Submodules of $L^2(\mathbb{R}^2)$

MICHIRO SETO

Abstract

In this paper, we study submodules over $\mathbb{R}^2$. We will give a Lax-type of theorem and a result analogous to Helson’s theory.


Keywords and phrases: Hardy submodules.

1 Introduction

$L^2(\mathbb{R}^2)$ will denote the Hilbert space of square-integrable measurable functions with respect to the usual Lebesgue measure $dx_1dx_2$ on the two dimensional Euclidean space $\mathbb{R}^2$. $H^2(\mathbb{R})$ denotes the usual Hardy space on $\mathbb{R}$, that is, $H^2(\mathbb{R})$ consists of all functions in $L^2(\mathbb{R})$ which can be extended analytically to the upper half plane $\mathbb{C}_+ = \{x + it : x \in \mathbb{R}, t > 0\}$. $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$, the Hilbert space tensor product of $H^2(\mathbb{R})$, is the space of all $f$ in $L^2(\mathbb{R}^2)$ whose Fourier transform

$$\hat{f}(\lambda_1, \lambda_2) = \hat{f}(\lambda_1, \lambda_2) = \int_{\mathbb{R}^2} f(x_1, x_2)e^{-i(\lambda_1 x_1 + \lambda_2 x_2)}dx_1dx_2$$

is 0 whenever at least one component of $(\lambda_1, \lambda_2)$ is negative, where $(\lambda_1, \lambda_2)$ and $(x_1, x_2)$ are in $\mathbb{R}^2$. In this paper, $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ is denoted by $H^2(\mathbb{R}^2)$, for short. Note that our $H^2(\mathbb{R}^2)$ is different from the usual Hardy space on $\mathbb{R}^2$.

Definition 1.1 A closed subspace $\mathcal{M}$ of $L^2(\mathbb{R}^2)$ is said to be a submodule of $L^2(\mathbb{R}^2)$ if $e^{ix_j}\mathcal{M} \subseteq \mathcal{M}$ for any $j = 1, 2$ and any $s \geq 0$. For $s \geq 0$, $S_j(s)$ denotes the restriction on $\mathcal{M}$ of the multiplication operator on $L^2(\mathbb{R}^2)$ by $e^{ix_j}$.

Submodules in one variable were completely described by Lax in [4]. In [1], Helson gave another point of view to the result of Lax. The purpose of our study is to consider Helson’s theory in the multi-variable setting. My interest in considering Helson’s theory in two variables is motivated by the study of Hardy submodules over the bidisk: Hardy submodules are invariant subspaces of Hardy space under multiplication operators by
bounded analytic functions. However, it is easy to see that a straightforward generalization of Helson’s theory fails in the multi-variable setting. In Section 2 of this paper, we give a Lax-type of theorem in two variables. To prove this we use Masani’s integral (cf. [6]). In Section 3, we consider Helson’s theory in two variables. We will give a result analogous to Helson’s result under the following condition: \( S_1(s)S_2(t)^* = S_2(t)^*S_1(s) \) for all \( s, t \geq 0 \).

2 A Lax-type of theorem in \( \mathbb{R}^2 \)

In [9], the author showed the following Lax-type of theorem analogous to the theorem proved by Mandrekar [5] and Nakazi [7] for the bitorus.

**Theorem 2.1** Let \( \mathcal{M} \) be a submodule of \( L^2(\mathbb{R}^2) \), \( H^2_1(\mathbb{R}^2) = L^2(\mathbb{R}, dx_1) \otimes H^2(\mathbb{R}, dx_2) \) and \( H^2_2(\mathbb{R}^2) = H^2(\mathbb{R}, dx_1) \otimes L^2(\mathbb{R}, dx_2) \). If \( S_1(s)S_2(t)^* = S_2(t)^*S_1(s) \) for all \( s, t \geq 0 \), then one and only one of the following occurs:

(i) \( \mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_F \varphi H^2_1(\mathbb{R}^2) \),

(ii) \( \mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_G \psi H^2_2(\mathbb{R}^2) \),

(iii) \( \mathcal{M} = q H^2(\mathbb{R}^2) \),

where \( \varphi, \psi \) and \( q \) are unimodular functions, \( \chi_E \) is the characteristic function of \( E \), \( \chi_F \) (resp. \( \chi_G \)) is the characteristic function of \( F \) (resp. \( G \)) which depends only on the variable \( x_1 \) (resp. \( x_2 \)).

