Submodules of $L^2(\mathbb{R}^2)$

MICHIO SETO

Abstract

In this paper, we study submodules over $\mathbb{R}^2$. We will give a Lax-type of theorem and a result analogous to Helson’s theory.


Keywords and phrases: Hardy submodules.

1 Introduction

$L^2(\mathbb{R}^2)$ will denote the Hilbert space of square-integrable measurable functions with respect to the usual Lebesgue measure $dx_1dx_2$ on the two dimensional Euclidean space $\mathbb{R}^2$. $H^2(\mathbb{R})$ denotes the usual Hardy space on $\mathbb{R}$, that is, $H^2(\mathbb{R})$ consists of all functions in $L^2(\mathbb{R})$ which can be extended analytically to the upper half plane $\mathbb{C}_+ = \{ x + it : x \in \mathbb{R}, t > 0 \}$. $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$, the Hilbert space tensor product of $H^2(\mathbb{R})$, is the space of all $f$ in $L^2(\mathbb{R}^2)$ whose Fourier transform

$$\hat{f}(\lambda_1, \lambda_2) = \hat{f}(\lambda_1, \lambda_2) = \int_{\mathbb{R}^2} f(x_1, x_2)e^{-i(\lambda_1 x_1 + \lambda_2 x_2)} dx_1dx_2$$

is 0 whenever at least one component of $(\lambda_1, \lambda_2)$ is negative, where $(\lambda_1, \lambda_2)$ and $(x_1, x_2)$ are in $\mathbb{R}^2$. In this paper, $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ is denoted by $H^2(\mathbb{R}^2)$, for short. Note that our $H^2(\mathbb{R}^2)$ is different from the usual Hardy space on $\mathbb{R}^2$.

Definition 1.1 A closed subspace $\mathcal{M}$ of $L^2(\mathbb{R}^2)$ is said to be a submodule of $L^2(\mathbb{R}^2)$ if $e^{is_j} \mathcal{M} \subseteq \mathcal{M}$ for any $j = 1, 2$ and any $s \geq 0$. For $s \geq 0$, $S_j(s)$ denotes the restriction on $\mathcal{M}$ of the multiplication operator on $L^2(\mathbb{R}^2)$ by $e^{is_j}$.

Submodules in one variable were completely described by Lax in [4]. In [1], Helson gave another point of view to the result of Lax. The purpose of our study is to consider Helson’s theory in the multi-variable setting. My interest in considering Helson’s theory in two variables is motivated by the study of Hardy submodules over the bidisk: Hardy submodules are invariant subspaces of Hardy space under multiplication operators by
bounded analytic functions. However, it is easy to see that a straightforward generalization of Helson’s theory fails in the multi-variable setting. In Section 2 of this paper, we give a Lax-type theorem in two variables. To prove this we use Masani’s integral (cf. [6]). In Section 3, we consider Helson’s theory in two variables. We will give a result analogous to Helson’s result under the following condition: $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$.

2 A Lax-type of theorem in $\mathbb{R}^2$

In [9], the author showed the following Lax-type of theorem analogous to the theorem proved by Mandrekar [5] and Nakazi [7] for the bitorus.

Theorem 2.1 Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$, $H^2_{x_1}(\mathbb{R}^2) = \mathcal{L}^2(\mathbb{R}, dx_1) \otimes H^2(\mathbb{R}, dx_2)$ and $H^2_{x_2}(\mathbb{R}^2) = H^2(\mathbb{R}, dx_2) \otimes \mathcal{L}^2(\mathbb{R}, dx_1)$. If $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$, then one and only one of the following occurs:

(i) $\mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_F \varphi H^2_{x_1}(\mathbb{R}^2)$,

(ii) $\mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_G \psi H^2_{x_2}(\mathbb{R}^2)$,

(iii) $\mathcal{M} = q H^2(\mathbb{R}^2)$,

where $\varphi, \psi$ and $q$ are unimodular functions, $\chi_E$ is the characteristic function of $E$, $\chi_F$ (resp. $\chi_G$) is the characteristic function of $F$ (resp. $G$) which depends only on the variable $x_1$ (resp. $x_2$).

