Submodules of $L^2(\mathbb{R}^2)$

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Abstract

In this paper, we study submodules over $\mathbb{R}^2$. We will give a Lax-type of theorem and a result analogous to Helson's theory.

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1 Introduction

$L^2(\mathbb{R}^2)$ will denote the Hilbert space of square-integrable measurable functions with respect to the usual Lebesgue measure dx_1dx_2 on the two dimensional Euclidean space $\mathbb{R}^2$. $H^2(\mathbb{R})$ denotes the usual Hardy space on $\mathbb{R}$, that is, $H^2(\mathbb{R})$ consists of all functions in $L^2(\mathbb{R})$ which can be extended analytically to the upper half plane $\mathbb{C}_+ = \{x + it : x \in \mathbb{R}, t > 0\}$. $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$, the Hilbert space tensor product of $H^2(\mathbb{R})$, is the space of all f in $L^2(\mathbb{R}^2)$ whose Fourier transform

$$\mathfrak{F}(f)(\lambda_1, \lambda_2) = \hat{f}(\lambda_1, \lambda_2) = \int_{\mathbb{R}^2} f(x_1, x_2)e^{-i(\lambda_1 x_1 + \lambda_2 x_2)}dx_1dx_2$$

is 0 whenever at least one component of (λ_1, λ_2) is negative, where (λ_1, λ_2) and (x_1, x_2) are in $\mathbb{R}^2$. In this paper, $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ is denoted by $H^2(\mathbb{R}^2)$, for short. Note that our $H^2(\mathbb{R}^2)$ is different from the usual Hardy space on $\mathbb{R}^2$.

Definition 1.1 *A closed subspace $\mathcal{M}$ of $L^2(\mathbb{R}^2)$ is said to be a submodule of $L^2(\mathbb{R}^2)$ if $e^{isx_j}\mathcal{M} \subseteq \mathcal{M}$ for any $j = 1, 2$ and any $s \geq 0$. For $s \geq 0$, $S_j(s)$ denotes the restriction on $\mathcal{M}$ of the multiplication operator on $L^2(\mathbb{R}^2)$ by e^{isx_j}.*

Submodules in one variable were completely described by Lax in [4]. In [1], Helson gave another point of view to the result of Lax. The purpose of our study is to consider Helson's theory in the multi-variable setting. My interest in considering Helson's theory in two variables is motivated by the study of Hardy submodules over the bidisk: Hardy submodules are invariant subspaces of Hardy space under multiplication operators by bounded analytic functions. However, it is easy to see that a straightforward generalization of Helson's theory fails in the multi-variable setting. In Section 2 of this paper, we give a Lax-type of theorem in two variables. To prove this we use Masani's integral (cf. [6]). In Section 3, we consider Helson's theory in two variables. We will give a result analogous to Helson's result under the following condition: $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$.

2 A Lax-type of theorem in $\mathbb{R}^2$

In [9], the author showed the following Lax-type of theorem analogous to the theorem proved by Mandrekar [5] and Nakazi [7] for the bitorus.

Theorem 2.1 *Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$, $H^2_{x_1}(\mathbb{R}^2) = L^2(\mathbb{R}, dx_1) \otimes H^2(\mathbb{R}, dx_2)$ and $H^2_{x_2}(\mathbb{R}^2) = H^2(\mathbb{R}, dx_1) \otimes L^2(\mathbb{R}, dx_2)$. If $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$, then one and only one of the following occurs:*

(i) $\mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_F \varphi H^2_{x_1}(\mathbb{R}^2)$,

(ii) $\mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_G \psi H^2_{x_2}(\mathbb{R}^2)$,

(iii) $\mathcal{M} = qH^2(\mathbb{R}^2)$,

where φ, ψ and q are unimodular functions, χ_E is the characteristic function of E, χ_F (resp. χ_G) is the characteristic function of F (resp. G) which depends only on the variable x_1 (resp. x_2).

We shall give a proof which differs from that given in [9]. To begin with, we briefly introduce Masani's integral which can be seen as a continuous Wold decomposition for a continuous semi-group of isometries, according to [6].

