Factorizations Of Functions In $H^p(T^n)$

By

Takahiko Nakazi

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan

2000 Mathematics Subject Classification : 32 A 35, 46 J 15

Key words and phrases : Hardy space, polydisc, factorization, extreme point
Abstract. We are interested in extremal functions in a Hardy space $H^p(T^n)$ ($1 \leq p \leq \infty$). For example, we study extreme points of the unit ball of $H^1(T^n)$ and give a factorization theorem. In particular, we show that any rational function can be factorized.
§1. Introduction

Let $D^n$ be the open unit polydisc in $C^n$ and $T^n$ be its distinguished boundary. The normalized Lebesgue measure on $T^n$ is denoted by $dm$. For $0 < p \leq \infty$, $H^p(D^n)$ is the Hardy space and $L^p(T^n)$ is the Lebesgue space on $T^n$. Let $N(D^n)$ denote the Nevanlinna class. Each $f$ in $N(D^n)$ has radial limits $f^*$ defined on $T^n$ a.e.$dm$. Moreover, there is a singular measure $d\sigma_f$ on $T^n$ determined by $f$ such that the least harmonic majorant $u(\log |f|)$ of $|f|$ is given by $u(\log |f|)(z) = P_z(\log |f^*| + d\sigma_f)$ where $P_z$ denotes Poisson integration and $z = (z_1, z_2, \cdots, z_n) \in D^n$. Put $N_*(D^n) = \{ f \in N(D^n) \mid d\sigma_f \leq 0 \}$, then $H^p(D^n) \subset N_*(D^n) \subset N(D^n)$ and $H^p(D^n) = N_*(D^n) \cap L^p(T^n) \subset N(D^n) \cap L^p(T^n)$.

These facts are shown in [5, Theorem 3.3.5].

Let $\mathcal{L}$ be a subset of $L^\infty(T^n)$. For a function $f$ in $H^p$, put

$$\mathcal{L}^f_p = \{ \phi \in \mathcal{L} \mid \phi f \in H^p \}.$$

When $\mathcal{L}^f_p \subset H^\infty$, $f$ is called an $\mathcal{L}$-extremal function for $H^p$. When $\mathcal{L} = L^\infty(T^n)$, $\mathcal{L} = L^\infty_R(T^n)$ or $\mathcal{L} = L^\infty_U(T^n)$ is the set of all unimodular functions, such $\mathcal{L}$-extremal functions have been considered in [3]. In [3], the author studied functions which have harmonic properties (A),(B),(C). For example, the property (A) is the following : If $f \in H^p$ and $|f| \geq |g|$ a.e. on $T^n$, then $|f| \geq |g|$ on $D^n$. It is easy to see that $f$ is an $\mathcal{L}$-extremal function for $H^p$ and $\mathcal{L} = L^\infty(T^n)$ if and only if $f$ has the property (A). The properties (B) and (C) are related to $\mathcal{L} = L^\infty_R(T^n)$ and $\mathcal{L} = L^\infty_U(T^n)$, respectively. In this paper, as $\mathcal{L}$ we consider only the above three sets.

Definition. When $f$ is not $\mathcal{L}$-extremal for $H^p$, if there exists a function $\phi$ in $\mathcal{L}$ such that $\phi f = h$ is an $\mathcal{L}$-extremal function for $H^p$, we say that $f$ is factorized as $f = \phi^{-1}h$.