We shall give a proof which differs from that given in [9]. To begin with, we briefly introduce Masani’s integral which can be seen as a continuous Wold decomposition for a continuous semi-group of isometries, according to [6].

Definition 2.1 (Masani [6]) Let $\{S(t) : t \geq 0\}$ be a strongly continuous semi-group of isometries on a Hilbert space $\mathcal{H}$. We introduce an operator-valued interval-measure. The measure $T_{ab}$ of the interval $[a, b]$ is defined by as follows:

$$T_{[a, b]} = T(b) - T(a),$$

where $T(t) = \frac{1}{\sqrt{2}} \left\{ S(t) - I - \int_0^t S(s) \, ds \right\}$, for $t \geq 0$.

Let $iH$ be the infinitesimal generator of $\{S(t) : t \geq 0\}$ and $V$ be the Cayley transform of $H$ and $R = V(\mathcal{H})$. For the step-function $x = \sum_{k=1}^n \alpha_k \chi_{J_k}$ on $[a, b]$, where $\alpha_k$ in $R^+$ and $\chi_{J_k}$ is the characteristic function of bounded interval $J_k$, we define

$$\int_a^b T_{\delta t}(x) := \sum_{k=1}^n T_{J_k}(\alpha_k).$$
For any \( x \) in \( L^2([a, b], \mathbb{R}^2) \), we define
\[
\int_a^b T_d(t) := \lim_{n \to \infty} \int_a^b T_d(x^{[n]}_t),
\]
where \( \{x^{[n]}_t, n \geq 1\} \) is any sequence of step-functions which is tending to \( x \) in the \( L^2 \)-topology.

We now define a direct integral as a set of vector-valued integrals:
\[
\int_a^b T_d(R^k) := \left\{ \xi : \xi = \int_a^b T_d(x_t), x \in L^2([a, b], \mathbb{R}^k) \right\}.
\]

**Theorem 2.2 (Masani [6])** Let \( \{S(t) : t \geq 0\} \) be a strongly continuous semi-group of isometries on a Hilbert space \( \mathcal{H} \), \( iH \) be its infinitesimal generator and let \( V \) be the Cayley transform of \( H \). Then, for \( a \geq 0 \),
\[
S(a)(\mathcal{H}) = \int_a^\infty T_d(R^k) \oplus \mathcal{H}_\infty,
\]
where \( R = V(\mathcal{H}) \) and \( \mathcal{H}_\infty = \bigcap_{t \geq 0} S(t)(\mathcal{H}) \).

This theorem can be seen as a continuous Wold decomposition.

**Example 2.1** Let \( T^{(k)}_{d_s} \) be the operator-valued measures defined by \( S_k(s) \) for \( k = 1, 2 \). Identifying bounded functions with multiplication operators, \( T^{(k)}(s) \) can be computed formally as follows:
\[
T^{(k)}(s) = \frac{1}{\sqrt{2}} \left\{ S_k(s) - \int_0^s S_k(t) \, dt \right\}
\]
\[
= \frac{1}{\sqrt{2}} \left\{ e^{ix_k} - 1 - \int_0^s e^{itx_k} \, dt \right\}
\]
\[
= \frac{1}{\sqrt{2}} \left\{ e^{ix_k} - 1 - \left[ \frac{1}{i x_k} \right]^{s} \right\}
\]
\[
= \frac{1}{\sqrt{2}} \left\{ e^{ix_k} - 1 - \frac{1}{ix_k} (e^{ix_k} - 1) \right\}
\]
\[
= \frac{1}{\sqrt{2}} (e^{ix_k} - 1) \left( 1 - \frac{1}{ix_k} \right)
\]
\[
= \frac{1}{\sqrt{2}} x_k (e^{ix_k} - 1)(x_k + i).
\]

Thus the operator valued measure \( T^{(k)}_{d_s} \) can be computed as follows:
\[
T^{(k)}_{d_s} = \frac{d}{ds} \left( \frac{1}{\sqrt{2} x_k} (e^{ix_k} - 1)(x_k + i) \right) \, ds
\]
\[
= \frac{1}{\sqrt{2}} i e^{ix_k}(x_k + i) \, ds.
\]
We are now in a position to prove Theorem 2.1.