Definition 2.1 (Masani [6]) Let $\{S(t) : t \geq 0\}$ be a strongly continuous semi-group of isometries on a Hilbert space $\mathcal{H}$. We introduce an operator-valued interval-measure. The measure T_{ab} of the interval $[a, b]$ is defined by as follows:

$$T_{[a,b]} = T(b) - T(a), \text{ where } T(t) = \frac{1}{\sqrt{2}} \left\{ S(t) - I - \int_0^t S(s)\, ds \right\}, \text{ for } t \geq 0.$$

Let iH be the infinitesimal generator of $\{S(t) : t \geq 0\}$ and V be the Cayley transform of H and $R = V(\mathcal{H})$. For the step-function $x = \sum_{k=1}^n \alpha_k \chi_{J_k}$ on $[a, b]$, where α_k in $R^\perp$ and χ_{J_k} is the characteristic function of bounded interval J_k, we define

$$\int_a^b T_{dt}(x_t) := \sum_{k=1}^n T_{J_k}(\alpha_k).$$

For any x in $L^2([a,b], R^\perp)$, we define

$$\int_a^b T_{dt}(x_t) := \lim_{n\to\infty} \int_a^b T_{dt}(x_t^{(n)}),$$

where $\{x_t^{(n)}, n \geq 1\}$ is any sequence of step-functions which is tending to x in the L^2-topology.

We now define a direct integral as a set of vector-valued integrals:

$$\int_a^b T_{dt}(R^\perp) := \left\{\xi : \xi = \int_a^b T_{dt}(x_t), x \in L^2\left([a,b], R^\perp\right)\right\}.$$

Theorem 2.2 (Masani [6]) *Let $\{S(t) : t \geq 0\}$ be a strongly continuous semi-group of isometries on a Hilbert space $\mathcal{H}$, iH be its infinitesimal generator and let V be the Cayley transform of H. Then, for $a \geq 0$,*

$$S(a)(\mathcal{H}) = \int_a^\infty T_{dt}(R^\perp) \oplus \mathcal{H}_\infty,$$

where $R = V(\mathcal{H})$ and $\mathcal{H}_\infty = \bigcap_{t\geq 0} S(t)(\mathcal{H})$.

This theorem can be seen as a continuous Wold decomposition.

Example 2.1 Let $T_{ds}^{(k)}$ be the operator-valued measures defined by $S_k(s)$ for $k = 1, 2$. Identifying bounded functions with multiplication operators, $T^{(k)}(s)$ can be computed formally as follows:

$$\begin{aligned}
T^{(k)}(s) &= \frac{1}{\sqrt{2}}\left\{S_k(s) - I_{\mathcal{M}} - \int_0^s S_k(t)\, dt\right\} \\
&= \frac{1}{\sqrt{2}}\left\{e^{isx_k} - 1 - \int_0^s e^{itx_k}\, dt\right\} \\
&= \frac{1}{\sqrt{2}}\left\{e^{isx_k} - 1 - \left[\frac{1}{ix_k}e^{itx_k}\right]_0^s\right\} \\
&= \frac{1}{\sqrt{2}}\left\{e^{isx_k} - 1 - \frac{1}{ix_k}\left(e^{isx_k} - 1\right)\right\} \\
&= \frac{1}{\sqrt{2}}(e^{isx_k} - 1)\left(1 - \frac{1}{ix_k}\right) \\
&= \frac{1}{\sqrt{2}\, x_k}(e^{isx_k} - 1)(x_k + i).
\end{aligned}$$

Thus the operator valued measure $T_{ds}^{(k)}$ can be computed as follows:

$$\begin{aligned}
T_{ds}^{(k)} &= \frac{d}{ds}\left(\frac{1}{\sqrt{2}\, x_k}(e^{isx_k} - 1)(x_k + i)\right)\, ds \\
&= \frac{1}{\sqrt{2}} i e^{isx_k}(x_k + i)\, ds.
\end{aligned}$$

We are now in a position to prove Theorem 2.1.

Proof (A proof of Theorem 2.1) Some parts of this proof are similar to those in the proof by Mandrekar [5] and Nakazi [7] for the bitorus (cf. Seto [9]).

Suppose that $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s,t \geq 0$. Let V_{x_k} be the isometry induced by $\{S_k(s) : s \geq 0\}$ as in Theorem 2.2 for $k = 1,2$. Since V_{x_k} is in the von Neumann algebra generated by $\{S_k(s) : s \geq 0\}$, we have $V_{x_1}^* V_{x_2} = V_{x_2} V_{x_1}^*$. It suffices to consider the following two cases:

- V_{x_1} and V_{x_2} are completely non-unitary,
- V_{x_1} is completely non-unitary and V_{x_2} is unitary.