In this paper, we are interested in when $f$ is factorized for $\mathcal{L} = L^\infty(T^n)$ or $\mathcal{L} = L^\infty_R(T^n)$. The function $h$ in $N(D^n)$ is called outer function if

$$\int_{T^n} \log |h| dm = \log \left| \int_{T^n} h dm \right| > -\infty.$$

The function $q$ in $N_*(D^n)$ is called inner function if $|q| = 1$ a.e.$dm$ on $T^n$. When $\mathcal{L} \subset \mathcal{L}'$, a $\mathcal{L}'$-extremal function is always $\mathcal{L}$-extremal. If $f$ is an outer function, then $f$ is $\mathcal{L}$-extremal for $\mathcal{L} = L^\infty(T^n)$. In fact, if $\phi f$ is in $H^p$ then $\phi$ belongs to $f^{-1}H^p$ and $f^{-1}H^p \subset N_*$. Hence if $\phi$ is bounded then $\phi$ belongs to $H^\infty$ because $N_* \cap L^\infty(T^n) = H^\infty$. When $n = 1$, $f$ is $\mathcal{L}$-extremal if and only if $f$ is an outer function. This is known because $f$ has an inner outer factorization.

In this paper, for a subset $S$ in $L^\infty$ we say that $S$ is of finite dimension if the linear span of $S$ is of finite dimension. We use the following notations.

$$z = (z_j, z'_j), z'_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n).$$
exists a real number non-zero coefficient. The degree of a rational function $P$ in $L$ is the maximum of the degrees of the monomials which occur in $P$.

In this section, we assume that $L = L_R^\infty$. When $n = 1$, any nonzero function in $H^p$ has a $L_R^\infty$-factorization in $H^p$ by Proposition 1. Even if $n > 1$, we have a lot of $L_R^\infty$-extremal functions for $H^p$.

**Proposition 1.** If $f = qh$ where $q$ is inner and $h$ is $L_R^\infty$-extremal for $H^p$, then $f$ has a $L_R^\infty$-factorization in $H^p$: $f = \phi^{-1}k$ where $\phi = q + \bar{q}$ and $k = (1 + q^2)h$ is $L_R^\infty$-extremal for $H^p$.

Proof. It is enough to show that $(1 + q^2)h$ is $L_R^\infty$-extremal for $H^p$. If $\psi \in L_R^\infty(T^n)$ and $\psi(1 + q^2)h \in H^p$ then $\psi(1 + q^2)$ belongs to $H^\infty$ because $h$ is $L_R^\infty$-extremal for $H^p$. Since $1 + q^2$ is outer and so $1 + q^2$ is $L_R^\infty$-extremal for $H^p$, $\psi$ belongs to $H^\infty$. □

**Proposition 2.** Suppose $f$ is a nonzero function in $H^1$. $f$ is $L_R^\infty$-extremal for $H^1$ if and only if $f/\|f\|_1$ is an extreme point of the unit ball of $H^1$.

Proof. It is well known. □

The degree of a monomial $z_{i_1}^{\alpha_1} \cdots z_{i_n}^{\alpha_n}$ (where $\alpha_i \in \mathbb{Z}_+$) is $\alpha_1 + \cdots + \alpha_n$. The degree of a polynomial $P$ is the maximum of the degrees of the monomials which occur in $P$ with non-zero coefficient. The degree of a rational function $f = P/Q$ is the maximum of deg $P$, deg $Q$, provided that all common factors of positive degree have first been cancelled.

**Theorem 3.** Let $0 < p \leq \infty$ and $L = L_R^\infty$. If $f$ is a nonzero function in $H^p$ and $L^f_p$ is of finite dimension then there exists a function $\psi$ in $L$ such that $f = \psi^{-1}h$ and $h$ is $L$-extremal for $H^p$.

Proof. Suppose that $L^f_p$ is of finite dimension. Then there exist $s_1, s_2, \ldots, s_n$ in $L_R^\infty$ such that $\{s_j\}_{j=1}^n$ is a basis of $L^f_p$, $s_1 = 1$ and $s_n^{-1} \notin L^\infty$. For if $s_n^{-1} \in L^\infty$ then there exists a real number $\lambda$ such that $(s_n - \lambda)^{-1} \notin L^\infty$. Then $\{s_1, s_2, \ldots, (s_n - \lambda)\}$ is also a basis.