**Proof (A proof of Theorem 2.1)** Some parts of this proof are similar to those in the proof by Mandrekar [5] and Nakazi [7] for the bitorus (cf. Seto [9]).

Suppose that \( S_1(s)S_2(t)^* = S_2(t)^*S_1(s) \) for all \( s, t \geq 0 \). Let \( V_{x_k} \) be the isometry induced by \( \{ S_k(s) : s \geq 0 \} \) as in Theorem 2.2 for \( k = 1, 2 \). Since \( V_{x_k} \) is in the von Neumann algebra generated by \( \{ S_k(s) : s \geq 0 \} \), we have \( V_{x_1}^*V_{x_2} = V_{x_2}V_{x_1}^* \). It suffices to consider the following two cases:

- \( V_{x_1} \) and \( V_{x_2} \) are completely non-unitary,
- \( V_{x_1} \) is completely non-unitary and \( V_{x_2} \) is unitary.

First, we suppose that \( V_{x_1} \) and \( V_{x_2} \) are completely non-unitary. Then

\[
\mathcal{M} = \int_0^\infty T_{ds}^{(1)} \left\{ \int_0^\infty T_{dt}^{(2)} (\mathcal{M} \ominus (V_{x_1} \mathcal{M} + V_{x_2} \mathcal{M})) \right\},
\]

by Theorem 2.2. Let \( f \) be in \( \mathcal{M} \ominus (V_{x_1} \mathcal{M} + V_{x_2} \mathcal{M}) \) such that \( \|f\| = 1 \). Then

\[
\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \frac{(x_1 - i)^k}{(x_1 + i)^k} \frac{(x_2 - i)^l}{(x_2 + i)^l} \, dx_1 \, dx_2 = 0,
\]

for all \((k, l) \neq (0, 0)\). Changing variables \( x_1 \) and \( x_2 \) to \( \theta_1 \) and \( \theta_2 \), we have

\[
\int_0^{2\pi} \int_0^{2\pi} |f(\theta_1, \theta_2)|^2 e^{ik\theta_1} e^{il\theta_2} \frac{1}{(\cos^2 \frac{\theta_1}{2})(\cos^2 \frac{\theta_2}{2})} \, d\theta_1 \, d\theta_2 = 0.
\]

Hence \( |f(\theta_1, \theta_2)|^2 (\cos^2 \frac{\theta_1}{2})^{-1} (\cos^2 \frac{\theta_2}{2})^{-1} = 1 \), equivalently \((x_1^2 + 1)(x_2^2 + 1)|f(x_1, x_2)|^2 = 1\). Therefore, there exists a unimodular function \( q \) such that

\[
f = \frac{q}{(x_1 + i)(x_2 + i)}.
\]

Hence we have

\[
\mathcal{M} \ominus (V_{x_1} \mathcal{M} + V_{x_2} \mathcal{M}) = C \frac{q}{(x_1 + i)(x_2 + i)}.
\]