We shall give a proof which differs from that given in [9]. To begin with, we briefly introduce Masani’s integral which can be seen as a continuous Wold decomposition for a continuous semi-group of isometries, according to [6].

**Definition 2.1 (Masani [6])** Let \( \{ S(t) : t \geq 0 \} \) be a strongly continuous semi-group of isometries on a Hilbert space \( \mathcal{H} \). We introduce an operator-valued interval-measure. The measure \( T_{ab} \) of the interval \([a, b]\) is defined by as follows:

\[
T_{[a,b]} = T(b) - T(a), \quad \text{where} \quad T(t) = \frac{1}{\sqrt{2}} \left\{ S(t) - I - \int_0^t S(s) \, ds \right\}, \quad \text{for} \ t \geq 0.
\]

Let \( iH \) be the infinitesimal generator of \( \{ S(t) : t \geq 0 \} \) and \( V \) be the Cayley transform of \( H \) and \( R = V(\mathcal{H}) \). For the step-function \( x = \sum_{k=1}^n \alpha_k \chi_{J_k} \) on \([a, b]\), where \( \alpha_k \) in \( R^1 \) and \( \chi_{J_k} \) is the characteristic function of bounded interval \( J_k \), we define

\[
\int_a^b T_{ti}(x_i) := \sum_{k=1}^n T_{J_k}(\alpha_k).
\]
For any $x$ in $L^2([a, b], R^2)$, we define

$$
\int_a^b T_{dt}(x_t) := \lim_{n \to \infty} \int_a^b T_{dt}(x_t^{[n]}),
$$

where $\{x_t^{[n]}, n \geq 1\}$ is any sequence of step-functions which is tending to $x$ in the $L^2$-topology.

We now define a direct integral as a set of vector-valued integrals:

$$
\int_a^b T_{dt}(R^2) := \left\{ \xi : \xi = \int_a^b T_{dt}(x_t), x \in L^2([a, b], R^2) \right\}.
$$

**Theorem 2.2 (Masani [6])** Let $\{S(t) : t \geq 0\}$ be a strongly continuous semi-group of isometries on a Hilbert space $\mathcal{H}$, $iH$ be its infinitesimal generator and let $V$ be the Cayley transform of $H$. Then, for $a \geq 0$,

$$
S(a)(\mathcal{H}) = \int_a^\infty T_{dt}(R^2) \oplus \mathcal{H}_\infty,
$$

where $R = V(\mathcal{H})$ and $\mathcal{H}_\infty = \bigcap_{t \geq 0} S(t)(\mathcal{H})$.

This theorem can be seen as a continuous Wold decomposition.

**Example 2.1** Let $T_{ds}^{(k)}$ be the operator-valued measures defined by $S_k(s)$ for $k = 1, 2$. Identifying bounded functions with multiplication operators, $T^{(k)}(s)$ can be computed formally as follows:

$$
T^{(k)}(s) = \frac{1}{\sqrt{2}} \left\{ S_k(s) - I_{\mathcal{M}} - \int_0^s S_k(t) \, dt \right\} = \frac{1}{\sqrt{2}} \left\{ e^{ix_k} - 1 - \int_0^s e^{itx_k} \, dt \right\} = \frac{1}{\sqrt{2}} \left\{ e^{ix_k} - 1 - \left[ \frac{1}{ix_k} e^{itx_k} \right]_0^s \right\} = \frac{1}{\sqrt{2}} \left\{ e^{ix_k} - 1 - \frac{1}{ix_k} \left( e^{ix_k} - 1 \right) \right\} = \frac{1}{\sqrt{2}} \left( e^{ix_k} - 1 \right) \left( 1 - \frac{1}{ix_k} \right) = \frac{1}{\sqrt{2}} x_k \left( e^{ix_k} - 1 \right) (x_k + i).
$$

Thus the operator valued measure $T_{ds}^{(k)}$ can be computed as follows:

$$
T_{ds}^{(k)} = \frac{d}{ds} \left( \frac{1}{\sqrt{2}} x_k \left( e^{ix_k} - 1 \right) (x_k + i) \right) \, ds = \frac{1}{\sqrt{2}} e^{ix_k} (x_k + i) \, ds.
$$
We are now in a position to prove Theorem 2.1.