First, we suppose that V_{x_1} and V_{x_2} are completely non-unitary. Then

$$\mathcal{M} = \int_0^\infty T_{ds}^{(1)} \left\{ \int_0^\infty T_{dt}^{(2)} \left(\mathcal{M} \ominus (V_{x_1}\mathcal{M} + V_{x_2}\mathcal{M}) \right) \right\},$$

by Theorem 2.2. Let f be in $\mathcal{M} \ominus (V_{x_1}\mathcal{M} + V_{x_2}\mathcal{M})$ such that $\|f\| = 1$. Then

$$\int_{\mathbb{R}^2} |f(x_1,x_2)|^2 \frac{(x_1-i)^k}{(x+i)^k} \frac{(x_2-i)^l}{(x_2+i)^l} \, dx_1 dx_2 = 0,$$

for all $(k,l) \neq (0,0)$. Changing variables x_1 and x_2 to θ_1 and θ_2, we have

$$\int_0^{2\pi} \int_0^{2\pi} |f(\theta_1,\theta_2)|^2 e^{ik\theta_1} e^{il\theta_2} \frac{1}{(\cos^2 \frac{\theta_1}{2})(\cos^2 \frac{\theta_2}{2})} \, d\theta_1 d\theta_2 = 0.$$

Hence $|f(\theta_1,\theta_2)|^2 (\cos^2 \frac{\theta_1}{2})^{-1} (\cos^2 \frac{\theta_2}{2})^{-1} = 1$, equivalently $(x_1^2+1)(x_2^2+1)|f(x_1,x_2)|^2 = 1$. Therefore, there exists a unimodular function q such that

$$f = \frac{q}{(x_1+i)(x_2+i).}$$

Hence we have

$$\mathcal{M} \ominus (V_{x_1}\mathcal{M} + V_{x_2}\mathcal{M}) = \mathbb{C}\frac{q}{(x_1+i)(x_2+i).}$$

By the Paley-Wiener theorem,

$$\begin{aligned}
\mathcal{M} &= \int_0^\infty T_{ds}^{(1)} \left\{ \int_0^\infty T_{dt}^{(2)} \left(\mathbb{C} \frac{q}{(x_1+i)(x_2+i)} \right) \right\} \\
&= \left\{ \xi : \xi = q \int_0^\infty e^{isx_1} \, ds \int_0^\infty e^{itx_2} f(s,t) \, dt \; ; f \in L^2\left((0,\infty) \times (0,\infty)\right) \right\} \\
&= q \left(H^2(\mathbb{R}) \otimes H^2(\mathbb{R}) \right) \\
&= qH^2(\mathbb{R}^2).
\end{aligned}$$

Next, we suppose that V_{x_1} is completely non-unitary and V_{x_2} is unitary. Then

$$\mathcal{M} = \int_0^\infty T_{ds}^{(1)}(\mathcal{M} \ominus V_{x_1}\mathcal{M}),$$

by Theorem 2.2. Let f be in $\mathcal{M} \ominus V_{x_1}\mathcal{M}$. Then

$$\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \frac{(x_1 - i)^k}{(x_1 + i)^k} \frac{(x_2 - i)^l}{(x_2 + l)^l} \; dx_2 dx_1 = 0,$$

for all $k \neq 0$ and l. By the same calculations as in the first case, we have

$$f(x_1, x_2) = g(x_1, x_2)/(x_1 + i)$$

for some g such that the function $|g|$ depends only on the variable x_2.
The following argument is known (cf. [3]). Let $\chi_{E(g)}$ be the support function of g, that is, $\chi_{E(g)}$ is the characteristic function of the set $E(g) = \{(x_1, x_2) \in \mathbb{R}^2 : g(x_1, x_2) \neq 0\}$, and ϕ_g be a unimodular function defined as follows:

$$\phi_g = \begin{cases} g/|g| & (g \neq 0) \\ 1 & (g = 0). \end{cases}$$

Then

$$\bigvee_{t \in \mathbb{R}} e^{itx_2} \frac{g}{x_1 + i} = \frac{\phi_g}{x_1 + i} \chi_{E(g)} L^2(\mathbb{R}, dx_2),$$

where $\vee$ denotes the closed vector span. Since there exists a function F in $\mathcal{M} \ominus V_{x_1}\mathcal{M}$ which has the maximal support in $\mathcal{M} \ominus V_{x_1}\mathcal{M}$, that is, $E(g) \subseteq E(F)$, for any g in $\mathcal{M} \ominus V_{x_1}\mathcal{M}$, we have

$$\mathcal{M} \ominus V_{x_1}\mathcal{M} = \frac{\phi_F}{x_1 + i} \chi_{E(F)} L^2(\mathbb{R}, dx_2).$$

