongs to $H^\infty$.

Proof. (1) If $g \in L^p_f$, then $\phi gf = g\phi f \in H^p$ and so $g \phi \in L^p_f$. (2) If $\phi^{-1} \in L^\infty$ then $L^p_f \supseteq \phi^{-1} f L^p_f \supseteq \phi L^p_f$ by (1). If $L^p_f = \phi L^p_f$ then $\phi^{-1} f L^p_f = \phi L^p_f$ and so $\phi^{-1} \in L^p_f$. This implies that $\phi^{-1} \in L^\infty$. (3) Suppose $L^p_f \supseteq L^p_\phi$. If $k \in L^p_f$ then $k \in L^p_\phi$ and so $k \phi f \in H^p$. Hence $\phi^2 f \in H^p$. Repeating this process, $\phi^n \in L^p_f$ and so $\phi^n f \in H^p$ for all $n \geq 1$. Thus $\phi$ belongs to $H^\infty$. If $L^p_f \supseteq \phi L^p_f$ and $\phi^{-1} \in L^\infty$, then $\phi^{-1}$ belongs to $H^\infty$. For apply what was proved above for $\phi^{-1}$ assuming $\phi^{-1}(\phi f) = f$.

When $f$ is a nonzero function in $H^p$, $f$ is factorable in $H^p$ if and only if there exists a nonzero function $h$ in $H^p$ such that $|f| \geq |h|$ a.e. on $T^2$ and $L^p_h = H^\infty$.

Proposition 7. Let $1 \leq p \leq \infty$ and $f$ be a nonzero function in $H^p$. Suppose $\phi$ is a nonzero function in $L^p_f$.

1. If $\phi^{-1}$ is in $L^\infty$ and $L^p_\phi = H^\infty$, then $\phi^{-1}$ belongs to $H^\infty$ and $L^p_f = \phi H^\infty$.
2. If $L^p_f = \phi H^\infty$, then $\phi^{-1}$ belongs to $H^\infty$ and $L^p_\phi = H^\infty$.
3. If $L^p_f$ is the weak * closure of $\phi H^\infty$ and $|\phi| = |h|$ a.e. for some function $h$ in $H^\infty$, then $L^p_\phi = H^\infty$ for some inner function $\phi_0$ and so $f$ is factorable.
4. There exist $f$ and $\phi$ such that $f$ is not the quotient of any two members of $H^\infty(T^n)$.

Proof. (1) By (2) of Proposition 6, $\phi L^p_\phi = L^p_f$. Since $L^p_\phi = H^\infty$, $\phi H^\infty = L^p_f \supset H^\infty$ and so $\phi^{-1}$ belongs to $H^\infty$. (2) Since $L^p_f \supseteq \phi L^p_\phi \supseteq \phi H^\infty$ by (1) of Proposition 6 and $L^p_f = \phi H^\infty$, $L^p_\phi = H^\infty$. It is clear that $\phi^{-1} \in H^\infty$. (3) Since $|\phi| = |h|$ a.e., $\phi = \phi_0 h$ and $|\phi_0| = 1$ a.e. Then $L^p_f = [\phi H^\infty]_v = \phi_0 [h H^\infty]_v \supset H^\infty$ and so $\phi_0$ is an inner function where $[S]_v$ is the weak * closure of $S$. (4) This is a result of [6].

Proposition 8. Let $1 \leq p \leq \infty$. Suppose $f$ and $g$ are nonzero functions in $H^p$.

1. If $L^p_f = H^\infty$ and $|f| \geq |g|$ a.e., then there exists a function $\phi$ in $H^\infty$ such that $g = \phi f$.
2. If $L^p_f = H^\infty$ and $|f| = |g|$ a.e., then there exists an inner function $\phi$ such that $g = \phi f$.

Proof. (1) Let $\phi = g/f$, then $\phi \in L^\infty$ because $|f| \geq |g|$ a.e.. By (1) of Proposition 6, $\phi$ belongs to $H^\infty$ because $H^\infty = L^p_f$. (2) follows from (1).

Proposition 9. Let $1 \leq p \leq \infty$. If $f$ is homogeneous polynomial such that $f(z_1, \ldots, z_n) = g(z, w)$ where $z = z_i, w = z_j$ and $i \neq j$ then $f$ is factorable in $H^p$.

Proof. Since $f(z_1, \ldots, z_n) = \sum_{j=0}^{\ell} a_j z^{\ell-j} w^j$, $f(z_1, \ldots, z_n) = z^{\ell} \sum_{j=0}^{\ell} a_j \left(\frac{w}{z}\right)^j = \sum_{j=0}^{\ell} a_j z^{\ell-j} w^j = g(z, w)$.
\[ c \prod_{j=0}^{\ell} (b_j w - c_j z) \text{ where } b_j = 1 \text{ or } c_j = 1, \text{ and } |b_j| \leq 1, \ |c_j| \leq 1. \] It is easy to see that \( \mathcal{L}_p^f = \phi H^\infty \) where \( \phi = \prod(\alpha z - \beta w)^{-1} \) and \( (\alpha, \beta) \in (\partial D \times D) \cup (D \times \partial D) \) (cf. [2],[4]). By (2) of Proposition 7, \( f \) is factorable.

**Question**

1. For any nonzero function \( f \) in \( H^p \), does there exist a function \( \phi \) such that \( \mathcal{L}_p^f \supset \overline{\mathcal{L}_p^{\phi f}} \) ?
2. Describe \( \phi \) in \( L^\infty \) such that \( \mathcal{L}_p^f \supset \mathcal{L}_p^{\phi f} \).
3. Describe \( \phi \) in \( L^\infty \) such that \( \left[ \phi H^\infty \right]_p \supset H^\infty \).

**References**


Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

E-mail : nakazi @ math.sci.hokudai.ac.jp