**Proof (A proof of Theorem 2.1)** Some parts of this proof are similar to those in the proof by Mandrekar [5] and Nakazi [7] for the bitorus (cf. Seto [9]).

Suppose that \( S_1(s)S_2(t)^* = S_2(t)^*S_1(s) \) for all \( s, t \geq 0 \). Let \( V_{x_k} \) be the isometry induced by \( \{ S_k(s) : s \geq 0 \} \) as in Theorem 2.2 for \( k = 1, 2 \). Since \( V_{x_k} \) is in the von Neumann algebra generated by \( \{ S_k(s) : s \geq 0 \} \), we have \( V_{x_1}^* V_{x_2} = V_{x_2} V_{x_1}^* \). It suffices to consider the following two cases:

- \( V_{x_1} \) and \( V_{x_2} \) are completely non-unitary,
- \( V_{x_1} \) is completely non-unitary and \( V_{x_2} \) is unitary.

First, we suppose that \( V_{x_1} \) and \( V_{x_2} \) are completely non-unitary. Then

\[
\mathcal{M} = \int_0^\infty T_{ds}^{(1)} \left\{ \int_0^\infty T_{dt}^{(2)} \left( \mathcal{M} \ominus (V_{x_1} \mathcal{M} + V_{x_2} \mathcal{M}) \right) \right\},
\]

by Theorem 2.2. Let \( f \) be in \( \mathcal{M} \ominus (V_{x_1} \mathcal{M} + V_{x_2} \mathcal{M}) \) such that \( \| f \| = 1 \). Then

\[
\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \frac{2(x_1 - i)^k (x_2 - i)^l}{(x + i)^k (x + i)^l} \, dx_1 \, dx_2 = 0,
\]

for all \( (k, l) \neq (0, 0) \). Changing variables \( x_1 \) and \( x_2 \) to \( \theta_1 \) and \( \theta_2 \), we have

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} |f(\theta_1, \theta_2)|^2 e^{i k \theta_1} e^{i l \theta_2} \frac{1}{(\cos^2 \frac{\theta_1}{2})(\cos^2 \frac{\theta_2}{2})} \, d\theta_1 \, d\theta_2 = 0.
\]

Hence \( |f(\theta_1, \theta_2)|^2 (\cos^2 \frac{\theta_1}{2})^{-1} (\cos^2 \frac{\theta_2}{2})^{-1} = 1 \), equivalently \( (x_1^2 + 1)(x_2^2 + 1)|f(x_1, x_2)|^2 = 1 \). Therefore, there exists a unimodular function \( q \) such that

\[
f = \frac{q}{(x_1 + i)(x_2 + i)}.
\]

Hence we have

\[
\mathcal{M} \ominus (V_{x_1} \mathcal{M} + V_{x_2} \mathcal{M}) = \mathbb{C} \frac{q}{(x_1 + i)(x_2 + i)}.
\]

By the Paley-Wiener theorem,

\[
\mathcal{M} = \left\{ \xi : \xi = q \int_0^\infty e^{i x_1 \xi} \, ds \right\} \left\{ \xi : \xi = q \int_0^\infty e^{i x_2 \xi} \, dt \right\} = q \left( H^2(\mathbb{R}) \otimes H^2(\mathbb{R}) \right) = q H^2(\mathbb{R}^2).
\]

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Next, we suppose that $V_{x_1}$ is completely non-unitary and $V_{x_2}$ is unitary. Then
\[ \mathcal{M} = \int_{0}^{\infty} T_{d_s}^{(1)}(\mathcal{M} \oplus V_{x_1} \mathcal{M}), \]
by Theorem 2.2. Let $f$ be in $\mathcal{M} \oplus V_{x_1} \mathcal{M}$. Then
\[ \int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \frac{(x_1 - i)^k (x_2 - i)^l}{(x_1 + i)^k (x_2 + l)^l} \, dx_2 \, dx_1 = 0, \]
for all $k \neq 0$ and $l$. By the same calculations as in the first case, we have
\[ f(x_1, x_2) = g(x_1, x_2)/(x_1 + i) \]
for some $g$ such that the function $|g|$ depends only on the variable $x_2$.