By the Paley-Wiener theorem,

\[
\mathcal{M} = \int_0^\infty T_{ds}^{(1)} \left\{ \int_0^\infty T_{dt}^{(2)} \left( \frac{C}{(x_1 + i)(x_2 + i)} \right) \right\} = \left\{ \xi : \xi = q \int_0^\infty e^{i\alpha x_1} ds \int_0^\infty e^{i\beta x_2} f(s, t) \, dt ; f \in L^2 \left( (0, \infty) \times (0, \infty) \right) \right\} = q \left( H^2(\mathbb{R}) \otimes H^2(\mathbb{R}) \right) = qH^2(\mathbb{R}^2).
\]
Next, we suppose that $V_{x_1}$ is completely non-unitary and $V_{x_2}$ is unitary. Then
\[ \mathcal{M} = \int_0^\infty T_{ds}^{(1)}(\mathcal{M} \ominus V_{x_1}, \mathcal{M}), \]
by Theorem 2.2. Let $f$ be in $\mathcal{M} \ominus V_{x_1}, \mathcal{M}$. Then
\[ \int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \frac{(x_1 - i)^k(x_2 - i)^l}{(x_1 + i)^k(x_2 + l)^l} \, dx_2 \, dx_1 = 0, \]
for all $k \neq 0$ and $l$. By the same calculations as in the first case, we have
\[ f(x_1, x_2) = g(x_1, x_2)/(x_1 + i) \]
for some $g$ such that the function $|g|$ depends only on the variable $x_2$.

The following argument is known (cf. [3]). Let $\chi_{E[g]}$ be the support function of $g$, that is, $\chi_{E[g]}$ is the characteristic function of the set $E(g) = \{(x_1, x_2) \in \mathbb{R}^2 : g(x_1, x_2) \neq 0\}$, and $\phi_g$ be a unimodular function defined as follows:
\[ \phi_g = \begin{cases} g/|g| & (g \neq 0) \\ 1 & (g = 0). \end{cases} \]
Then
\[ \bigvee_{i \in \mathbb{R}} e^{ix_2} \frac{g}{x_1 + i} = \frac{\phi_g}{x_1 + i} \chi_{E[g]} L^2(\mathbb{R}, dx_2), \]
where $\bigvee$ denotes the closed vector span. Since there exists a function $F$ in $\mathcal{M} \ominus V_{x_1}, \mathcal{M}$ which has the maximal support in $\mathcal{M} \ominus V_{x_1}, \mathcal{M}$, that is, $E(g) \subseteq E(F)$, for any $g$ in $\mathcal{M} \ominus V_{x_1}, \mathcal{M}$, we have
\[ \mathcal{M} \ominus V_{x_1}, \mathcal{M} = \frac{\phi_F}{x_1 + i} \chi_{E[F]} L^2(\mathbb{R}, dx_2). \]

Let $\chi_G = \chi_{E(F)}$ and $\psi = \phi_F$. By the Paley-Wiener theorem, we have the following:
\[ \mathcal{M} = \int_0^\infty T_{ds}^{(1)} \left( \frac{1}{x_1 + i} \chi_G \psi L^2(\mathbb{R}, dx_2) \right) \]
\[ = \left\{ \xi : \xi = \chi_G \psi \int_0^\infty e^{ix_2} f(s, x_2) \, ds ; f \in L^2((0, \infty) \times \mathbb{R}) \right\} \]
\[ = \chi_G \psi H^2(\mathbb{R}, dx_1 \otimes L^2(\mathbb{R}, dx_2) \]
\[ = \chi_G \psi H^2_{x_2}(\mathbb{R}^2). \]

The converse is easy to verify.

A function $q$ is said to be inner if $q$ is in $H^2(\mathbb{R}^2)$ and $|q(x_1, x_2)| = 1$ a.e.

**Corollary 2.1** Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{R}^2)$. Then $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ if and only if $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some inner function $q$. 

5
3 Helson’s theory under the double commuting condition in $L^2(\mathbb{R}^2)$

In this section, we discuss Helson’s theory in $L^2(\mathbb{R}^2)$ under the double commuting condition: $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$. Then, it is parallel to Helson’s argument for the one-variable case in [1].

