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Title

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Citation

Hokkaido University Preprint Series in Mathematics, 627, 1-9

Issue Date

2004

DOI

10.14943/83781

Doc URL

http://hdl.handle.net/2115/69435

Type

bulletin (article)

File Information

pre627.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP
Factorizations Of Functions In $H^p(T^n)$

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* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan

2000 Mathematics Subject Classification : 32 A 35, 46 J 15

Key words and phrases : Hardy space, polydisc, factorization, extreme point
Abstract. We are interested in extremal functions in a Hardy space $H^p(T^n)$ ($1 \leq p \leq \infty$). For example, we study extreme points of the unit ball of $H^1(T^n)$ and give a factorization theorem. In particular, we show that any rational function can be factorized.
§1. Introduction

Let \( D^n \) be the open unit polydisc in \( \mathbb{C}^n \) and \( T^n \) be its distinguished boundary. The normalized Lebesgue measure on \( T^n \) is denoted by \( dm \). For \( 0 < p \leq \infty \), \( H^p(D^n) \) is the Hardy space and \( L^p(T^n) \) is the Lebesgue space on \( T^n \). Let \( N(D^n) \) denote the Nevanlinna class. Each \( f \) in \( N(D^n) \) has radial limits \( f^* \) defined on \( T^n \) a.e.\( dm \). Moreover, there is a singular measure \( df \) on \( T^n \) determined by \( f \) such that the least harmonic majorant \( u(\log |f|) \) of \( |f| \) is given by \( u(\log |f|)(z) = P_\mathbb{Z}(\log |f^*| + \sigma_f) \) where \( P_\mathbb{Z} \) denotes Poisson integration and \( z = (z_1, z_2, \ldots, z_n) \in D^n \). Put \( N_*(D^n) = \{ f \in N(D^n) ; \sigma_f \leq 0 \} \), then \( H^p(D^n) \subset N_*(D^n) \subset N(D^n) \) and \( H^p(D^n) = N_*(D^n) \cap L^p(T^n) \subset N(D^n) \cap L^p(T^n) \). These facts are shown in [5, Theorem 3.3.5].

Let \( \mathcal{L} \) be a subset of \( L^\infty(T^n) \). For a function \( f \) in \( H^p \), put

\[ \mathcal{L}_p^f = \{ \phi \in \mathcal{L} ; \phi f \in H^p \}. \]

When \( \mathcal{L}_p^f \subset H^\infty \), \( f \) is called an \( \mathcal{L} \)-extremal function for \( H^p \). When \( \mathcal{L} = L^\infty(T^n) \), \( \mathcal{L} = L^\infty_\mathbb{R}(T^n) \) or \( \mathcal{L} = L^\infty_\mathbb{U}(T^n) \) is the set of all unimodular functions, such \( \mathcal{L} \)-extremal functions have been considered in [3]. In [3], the author studied functions which have harmonic properties (A),(B),(C). For example, the property (A) is the following : If \( f \in H^p \) and \( |f| \geq |g| \) a.e. on \( T^n \), then \( |f| \geq |g| \) on \( D^n \). It is easy to see that \( f \) is an \( \mathcal{L} \)-extremal function for \( H^p \) and \( \mathcal{L} = L^\infty(T^n) \) if and only if \( f \) has the property (A). The properties (B) and (C) are related to \( \mathcal{L} = L^\infty_\mathbb{R}(T^n) \) and \( \mathcal{L} = L^\infty_\mathbb{U}(T^n) \), respectively. In this paper, as \( \mathcal{L} \) we consider only the above three sets.

Definition. When \( f \) is not \( \mathcal{L} \)-extremal for \( H^p \), if there exists a function \( \phi \) in \( \mathcal{L} \) such that \( \phi f = h \) is an \( \mathcal{L} \)-extremal function for \( H^p \), we say that \( f \) is factorized as \( f = \phi^{-1}h \).