The following argument is known (cf. [3]). Let $\chi_{E(g)}$ be the support function of $g$, that is, $\chi_{E(g)}$ is the characteristic function of the set $E(g) = \{(x_1, x_2) \in \mathbb{R}^2 : g(x_1, x_2) \neq 0\}$, and $\phi_g$ be a unimodular function defined as follows:
\[ \phi_g = \begin{cases} \frac{g}{|g|} & (g \neq 0) \\ 1 & (g = 0). \end{cases} \]

Then
\[ \int_{\mathbb{R}} e^{ix_2} \frac{g}{x_1 + i} = \phi_g \frac{1}{x_1 + i} \chi_{E(g)} L^2(\mathbb{R}, dx_2), \]
where $\vee$ denotes the closed vector span. Since there exists a function $F$ in $\mathcal{M} \oplus V_{x_1} \mathcal{M}$ which has the maximal support in $\mathcal{M} \oplus V_{x_1} \mathcal{M}$, that is, $E(g) \subseteq E(F)$, for any $g$ in $\mathcal{M} \oplus V_{x_1} \mathcal{M}$, we have
\[ \mathcal{M} \oplus V_{x_1} \mathcal{M} = \phi_F \frac{1}{x_1 + i} \chi_{E(F)} L^2(\mathbb{R}, dx_2). \]

Let $\chi_G = \chi_{E(F)}$ and $\psi = \phi_F$. By the Paley-Wiener theorem, we have the following:
\[ \mathcal{M} = \int_{0}^{\infty} T_{d_s}^{(1)} \left( \frac{1}{x_1 + i} \chi_G \psi L^2(\mathbb{R}, dx_2) \right) \]
\[ = \left\{ \xi : \xi = \chi_G \psi \int_{0}^{\infty} e^{ix_1} f(s, x_2) \, ds ; f \in L^2((0, \infty) \times \mathbb{R}) \right\} \]
\[ = \chi_G \psi H^2(\mathbb{R}, dx_1) \otimes L^2(\mathbb{R}, dx_2) \]
\[ = \chi_G \psi H_{x_2}^2(\mathbb{R}^2). \]

The converse is easy to verify.

A function $q$ is said to be inner if $q$ is in $H^2(\mathbb{R}^2)$ and $|q(x_1, x_2)| = 1$ a.e.

**Corollary 2.1** Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{R}^2)$. Then $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ if and only if $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some inner function $q$.  

5
3 Helson’s theory under the double commuting condition in $L^2(\mathbb{R}^2)$

In this section, we discuss Helson’s theory in $L^2(\mathbb{R}^2)$ under the double commuting condition: $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$. Then, it is parallel to Helson’s argument for the one-variable case in [1].

**Definition 3.1** Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. For any $\lambda, \mu$ in $\mathbb{R}$, we define one-parameter unitary groups $\{\alpha_\lambda\}$, $\{\beta_\mu\}$ and projections $\{P_\lambda\}$, $\{Q_\mu\}$ on $L^2(\mathbb{R}^2)$ as follows: for any $f$ in $L^2(\mathbb{R}^2)$, $\alpha_\lambda f = e^{i\lambda x} f$, $\beta_\mu f = e^{i\mu y} f$, and $P_\lambda = \alpha_\lambda^* P_\lambda \alpha_\lambda$, $Q_\mu = \beta_\mu^* P_\mu \beta_\mu$, that is, $P_\lambda$ and $Q_\mu$ are the orthogonal projections of $L^2(\mathbb{R}^2)$ onto $\alpha_\lambda^* \mathcal{M}$ and $\beta_\mu^* \mathcal{M}$, respectively.

**Lemma 3.1** Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ if and only if $P_\lambda Q_\mu \alpha_\lambda P_\lambda = P_\lambda \beta_\mu P_\lambda \alpha_\lambda P_\lambda$ for all $\lambda, \mu$ in $\mathbb{R}$.

**Proof** It is easy to verify.

**Definition 3.2** A submodule $\mathcal{M}$ of $L^2(\mathbb{R}^2)$ is said to be simple if $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ and $\left(\cap_\lambda \alpha_\lambda \mathcal{M} + \cap_\mu \beta_\mu \mathcal{M}\right) = \{0\}$ (this is equivalent to that $P_{-\infty} = \lim_{\lambda \to -\infty} P_\lambda = O$ and $Q_{-\infty} = \lim_{\mu \to -\infty} Q_\mu = O$).

Note that a submodule $\mathcal{M}$ is simple if and only if $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some unimodular function $q$ by Theorem 2.2.