When $L^e_p R_n f = R$, put $\phi = s_n$ and $h = s_n f$, then the theorem is proved. Suppose that $L^e_p R_n f \neq R$. If $\ell_1$ is a nonconstant function in $L^e_p R_n f$ then $\ell_1 s_n$ is nonconstant because $s_n^{-1} \notin L^\infty$. We may assume that $\ell_1^{-1} \notin L^\infty$. When $L^e_{\ell_1 s_n} f = R$, put $\phi = \ell_1 s_n$ and $h = \ell_1 s_n f$, then the theorem is proved. Suppose that $L^e_{\ell_1 s_n} f \neq R$. Then there exists $\ell_2$ in $L^e_{\ell_1 s_n} f$ such that $\ell_2 \ell_1 s_n$ is nonconstant and $\ell_2^{-1} \notin L^\infty$. When $L^e_p \ell_1 s_n f \neq R$, we can proceed similarly. Put

$$k_j = \ell_j \ell_{j-1} \cdots \ell_1 \quad (j = 1, 2, \ldots, n)$$
where $\ell_i^{-1} \notin L^\infty$ ($1 \leq i \leq j$). Suppose that $L_p^{k_j s_n} \neq R$ for $j = 1, 2, \ldots, n$. Hence
\[ k_j s_n = \sum_{i=1}^{n} \alpha_{ij} s_i \quad (j = 1, 2, \ldots, n) \]
and so for $j = 1, 2, \ldots, n$
\[ \sum_{i=1}^{n-1} \alpha_{ij} s_i + (\alpha_{nj} - k_j) s_n = 0. \]
Hence
\[
\begin{vmatrix}
\alpha_{11} & \cdots & \alpha_{n-1, 1} & \alpha_{n1} - k_1 \\
\alpha_{12} & \cdots & \alpha_{n-1, 2} & \alpha_{n2} - k_2 \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_{1n} & \cdots & \alpha_{n-1, n} & \alpha_{nn} - k_n
\end{vmatrix} = 0
\]
and so there exist $\gamma_1, \ldots, \gamma_n$ in $\mathcal{C}$ such that
\[ \gamma_1 (\alpha_{n1} - k_1) + \gamma_2 (\alpha_{n2} - k_2) + \cdots + \gamma_n (\alpha_{nn} - k_n) = 0 \]
where
\[
\gamma_j = \begin{vmatrix}
\alpha_{11} & \cdots & \alpha_{n-1, 1} \\
\vdots & \ddots & \vdots \\
\alpha_{1, j-1} & \cdots & \alpha_{n-1, j-1} \\
\alpha_{1, j+1} & \cdots & \alpha_{n-1, j+1} \\
\vdots & \ddots & \vdots \\
\alpha_{1n} & \cdots & \alpha_{n-1, n}
\end{vmatrix} < j
\]
Hence
\[ \sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j. \]
Here we need the following claim.

**Claim** For any $t$ ($1 \leq t \leq n$), if $(\delta_1, \ldots, \delta_t) \neq (0, \ldots, 0)$ then $\delta = \sum_{j=1}^{t} \delta_j k_j$ can not be constant.

Proof. Let $s$ be the smallest integer such that $\delta_s \neq 0$ and $1 \leq s \leq t$. Then
\[ \delta = \sum_{j=s}^{t} \delta_j k_j. \]
Hence
\[ \delta = \delta_s (\ell_1 \cdots \ell_s) + \cdots + \delta_t (\ell_1 \cdots \ell_s) \ell_{s+1} \cdots \ell_t. \]
If $\delta = 0$, then $0 = \delta_s + \delta_{s+1} \ell_{s+1} + \cdots + \delta_t \ell_{s+1} \cdots \ell_t$ and this contradicts that $\ell_{s+1}^{-1} \notin L^\infty$ because $\delta_s \neq 0$. If $\delta$ is a nonzero constant, then this contradicts that $(\ell_1 \cdots \ell_s)^{-1} \notin L^\infty$.