In this paper, we are interested in when \( f \) is factorized for \( \mathcal{L} = L^\infty(T^n) \) or \( \mathcal{L} = L^\infty_\mathbb{R}(T^n) \). The function \( h \) in \( N(D^n) \) is called outer function if

\[ \int_{T^n} \log |h| dm = \log \left| \int_{T^n} h dm \right| > -\infty. \]

The function \( q \) in \( N_*(D^n) \) is called inner function if \( |q| = 1 \) a.e.\( dm \) on \( T^n \). When \( \mathcal{L} \subset \mathcal{L}' \), a \( \mathcal{L}' \)-extremal function is always \( \mathcal{L} \)-extremal. If \( f \) is an outer function, then \( f \) is \( \mathcal{L} \)-extremal for \( \mathcal{L} = L^\infty(T^n) \). In fact, if \( \phi f \) is in \( H^p \) then \( \phi \) belongs to \( f^{-1}H^p \) and \( f^{-1}H^p \subset N_* \). Hence if \( \phi \) is bounded then \( \phi \) belongs to \( H^\infty \) because \( N_* \cap L^\infty(T^n) = H^\infty \). When \( n = 1 \), \( f \) is \( \mathcal{L} \)-extremal if and only if \( f \) is an outer function. This is known because \( f \) has an inner outer factorization.

In this paper, for a subset \( S \) in \( L^\infty \) we say that \( S \) is of finite dimension if the linear span of \( S \) is of finite dimension. We use the following notations.

\[ z = (z_j, z'_j), \quad z'_j = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n). \]
\[ D^n = D_j \times D'_j, \quad D' = \prod_{\ell \neq j} D_\ell \text{ where } D^n = \prod_{\ell=1}^n D_\ell \text{ and } D_\ell = D. \]
\[ T^n = T_j \times T'_j, \quad T'_j = \prod_{\ell \neq j} T_\ell \text{ where } T^n = \prod_{\ell=1}^n T_\ell \text{ and } T_\ell = T. \]
\[ m = m_j \times m'_j, \quad m'_j = \prod_{\ell \neq j} m_\ell \text{ where } m = \prod_{\ell=1}^n m_\ell \text{ and } m_\ell \text{ is the normarized } \]
Lebesgue measure on \( T_\ell \).

\[ \mathcal{L} = L_R^\infty \]

In this section, we assume that \( \mathcal{L} = L_R^\infty \). When \( n = 1 \), any nonzero function in \( H^p \) has a \( L_R^\infty \)-factorization in \( H^p \) by Proposition 1. Even if \( n > 1 \), we have a lot of \( L_R^\infty \)-extremal functions for \( H^p \).

**Proposition 1.** If \( f = qh \) where \( q \) is inner and \( h \) is \( L_R^\infty \)-extremal for \( H^p \), then \( f \) has a \( L_R^\infty \)-factorization in \( H^p \) : \( f = \phi^{-1} k \) where \( \phi = q + \bar{q} \) and \( k = (1 + q^2) h \) is \( L_R^\infty \)-extremal for \( H^p \).

Proof. It is enough to show that \((1 + q^2) h \) is \( L_R^\infty \)-extremal for \( H^p \). If \( \psi \in L_R^\infty(T^n) \) and \( \psi(1 + q^2) h \in H^p \) then \( \psi(1 + q^2) \) belongs to \( H^\infty \) because \( h \) is \( L_R^\infty \)-extremal for \( H^p \). Since \( 1 + q^2 \) is outer and so \( 1 + q^2 \) is \( L_R^\infty \)-extremal for \( H^p \), \( \psi \) belongs to \( H^\infty \). \( \square \)

**Proposition 2.** Suppose \( f \) is a nonzero function in \( H^1 \). \( f \) is \( L_R^\infty \)-extremal for \( H^1 \) if and only if \( f/\|f\|_1 \) is an extreme point of the unit ball of \( H^1 \).

Proof. It is well known. \( \square \)

The degree of a monomial \( z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) (where \( \alpha_i \in \mathbb{Z}_+ \)) is \( \alpha_1 + \cdots + \alpha_n \). The degree of a polynomial \( P \) is the maximum of the degrees of the monomials which occur in \( P \) with non-zero coefficient. The degree of a rational function \( f = P/Q \) is the maximum of deg \( P \), deg \( Q \), provided that all common factors of positive degree have first been cancelled.

**Theorem 3.** Let \( 0 < p \leq \infty \) and \( \mathcal{L} = L_R^\infty \). If \( f \) is a nonzero function in \( H^p \) and \( \mathcal{L}_p^f \) is of finite dimension then there exists a function \( \phi \) in \( \mathcal{L} \) such that \( f = \phi^{-1} h \) and \( h \) is \( \mathcal{L} \)-extremal for \( H^p \).