Next, we define two sequences of projections, and show that these are the spectral measures of $L^2(\mathbb{R}^2)$. Let $E_\lambda$ and $F_\mu$ be projections defined as follows:

$$E_\lambda = \alpha_\lambda^* Q_{+\infty} \alpha_\lambda \text{ and } F_\mu = \beta_\mu^* P_{+\infty} \beta_\mu.$$  

**Lemma 3.2** Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. If $\mathcal{M}$ is simple, then $\{E_\lambda\}$ and $\{F_\mu\}$ are spectral families. Moreover $E_\lambda F_\mu = F_\mu E_\lambda = \alpha_\lambda^* \beta_\mu^* P_\lambda \alpha_\lambda \beta_\mu$ for all $\lambda, \mu$ in $\mathbb{R}$.

**Proof** Since, for $\gamma \geq \lambda, \mu$,

$$E_\lambda F_\mu = \alpha_\lambda^* Q_{+\infty} \alpha_\lambda \beta_\mu^* P_{+\infty} \beta_\mu = \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\gamma^* P_\lambda \beta_\gamma \alpha_\lambda \beta_\mu^* \alpha_\gamma^* P_\lambda \alpha_\gamma \beta_\mu \right) = \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\gamma^* P_\lambda \alpha_\gamma \beta_\mu \beta_\gamma^* P_\lambda \alpha_\gamma \beta_\mu \right) = \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\gamma^* P_\lambda \beta_\gamma^* P_\lambda \alpha_\gamma \beta_\mu \beta_\gamma^* P_\lambda \alpha_\gamma \beta_\mu \right) = \alpha_\lambda^* \beta_\mu^* P_\lambda \alpha_\lambda \beta_\mu.$$
we have $E_{\lambda} F_{\mu} = \alpha_{\lambda}^{*} \beta_{\mu}^{*} P_{M} \alpha_{\lambda} \beta_{\mu} = F_{\mu} E_{\lambda}$ for all $\lambda, \mu$ in $\mathbb{R}$.

Next, suppose that

$$\chi_{G} L^{2}(\mathbb{R}^{2}) = \bigcup_{\lambda, \mu} \alpha_{\lambda} \beta_{\mu} \mathcal{M} \bigcap \infty \left( \alpha_{\lambda} \bigcup_{\mu} \beta_{\mu} \mathcal{M} + \bigcap_{\mu} \alpha_{\lambda} \mathcal{M} \right),$$

where the bar denotes the closure. We shall show $\chi_{G} = 1$. The following argument is the same as in [1]. Let $U_{s,0} = \int_{\mathbb{R}} e^{is \lambda} \, dE_{\lambda}$. Then, since $\alpha_{\lambda_{0}} \beta_{\mu_{0}} \alpha_{\lambda}^{*} \beta_{\mu}^{*} = E_{\lambda - \lambda_{0}}$, we have

$$\alpha_{\lambda_{0}} \beta_{\mu_{0}} U_{s,0} = \int_{\mathbb{R}} e^{is \lambda} \, dE_{\lambda},$$

$$= \int_{\mathbb{R}} e^{is \lambda} \, dE_{\lambda - \lambda_{0}} \alpha_{\lambda_{0}} \beta_{\mu_{0}},$$

$$= e^{is \lambda_{0}} \int_{\mathbb{R}} e^{is(\lambda_{0} - \lambda)} \, dE_{\lambda - \lambda_{0}} \alpha_{\lambda_{0}} \beta_{\mu_{0}},$$

$$= e^{is \lambda_{0}} U_{s,0} \alpha_{\lambda_{0}} \beta_{\mu_{0}}.$$

Therefore

$$U_{s,0} T_{-s,0} \alpha_{\lambda} \beta_{\mu} = U_{s,0} e^{is \lambda} \alpha_{\lambda} \beta_{\mu} T_{-s,0},$$