Let $\chi_G = \chi_{E(F)}$ and $\psi = \phi_F$. By the Paley-Wiener theorem, we have the following:

$$\begin{aligned}
\mathcal{M} &= \int_0^\infty T_{ds}^{(1)} \left(\frac{1}{x_1 + i} \chi_G \psi L^2(\mathbb{R}, dx_2) \right) \\
&= \left\{ \xi : \xi = \chi_G \psi \int_0^\infty e^{isx_1} f(s, x_2) \; ds \; ; f \in L^2 \left((0, \infty) \times \mathbb{R} \right) \right\} \\
&= \chi_G \psi H^2(\mathbb{R}, dx_1) \otimes L^2(\mathbb{R}, dx_2) \\
&= \chi_G \psi H^2_{x_2}(\mathbb{R}^2).
\end{aligned}$$

The converse is easy to verify.

A function q is said to be inner if q is in $H^2(\mathbb{R}^2)$ and $|q(x_1, x_2)| = 1$ a.e.

Corollary 2.1 *Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{R}^2)$. Then $S_1(s)S_2(t)^* = S_2(t)^* S_1(s)$ for all $s, t \geq 0$ if and only if $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some inner function q.*

3 Helson's theory under the double commuting condition in $L^2(\mathbb{R}^2)$

In this section, we discuss Helson's theory in $L^2(\mathbb{R}^2)$ under the double commuting condition: $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$. Then, it is parallel to Helson's argument for the one-variable case in [1].

Definition 3.1 Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. For any λ, μ in $\mathbb{R}$, we define one-parameter unitary groups $\{\alpha_\lambda\}$, $\{\beta_\mu\}$ and projections $\{P_\lambda\}$, $\{Q_\mu\}$ on $L^2(\mathbb{R}^2)$ as follows: for any f in $L^2(\mathbb{R}^2)$, $\alpha_\lambda f = e^{i\lambda x} f$, $\beta_\mu f = e^{i\mu y} f$, and $P_\lambda = \alpha_\lambda^* P_{\mathcal{M}} \alpha_\lambda$, $Q_\mu = \beta_\mu^* P_{\mathcal{M}} \beta_\mu$, that is, P_λ and Q_μ are the orthogonal projections of $L^2(\mathbb{R}^2)$ onto $\alpha_\lambda^* \mathcal{M}$ and $\beta_\mu^* \mathcal{M}$, respectively.

Lemma 3.1 Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ if and only if $P_{\mathcal{M}}\alpha_\lambda P_{\mathcal{M}}\beta_\mu P_{\mathcal{M}} = P_{\mathcal{M}}\beta_\mu P_{\mathcal{M}}\alpha_\lambda P_{\mathcal{M}}$ for all λ, μ in $\mathbb{R}$.

Proof It is easy to verify.

Definition 3.2 *A submodule $\mathcal{M}$ of $L^2(\mathbb{R}^2)$ is said to be simple if $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ and $\left(\bigcap_\lambda \alpha_\lambda \mathcal{M} + \bigcap_\mu \beta_\mu \mathcal{M}\right) = \{o\}$ (this is equivalent to that $P_{-\infty} = \lim_{\lambda \to -\infty} P_\lambda = O$ and $Q_{-\infty} = \lim_{\mu \to -\infty} Q_\mu = O$).*

Note that a submodule $\mathcal{M}$ is simple if and only if $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some unimodular function q by Theorem 2.2.