Proof. Suppose that \( \mathcal{L}_p^f \) is of finite dimension. Then there exist \( s_1, s_2, \ldots, s_n \) in \( L_R^\infty \) such that \( \{s_j\}_{j=1}^n \) is a basis of \( \mathcal{L}_p^f \), \( s_1 = 1 \) and \( s_n^{-1} \notin L^\infty \). For if \( s_n^{-1} \in L^\infty \) then there exists a real number \( \lambda \) such that \( (s_n - \lambda)^{-1} \notin L^\infty \). Then \( \{s_1, s_2, \ldots, (s_n - \lambda)\} \) is also a basis.

When \( \mathcal{L}_p^{s_1 f} = R \), put \( \phi = s_n \) and \( h = s_n f \), then the theorem is proved. Suppose that \( \mathcal{L}_p^{s_1 f} \neq R \). If \( \ell_1 \) is a nonconstant function in \( \mathcal{L}_p^{s_1 f} \) then \( \ell_1 s_n \) is nonconstant because \( s_n^{-1} \notin L^\infty \). We may assume that \( \ell_1^{-1} \notin L^\infty \). When \( \mathcal{L}_p^{\ell_1 s_n f} = R \), put \( \phi = \ell_1 s_n \) and \( h = \ell_1 s_n f \), then the theorem is proved. Suppose that \( \mathcal{L}_p^{\ell_1 s_n f} \neq R \). Then there exists \( \ell_2 \) in \( \mathcal{L}_p^{\ell_1 s_n f} \) such that \( \ell_2 \ell_1 s_n \) is nonconstant and \( \ell_2^{-1} \notin L^\infty \). When \( \mathcal{L}_p^{\ell_2 \ell_1 s_n f} \neq R \), we can proceed similarly. Put

\[ k_j = \ell_j \ell_{j-1} \cdots \ell_1 \quad (j = 1, 2, \ldots, n) \]
where $\ell_i^{-1} \notin L^\infty$ ($1 \leq i \leq j$). Suppose that $L_\ell^{k_j s_n} \neq R$ for $j = 1, 2, \cdots, n$. Hence

$$k_j s_n = \sum_{i=1}^{n} \alpha_{ij} s_i \quad (j = 1, 2, \cdots, n)$$

and so for $j = 1, 2, \cdots, n$

$$\sum_{i=1}^{n-1} \alpha_{ij} s_i + (\alpha_{nj} - k_j) s_n = 0.$$ 

Hence

$$\left| \begin{array}{cccccc}
\alpha_{11} & \cdots & \alpha_{n-1} & \alpha_{n1} - k_1 \\
\alpha_{12} & \cdots & \alpha_{n-1} & \alpha_{n2} - k_2 \\
\vdots & \cdots & \vdots & \vdots \\
\alpha_{1n} & \cdots & \alpha_{n-1} & \alpha_{nn} - k_n \\
\end{array} \right| = 0$$

and so there exist $\gamma_1, \cdots, \gamma_n$ in $C$ such that

$$\gamma_1 (\alpha_{n1} - k_1) + \gamma_2 (\alpha_{n2} - k_2) + \cdots + \gamma_n (\alpha_{nn} - k_n) = 0$$

where

$$\gamma_j = \left| \begin{array}{cccccc}
\alpha_{11} & \cdots & \alpha_{n-1} & \\
\vdots & \cdots & \vdots & \vdots \\
\alpha_{1j-1} & \cdots & \alpha_{n-1} & \alpha_{nj} - k_j \\
\alpha_{1j+1} & \cdots & \alpha_{n-1} & \alpha_{nj} - k_j \\
\vdots & \cdots & \vdots & \vdots \\
\alpha_{1n} & \cdots & \alpha_{n-1} & \\
\end{array} \right| < j$$

Hence $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$. Here we need the following claim.

**Claim** For any $t$ ($1 \leq t \leq n$), if $(\delta_1, \cdots, \delta_t) \neq (0, \cdots, 0)$ then $\delta = \sum_{j=1}^{t} \delta_j k_j$ cannot be constant.