$$= \alpha_{\lambda} \beta_{\mu} U_{s,0} T_{-s,0},$$

where $T_{s,t}$ is the translation operator such that $(T_{s,t} f)(x, y) = f(x - s, y - t)$. Hence $U_{s,0} T_{-s,0}$ is a multiplication operator on $L^{2}(\mathbb{R}^{2})$. Since $U_{s,0} T_{-s,0}$ maps $T_{s,0} \chi_{G} L^{2}(\mathbb{R}^{2})$ to $\chi_{G} L^{2}(\mathbb{R}^{2})$, we have $T_{s,0} \chi_{G} L^{2}(\mathbb{R}^{2}) = \chi_{G} L^{2}(\mathbb{R}^{2})$. By the same argument for $\beta_{\mu}$, we have $T_{0,t} \chi_{G} L^{2}(\mathbb{R}^{2}) = \chi_{G} L^{2}(\mathbb{R}^{2})$, that is, $T_{s,t} \chi_{G} L^{2}(\mathbb{R}^{2}) = L^{2}(\mathbb{R}^{2})$ for all $s, t$ in $\mathbb{R}$. Hence $G$ is a null set or $G = \mathbb{R}^{2}$, and we have

$$\lim_{\lambda \to \infty} E_{\lambda} = \lim_{\mu \to \infty} F_{\mu} = \bigcup_{\lambda, \mu} \alpha_{\lambda} \beta_{\mu} \mathcal{M} = L^{2}(\mathbb{R}^{2}),$$

$$\lim_{\lambda \to -\infty} E_{\lambda} = \bigcap_{\lambda} \alpha_{\lambda} \bigcup_{\mu} \beta_{\mu} \mathcal{M} = \{0\},$$

$$\lim_{\mu \to -\infty} F_{\mu} = \bigcap_{\mu} \beta_{\mu} \bigcup_{\lambda} \alpha_{\lambda} \mathcal{M} = \{0\}.$$

Therefore $\{E_{\lambda}\}$ and $\{F_{\mu}\}$ are the spectral families.

By virtue of Lemma 3.2, for any simple submodule of $L^{2}(\mathbb{R}^{2})$, there exists a spectral measure $dE_{\lambda,\mu} = dE_{\lambda} dF_{\mu}$ on $\mathbb{R}^{2}$ and we have a two-parameter continuous unitary group $\{U_{s,t}\}$ on $L^{2}(\mathbb{R}^{2})$ as follows:

$$U_{s,t} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s \lambda + t \mu)} \, dE_{\lambda} dF_{\mu} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s \lambda + t \mu)} \, dE_{\lambda,\mu}.$$
Definition 3.3 A family \( \{A_{s,t}\} \) of functions on \( \mathbb{R}^2 \) which are individually measurable is said to be a cocycle of \( \mathbb{R}^2 \) if

(i) \( |A_{s,t}(x, y)| = 1 \) almost everywhere in \( x, y \), for each \( s, t \),

(ii) \( A_{s,t}f \) moves continuously in \( L^2(\mathbb{R}^2) \) as \( s \) and \( t \) varies, for each \( f \) in \( L^2(\mathbb{R}^2) \),

(iii) \( A_{s+u,t+v} = A_{s,t}T_{s,t} A_{u,v} \) almost everywhere, for each \( s, t, u \) and \( v \).

Example 3.1 (cf. [1]) In Lemma 5.3, we showed the following commutation relation:

\[
U_{s,0}T_{s,0} \alpha \beta \mu = \alpha \beta \mu U_{s,0} T_{s,0}.
\]

Using the same argument with respect to the variable \( x_2 \), we have

\[
U_{s,t} T_{-s,-t} \alpha \beta \mu = \alpha \beta \mu U_{s,t} T_{-s,-t}.
\]

Therefore \( U_{s,t} T_{-s,-t} \) is the multiplication operator by some unimodular function \( A_{s,t} \). We shall show \( \{A_{s,t}\} \) is a cocycle of \( \mathbb{R}^2 \). Identifying bounded functions with multiplication operators, we have

\[
A_{s+u,t+v} = U_{s+u,t+v} T_{-s-u,-t-v} = U_{s,u}T_{-u,-v} T_{-s,-t} = U_{s,t} A_{u,v} T_{-s,-t} = A_{s,t} T_{s,t} A_{u,v} T_{-s,-t}.
\]

Hence

\[
A_{s+u,t+v}(x, y) = A_{s,t}(x, y) A_{u,v}(x - s, y - t).
\]

Proposition 3.1 There exists a one-to-one correspondence between simple submodules of \( L^2(\mathbb{R}^2) \) and cocycles of \( \mathbb{R}^2 \).