**Definition 3.1** Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. For any $\lambda, \mu$ in $\mathbb{R}$, we define one-parameter unitary groups $\{\alpha_{\lambda}\}$, $\{\beta_{\mu}\}$ and projections $\{P_{\lambda}\}$, $\{Q_{\mu}\}$ on $L^2(\mathbb{R}^2)$ as follows: for any $f$ in $L^2(\mathbb{R}^2)$, $\alpha_{\lambda}f = e^{i\lambda x}f$, $\beta_{\mu}f = e^{i\mu y}f$, and $P_{\lambda} = \alpha_{\lambda}^*P_{\mathcal{M}}\alpha_{\lambda}$, $Q_{\mu} = \beta_{\mu}^*P_{\mathcal{M}}\beta_{\mu}$, that is, $P_{\lambda}$ and $Q_{\mu}$ are the orthogonal projections of $L^2(\mathbb{R}^2)$ onto $\alpha_{\lambda}^*\mathcal{M}$ and $\beta_{\mu}^*\mathcal{M}$, respectively.

**Lemma 3.1** Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ if and only if $P_{\mathcal{M}}\alpha_{\lambda}P_{\mathcal{M}}\beta_{\mu}P_{\mathcal{M}} = P_{\mathcal{M}}\beta_{\mu}P_{\mathcal{M}}\alpha_{\lambda}P_{\mathcal{M}}$ for all $\lambda, \mu$ in $\mathbb{R}$.

**Proof** It is easy to verify.

**Definition 3.2** A submodule $\mathcal{M}$ of $L^2(\mathbb{R}^2)$ is said to be simple if $S_1(s)S_2(t)^* = S_2(t)^*S_1(s)$ for all $s, t \geq 0$ and $(\cap_\lambda \alpha_{\lambda}\mathcal{M} + \cap_\mu \beta_{\mu}\mathcal{M}) = \{0\}$ (this is equivalent to that $P_{-\infty} = \lim_{\lambda \to -\infty} P_{\lambda} = 0$ and $Q_{-\infty} = \lim_{\mu \to -\infty} Q_{\mu} = 0$).

Note that a submodule $\mathcal{M}$ is simple if and only if $\mathcal{M} = qH^2(\mathbb{R}^2)$ for some unimodular function $q$ by Theorem 2.2.

Next, we define two sequences of projections, and show that these are the spectral measures of $L^2(\mathbb{R}^2)$. Let $E_{\lambda}$ and $F_{\mu}$ be projections defined as follows:

$$E_{\lambda} = \alpha_{\lambda}^*Q_{\infty} \alpha_{\lambda} \text{ and } F_{\mu} = \beta_{\mu}^*P_{\infty} \beta_{\mu}. $$

**Lemma 3.2** Let $\mathcal{M}$ be a submodule of $L^2(\mathbb{R}^2)$. If $\mathcal{M}$ is simple, then $\{E_{\lambda}\}$ and $\{F_{\mu}\}$ are spectral families. Moreover $E_{\lambda}F_{\mu} = F_{\mu}E_{\lambda} = \alpha_{\lambda}^*\beta_{\mu}^*P_{\mathcal{M}}\alpha_{\lambda}\beta_{\mu}$ for all $\lambda, \mu$ in $\mathbb{R}$.

**Proof** Since, for $\gamma \geq \lambda, \mu$,

$$E_{\lambda}F_{\mu} = \alpha_{\lambda}^*Q_{\infty} \alpha_{\lambda}\beta_{\mu}^*P_{\infty} \beta_{\mu} = \lim_{\gamma \to +\infty} (\alpha_{\lambda}^*\beta_{\gamma}^*P_{\mathcal{M}}\beta_{\gamma}\alpha_{\lambda}\beta_{\mu}^*\alpha_{\gamma}^*P_{\mathcal{M}}\alpha_{\gamma}\beta_{\mu})$$

$$= \lim_{\gamma \to +\infty} (\alpha_{\lambda}^*\beta_{\gamma}^*P_{\mathcal{M}}\alpha_{\gamma-\lambda}\alpha_{\lambda}\beta_{\mu}^*P_{\mathcal{M}}\alpha_{\gamma}\beta_{\mu})$$