Now we will prove that the equality : $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$ contradicts the definition of $k_j$ ($1 \leq j \leq n$). If $\gamma_n = 0$, then there exist $(\delta_1, \ldots, \delta_{n-1}) \neq (0, \ldots, 0)$ such that
\[ \delta(\alpha_{11}, \cdots, \alpha_{n-1}1) + \cdots + \delta_{n-1}(\alpha_{1n-1}, \cdots, \alpha_{n-1}n-1) = (0, \cdots, 0). \]

Hence
\[
\left( \sum_{j=1}^{n-1} \delta_j \alpha_{nj} \right) s_n = \left( \sum_{j=1}^{n-1} \delta_j k_j \right) s_n
\]

because \[ \sum_{i=1}^{n} \alpha_{ij}s_i + \alpha_{nj}s_n = k_j s_n \] for \( j = 1, 2, \cdots, n \). Hence \[ \sum_{j=1}^{n-1} \delta_j \alpha_{nj} = \sum_{j=1}^{n-1} \delta_j k_j \] because \[ |s_n| > 0 \]. This contradicts the claim. Hence \( \gamma_n \neq 0 \). Thus \( (\gamma_1, \cdots, \gamma_n) \neq (0, \cdots, 0) \) and \[ \sum_{j=1}^{n} \gamma_j k_j \] is constant. This also contradicts the claim. Thus \( L^{k_n s_n f} = R \) and so the theorem is proved.

**Lemma 1.** Let \( p \geq 1 \) and \( f \) be in \( H^p \). If \( f_\zeta(\zeta) = f(\zeta z) \) is a rational function (of one variable) of degree \( \leq k_0 < \infty \), for almost all \( z \in T^n \) then \( f \) is a rational function (of \( n \) variables) of degree \( k \) and \( k \leq k_0 \).

Proof. There exist a nonnegative integer \( k \leq k_0 \) and a closed set \( E_k \) such that \( f_\zeta(\zeta) \) is a rational function (of one variable) of degree \( k \) for all \( z \in E_k \) and \( E_k \) is a nonempty interior. We will use [5, Theorem 5.2.2]. In Theorem 5.2.2 in [5], we put \( \Omega = D^u \) and \( E = E_k \). If \( f_\zeta(\zeta) \) is a rational function (of one variable) of degree \( k \) for all \( z \in E \), then \( f \) belongs to \( Y \) in Theorem 5.2.2 in [5]. For \( f_\zeta \) is in \( H^p(D) \), \( p \geq 1 \) and so \( f_\zeta \) is continuous on \( \partial D \). Now Theorem 5.2.2 in [5] implies the lemma.

**Proposition 4.** Suppose \( 1 \leq p \leq \infty \). \( L^f_p \) is of finite dimension if \( f \) is a rational function.

Proof. Suppose \( f = P/Q \) is a nonzero function in \( H^p \) where \( P \) and \( Q \) are polynomials. If \( s \in L^f_p \) then \( sP/Q \in H^p \) and so \( sP \in H^p \). Hence \( s \) belongs to \( L^P_p \) and so \( L^f_p \subseteq L^P_p \). It is enough to prove that \( L^P_p \) is of finite dimension.

**Case** \( n = 1 \). We have the inner outer factorization for \( n = 1 \), that is, \( P = qh \) where \( q \) is a finite Blaschke product and \( h \) is an outer function in \( H^p \). Then it is easy to see that \( L^P_p = L^q_p \). Since \( \mathcal{L}^q_p \subseteq \mathcal{Q}H^p \) \( \cap \mathcal{Q}H^2 \) and \( \mathcal{L}^q_p \subset L^\infty \), \( \mathcal{L}^q_p \subset \mathcal{Q}H^2 \cap \mathcal{Q}H^2 = \mathcal{Q}(H^2 \cap \mathcal{Q}H^2) = \mathcal{Q}(H^2 \cup \mathcal{Q}H^2) \) is of finite dimension because \( q \) is a finite Blaschke product.