Next, we define two sequences of projections, and show that these are the spectral measures of $L^2(\mathbb{R}^2)$. Let E_λ and F_μ be projections defined as follows:

$$E_\lambda = \alpha_\lambda^* Q_{+\infty} \alpha_\lambda \text{ and } F_\mu = \beta_\mu^* P_{+\infty} \beta_\mu.$$

Lemma 3.2 *Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. If $\mathcal{M}$ is simple, then $\{E_\lambda\}$ and $\{F_\mu\}$ are spectral families. Moreover $E_\lambda F_\mu = F_\mu E_\lambda = \alpha_\lambda^* \beta_\mu^* P_{\mathcal{M}} \alpha_\lambda \beta_\mu$ for all λ, μ in $\mathbb{R}$.*

Proof Since, for $\gamma \geq \lambda, \mu$,

$$\begin{aligned}
E_\lambda F_\mu &= \alpha_\lambda^* Q_{+\infty} \alpha_\lambda \beta_\mu^* P_{+\infty} \beta_\mu \\
&= \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\gamma^* P_{\mathcal{M}} \beta_\gamma \alpha_\lambda \beta_\mu^* \alpha_\gamma^* P_{\mathcal{M}} \alpha_\gamma \beta_\mu\right) \\
&= \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\gamma^* P_{\mathcal{M}} \alpha_{\gamma-\alpha}^* P_{\mathcal{M}} \beta_{\gamma-\mu} P_{\mathcal{M}} \alpha_\gamma \beta_\mu\right) \\
&= \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\gamma^* P_{\mathcal{M}} \beta_{\gamma-\mu} P_{\mathcal{M}} \alpha_{\gamma-\lambda}^* P_{\mathcal{M}} \alpha_\gamma \beta_\mu\right) \\
&= \lim_{\gamma \to +\infty} \left(\alpha_\lambda^* \beta_\mu^* \beta_{\gamma-\mu}^* P_{\mathcal{M}} \beta_{\gamma-\mu} P_{\mathcal{M}} \alpha_{\gamma-\lambda}^* P_{\mathcal{M}} \alpha_{\gamma-\lambda} \alpha_\lambda \beta_\mu\right) \\
&= \alpha_\lambda^* \beta_\mu^* P_{\mathcal{M}} \alpha_\lambda \beta_\mu,
\end{aligned}$$

we have $E_\lambda F_\mu = \alpha_\lambda^* \beta_\mu^* P_{\mathcal{M}} \alpha_\lambda \beta_\mu = F_\mu E_\lambda$ for all λ, μ in $\mathbb{R}$.
Next, suppose that

$$\chi_G L^2(\mathbb{R}^2) = \overline{\bigcup_{\lambda,\mu} \alpha_\lambda \beta_\mu \mathcal{M}} \ominus \overline{\bigcap_\lambda \alpha_\lambda \bigcup_\mu \beta_\mu \mathcal{M} + \bigcap_\mu \beta_\mu \bigcup_\lambda \alpha_\lambda \mathcal{M}},$$

where the bar denotes the closure. We shall show $\chi_G = 1$. The following argument is the same as in [1]. Let $U_{s,0} = \int_{\mathbb{R}} e^{it\lambda}\, dE_\lambda$. Then, since $\alpha_{\lambda_0}\beta_{\mu_0} E_\lambda \alpha_{\lambda_0}^* \beta_{\mu_0}^* = E_{\lambda-\lambda_0}$, we have

$$\begin{aligned}
\alpha_{\lambda_0}\beta_{\mu_0} U_{s,0} &= \alpha_{\lambda_0}\beta_{\mu_0} \int e^{is\lambda}\, dE_\lambda \\
&= \int e^{is\lambda}\, dE_{\lambda-\lambda_0} \alpha_{\lambda_0}\beta_{\mu_0} \\
&= e^{is\lambda_0} \int e^{is(\lambda-\lambda_0)}\, dE_{\lambda-\lambda_0} \alpha_{\lambda_0}\beta_{\mu_0} \\
&= e^{is\lambda_0} U_{s,0} \alpha_{\lambda_0}\beta_{\mu_0}.
\end{aligned}$$