$$= \lim_{\gamma \to +\infty} (\alpha_{\lambda}^*\beta_{\gamma}^*P_{\mathcal{M}}\alpha_{\gamma-\lambda}\alpha_{\lambda}\beta_{\mu}^*P_{\mathcal{M}}\alpha_{\gamma-\lambda}\alpha_{\lambda}\beta_{\mu})$$

$$= \alpha_{\lambda}^*\beta_{\mu}^*P_{\mathcal{M}}\alpha_{\lambda}\beta_{\mu}.$$
we have \( E_\lambda F_\mu = \alpha_\lambda^* \beta_\mu^* P_\lambda \alpha_\lambda \beta_\mu = F_\mu E_\lambda \) for all \( \lambda, \mu \) in \( \mathbb{R} \).

Next, suppose that
\[
\chi_G L^2(\mathbb{R}^2) = \bigcup_{\lambda, \mu} \alpha_\lambda \beta_\mu^* M \sqcup \bigcap_{\lambda} \alpha_\lambda \bigcup_{\mu} \beta_\mu M + \bigcap_{\mu} \beta_\mu \bigcup_{\lambda} \alpha_\lambda M,
\]
where the bar denotes the closure. We shall show \( \chi_G = 1 \). The following argument is the same as in [1]. Let \( U_{s,0} = \int_{\mathbb{R}} e^{is\lambda} \, dE_\lambda \). Then, since \( \alpha_{\lambda_0} \beta_{\mu_0} E_\lambda \alpha_{\lambda_0}^* \beta_{\mu_0}^* = E_{\lambda - \lambda_0} \), we have
\[
\alpha_{\lambda_0} \beta_{\mu_0} U_{s,0} = \alpha_{\lambda_0}^* \beta_{\mu_0} \int e^{is\lambda} \, dE_\lambda
= \int e^{is\lambda} \, dE_{\lambda - \lambda_0} \alpha_{\lambda_0} \beta_{\mu_0}
= e^{is\lambda_0} \int e^{is(\lambda - \lambda_0)} \, dE_{\lambda - \lambda_0} \alpha_{\lambda_0} \beta_{\mu_0}
= e^{is\lambda_0} U_{s,0} \alpha_{\lambda_0} \beta_{\mu_0}^*.
\]

Therefore
\[
U_{s,0} T_{-s,0} \alpha_{\lambda} \beta_{\mu} = U_{s,0} e^{is\lambda} \alpha_{\lambda} \beta_{\mu} T_{-s,0}
= \alpha_{\lambda} \beta_{\mu} U_{s,0} T_{-s,0},
\]
where \( T_{s,t} \) is the translation operator such that \( (T_{s,t}f)(x, y) = f(x - s, y - t) \). Hence \( U_{s,0} T_{-s,0} \) is a multiplication operator on \( L^2(\mathbb{R}^2) \). Since \( U_{s,0} T_{-s,0} \) maps \( T_{s,0} \chi_G L^2(\mathbb{R}^2) \) to \( \chi_G L^2(\mathbb{R}^2) \), we have \( T_{s,0} \chi_G L^2(\mathbb{R}^2) = \chi_G L^2(\mathbb{R}^2) \). By the same argument for \( \beta_\mu \), we have \( T_{0,t} \chi_G L^2(\mathbb{R}^2) = \chi_G L^2(\mathbb{R}^2) \), that is, \( T_{s,t} \chi_G L^2(\mathbb{R}^2) = L^2(\mathbb{R}^2) \) for all \( s, t \) in \( \mathbb{R} \). Hence \( G \) is a null set or \( G = \mathbb{R}^2 \), and we have
\[
\text{ran} \left( \lim_{\lambda \to +\infty} E_\lambda \right) = \text{ran} \left( \lim_{\mu \to +\infty} F_\mu \right) = \bigcup_{\lambda, \mu} \alpha_\lambda \beta_\mu^* M = L^2(\mathbb{R}^2),
\]
\[
\text{ran} \left( \lim_{\lambda \to -\infty} E_\lambda \right) = \bigcap_{\lambda} \alpha_\lambda \bigcup_{\mu} \beta_\mu M = \{ 0 \},
\]
\[
\text{ran} \left( \lim_{\mu \to -\infty} F_\mu \right) = \bigcap_{\mu} \beta_\mu \bigcup_{\lambda} \alpha_\lambda M = \{ 0 \}.
\]

Therefore \( \{ E_\lambda \} \) and \( \{ F_\mu \} \) are the spectral families.