**Case** \( n \neq 1 \). If \( s \in L^P_p \) then \( sP \in H^p \) and by Case \( n = 1 \) \( (sP)_z(\zeta) \) is a rational function (of one variable) of degree \( \leq k_0 \) for almost all \( z \in T^n \). By Lemma 1, \( sP \) is a rational function (of \( n \) variables) of degree \( k \) and \( k \leq k_0 \). This implies that \( L^P_p \) is of finite dimension.

When \( f \) is a rational function in \( H^p \), by Theorem 3 and Proposition 4 \( f \) has our factorization. The function \( h \) in \( N(D^n) \) is called \( z_j \)-outer if
\[
\int_{T_j \times T_j} \log |h(z_j, z'_j)| dm = \int_{T_j} (\int_{T_j} h(z_j, z'_j) dm_j) dm'_j > -\infty.
\]
Proposition 5. Fix $1 \leq j \leq n$. If $f(z_1, \ldots, z_n)$ is $z_i$-outer in $H^p$ for $i \neq j$ and $1 \leq i \leq n$, then $f$ has a factorization in $H^p$.

Proof. We will generalize Theorem 2 in [3]. That is, when $h$ is $z_i$-outer in $H^p$ for $i \neq j$, $\mathcal{L}_p^h = R$ if and only if the common inner divisor of $\{h(z_j)\}_\alpha$ is constant, where

$$h_{\alpha}(z_j) = \int_{T_j} h(z_j, z_j') \overline{z} \, dm_j$$

, $\alpha = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n)$ and $\overline{z}^\alpha_j = \overline{z}_{j-1}^{\alpha_1} \cdots \overline{z}_{j+1}^{\alpha_{j-1}} \cdots \overline{z}_n^{\alpha_n}$. Note that $h_{\alpha}(z_j)$ belongs to $H^p(T_j)$. For the proof, we use the following notation: $\mathcal{H}^p_{(j)} = \{ f \in L^p(T^n) ; \hat{f}(m_1, \ldots, m_n) = 0 \text{ if } m_i < 0 \text{ for all } i \neq j \}$ and $\mathcal{H}^p_{(j)} \cap \overline{\mathcal{H}^p_{(j)}} = \mathcal{L}_p^h$ is the Lebesgue space on $T_j$.

If $\phi \in \mathcal{L}_p^h$ then $g = \phi h$ and $\phi$ belongs to $\mathcal{H}^p_{(j)}$ because $h$ is $z_i$-outer in $H^p$ for $i \neq j$. Since $\phi$ is real-valued, $\phi \in \mathcal{L}_p^h$ and so $\phi = \phi(z_j)$. If the common inner divisor of $\{h(z_j)\}_\alpha$ is constant, then for each $\alpha$

$$\phi(z_j)h_{\alpha}(z_j) = \int_{T_j} \phi(z_j)h(z_j, z_j') \overline{z} \, dm_j = \int_{T_j} g(z_j, z_j') \overline{z} \, dm_j$$

belongs to $H^p(T_j)$ and hence $\phi \in H^\infty(T_j)$. Therefore $\phi$ is constant. This implies that $\mathcal{L}_p^h = R$. Conversely suppose that $\mathcal{L}_p^h = R$. If $\{h_{\alpha}(z_j)\}_\alpha$ has a non-constant common inner divisor $q(z_j)$, put $\phi(z_j, z_j') = q(z_j) + q(z_j)$, then $g = \phi h$ belongs to $H^p$. This contradiction shows the ‘only if’ part.

Now we will prove that $f$ has a factorization in $H^p$. If $\{f_{\alpha}(z_j)\}_\alpha$ does not have common inner divisors, then by what was just proved $\mathcal{L}_p^f = R$ and so we need not prove. If $\{f_{\alpha}(z_j)\}_\alpha$ have common inner divisors, let $q(z_j)$ be the greatest common inner divisor. Put $\phi = \overline{q}(z_j) + q(z_j)$ and $h = \phi f$, then $h$ belongs to $H^p$ and $\mathcal{L}_p^h = R$. This completes the proof.