Proof. Let $s$ be the smallest integer such that $\delta_s \neq 0$ and $1 \leq s \leq t$. Then $\delta = \sum_{j=s}^{t} \delta_j k_j$. Hence

$$\delta = \delta_s (\ell_1 \cdots \ell_s) + \cdots + \delta_t (\ell_1 \cdots \ell_s) \ell_{s+1} \cdots \ell_t.$$ 

If $\delta = 0$, then $0 = \delta_s + \delta_{s+1} \ell_{s+1} + \cdots + \delta_t \ell_{s+1} \cdots \ell_t$ and this contradicts that $\ell_{s+1}^{-1} \notin L^\infty$ because $\delta_s \neq 0$. If $\delta$ is a nonzero constant, then this contradicts that $(\ell_1 \cdots \ell_s)^{-1} \notin L^\infty$.

Now we will prove that the equality $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$ contradicts the definition of $k_j$ ($1 \leq j \leq n$). If $\gamma_n = 0$, then there exist $(\delta_1, \cdots, \delta_{n-1}) \neq (0, \cdots, 0)$ such that
\[ \delta_1(\alpha_{11}, \cdots, \alpha_{n-1\,1}) + \cdots + \delta_{n-1}(\alpha_{1\,n-1}, \cdots, \alpha_{n-1\,n-1}) = (0, \cdots, 0). \]

Hence

\[ \left( \sum_{j=1}^{n-1} \delta_j \alpha_{nj} \right) s_n = \left( \sum_{j=1}^{n-1} \delta_j k_j \right) s_n \]

because \( \sum_{i=1}^n \alpha_{ij} s_i + \alpha_{nj} s_n = k_j s_n \) for \( j = 1, 2, \cdots, n \). Hence \( \sum_{j=1}^{n-1} \delta_j \alpha_{nj} = \sum_{j=1}^{n-1} \delta_j k_j \) because \( |s_n| > 0 \). This contradicts the claim. Hence \( \gamma_n \neq 0 \). Thus \( (\gamma_1, \cdots, \gamma_n) \neq (0, \cdots, 0) \) and \( \sum_{j=1}^{n-1} k_j \) is constant. This also contradicts the claim. Thus \( L^{k_n, s_n} = R \) and so the theorem is proved.

**Lemma 1.** Let \( p \geq 1 \) and \( f \) be in \( H^p \). If \( f_z(\zeta) = f(\zeta z) \) is a rational function (of one variable) of degree \( \leq k_0 < \infty \), for almost all \( z \in T^n \) then \( f \) is a rational function (of \( n \) variables) of degree \( k \) and \( k \leq k_0 \).

Proof. There exist a nonnegative integer \( k \leq k_0 \) and a closed set \( E_k \) such that \( f_z(\zeta) \) is a rational function (of one variable) of degree \( k \) for all \( z \in E_k \) and \( E_k \) is a nonempty interior. We will use [5, Theorem 5.2.2]. In Theorem 5.2.2 in [5], we put \( \Omega = D^n \) and \( E = E_k \). If \( f_z(\zeta) \) is a rational function (of one variable) of degree \( k \) for all \( z \in E \), then \( f_z \) belongs to \( Y \) in Theorem 5.2.2 in [5]. For \( f_z \) is in \( H^p(D) \), \( p \geq 1 \) and so \( f_z \) is continuous on \( \partial D \). Now Theorem 5.2.2 in [5] implies the lemma. \( \Box \)

**Proposition 4.** Suppose \( 1 \leq p \leq \infty \). \( L^P_p \) is of finite dimension if \( f \) is a rational function.

Proof. Suppose \( f = P/Q \) is a nonzero function in \( H^p \) where \( P \) and \( Q \) are polynomials. If \( s \in L^P_p \) then \( sP/Q \in H^p \) and so \( sP \in H^p \). Hence \( s \) belongs to \( L^P_p \) and so \( L^P_p \subseteq L^P_p \). It is enough to prove that \( L^P_p \) is of finite dimension.