By virtue of Lemma 3.2, for any simple submodule of \( L^2(\mathbb{R}^2) \), there exists a spectral measure \( dE_{\lambda,\mu} = dE_{\lambda} dF_\mu \) on \( \mathbb{R}^2 \) and we have a two-parameter continuous unitary group \( \{ U_{s,t} \} \) on \( L^2(\mathbb{R}^2) \) as follows:
\[
U_{s,t} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s\lambda + t\mu)} \, dE_{\lambda} dF_\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s\lambda + t\mu)} \, dE_{\lambda,\mu}.
\]
**Definition 3.3** A family \( \{A_{s,t}\} \) of functions on \( \mathbb{R}^2 \) which are individually measurable is said to be a cocycle of \( \mathbb{R}^2 \) if

(i) \( |A_{s,t}(x, y)| = 1 \) almost everywhere in \( x, y \), for each \( s, t \),

(ii) \( A_{s,t}f \) moves continuously in \( L^2(\mathbb{R}^2) \) as \( s \) and \( t \) varies, for each \( f \) in \( L^2(\mathbb{R}^2) \),

(iii) \( A_{s+u,t+v} = A_{s,t}A_{u,v} \) almost everywhere, for each \( s, t, u \) and \( v \).

**Example 3.1** (cf. [1]) In Lemma 5.3, we showed the following commutation relation:

\[
U_{s,0}T_{-s,0} \alpha_{\lambda} \beta_{\mu} = \alpha_{\lambda} \beta_{\mu} U_{s,0}T_{-s,0}.
\]

Using the same argument with respect to the variable \( x_2 \), we have

\[
U_{s,t}T_{-s,-t} \alpha_{\lambda} \beta_{\mu} = \alpha_{\lambda} \beta_{\mu} U_{s,t}T_{-s,-t}.
\]

Therefore \( U_{s,t}T_{-s,-t} \) is the multiplication operator by some unimodular function \( A_{s,t} \). We shall show \( \{A_{s,t}\} \) is a cocycle of \( \mathbb{R}^2 \). Identifying bounded functions with multiplication operators, we have

\[
A_{s+u,t+v} = U_{s+u,t+v}T_{-s-u,-t-v} = U_{s,t}U_{u,v}T_{-u,-v}T_{-s,-t} = U_{s,t}A_{u,v}T_{-s,-t} = A_{s,t}A_{u,v}T_{-s,-t}.
\]

Hence

\[
A_{s+u,t+v}(x, y) = A_{s,t}(x, y)A_{u,v}(x - s, y - t).
\]

**Proposition 3.1** There exists a one-to-one correspondence between simple submodules of \( L^2(\mathbb{R}^2) \) and cocycles of \( \mathbb{R}^2 \).

**Proof** Suppose that \( \{A_{s,t}\} \) is a cocycle of \( \mathbb{R}^2 \). Let \( U_{s,t} = A_{s,t}T_{s,t} \). Then \( \{U_{s,t}\} \) is a two-parameter unitary group on \( L^2(\mathbb{R}^2) \). By Stone’s theorem for \( \mathbb{R}^2 \), there exists a unique spectral measure of \( L^2(\mathbb{R}^2) \) such that

\[
U_{s,t} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x\lambda + y\mu)} \, dE_{\lambda,\mu}.
\]