§3 $\mathcal{L} = L^\infty$

In this section, we assume that $\mathcal{L} = L^\infty$. If $f$ is a $L^\infty$-extremal function for $H^p$ then $f$ is also a $L^\infty_R$-extremal function for $H^p$. When $n = 1$, the converse is true. However this is not true for $n \neq 1$. For example, $z - 2w$ is a $L^\infty_R$-extremal function but not a $L^\infty$-extremal. We can prove an analogy of Proposition 1 for $L^\infty$. Let $M_f$ be an invariant closed subspace generated by $f$ in $H^p$ and $\mathcal{M}(M_f)$ the set of multipliers of $M_f$ (see [1]). Then $\mathcal{M}(M_f) = \mathcal{L}_p^f$ for $\mathcal{L} = L^\infty$. It is easy to see that

$$(L^\infty)^f_p \cap \overline{(L^\infty)^f_p} = (L^\infty_R)^f_p + i(L^\infty_R)^f_p.$$ 

It is easy to see that $(L^\infty)^f_p$ is a weak * closed invariant subspace which contains $H^\infty$. Then $(L^\infty)^f_p/H^\infty$ is of infinite dimension (see [4, Theorem 1]). Thus we can not expect the analogy of Theorem 3.
Proposition 6. Let $1 \leq p \leq \infty$ and $f$ a nonzero function in $H^p$. Suppose $\phi$ is in $\mathcal{L}_p^f$.

(1) $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^\phi \supseteq \phi H^\infty$.
(2) $\phi^{-1}$ is in $L^\infty$ if and only if $\mathcal{L}_p^f = \phi \mathcal{L}_p^\phi$.
(3) If $\mathcal{L}_p^\phi \supseteq \mathcal{L}_p^f$ then $\phi$ belongs to $H^\infty$. If $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^\phi$ and $\phi^{-1}$ is in $L^\infty$ then $\phi^{-1}$ belongs to $H^\infty$.

Proof. (1) If $g \in \mathcal{L}_p^\phi$ then $\phi g f = g \phi f \in H^p$ and so $g \in \mathcal{L}_p^\phi$. (2) If $\phi^{-1} \in L^\infty$ then $\mathcal{L}_p^\phi \supseteq \phi^{-1} \mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^\phi$ by (1). If $\mathcal{L}_p^f = \phi \mathcal{L}_p^\phi$ then $\phi^{-1} \mathcal{L}_p^f = \mathcal{L}_p^\phi$ and so $\phi^{-1} \in \mathcal{L}_p^\phi$. This implies that $\phi^{-1} \in L^\infty$. (3) Suppose $\mathcal{L}_p^\phi \supseteq \mathcal{L}_p^f$. If $k \in \mathcal{L}_p^\phi$ then $k \in \mathcal{L}_p^\phi$ and so $k \phi f \in H^p$. Hence $\phi^2 f \in H^p$. Repeating this process, $\phi^n \in \mathcal{L}_p^f$ and so $\phi^n f \in H^p$ for all $n \geq 1$. Thus $\phi$ belongs to $H^\infty$. If $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^\phi$ and $\phi^{-1} \in L^\infty$, then $\phi^{-1}$ belongs to $H^\infty$. For apply what was proved above for $\phi^{-1}$ assuming $\phi^{-1}(\phi f) = f$.

When $f$ is a nonzero function in $H^p$, $f$ is factorable in $H^p$ if and only if there exists a nonzero function $h$ in $H^p$ such that $|f| \geq |h|$ a.e. on $T^2$ and $\mathcal{L}_p^h = H^\infty$.

Proposition 7. Let $1 \leq p \leq \infty$ and $f$ be a nonzero function in $H^p$. Suppose $\phi$ is a nonzero function in $\mathcal{L}_p^f$.