**Case** \( n = 1 \). We have the inner outer factorization for \( n = 1 \), that is, \( P = qh \) where \( q \) is a finite Blaschke product and \( h \) is an outer function in \( H^p \). Then it is easy to see that \( L^P_p = L^q_p \). Since \( L^q_p \subseteq \bar{q} H^p \cap H^q \) and \( L^q_p \subseteq \bar{q} H^2 \cap q H^2 = \bar{q}(H^2 \cap q H^2) = \bar{q}(H^2 \oplus q^2 H_0^2) \), \( H^2 \oplus q^2 H_0^2 \) is of finite dimension because \( q \) is a finite Blaschke product.

**Case** \( n \neq 1 \). If \( s \in L^P_p \) then \( sP \in H^p \) and by Case \( n = 1 \) \( (sP)_z(\zeta) \) is a rational function (of one variable) of degree \( \leq k_0 \) for almost all \( z \in T^n \). By Lemma 1, \( sP \) is a rational function (of \( n \) variables) of degree \( k \) and \( k \leq k_0 \). This implies that \( L^P_p \) is of finite dimension. \( \Box \)

When \( f \) is a rational function in \( H^p \), by Theorem 3 and Proposition 4 \( f \) has our factorization. The function \( h \) in \( N(D^n) \) is called \( z_j \)-outer if

\[ \int_{T_j \times T_j} \log |h(z_j, z_j')|dm = \int_{T_j} (\log |\int_{T_j} h(z_j', z_j')dm_j|)dm_j' > -\infty. \]
**Proposition 5.** Fix $1 \leq j \leq n$. If $f(z_1, \ldots, z_n)$ is $z_i$-outer in $H^p$ for $i \neq j$ and $1 \leq i \leq n$, then $f$ has a factorization in $H^p$.

**Proof.** We will generalize Theorem 2 in [3]. That is, when $h$ is $z_i$-outer in $H^p$ for $i \neq j$, $\mathcal{L}^h_p = R$ if and only if the common inner divisor of $\{h_\alpha(z_j)\}_\alpha$ is constant, where

$$h_\alpha(z_j) = \int_{T_j} h(z_j, z'_j) \overline{z_j^\alpha} dm_j'$$

, $\alpha = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n)$ and $\overline{z_j^\beta} = \overline{z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}} z_{j+1}^{\alpha_{j+1}} \cdots z_n^{\alpha_n}}$. Note that $h_\alpha(z_j)$ belongs to $H^p(T_j)$. For the proof, we use the following notation: $\mathcal{H}^p_{(j)} = \{f \in L^p(T^n); \hat{f}(m_1, \ldots, m_n) = 0 \text{ if } m_i < 0 \text{ for all } i \neq j\}$ and $\mathcal{H}^p_{(j)} \cap \mathcal{H}^p_{(j)} = \mathcal{L}^p = \text{the Lebesgue space on } T_j$.

If $\phi \in \mathcal{L}^h_p$ then $g = \phi h$ and $\phi$ belongs to $\mathcal{H}^p_{(j)}$ because $h$ is $z_i$-outer in $H^p$ for $i \neq j$. Since $\phi$ is real-valued, $\phi \in \mathcal{L}^p$ and so $\phi = \phi(z_j)$. If the common inner divisor of $\{h_\alpha(z_j)\}_\alpha$ is constant, then for each $\alpha$

$$\phi(z_j) h_\alpha(z_j) = \int_{T_j} \phi(z_j) h(z_j, z'_j) \overline{z_j^\alpha} dm_j' = \int_{T_j} g(z_j, z'_j) \overline{z_j^\alpha} dm_j'$$

belongs to $H^p(T_j)$ and hence $\phi \in H^\infty(T_j)$. Therefore $\phi$ is constant. This implies that $\mathcal{L}^h_p = R$. Conversely suppose that $\mathcal{L}^h_p = R$. If $\{h_\alpha(z_j)\}_\alpha$ has a non-constant common inner divisor $q(z_j)$, put $\phi(z_j, z'_j) = q(z_j) + q(z_j)$, then $g = \phi h$ belongs to $H^p$. This contradiction shows the ‘only if’ part.

Now we will prove that $f$ has a factorization in $H^p$. If $\{f_\alpha(z_j)\}_\alpha$ does not have common inner divisors, then by what was just proved $\mathcal{L}^f_p = R$ and so we need not prove. If $\{f_\alpha(z_j)\}_\alpha$ have common inner divisors, let $q(z_j)$ be the greatest common inner divisor. Put $\phi = \bar{q}(z_j) + q(z_j)$ and $h = \phi f$, then $h$ belongs to $H^p$ and $\mathcal{L}^h_p = R$. This completes the proof.