Let \( \mathcal{M} = \text{ran} \, E_{0,0} \). Then

\[
\int_{\mathbb{R}^2} e^{i(x\lambda + y\mu)} \, dE_{\lambda+\tau_1,\mu+\tau_2} = e^{-i(x\tau_1 + y\tau_2)} \int_{\mathbb{R}^2} e^{i(x(\lambda + \tau_1) + y(\mu + \tau_2))} \, dE_{\lambda+\tau_1,\mu+\tau_2} = e^{-i(x\tau_1 + y\tau_2)} \int_{\mathbb{R}^2} e^{i(x\lambda + y\mu)} \, dE_{\lambda,\mu} = \alpha_{\tau_1}^* \beta_{\tau_2}^* U_{s,t} \alpha_{\tau_1} \beta_{\tau_2} = \int_{\mathbb{R}^2} e^{i(x\lambda + y\mu)} \, d\left(\alpha_{\tau_1}^* \beta_{\tau_2}^* E_{\lambda,\mu} \alpha_{\tau_1} \beta_{\tau_2}\right).
\]

\[
8
\]
Hence we have

\[ E_{\lambda+\tau_1, \mu+\tau_2} = \alpha_{\tau_1}^{\ast} \beta_{\tau_2}^{\ast} E_{\lambda, \mu} \alpha_{\tau_1} \beta_{\tau_2}. \]

Therefore \( \mathcal{M} \) is a submodule of \( L^2(\mathbb{R}^2) \).

Next, we shall show that \( \mathcal{M} \) satisfies the double commuting condition. It suffices to consider the case where \( \lambda \geq 0 \) and \( \mu \leq 0 \).

\[
P_{\mathcal{M}\alpha_{\lambda}} P_{\mathcal{M}\beta_{\mu}} P_{\mathcal{M}} &= E_{0,0} \alpha_{\lambda} E_{0,0} \beta_{\mu} E_{0,0} \\
&= \alpha_{\lambda} E_{0,0} E_{0,0} \beta_{\mu} \\
&= \alpha_{\lambda} E_{0,0} \beta_{\mu},
\]

and

\[
P_{\mathcal{M}\beta_{\mu}} P_{\mathcal{M}\alpha_{\lambda}} P_{\mathcal{M}} &= E_{0,0} \beta_{\mu} E_{0,0} \alpha_{\lambda} E_{0,0} \\
&= E_{0,0} E_{0,0} \beta_{\mu} \alpha_{\lambda} E_{0,0} \\
&= E_{0,0} \alpha_{\lambda} \beta_{\mu} E_{0,0} \\
&= \alpha_{\lambda} E_{0,0} E_{0,0} \beta_{\mu} \\
&= \alpha_{\lambda} E_{0,0} \beta_{\mu}.
\]

Therefore \( P_{\mathcal{M}\alpha_{\lambda}} P_{\mathcal{M}\beta_{\mu}} P_{\mathcal{M}} = P_{\mathcal{M}\beta_{\mu}} P_{\mathcal{M}\alpha_{\lambda}} P_{\mathcal{M}} \). This concludes the proof by Lemma 3.1.

**Example 3.2 (cf. [1])** Suppose that \( \mathcal{M} = qH^2(\mathbb{R}^2) \) for some unimodular function \( q \). Then its cocycle is \( \{qT_s q^{-1}\} \).

A cocycle of the form \( A_{s,t} = qT_{s,t} q^{-1} \), for some unimodular function, is called a coboundary of \( \mathbb{R}^2 \).

**Corollary 3.1** Every cocycle of \( \mathbb{R}^2 \) is a coboundary of \( \mathbb{R}^2 \).

**Proof** By Theorem 2.1, for any simple submodule \( \mathcal{M} \) of \( L^2(\mathbb{R}^2) \), there is a unimodular function \( q \) such that \( \mathcal{M} = qH^2(\mathbb{R}^2) \). Hence the cocycle of \( \mathcal{M} \) is a coboundary.

**References**


Department of Mathematics  
Faculty of Science  
Hokkaido University  
Sapporo 060-0810  
Japan  
e-mail: seto@math.sci.hokudai.ac.jp