(1) If $\phi^{-1}$ is in $L^\infty$, $\phi \mathcal{L}_p^\phi = H^\infty$, then $\phi^{-1}$ belongs to $H^\infty$ and $\mathcal{L}_p^f = \phi H^\infty$.
(2) If $\mathcal{L}_p^f$ is the weak * closure of $\phi^* H^\infty$ and $|\phi| = |h|$ a.e. for some function $h$ in $H^\infty$, then $\mathcal{L}_p^\phi = H^\infty$ for some inner function $\phi_0$ and so $f$ is factorable.

(4) There exist $f$ and $\phi$ such that $\phi$ is not the quotient of any two members of $H^\infty(T^n)$.

Proof. (1) By (2) of Proposition 6, $\phi \mathcal{L}_p^\phi = \mathcal{L}_p^f$. Since $\mathcal{L}_p^\phi = H^\infty$, $\phi H^\infty = \mathcal{L}_p^f \supseteq H^\infty$ and so $\phi^{-1}$ belongs to $H^\infty$. (2) Since $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^\phi \supseteq \phi H^\infty$ by (1) of Proposition 6 and $\mathcal{L}_p^\phi = \phi H^\infty$, $\mathcal{L}_p^\phi = H^\infty$. It is clear that $\phi^{-1} \in H^\infty$. (3) Since $|\phi| = |h|$ a.e., $\phi = \phi_0 h$ and $|\phi_0| = 1$ a.e. Then $\mathcal{L}_p^f = \phi H^\infty$, $\phi_0 h H^\infty$, $\mathcal{L}_p^\phi = H^\infty$ and so $\phi_0$ is an inner function where $[S]_\ast$ is the weak * closure of $S$. (4) This is a result of [6].

Proposition 8. Let $1 \leq p \leq \infty$. Suppose $f$ and $g$ are nonzero functions in $H^p$.

(1) If $\mathcal{L}_p^f = H^\infty$ and $|f| \geq |g|$ a.e., then there exists a function $\phi$ in $H^\infty$ such that $g = \phi f$.
(2) If $\mathcal{L}_p^f = H^\infty$ and $|f| = |g|$ a.e., then there exists an inner function $\phi$ such that $g = \phi f$.

Proof. (1) Let $\phi = g / f$, then $\phi \in L^\infty$ because $|f| \geq |g|$ a.e. By (1) of Proposition 6, $\phi$ belongs to $H^\infty$ because $H^\infty = \mathcal{L}_p^f$. (2) follows from (1).

Proposition 9. Let $1 \leq p \leq \infty$. If $f$ is homogeneous polynomial such that $f(z_1, \ldots, z_n) = g(z, w)$ where $z = z_i$, $w = z_j$ and $i \neq j$ then $f$ is factorable in $H^p$.

Proof. Since $f(z_1, \ldots, z_n) = \sum_{j=0}^\ell a_j \bar{z}^{-j} w^j$, $f(z_1, \ldots, z_n) = z^\ell \sum_{j=0}^\ell a_j \left( \frac{w}{z} \right)^j = \sum_{j=0}^\ell a_j z^{\ell-j} w^j$.
\[ c \prod_{j=0}^{\ell} (b_j w - c_j z) \] where \( b_j = 1 \) or \( c_j = 1 \), and \( |b_j| \leq 1, \ |c_j| \leq 1 \). It is easy to see that \( L_f^p = \phi H^\infty \) where \( \phi = \prod (\alpha z - \beta w)^{-1} \) and \((\alpha, \beta) \in (\partial D \times D) \cup (D \times \partial D) \) (cf. [2],[4]). By (2) of Proposition 7, \( f \) is factorable.

**Question**

1. For any nonzero function \( f \) in \( H^p \), does there exists a function \( \phi \) such that \( L_f^p \supset L_\phi^p \)?
2. Describe \( \phi \) in \( L^\infty \) such that \( L_f^p \supset L_\phi^p \).
3. Describe \( \phi \) in \( L^\infty \) such that \( [\phi H^\infty]_p \supset H^\infty \).

**References**


Takahiko Nakazi  
Department of Mathematics  
Hokkaido University  
Sapporo 060-0810, Japan  
E-mail : nakazi @ math.sci.hokudai.ac.jp