§3 $\mathcal{L} = L^\infty$

In this section, we assume that $\mathcal{L} = L^\infty$. If $f$ is a $L^\infty$-extremal function for $H^p$ then $f$ is also a $L^{\infty}_R$-extremal function for $H^p$. When $n = 1$, the converse is true. However this is not true for $n \neq 1$. For example, $z - 2w$ is a $L^{\infty}_R$-extremal function but not a $L^\infty$-extremal. We can prove an analogy of Proposition 1 for $L^\infty$. Let $M_f$ be an invariant closed subspace generated by $f$ in $H^p$ and $\mathcal{M}(M_f)$ the set of multipliers of $M_f$ (see [1]). Then $\mathcal{M}(M_f) = \mathcal{L}^f_p$ for $\mathcal{L} = L^\infty$. It is easy to see that

$$(L^\infty)^f_p \cap \overline{(L^\infty)^f_p} = (L^\infty)^f_p + i(L^\infty)^f_p.$$ 

It is easy to see that $(L^\infty)^f_p$ is a weak * closed invariant subspace which contains $H^\infty$. Then $(L^\infty)^f_p/H^\infty$ is of infinite dimension (see [4, Theorem 1]). Thus we can not expect the analogy of Theorem 3.
Proposition 6. Let $1 \leq p \leq \infty$ and $f$ a nonzero function in $H^p$. Suppose $\phi$ is in $\mathcal{L}_p^f$.

1. $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^{\phi f} \supseteq \phi H^\infty$.
2. $\phi^{-1}$ is in $L^\infty$ if and only if $\mathcal{L}_p^f = \phi \mathcal{L}_p^{\phi f}$.
3. If $\mathcal{L}_p^{\phi f} \supseteq \mathcal{L}_p^f$ then $\phi$ belongs to $H^\infty$. If $\mathcal{L}_p^{\phi f} \supseteq \mathcal{L}_p^f$ and $\phi^{-1}$ is in $L^\infty$ then $\phi^{-1}$ belongs to $H^\infty$.

Proof. (1) If $g \in \mathcal{L}_p^{\phi f}$ then $\phi gf = \phi f \in H^p$ and so $\phi g \in \mathcal{L}_p^f$. (2) If $\phi^{-1} \in L^\infty$ then $\mathcal{L}_p^{\phi f} \supseteq \phi^{-1} \mathcal{L}_p^f \supseteq \mathcal{L}_p^{\phi f}$ by (1). If $\mathcal{L}_p^f = \phi \mathcal{L}_p^{\phi f}$ then $\phi^{-1} \mathcal{L}_p^f = \mathcal{L}_p^{\phi f}$ and so $\phi^{-1} \in \mathcal{L}_p^{\phi f}$. This implies that $\phi^{-1} \in L^\infty$. (3) Suppose $\mathcal{L}_p^{\phi f} \supseteq \mathcal{L}_p^f$. If $k \in \mathcal{L}_p^f$ then $k \in \mathcal{L}_p^{\phi f}$ and so $k \phi f \in H^p$. Hence $\phi^2 f \in H^p$. Repeating this process, $\phi^n \in \mathcal{L}_p^f$ and so $\phi^n f \in H^p$ for all $n \geq 1$. Thus $\phi$ belongs to $H^\infty$. If $\mathcal{L}_p^f \supseteq \mathcal{L}_p^{\phi f}$ and $\phi^{-1} \in L^\infty$, then $\phi^{-1}$ belongs to $H^\infty$. For apply what was proved above for $\phi^{-1}$ assuming $\phi^{-1}(\phi f) = f$.

When $f$ is a nonzero function in $H^p$, $f$ is factorable in $H^p$ if and only if there exists a nonzero function $h$ in $H^p$ such that $|f| \geq |h|$ a.e. on $T^2$ and $\mathcal{L}_p^h = H^\infty$.

Proposition 7. Let $1 \leq p \leq \infty$ and $f$ be a nonzero function in $H^p$. Suppose $\phi$ is a nonzero function in $\mathcal{L}_p^f$.