Proof Suppose that \( \{A_{s,t}\} \) is a cocycle of \( \mathbb{R}^2 \). Let \( U_{s,t} = A_{s,t} T_{s,t} \). Then \( \{U_{s,t}\} \) is a two-parameter unitary group on \( L^2(\mathbb{R}^2) \). By Stone’s theorem for \( \mathbb{R}^2 \), there exists a unique spectral measure of \( L^2(\mathbb{R}^2) \) such that

\[
U_{s,t} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s\lambda + t\mu)} \, dE_{\lambda,\mu}.
\]

Let \( \mathcal{M} = \text{ran } E_{0,0} \). Then

\[
\int_{\mathbb{R}^2} e^{i(s\lambda + t\mu)} \, dE_{\lambda+\tau_1,\mu+\tau_2} = e^{-i(s\tau_1 + t\tau_2)} \int_{\mathbb{R}^2} e^{i(s\lambda + t\mu)} \, dE_{\lambda+\tau_1,\mu+\tau_2} = e^{-i(s\tau_1 + t\tau_2)} \int_{\mathbb{R}^2} e^{i(s\lambda + t\mu)} \, dE_{\lambda,\mu} = \alpha_{\tau_1}^* \beta_{\tau_2}^* U_{s,t} A_{\tau_1} \beta_{\tau_2} = \int_{\mathbb{R}^2} e^{i(s\lambda + t\mu)} \, d \left( \alpha_{\tau_1}^* \beta_{\tau_2}^* E_{\lambda,\mu} A_{\tau_1} \beta_{\tau_2} \right)
\]
Hence we have
\[ E_{\lambda + \tau_1, \mu + \tau_2} = \alpha^*_{\tau_1} \beta^*_{\tau_2} E_{\lambda, \mu} \alpha_{\tau_1} \beta_{\tau_2}. \]

Therefore \( \mathcal{M} \) is a submodule of \( L^2(\mathbb{R}^2) \).

Next, we shall show that \( \mathcal{M} \) satisfies the double commuting condition. It suffices to consider the case where \( \lambda \geq 0 \) and \( \mu \leq 0 \).

\[
P_{\mathcal{M}} \alpha_{\lambda} P_{\mathcal{M}} \beta_{\mu} P_{\mathcal{M}} = E_{0,0} \alpha_{\lambda} E_{0,0} \beta_{\mu} E_{0,0}
= \alpha_{\lambda} E_{0,0} E_{0,0} - \mu \beta_{\mu}
= \alpha_{\lambda} E_{0,0} \beta_{\mu},
\]

and
\[
P_{\mathcal{M}} \beta_{\mu} P_{\mathcal{M}} \alpha_{\lambda} P_{\mathcal{M}} = E_{0,0} \beta_{\mu} E_{0,0} \alpha_{\lambda} E_{0,0}
= E_{0,0} E_{0,0} - \mu \beta_{\mu} \alpha_{\lambda} E_{0,0}
= E_{0,0} \alpha_{\lambda} \beta_{\mu} E_{0,0}
= \alpha_{\lambda} E_{0,0} E_{0,0} - \mu \beta_{\mu}
= \alpha_{\lambda} E_{0,0} \beta_{\mu}.
\]

Therefore \( P_{\mathcal{M}} \alpha_{\lambda} P_{\mathcal{M}} \beta_{\mu} P_{\mathcal{M}} = P_{\mathcal{M}} \beta_{\mu} P_{\mathcal{M}} \alpha_{\lambda} P_{\mathcal{M}} \). This concludes the proof by Lemma 3.1.

**Example 3.2** (cf. [1]) Suppose that \( \mathcal{M} = qH^2(\mathbb{R}^2) \) for some unimodular function \( q \).

Then its cocycle is \( \{qT_{s,t}q^{-1}\} \).

A cocycle of the form \( A_{s,t} = qT_{s,t}q^{-1} \), for some unimodular function, is called a coboundary of \( \mathbb{R}^2 \).

**Corollary 3.1** Every cocycle of \( \mathbb{R}^2 \) is a coboundary of \( \mathbb{R}^2 \).

**Proof** By Theorem 2.1, for any simple submodule \( \mathcal{M} \) of \( L^2(\mathbb{R}^2) \), there is a unimodular function \( q \) such that \( \mathcal{M} = qH^2(\mathbb{R}^2) \). Hence the cocycle of \( \mathcal{M} \) is a coboundary.

**References**


Department of Mathematics  
Faculty of Science  
Hokkaido University  
Sapporo 060-0810  
Japan  
e-mail: seto@math.sci.hokudai.ac.jp