Therefore

$$\begin{aligned}
U_{s,0} T_{-s,0} \alpha_\lambda \beta_\mu &= U_{s,0} e^{is\lambda} \alpha_\lambda \beta_\mu T_{-s,0} \\
&= \alpha_\lambda \beta_\mu U_{s,0} T_{(-s,0)},
\end{aligned}$$

where $T_{s,t}$ is the translation operator such that $(T_{s,t} f)(x, y) = f(x - s, y - t)$. Hence $U_{s,0}T_{-s,0}$ is a multiplication operator on $L^2(\mathbb{R}^2)$. Since $U_{s,0}T_{-s,0}$ maps $T_{s,0}\chi_G L^2(\mathbb{R}^2)$ to $\chi_G L^2(\mathbb{R}^2)$, we have $T_{s,0}\chi_G L^2(\mathbb{R}^2) = \chi_G L^2(\mathbb{R}^2)$. By the same argument for β_μ, we have $T_{0,t}\chi_G L^2(\mathbb{R}^2) = \chi_G L^2(\mathbb{R}^2)$, that is, $T_{s,t}\chi_G L^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)$ for all s, t in $\mathbb{R}$. Hence G is a null set or $G = \mathbb{R}^2$, and we have

$$\begin{aligned}
\operatorname{ran}\left(\lim_{\lambda\to+\infty} E_\lambda\right) &= \operatorname{ran}\left(\lim_{\mu\to+\infty} F_\mu\right) = \overline{\bigcup_{\lambda,\mu} \alpha_\lambda\beta_\mu\mathcal{M}} = L^2(\mathbb{R}^2), \\
\operatorname{ran}\left(\lim_{\lambda\to-\infty} E_\lambda\right) &= \overline{\bigcap_\lambda \alpha_\lambda \bigcup_\mu \beta_\mu \mathcal{M}} = \{o\}, \\
\operatorname{ran}\left(\lim_{\mu\to-\infty} F_\mu\right) &= \overline{\bigcap_\mu \beta_\mu \bigcup_\lambda \alpha_\lambda \mathcal{M}} = \{o\}.
\end{aligned}$$

Therefore $\{E_\lambda\}$ and $\{F_\mu\}$ are the spectral families.

By virtue of Lemma 3.2, for any simple submodule of $L^2(\mathbb{R}^2)$, there exists a spectral measure $dE_{\lambda,\mu} = dE_\lambda dF_\mu$ on $\mathbb{R}^2$ and we have a two-parameter continuous unitary group $\{U_{s,t}\}$ on $L^2(\mathbb{R}^2)$ as follows:

$$U_{s,t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda+t\mu)}\, dE_\lambda dF_\mu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda+t\mu)}\, dE_{\lambda,\mu}.$$

Definition 3.3 A family $\{A_{s,t}\}$ of functions on $\mathbb{R}^2$ which are individually measurable is said to be a cocycle of $\mathbb{R}^2$ if

(i) $|A_{s,t}(x,y)| = 1$ almost everywhere in x, y, for each s, t,

(ii) $A_{s,t}f$ moves continuously in $L^2(\mathbb{R}^2)$ as s and t varies, for each f in $L^2(\mathbb{R}^2)$,

(iii) $A_{s+u,t+v} = A_{s,t}T_{s,t}A_{u,v}$ almost everywhere, for each s, t, u and v.

Example 3.1 (cf. [1]) In Lemma 5.3, we showed the following commutation relation:

$$U_{s,0}T_{-s,0}\alpha_\lambda\beta_\mu = \alpha_\lambda\beta_\mu U_{s,0}T_{-s,0}.$$

Using the same argument with respect to the variable x_2, we have

$$U_{s,t}T_{-s,-t}\alpha_\lambda\beta_\mu = \alpha_\lambda\beta_\mu U_{s,t}T_{-s,-t}.$$

Therefore $U_{s,t}T_{-s,-t}$ is the multiplication operator by some unimodular function $A_{s,t.}$ We shall show $\{A_{s,t}\}$ is a cocycle of $\mathbb{R}^2$. Identifying bounded functions with multiplication operators, we have

$$\begin{aligned} A_{s+u,t+v} &= U_{s+u,t+v}T_{-s-u,-t-v} \\ &= U_{s,t}U_{u,v}T_{-u,-v}T_{-s,-t} \\ &= U_{s,t}A_{u,v}T_{-s,-t} \\ &= A_{s,t}T_{s,t}A_{u,v}T_{-s,-t}. \end{aligned}$$

Hence

$$A_{s+u,t+v}(x,y) = A_{s,t}(x,y)A_{u,v}(x-s, y-t).$$

Proposition 3.1 *There exists a one-to-one correspondence between simple submodules of $L^2(\mathbb{R}^2)$ and cocycles of $\mathbb{R}^2$.*