1. If $\phi^{-1}$ is in $L^\infty$ and $\mathcal{L}_p^{\phi f} = H^\infty$, then $\phi^{-1}$ belongs to $H^\infty$ and $\mathcal{L}_p^f = \phi H^\infty$.
2. If $\mathcal{L}_p^f = \phi H^\infty$, then $\phi^{-1}$ belongs to $H^\infty$ and $\mathcal{L}_p^{\phi f} = H^\infty$.
3. If $\mathcal{L}_p^f$ is the weak * closure of $\phi H^\infty$ and $|\phi| = |h|$ a.e. for some function $h$ in $H^\infty$, then $\mathcal{L}_p^{\phi f} = H^\infty$ for some inner function $\tilde{\phi}_0$ and so $f$ is factorable.

4. There exist $f$ and $\phi$ such that $\phi$ is not the quotient of any two members of $H^\infty(T^n)$.

Proof. (1) By (2) of Proposition 6, $\mathcal{L}_p^{\phi f} = \mathcal{L}_p^f$. Since $\mathcal{L}_p^{\phi f} = H^\infty$, $\phi H^\infty = \mathcal{L}_p^f \supset H^\infty$ and so $\phi^{-1}$ belongs to $H^\infty$. (2) Since $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^{\phi f} \supseteq \phi H^\infty$ by (1) of Proposition 6 and $\mathcal{L}_p^{\phi f} = \phi H^\infty$, $\mathcal{L}_p^{\phi f} = H^\infty$. It is clear that $\phi^{-1} \in H^\infty$. (3) Since $|\phi| = |h|$ a.e., $\phi = \phi_0 h$ and $|\phi_0| = 1$ a.e.. Then $\mathcal{L}_p^{\phi f} = [\phi H^\infty]_* = \phi_0[h H^\infty]_* \supset H^\infty$ and so $\phi_0$ is an inner function where $[S]_*$ is the weak * closure of $S$. (4) This is a result of [6].

Proposition 8. Let $1 \leq p \leq \infty$. Suppose $f$ and $g$ are nonzero functions in $H^p$.

1. If $\mathcal{L}_p^f = H^\infty$ and $|f| \geq |g|$ a.e., then there exists a function $\phi$ in $H^\infty$ such that $g = \phi f$.
2. If $\mathcal{L}_p^f = H^\infty$ and $|f| = |g|$ a.e., then there exists an inner function $\phi$ such that $g = \phi f$.

Proof. (1) Let $\phi = g/f$, then $\phi \in L^\infty$ because $|f| \geq |g|$ a.e.. By (1) of Proposition 6, $\phi$ belongs to $H^\infty$ because $H^\infty = \mathcal{L}_p^f$. (2) follows from (1).

Proposition 9. Let $1 \leq p \leq \infty$. If $f$ is homogeneous polynomial such that $f(z_1, \cdots, z_n) = g(z, w)$ where $z = z_i$, $w = z_j$ and $i \neq j$ then $f$ is factorable in $H^p$.

Proof. Since $f(z_1, \cdots, z_n) = \sum_{j=0}^{\ell} a_j z^{-j} w^j$, $f(z_1, \cdots, z_n) = z^\ell \sum_{j=0}^{\ell} a_j \left(\frac{w}{z}\right)^j = \cdots$
\[
c_{\prod_{j=0}^{\ell}(b_jw - c_jz)}\text{ where } b_j = 1 \text{ or } c_j = 1, \text{ and } |b_j| \leq 1, \ |c_j| \leq 1. \text{ It is easy to see that } L_f^p = \phi H^\infty \text{ where } \phi = \prod(\alpha z - \beta w)^{-1} \text{ and } (\alpha, \beta) \in (\partial D \times D) \cup (D \times \partial D) \text{ (cf. } [2],[4]). \text{ By } (2) \text{ of Proposition 7, } f \text{ is factorable.}
\]

**Question**

1. For any nonzero function \( f \) in \( H^p \), does there exists a function \( \phi \) such that \( L_f^p \supset \neq L_\phi^p \)?

2. Describe \( \phi \) in \( L^\infty \) such that \( L_f^p \supset L_\phi^p \).

3. Describe \( \phi \) in \( L^\infty \) such that \( [\phi H^\infty]_\phi \supset H^\infty \).

**References**


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