Proof Suppose that $\{A_{s,t}\}$ is a cocycle of $\mathbb{R}^2$. Let $U_{s,t} = A_{s,t}T_{s,t}$. Then $\{U_{s,t}\}$ is a two-parameter unitary group on $L^2(\mathbb{R}^2)$. By Stone's theorem for $\mathbb{R}^2$, there exists a unique spectral measure of $L^2(\mathbb{R}^2)$ such that

$$U_{s,t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda+t\mu)}\, dE_{\lambda,\mu}.$$

Let $\mathcal{M} = \text{ran } E_{0,0,}$. Then

$$\begin{aligned} \int_{\mathbb{R}^2} e^{i(s\lambda+t\mu)}\, dE_{\lambda+\tau_1,\mu+\tau_2} &= e^{-i(s\tau_1+t\tau_2)}\int_{\mathbb{R}^2} e^{i(s(\lambda+\tau_1)+t(\mu+\tau_2))}\, dE_{\lambda+\tau_1,\mu+\tau_2} \\ &= e^{-i(s\tau_1+t\tau_2)}\int_{\mathbb{R}^2} e^{i(s\lambda+t\mu)}\, dE_{\lambda,\mu} \\ &= \alpha^*_{\tau_1}\beta^*_{\tau_2}U_{s,t}\alpha_{\tau_1}\beta_{\tau_2} \\ &= \int_{\mathbb{R}^2} e^{i(s\lambda+t\mu)}\, d\left(\alpha^*_{\tau_1}\beta^*_{\tau_2}E_{\lambda,\mu}\alpha_{\tau_1}\beta_{\tau_2}\right) \end{aligned}$$

Hence we have

$$E_{\lambda+\tau_1,\mu+\tau_2} = \alpha^*_{\tau_1}\beta^*_{\tau_2}E_{\lambda,\mu}\alpha_{\tau_1}\beta_{\tau_2}.$$

Therefore $\mathcal{M}$ is a submodule of $L^2(\mathbb{R}^2)$.
Next, we shall show that $\mathcal{M}$ satisfies the double commuting condition. It suffices to consider the case where $\lambda \geq 0$ and $\mu \leq 0$.

$$\begin{aligned} P_{\mathcal{M}}\alpha_\lambda P_{\mathcal{M}}\beta_\mu P_{\mathcal{M}} &= E_{0,0}\alpha_\lambda E_{0,0}\beta_\mu E_{0,0} \\ &= \alpha_\lambda E_{\lambda,0}E_{0,0}E_{0,-\mu}\beta_\mu \\ &= \alpha_\lambda E_{0,0}\beta_\mu, \end{aligned}$$

and

$$\begin{aligned} P_{\mathcal{M}}\beta_\mu P_{\mathcal{M}}\alpha_\lambda P_{\mathcal{M}} &= E_{0,0}\beta_\mu E_{0,0}\alpha_\lambda E_{0,0} \\ &= E_{0,0}E_{0,-\mu}\beta_\mu\alpha_\lambda E_{0,0} \\ &= E_{0,0}\alpha_\lambda\beta_\mu E_{0,0} \\ &= \alpha_\lambda E_{\lambda,0}E_{0,-\mu}\beta_\mu \\ &= \alpha_\lambda E_{0,0}\beta_\mu. \end{aligned}$$

Therefore $P_{\mathcal{M}}\alpha_\lambda P_{\mathcal{M}}\beta_\mu P_{\mathcal{M}} = P_{\mathcal{M}}\beta_\mu P_{\mathcal{M}}\alpha_\lambda P_{\mathcal{M}}$. This concludes the proof by Lemma 3.1.

Example 3.2 (cf. [1]) Suppose that $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some unimodular function q. Then its cocycle is $\{qT_{s,t}q^{-1}\}$.

A cocycle of the form $A_{s,t} = qT_{s,t}q^{-1}$, for some unimodular function, is called a coboundary of $\mathbb{R}^2$.

Corollary 3.1 *Every cocycle of $\mathbb{R}^2$ is a coboundary of $\mathbb{R}^2$.*

Proof By Theorem 2.1, for any simple submodule $\mathcal{M}$ of $L^2(\mathbb{R}^2)$, there is a unimodular function q such that $\mathcal{M} = qH^2(\mathbb{R}^2)$. Hence the cocycle of $\mathcal{M}$ is a coboundary.

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